# Preservation of Craig interpolation by the product of matrix logics

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**Abstract**

The product of matrix logics, possibly with additional interaction axioms, is shown to preserve a slightly relaxed notion of Craig interpolation. The result is established symbolically, capitalizing on the complete axiomatization of the product of matrix logics provided by their meet-combination. Along the way preservation of the metatheorem of deduction is also proved. The computation of the interpolant in the resulting logic is proved to be polynomially reducible to the computation of the interpolants in the two given logics. Illustrations are provided for classical, intuitionistic and modal propositional logics.

**Keywords**: interpolation, matrix semantics, matrix-logic product.

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## 1 Introduction

After the seminal paper [5] by Craig, interpolation has been investigated in many logics and variants, with applications in definability and automated reasoning. More recently, Craig interpolation has been applied in modular specification [2] and model checking [16, 12, 17] of computer applications.

The property of Craig interpolation has been established by model-theoretic means, namely in [11, 6], and using proof-theoretic techniques, as in the original paper [5] and in [3]. Some negative examples are also reported in the literature, namely concerning modal and relevant logics in [19, 21].

In the field of combination of logics, the preservation of the Craig interpolation property has been established for fusion [13] and, providing that there is a suitable bridge between the component logics, for fibring [4], model-theoretically and proof-theoretically, respectively. Some negative results are also reported in the literature, namely concerning the product of Kripke semantics in [15].

Herein, we investigate if Craig interpolation is preserved by the product of two logics endowed with matrix semantics, capitalizing on its axiomatization provided by their meet-combination, a new truly conservative way of combining matrix logics proposed and shown to preserve soundness and completeness in [20]. Furthermore, we also study preservation of interpolation in the presence of additional interaction axioms with connectives from both logics.

The product of logics endowed with matrix semantics is relevant for expressing properties of the two logics at the same time. For instance, assuming that we have a logic for reasoning about time and a logic for reasoning about space we may want to express properties involving time and space. On the other hand, the meet-combination of two connectives captures the common properties of both.

In the product logic one finds combined propositional symbols that are not independent of each other. The presence of such propositional symbols requires a natural relaxation of the interpolation property. For instance if

$$\gamma \vdash \varphi$$

and the combined propositional symbol $\lceil q_1 q_2 \rceil$ occurs in $\gamma$ but not in $\varphi$ and $q_2$ occurs in $\varphi$, we allow the interpolant to use $q_2$.

After a brief summary of meet-combination of logics and its main properties in Section 2, the proposed variant of Craig interpolation is presented in Section 3. The enrichment of the meet-combination with interaction axioms is also introduced in Section 2. Therein we also provide a sufficient condition for the preservation of the metatheorem of deduction that will be needed for proving preservation of interpolation in the presence of interaction.

The preservation of the interpolation property by the product of matrix logics assumes that the two given logics have the identity connective, in addition to verum and falsum, as explained at the end of Section 2. Clearly, this assumption is not restrictive because, if missing, adding identity changes nothing of import.

After some technical lemmas, the main preservation result is constructively established by proof-theoretic means in Section 4, taking advantage of the axiomatization of the product of matrix logics provided by their meet-combination. As a corollary, we also establish a couple of weaker results on the preservation of interpolation in the presence of interaction axioms. Examples are delayed until Section 5.

In Section 6, an algorithm for computing the interpolant in the product logic is extracted from the preservation proof. Its worst-case complexity is established and it is shown that the computation of the interpolant in the resulting logic has only a polynomial penalty over the computation in the two given logics.

## 2 Meet-combination of logics

For the convenience of the reader we provide here a brief review of [20]. By a *matrix logic* over a given set $Q$ of *propositional symbols* we mean a triple[1]

$$\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$$

where:

- The *signature* $\Sigma$ is a family $\{\Sigma_n\}_{n \in \mathbb{N}}$ with each $\Sigma_n$ being a set of $n$-ary language *constructors*[2] and such that

  $$Q \subseteq \Sigma_0.$$

  Formulas are built as usual with the constructors and the *propositional or schema variables* in $\Xi = \{\xi_k \mid k \in \mathbb{N}\}$. We use $L$ and $L(\Xi)$ for denoting the set of *concrete formulas*[3] and the set of all *formulas*, respectively. If a formula contains schema variables we may emphasize this fact by saying that it is a *schema formula*. Schema formulas are useful for writing schema inference rules so that in a combined logic they can be instantiated with formulas from outside the original logic.

- The Hilbert *calculus* $\Delta$ is a set of finitary rules of the form

  $$\frac{\alpha_1 \quad \ldots \quad \alpha_m}{\beta}$$

[1] Taking a logic as a triple allows to deal with logics where the calculus and the semantics were developed independently of each other. For instance, in the case of a modal logic endowed with a matrix semantics $\mathcal{M}$ induced by its Kripke semantics.

[2] We use the term constructor to refer to any constant, like $\mathtt{t\!t}$, connective, like $\wedge$, and operator, like $\Box$.

[3] Formulas without schema variables.

where formulas $\alpha_1, \ldots, \alpha_m$ are said to be the *premises* of the rule and formula $\beta$ is said to be its *conclusion*. A rule without premises is said to be axiomatic and its conclusion is said to be an *axiom*. Derivability and derivation sequences are defined as usual for Hilbert calculi. We write

$$\Gamma \vdash \varphi$$

for stating that there is a derivation sequence of formula $\varphi$ from set $\Gamma$ of hypotheses. When $\emptyset \vdash \varphi$ we say that $\varphi$ is a *theorem* and write simply $\vdash \varphi$.

- The matrix *semantics*[4] $\mathcal{M}$ is a non empty class of matrices over $\Sigma$. Recall that each such matrix $M$ is a pair $(\mathfrak{A}, D)$ where $\mathfrak{A}$ is an algebra over $\Sigma$ and $D$ is a non-empty subset of its carrier set $A$. Denotation, satisfaction, entailment and validity are defined as usual for matrix semantics. Namely, we write

  $$\Gamma \vDash \varphi$$

  for stating that, for each matrix $M = (\mathfrak{A}, D)$ and assignment $\rho : \Xi \to A$, if $[\![\gamma]\!]_{M\rho} \in D$ for every $\gamma \in \Gamma$, then $[\![\varphi]\!]_{M\rho}$ is also a distinguished value.

We need to work with logics fulfilling some additional assumptions. By a *suitable logic* we mean a logic such that (i) there is a concrete formula, that we call *verum* and denote by $\mathtt{tt}$, which is a theorem, (ii) there is a concrete formula, that we call *falsum* and denote by $\mathtt{ff}$, from which all formulas are derived, and (iii) for each $n \geq 1$, there is a formula $\phi^{(n)}$, with schema variables $\xi_1, \ldots, \xi_n$, which is a theorem.[5]

In the context of a suitable logic, for each $n \geq 1$, we introduce the $n$-ary connective $\mathtt{tt}^{(n)}$ as follows:

$$\mathtt{tt}^{(n)}(\varphi_1, \ldots, \varphi_n) =_{\text{abbv.}} \phi^{(n)}|^{\xi_1,\ldots,\xi_n}_{\varphi_1,\ldots,\varphi_n}$$

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[4]Matrix semantics was introduced by Tarski (although implicit in previous works of Łukasiewicz, Bernays and Post among others) and has the advantage of providing a uniform general semantics for a wide variety of logics namely intuitionistic and modal logics as well as many-valued logics and some paraconsistent logics.

[5]For instance, in the context of classical and intuitionistic logics we can take

$$(\xi_1 \supset \xi_1) \wedge \ldots \wedge (\xi_n \supset \xi_n)$$

as $\phi^{(n)}$.

where $\phi^{(n)}|^{\xi_1,\ldots,\xi_n}_{\varphi_1,\ldots,\varphi_n}$ is the formula obtained from $\phi^{(n)}$ by the uniform and simultaneous substitution of $\varphi_1, \ldots, \varphi_n$ for $\xi_1, \ldots, \xi_n$, respectively. Moreover, we may write $\mathtt{tt}^{(0)}$ for $\mathtt{tt}$.

Given a suitable logic $\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$ over $Q$, we assume without loss of generality that $\Sigma_0 \setminus Q$ contains the constructors $\mathtt{tt}$, $\mathtt{ff}$ and, in general, $\Sigma_n$ contains $\mathtt{tt}^{(n)}$ for each $n \in \mathbb{N}^+$, as introduced above.

Given two suitable logics $\mathcal{L}_1 = (\Sigma_1, \Delta_1, \mathcal{M}_1)$ and $\mathcal{L}_2 = (\Sigma_2, \Delta_2, \mathcal{M}_2)$ over $Q_1$ and $Q_2$, respectively, with $\Sigma_1 = \{\Sigma_{1n}\}_{n\in\mathbb{N}}$ and $\Sigma_2 = \{\Sigma_{2n}\}_{n\in\mathbb{N}}$, their *meet-combination* is the logic

$$\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil = (\Sigma_{\lceil 12 \rceil}, \Delta_{\lceil 12 \rceil}, \mathcal{M}_{\lceil 12 \rceil})$$

over

$$Q_{\lceil 12 \rceil} = \{\lceil q_1 \mathtt{tt}_2 \rceil \,|q_1 \in Q_1\} \cup \{\lceil \mathtt{tt}_1 q_2 \rceil \,|q_2 \in Q_2\} \cup \{\lceil q_1 q_2 \rceil \,|q_1 \in Q_1, q_2 \in Q_2\}$$

where $\Sigma_{\lceil 12 \rceil}$, $\Delta_{\lceil 12 \rceil}$ and $\mathcal{M}_{\lceil 12 \rceil}$ are as follows. In the sequel, we denote by $\mathtt{tt}_k^{(n)}$ the n-ary constructor in $\Sigma_{kn}$ for $k = 1, 2$.

The signature $\Sigma_{\lceil 12 \rceil}$ is such that, for each $n \in \mathbb{N}$,

$$\Sigma_{\lceil 12 \rceil n} = \{\lceil c_1 c_2 \rceil \mid c_1 \in \Sigma_{1n}, c_2 \in \Sigma_{2n}\}.$$

The constructor $\lceil c_1 c_2 \rceil$ is said to be the *meet-combination* of $c_1$ and $c_2$. As expected, we use $L_{\lceil 12 \rceil}$ and $L_{\lceil 12 \rceil}(\Xi)$ for denoting the set of concrete formulas and the set of all formulas over $\Sigma_{\lceil 12 \rceil}$, respectively. Observe that we look at signature $\Sigma_{\lceil 12 \rceil}$ as an enrichment of $\Sigma_1$ via the embedding

$$\eta_1 : c_1 \mapsto \lceil c_1 \mathtt{tt}_2^{(n)} \rceil \quad \text{for each } c_1 \in \Sigma_{1n}.$$

Similarly, for $\Sigma_2$ we use the embedding $\eta_2 : c_2 \mapsto \lceil \mathtt{tt}_1^{(n)} c_2 \rceil$ for each $c_2 \in \Sigma_{2n}$. For the sake of lightness of notation, in the context of $\Sigma_{\lceil 12 \rceil}$, from now on, we may write

$$c_1 \text{ for } \lceil c_1 \mathtt{tt}_2^{(n)} \rceil \text{ when } c_1 \in \Sigma_{1n}$$

and $c_2$ for $\lceil \mathtt{tt}_1^{(n)} c_2 \rceil$ when $c_2 \in \Sigma_{2n}$. We refer to these constructors as the *inherited constructors* and refer to the other constructors in $\Sigma_{\lceil 12 \rceil}$ as the *proper combined constructors*. In this vein, for $k = 1, 2$, we look at $Q_k$ as a subset of $Q_{\lceil 12 \rceil}$, at $L_k$ as a subset of $L_{\lceil 12 \rceil}$ and at $L_k(\Xi)$ as a subset of $L_{\lceil 12 \rceil}(\Xi)$. Given a formula $\varphi$ over $\Sigma_{\lceil 12 \rceil}$ and $k \in \{1, 2\}$, we denote by

$$\varphi|_k$$

the formula obtained from $\varphi$ by replacing every occurrence of each combined constructor (proper and inherited) by its $k$-th component. Such a formula is called the *projection* of $\varphi$ to $k$.

The calculus $\Delta_{\lceil 12 \rceil}$ is composed of the rules inherited from $\Delta_1$ (via the implicit embedding $\eta_1$) and the rules inherited from $\Delta_2$ (via the implicit embedding $\eta_2$), plus the rules imposing that each combined connective enjoys the common properties of its components and the rules for propagating falsum. More precisely, $\Delta_{\lceil 12 \rceil}$ contains the following rules:

- for $k = 1, 2$, the *inherited rules* from $\Delta_k$:
  - every non-liberal rule (that is, a rule where the conclusion is not a schema variable) in $\Delta_k$;
  - every tagging of every liberal rule $r$ of the form

$$\frac{\alpha_1 \quad \dots \quad \alpha_m}{\xi}$$

    in $\Delta_k$, that is, for each $c \in \Sigma_{kn}$ and $n \in \mathbb{N}$, the rule $r_c$ of the form

$$\frac{\alpha_1|^{\xi}_{\beta_c} \quad \dots \quad \alpha_m|^{\xi}_{\beta_c}}{\beta_c}$$

    where $\beta_c = c(\xi_{j+1}, \dots, \xi_{j+n})$ with $j$ being the maximum of the indexes of the schema variables occurring in $r$;

- the *lifting rule* (in short LFT)

$$\frac{\varphi|_1 \quad \varphi|_2}{\varphi},$$

  for each formula $\varphi \in L_{\lceil 12 \rceil}(\Xi)$;

- the *co-lifting rule* (in short cLFT)

$$\frac{\varphi}{\varphi|_k},$$

  for each formula $\varphi \in L_{\lceil 12 \rceil}(\Xi)$ and $k = 1, 2$;

- the *falsum propagation rules* (in short FX) of the form

$$\frac{\mathsf{ff}_1}{\mathsf{ff}_2} \quad \text{and} \quad \frac{\mathsf{ff}_2}{\mathsf{ff}_1}.$$

At first sight one might be tempted to include in $\Delta_{\lceil 12\rceil}$ every rule in $\Delta_1 \cup \Delta_2$. For instance, if modus ponens is a rule in $\Delta_1$ one would expect to find in $\Delta_{\lceil 12\rceil}$ the rule

$$\frac{\xi_1 \quad (\xi_1 \supset_1 \xi_2)}{\xi_2}.$$

However, as discussed in [20], this rule is not sound. Instead, we tag such a liberal rule, including in $\Delta_{\lceil 12\rceil}$, for each $c \in \Sigma_{1n}$ and $n \in \mathbb{N}$, the $c$-tagged modus ponens rule

$$\frac{\xi_1 \quad (\xi_1 \supset_1 c(\xi_3, \ldots, \xi_{2+n}))}{c(\xi_3, \ldots, \xi_{2+n})}.$$

The lifting rule is motivated by the idea that $\lceil c_1 c_2\rceil$ inherits the common properties of $c_1$ and $c_2$. The co-lifting rule is motivated by the idea that $\lceil c_1 c_2\rceil$ should enjoy only the common properties of $c_1$ and $c_2$.

Observe that although we may write, for example, $\supset_1$ for $\lceil \supset_1 \mathtt{tt}_2^{(2)}\rceil$, the lifting and co-lifting rules also apply to such inherited constructors. For example, in the calculus of the meet-combination,

$$\frac{\neg_1(\xi_1 \supset_1 \xi_2) \quad \neg_2(\xi_1 \mathtt{tt}_2^{(2)} \xi_2)}{\lceil \neg_1 \neg_2\rceil(\xi_1 \supset_1 \xi_2)}$$

is an instance of LFT.

The semantics $\mathcal{M}_{\lceil 12\rceil}$ is the class of product matrices

$$\{M_1 \times M_2 \mid M_1 \in \mathcal{M}_1 \text{ and } M_2 \in \mathcal{M}_2\}$$

over $\Sigma_{\lceil 12\rceil}$ such that each

$$M_1 \times M_2 = (\mathfrak{A}_1 \times \mathfrak{A}_2, D_1 \times D_2)$$

where

$$\mathfrak{A}_1 \times \mathfrak{A}_2 = (A_1 \times A_2, \{\underline{\lceil c_1 c_2\rceil} : (A_1 \times A_2)^n \to A_1 \times A_2 \mid \lceil c_1 c_2\rceil \in \Sigma_{\lceil 12\rceil n}\}_{n \in \mathbb{N}})$$

with

$$\underline{\lceil c_1 c_2\rceil}((a_1, b_1), \ldots, (a_n, b_n)) = (\underline{c_1}(a_1, \ldots, a_n), \underline{c_2}(b_1, \ldots, b_n)).$$

Observe that the meet-combination $\lceil \mathcal{L}_1 \mathcal{L}_2\rceil$ of two given suitable, sound and complete matrix logics $\mathcal{L}_1$ and $\mathcal{L}_2$ provides an axiomatization of the product of their matrix semantics since it preserves soundness and completeness, as shown in [20].

It should also be stressed that we were able to define the meet-combination of two given logics only when they are suitable. In fact, the verum is needed for setting up the combined connectives and the falsum is needed in the calculus.

For this reason, from now on, when discussing the combination of $\mathcal{L}_1$ and $\mathcal{L}_2$, we assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are given suitable logics over $Q_1$ and $Q_2$, respectively. As shown in [20], their combination $\lceil\mathcal{L}_1\mathcal{L}_2\rceil$ is a suitable logic over $Q_{\lceil 12\rceil}$.

In some cases we may want to impose some interaction between connectives of the two logics. Interaction is stated by axioms.

Given logics $\mathcal{L}_1$ and $\mathcal{L}_2$ and a set Ax of interaction axioms in $L_{\lceil 12\rceil}(\Xi)$, we denote by

$$\lceil\mathcal{L}_1\mathcal{L}_2\rceil_{\mathrm{Ax}} = (\Sigma_{\lceil 12\rceil}, \Delta_{\lceil 12\rceil+\mathrm{Ax}}, \mathcal{M}_{\lceil 12\rceil+\mathrm{Ax}})$$

the logic obtained by enriching $\lceil\mathcal{L}_1\mathcal{L}_2\rceil$ with Ax as follows:

- $\Delta_{\lceil 12\rceil+\mathrm{Ax}} = \Delta_{\lceil 12\rceil} \cup \mathrm{Ax}$;
- $\mathcal{M}_{\lceil 12\rceil+\mathrm{Ax}} = \{M_1 \times M_2 : M_1 \times M_2 \in \mathcal{M}_{\lceil 12\rceil} \text{ and } M_1 \times M_2 \Vdash_{\lceil 12\rceil} \mathrm{Ax}\}$.

Moreover we write $\Gamma \vdash_{\lceil 12\rceil+\mathrm{Ax}} \varphi$ whenever there is a derivation sequence of $\varphi$ from $\Gamma$ in $\lceil\mathcal{L}_1\mathcal{L}_2\rceil_{\mathrm{Ax}}$.

For instance, given the meet-combination of two modal logics $\mathcal{L}_1$ and $\mathcal{L}_2$, assume that Ax includes the following interaction axiom:

$$(\Box_1\xi)\lceil\supset_1\supset_2\rceil(\Box_2\xi)$$

stating that necessitation $\Box_1$ in logic $\mathcal{L}_1$ implies necessitation $\Box_2$ in logic $\mathcal{L}_2$. Hence, if logic $\mathcal{L}_2$ has axiom 4 $(\Box_2\,\xi) \supset_2 (\Box_2\,\Box_2\,\xi)$ then one should expect to be able to obtain that

$$\vdash_{\lceil 12\rceil+\mathrm{Ax}} (\Box_1\xi)\lceil\supset_1\supset_2\rceil(\Box_2\Box_2\xi).$$

Until the end of this section, we recall the following two results (the proofs can be seen in [20]) and prove some additional technical lemmas that are needed later on. Namely we prove a sufficient condition for the preservation by meet-combination of the metatheorem of deduction.

**Proposition 2.1** For each $k = 1, 2$, let $\Gamma \cup \{\varphi\}$ be a set of formulas in $L_k$ such that $\Gamma \vdash_k \varphi$. Then, $\Gamma \vdash_{\lceil 12\rceil} \varphi$.

**Proposition 2.2** For each $k = 1, 2$ and $\varphi \in L_{\lceil 12\rceil}$, $\mathsf{ff}_k \vdash_{\lceil 12\rceil} \varphi$.

Proposition 2.1 means that $\lceil \mathcal{L}_1\mathcal{L}_2 \rceil$ is an extension of each of the given two logics with respect to concrete formulas. Proposition 2.2 states that the falsum of each of the given logics is also a falsum in $\lceil \mathcal{L}_1\mathcal{L}_2 \rceil$. It is worth mentioning, albeit not used in this paper, that $\lceil \mathcal{L}_1\mathcal{L}_2 \rceil$ is a conservative extension of the component logics and that if the latter are both sound and complete, then so is the former (see [20]).

The following result establishes a relationship between substitution in the combined logic and substitution in each of the component logics.

**Proposition 2.3** For each $k = 1, 2$ let $\sigma : \Xi \to L_{\lceil 12 \rceil}$ and $\sigma_k : \Xi \to L_k$ be substitutions such that $\sigma_k(\xi) = \sigma(\xi)|_k$. Then

$$\sigma_k(\psi) = \sigma(\psi)|_k \quad \text{for every } \psi \in L_k.$$

**Proof:** The proof follows by a straightforward induction on $\psi$. QED

The following useful result relates derivability in the combined logic with derivability in each component logic.

**Proposition 2.4** Let $\Gamma \cup \{\varphi\} \subseteq L_{\lceil 12 \rceil}$ be such that $\Gamma \vdash_{\lceil 12 \rceil} \varphi$ with a derivation sequence not using the FX rules. Then

$$\Gamma|_k \vdash_k \varphi|_k \ \text{ for } k = 1, 2.$$

**Proof:** Let $\psi_1 \ldots \psi_n$ be a derivation sequence in $\lceil \mathcal{L}_1\mathcal{L}_2 \rceil$ of $\varphi$ from $\Gamma$ not using the FX rule. We prove the result by induction on $n$:

(1) $\varphi \in \Gamma$. In this case, it is straightforward to obtain the thesis.

(2) $\varphi$ is an instance of an axiom $\alpha$ in $\lceil \mathcal{L}_1\mathcal{L}_2 \rceil$ with substitution $\sigma : \Xi \to L_{\lceil 12 \rceil}$. Suppose without loss of generality that $\alpha$ is inherited from $\alpha_1$ in $\mathcal{L}_1$. Then:
(i) $k = 1$. If $\alpha_1$ is a schema variable then $\vdash_1 \varphi|_1$ immediately. Otherwise, take $\sigma_1(\xi) = \sigma(\xi)|_1$ for every $\xi \in \Xi$. Then, $\varphi|_1 = \sigma(\alpha_1)|_1 = \sigma_1(\alpha_1)$, by Proposition 2.3. Hence, $\varphi|_1$ is an instance of $\alpha_1$, that is $\alpha$, by $\sigma_1$.
(ii) $k = 2$. Since the main constructor of $\varphi$ is from $\Sigma_1$ then the main constructor of $\varphi|_2$ is $\mathtt{t}_2^{(n)}$ for some $n$. The result follows straightforwardly.

(3) $\varphi$ is obtained from $\psi_{i_1} \ldots \psi_{i_m}$ using an inherited rule $r = (\{\alpha_1, \ldots, \alpha_m\}, \beta)$ with substitution $\sigma : \Xi \to L_{\lceil 12 \rceil}$. Then, $\Gamma \vdash_{\lceil 12 \rceil} \psi_{i_j}$ for $j = 1, \ldots, m$ and so, by the induction hypothesis

$$\Gamma|_1 \vdash_1 \psi_{i_j}|_1 \quad \text{and} \quad \Gamma|_2 \vdash_2 \psi_{i_j}|_2$$

for $j = 1, \ldots, m$. Suppose without loss of generality that $r$ is inherited from rule $r_1 = (\{\alpha'_1, \ldots, \alpha'_m\}, \beta')$ of $\mathcal{L}_1$. Then:

(i) $k = 1$. If $r_1$ is liberal then let $\sigma' : \Xi \to L_1(\Xi)$ be a substitution such that $\sigma'(\beta') = \beta$ and $\sigma'(\xi) = \xi$ for every $\xi \neq \beta'$, otherwise let $\sigma'$ be the identity. Observe that $\alpha_j = \sigma'(\alpha'_j)$ for $j = 1, \ldots, m$ and $\beta = \sigma'(\beta')$. Take $\sigma_1(\xi) = \sigma(\xi)|_1$ for every $\xi \in \Xi$. Then, $\psi_{i_j}|_1 = \sigma(\alpha_j)|_1 = \sigma_1(\alpha_j) = \sigma_1(\sigma'(\alpha'_j)) = (\sigma_1 \circ \sigma')(\alpha'_j)$ for $j = 1, \ldots, m$, by Proposition 2.3. Then, by rule $r_1$,

$$\Gamma|_1 \vdash_1 (\sigma_1 \circ \sigma')(\beta').$$

The result follows since $(\sigma_1 \circ \sigma')(\beta') = \sigma_1(\beta) = \sigma(\beta)|_1 = \varphi|_1$, by Proposition 2.3.

(ii) $k = 2$. Since the main constructor of $\varphi$ is from $\Sigma_1$ then the main constructor of $\varphi|_2$ is $\mathtt{t}_2^{(n)}$ for some $n$. The result follows straightforwardly.

(4) $\varphi$ is obtained from $\varphi|_1$ and $\varphi|_2$ by rule LFT. Then, by the induction hypothesis,

$$\Gamma|_1 \vdash_1 (\varphi|_1)|_1 \quad \text{and} \quad \Gamma|_2 \vdash_2 (\varphi|_1)|_2$$

and

$$\Gamma|_1 \vdash_1 (\varphi|_2)|_1 \quad \text{and} \quad \Gamma|_2 \vdash_2 (\varphi|_2)|_2.$$

The result follows since $(\varphi|_k)|_k = \varphi|_k$ for $k = 1, 2$.

(5) $\varphi$ is $\psi|_1$ using rule cLFT. Then, by the induction hypothesis,

$$\Gamma|_1 \vdash_1 \psi|_1 \quad \text{and} \quad \Gamma|_2 \vdash_2 \psi|_2$$

and, so:

(i) $k = 1$. Clearly, $\Gamma|_1 \vdash_1 \varphi|_1$ since $\varphi|_1$ is $(\psi|_1)|_1$, that is, $\psi|_1$.

(ii) $k = 2$. The result follows since the main constructor of $\varphi$ is from $L_1$.

(6) $\varphi$ is $\psi|_2$ using rule cLFT. The proof is similar to case (5). QED

We now investigate sufficient conditions for the preservation of the metatheorem of deduction. The results we get will be used to prove preservation of interpolation in the presence of interaction (see Theorem 4.5). For this purpose we start by introducing the following notion. A calculus $\Delta$ is said to be a *calculus with implication* $\supset$ when $\supset \in \Sigma_2$ and the following holds:

- *metatheorem of deduction* (in short MTD) with respect to $\supset$:

$$\text{if} \quad \Gamma, \eta \vdash \varphi \quad \text{then} \quad \Gamma \vdash \eta \supset \varphi;$$

- *modus ponens* (in short MP) with respect to $\supset$:

$$\text{if} \quad \Gamma \vdash \eta \text{ and } \Gamma \vdash \eta \supset \varphi \quad \text{then} \quad \Gamma \vdash \varphi.$$

Observe that imposing MP as above is equivalent to stating the following:

$$\text{if} \quad \Gamma \vdash \eta \supset \varphi \quad \text{then} \quad \Gamma, \eta \vdash \varphi.$$

In the proof of the following result we use the notion of a formula depending on an hypothesis in a derivation.

Given a derivation $\psi_1 \ldots \psi_n$ of $\varphi$ from $\Gamma \cup \{\eta\}$, we say that $\psi_i$ *depends* on $\eta$ in this derivation if either $\psi_i$ is $\eta$ or $\psi_i$ is obtained using a rule with at least one of the premises depending on $\eta$.

Notwithstanding the fact that the metatheorem of deduction may have been proved in the original logics in a standard way (for example, using Frege's syllogism, the simplification axiom and modus ponens, as in [8]), the proof of the metatheorem of deduction in their meet combination will always be more complicated since there are new rules (e.g. the LFT and the cLFT). Moreover, the following preservation result is also applicable to logics with other rules besides modus ponens.

**Theorem 2.5 (Preservation of metatheorem of deduction)**
Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ have MP and MTD with respect to $\supset_1$ and $\supset_2$, respectively, and let $\Gamma \cup \{\eta, \varphi\} \subseteq L_{\lceil 12 \rceil}$. If

$$\Gamma, \eta \vdash_{\lceil 12 \rceil} \varphi$$

with a derivation sequence not using the FX rules, then

$$\Gamma \vdash_{\lceil 12 \rceil} \eta \lceil \supset_1 \supset_2 \rceil \varphi$$

with a derivation sequence not using the FX rules.

**Proof:** Let $\psi_1 \ldots \psi_n$ be a derivation sequence for $\Gamma, \eta \vdash_{\lceil 12 \rceil} \varphi$ not using FX rules. We consider two cases:
(1) $\varphi$ does not depend on $\eta$. Then $\Gamma \vdash_{\lceil 12 \rceil} \varphi$. Observe that

$$\varphi|_k, \eta|_k \vdash_k \varphi|_k, \text{ for } k = 1, 2.$$

Hence, by MTD for $\mathcal{L}_k$,

$$\varphi|_k \vdash_k \eta|_k \supset_k \varphi|_k, \text{ for } k = 1, 2$$

and so, by Proposition 2.1,

$$\varphi|_k \vdash_{\lceil 12\rceil} \eta|_k \supset_k \varphi|_k, \text{ for } k = 1, 2.$$

Thus, by cLFT and transitivity,

$$\varphi \vdash_{\lceil 12\rceil} \eta|_k \supset_k \varphi|_k, \text{ for } k = 1, 2$$

and, by LFT

$$\varphi \vdash_{\lceil 12\rceil} \eta \lceil\supset_1\supset_2\rceil\, \varphi.$$

The thesis follows by transitivity.

(2) $\varphi$ is either $\eta$ (the proof follows straightforwardly) or is the conclusion of an instance of a rule $r$, other than FX, with a non empty set of premises and where at least a premise depends on $\eta$.

(a) Either $r \in \Delta_k$ is a non-liberal rule or $r$ is the tagging of a liberal rule in $\Delta_k$. We assume without loss of generality that $k = 1$ and that $r$ has $m$ premises.

Then, $\Gamma, \eta \vdash_{\lceil 12\rceil} \varphi_j$ where $\varphi_j$ is a premise of the instance of rule $r$ for $j = 1, \ldots, m$. Then, by the induction hypothesis,

$$\Gamma \vdash_{\lceil 12\rceil} \eta \lceil\supset_1\supset_2\rceil\, \varphi_j$$

with a derivation sequence not using the FX rules for $j = 1, \ldots, m$. Then, by Proposition 2.4,

$$\Gamma|_1 \vdash_1 \eta|_1 \supset_1 \varphi_j|_1$$

for $j = 1, \ldots, m$. Note that

$$\eta|_1 \supset_1 \varphi_1|_1, \ldots, \eta|_1 \supset_1 \varphi_m|_1, \eta|_1 \vdash_1 \varphi|_1$$

and so by MTD for $\mathcal{L}_1$

$$\eta|_1 \supset_1 \varphi_1|_1, \ldots, \eta|_1 \supset_1 \varphi_m|_1 \vdash_1 \eta|_1 \supset_1 \varphi|_1.$$

Thus, by transitivity,

$$\Gamma|_1 \vdash_1 \eta|_1 \supset_1 \varphi|_1.$$

On the other hand,

$$\Gamma|_2, \eta|_2 \vdash_2 \varphi|_2$$

since the head of $\varphi|_2$ is a verum connective and so by the MTD over $\mathcal{L}_2$

$$\Gamma|_2 \vdash_2 \eta|_2 \supset_2 \varphi|_2.$$

Therefore, by Proposition 2.1 and monotonicity,

$$\Gamma|_1, \Gamma|_2 \vdash_{\lceil 12 \rceil} \eta|_1 \supset_1 \varphi|_1 \quad \text{and} \quad \Gamma|_1, \Gamma|_2 \vdash_{\lceil 12 \rceil} \eta|_2 \supset_2 \varphi|_2.$$

Thus, by cLFT and transitivity,

$$\Gamma \vdash_{\lceil 12 \rceil} \eta|_1 \supset_1 \varphi|_1 \quad \text{and} \quad \Gamma \vdash_{\lceil 12 \rceil} \eta|_2 \supset_2 \varphi|_2.$$

Finally, using LFT, the thesis follows.

(b) $r$ is LFT. Then, $\Gamma, \eta \vdash_{\lceil 12 \rceil} \varphi|_j$, for $j = 1, 2$ and so, by the induction hypothesis, $\Gamma \vdash_{\lceil 12 \rceil} \eta \lceil \supset_1 \supset_2 \rceil \varphi|_j$, for $j = 1, 2$. Hence, by cLFT and transitivity,

$$\Gamma \vdash_{\lceil 12 \rceil} \eta|_1 \supset_1 \varphi|_1 \text{ and } \Gamma \vdash_{\lceil 12 \rceil} \eta|_2 \supset_2 \varphi|_2$$

since $\varphi|_1|_1$ is $\varphi|_1$ and $\varphi|_2|_2$ is $\varphi|_2$. The result follows by LFT.

(c) $r$ is cLFT. Then, $\Gamma, \eta \vdash_{\lceil 12 \rceil} \psi$ and $\varphi$ is $\psi|_k$. Assume without loss of generality, that $k = 1$. Hence, by the induction hypothesis,

$$\Gamma \vdash_{\lceil 12 \rceil} \eta \lceil \supset_1 \supset_2 \rceil \psi.$$

We consider two cases.

(i) $\varphi$ is not in $\Xi$. By cLFT

$$\Gamma \vdash_{\lceil 12 \rceil} \eta|_1 \supset_1 \psi|_1$$

and so

$$\Gamma \vdash_{\lceil 12 \rceil} \eta|_1 \supset_1 (\psi|_1)|_1.$$

On the other hand,

$$\eta|_2 \vdash_2 (\psi|_1)|_2$$

since the conclusion is a verum formula. Hence, by MTD for $\mathcal{L}_2$,

$$\vdash_2 \eta|_2 \supset_2 (\psi|_1)|_2.$$

Therefore, by Proposition 2.1 and monotonicity,

$$\Gamma \vdash_{\lceil 12 \rceil} \eta|_2 \supset_2 (\psi|_1)|_2.$$

Finally, by LFT, we get the result.

(ii) $\varphi$ is in $\Xi$. The thesis follows immediately by the induction hypothesis since $\psi|_1$ is $\psi$. QED

Note that metatheorem of deduction is also preserved in $\lceil \mathcal{L}_1\mathcal{L}_2 \rceil_{\mathrm{Ax}}$ for any set Ax of interaction axioms but we shall not need this result in the sequel. Modus ponens is also preserved under similar conditions as we now discuss.

**Theorem 2.6 (Preservation of modus ponens)**
Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ have MP with respect to $\supset_1$ and $\supset_2$, respectively, and let $\Gamma \cup \{\eta, \varphi\} \subseteq L_{\lceil 12 \rceil}$. If

$$\Gamma \vdash_{\lceil 12 \rceil} \eta \lceil \supset_1 \supset_2 \rceil \varphi$$

with a derivation sequence not using the FX rules, then

$$\Gamma, \eta \vdash_{\lceil 12 \rceil} \varphi$$

with a derivation sequence not using the FX rules.

**Proof:** Assume that $\Gamma \vdash_{\lceil 12 \rceil} \eta \lceil \supset_1 \supset_2 \rceil \varphi$ with a derivation sequence not using the FX rules. Hence, by Proposition 2.4,

$$\Gamma|_1 \vdash_1 \eta|_1 \supset_1 \varphi|_1 \quad \text{and} \quad \Gamma|_2 \vdash_2 \eta|_2 \supset_2 \varphi|_2.$$

Then, by MP for $\mathcal{L}_1$ and $\mathcal{L}_2$,

$$\Gamma|_1, \eta|_1 \vdash_1 \varphi|_1 \quad \text{and} \quad \Gamma|_2, \eta|_2 \vdash_2 \varphi|_2.$$

The result follows by Proposition 2.1, cLFT and LFT. QED

We observe that modus ponens is also preserved in $\lceil \mathcal{L}_1\mathcal{L}_2 \rceil_{\mathrm{Ax}}$ for any set Ax of interaction axioms but we shall not need this result in the sequel.

In order to establish the preservation of interpolation, we need in addition to consider logics endowed with identity. The identity constructor plays an important role when transforming an interpolant in a component logic to an interpolant in the combined logic, as made clear in the proof of the main theorem of Section 4 and illustrated in one of the examples of Section 5.

More concretely, we say that a given logic $\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$ is endowed *with identity* if it contains a unary constructor, say $\mathsf{id}$, in the signature such that:

- its denotation is the identity map $\mathsf{id}_M : b \mapsto b$ in every matrix $M$ in $\mathcal{M}$;
- $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$ where $\varphi'$ is a formula obtained from $\varphi$ by removing 0 or more applications of each $\mathsf{id}$.

In the sequel, for such a logic with identity, we denote by

$$\psi^{\text{-id}}$$

the formula obtained from the formula $\psi$ by removing every application of $\mathsf{id}$. The following result becomes handy in Section 4:

**Proposition 2.7** Let $\Gamma \cup \{\varphi\}$ be a set of formulas in a logic with identity. Then,

$$\Gamma \vdash \varphi \quad \text{iff} \quad \Gamma^{\text{-id}} \vdash \varphi^{\text{-id}}.$$

**Proof:** The result follows by a straightforward induction on the length of the derivation sequence. QED

It is possible to introduce $\mathsf{id}$ as an abbreviation in most logics. Otherwise, it is feasible to enrich the signature, the calculus and the matrix semantics in order to make it available, while preserving soundness, completeness and Craig interpolation.

# 3 A relaxed notion of Craig interpolation

For the sake of generality, we address turnstile interpolation, instead of the more common theoremhood interpolation using implication, since it may be the case that the logics at hand do not have implication (observe that implication is not required for preservation of interpolation when there is no interaction). Let $\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$ be a logic over $Q$ and

$$\mathsf{symb} : L \to \wp Q$$

be such that $\mathsf{symb}(\psi)$ is the set of propositional symbols occurring in $\psi$.

Recall that the "standard" turnstile notion of Craig interpolation (see [6]) is as follows. Logic $\mathcal{L}$ is said to enjoy the *Craig interpolation property* if, for every $\Gamma \cup \{\varphi\} \subseteq L$ with $\Gamma \vdash \varphi$,

$$\begin{array}{l} \text{there is } \Theta \subseteq L \text{ such that} \\ \quad (1)\ \mathsf{symb}(\Theta) \subseteq \mathsf{symb}(\Gamma) \cap \mathsf{symb}(\varphi) \\ \quad (2)\ \Gamma \vdash \theta \text{ for each } \theta \in \Theta \text{ and } \Theta \vdash \varphi \\ \text{whenever } \mathsf{symb}(\Gamma) \cap \mathsf{symb}(\varphi) \neq \emptyset, \\ \text{otherwise either } \Gamma \vdash \mathsf{ff} \text{ or } \vdash \varphi. \end{array}$$

When $\mathsf{symb}(\Gamma) \cap \mathsf{symb}(\varphi) \neq \emptyset$, such a $\Theta$ is said to be an *interpolant* for $\Gamma \vdash \varphi$.

As explained in Section 1, we need to relax this notion in order to be able to address the existence of pairs of propositional symbols in meet-combinations. To this end, it is convenient to introduce the following relation between propositional symbols of the combined logic.

Given $\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil$ over $Q_{\lceil 12 \rceil}$, let $\sqsubseteq$ be the *componentship relation* defined as follows:

- $c \sqsubseteq c$ for every $c \in Q_{\lceil 12 \rceil}$;
- $c_k \sqsubseteq \lceil c_1 c_2 \rceil$ for every $c_k, \lceil c_1 c_2 \rceil \in Q_{\lceil 12 \rceil}$ and $k = 1, 2$.

Furthermore, let

$$\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil} : L_{\lceil 12 \rceil} \to \wp Q_{\lceil 12 \rceil}$$

be such that

$$\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\psi) = \{c \in Q_{\lceil 12 \rceil} : \text{ there is } c' \in \mathsf{symb}_{\lceil 12 \rceil}(\psi) \text{ such that } c \sqsubseteq c'\}.$$

The meet-combination $\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil$ is said to enjoy the *relaxed Craig interpolation property* if, for every $\Gamma \cup \{\varphi\} \subseteq L_{\lceil 12 \rceil}$ with $\Gamma \vdash_{\lceil 12 \rceil} \varphi$,

$$\begin{array}{l} \text{there is } \Theta \subseteq L_{\lceil 12 \rceil} \text{ such that} \\ \quad (1)\ \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta) \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \\ \quad (2)\ \Gamma \vdash_{\lceil 12 \rceil} \theta \text{ for each } \theta \in \Theta \text{ and } \Theta \vdash_{\lceil 12 \rceil} \varphi \\ \text{whenever } \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \neq \emptyset, \\ \text{otherwise either } \Gamma \vdash_{\lceil 12 \rceil} \text{ff} \text{ or } \vdash_{\lceil 12 \rceil} \varphi. \end{array}$$

When $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \neq \emptyset$, such a $\Theta$ is said to be a *relaxed interpolant* for $\Gamma \vdash_{\lceil 12 \rceil} \varphi$.

The definition of relaxed Craig interpolation is easily extended for meet-combination of logics with interaction. More precisely the definition is the same except that we consider $\vdash_{\lceil 12 \rceil + \text{Ax}}$ instead of $\vdash_{\lceil 12 \rceil}$, for a given set of interaction axioms Ax.

Observe that, in the presence of pairs of propositional symbols, it is only natural that their components should also be allowed in the interpolant.

Note also that the relaxed notion of interpolation collapses into the standard one if the second condition of the componentship relation is deleted. To this end, one needs to extend the relaxed notion of interpolation to any logic, by confusing each $q$ with the pair $\lceil qq \rceil$.

## 4 Preservation of interpolation

From now on we assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are suitable matrix logics with identity over $Q_1$ and $Q_2$, respectively.

In the sequel we denote by $\tau_1 : L_{\lceil 12 \rceil} \to L_{\lceil 12 \rceil}$ the map such that

$$\tau_1(\psi) = \begin{cases} (\lceil \mathsf{id}_1 \mathsf{tt}_2^{(1)} \rceil\, \psi) & \text{if } \psi \in \Sigma_{\lceil 12 \rceil 0} \\ \psi & \text{otherwise,} \end{cases}$$

and define $\tau_2$ similarly. Observe that, if $\alpha$ is an atom of $L_{\lceil 12 \rceil}$, then:

$$\tau_1(\alpha) = (\lceil \mathsf{id}_1 \mathsf{tt}_2^{(1)} \rceil\, \alpha)$$

while

$$\tau_2(\alpha) = (\lceil \mathsf{tt}_1^{(1)} \mathsf{id}_2 \rceil\, \alpha).$$

The non-atomic formulas are not affected.

Furthermore, for each $k = 1, 2$, by a *right inverse of the $k$-th projection* or simply a *right $k$-inverse* we mean a map

$$f_k : Q_k \to Q_{\lceil 12 \rceil}$$

such that $f_k(q)|_k = q$ for each $q \in Q_k$. Such a right $k$-inverse is canonically extended to $L_k$ as expected: for each $\theta_k \in L_k$, $f_k(\theta_k)$ is the formula obtained by replacing in $\theta_k$ each propositional symbol $c_k \in Q_k$ by $f_k(c_k)$.

Towards the envisaged preservation result, we establish the following technical lemmas where $\mathsf{id}$ and the maps defined above play an important role.

**Proposition 4.1** For each $k = 1, 2$, let $\theta_k \in L_k$ and $f_k$ be a right $k$-inverse. Then,

$$\tau_k(f_k(\theta_k))|_k \vdash_{\lceil 12 \rceil} \theta_k \quad \text{and} \quad \theta_k \vdash_{\lceil 12 \rceil} \tau_k(f_k(\theta_k))|_k.$$

**Proof:** Without loss of generality, we assume that $k = 1$. Consider two cases.

(1) $f_1(\theta_1) \in \Sigma_{\lceil 12 \rceil 0}$. Hence

$$\tau_1(f_1(\theta_1)) = (\lceil \mathsf{id}_1 \mathsf{tt}_2^{(1)} \rceil\, f_1(\theta_1)).$$

Then

$$\tau_1(f_1(\theta_1))|_1 = (\lceil \mathsf{id}_1 \mathsf{tt}_2^{(1)} \rceil\, f_1(\theta_1))|_1 = (\mathsf{id}_1 \theta_1).$$

Thus $\tau_1(f_1(\theta_1))|_1 \vdash_{\lceil 12 \rceil} \theta_1$ and $\theta_1 \vdash_{\lceil 12 \rceil} \tau_1(f_1(\theta_1))|_1$ by Proposition 2.1.
(2) $f_1(\theta_1) \notin \Sigma_{\lceil 12 \rceil 0}$. Then, $\tau_1(f_1(\theta_1)) = f_1(\theta_1)$. Since it is easy to show that

$$\tau_1(f_1(\theta_1))|_1 = \theta_1,$$

the result follows immediately. QED

**Proposition 4.2** For each $k = 1, 2$, let $\theta_k \in L_k$ and $f_k$ be a right $k$-inverse. Then,

$$\vdash_{\lceil 12 \rceil} \tau_k(f_k(\theta_k))|_{\overline{k}}$$

where $\overline{1} = 2$ and $\overline{2} = 1$.

**Proof:** Assume without loss of generality that $k = 1$. We consider two cases.
(1) $f_1(\theta_1) \in \Sigma_{\lceil 12 \rceil 0}$. Hence

$$\tau_1(f_1(\theta_1)) = (\lceil \mathsf{id}_1 \mathsf{tt}_2^{(1)} \rceil \, f_1(\theta_1)).$$

Then

$$\tau_1(f_1(\theta_1))|_2 = (\lceil \mathsf{id}_1 \mathsf{tt}_2^{(1)} \rceil \, f_1(\theta_1))|_2 = \mathsf{tt}_2^{(1)}(f_1(\theta_1)|_2).$$

The result follows since $\vdash_{\lceil 12 \rceil} \mathsf{tt}_2^{(1)}(f_1(\theta_1)|_2)$.
(2) $f_1(\theta_1) \notin \Sigma_{\lceil 12 \rceil 0}$. Then, $\tau_1(f_1(\theta_1)) = f_1(\theta_1)$. Observing that the main constructor of $\theta_1$ is in $\Sigma_1$, then the result follows since the main constructor of $\tau_1(f_1(\theta_1))|_2$ is $\mathsf{tt}_2^{(n)}$ for some $n$. QED

**Proposition 4.3** Let $\Gamma \cup \{\varphi\} \subseteq L_{\lceil 12 \rceil}$. Then,

$$\mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1) \neq \emptyset \quad \text{or} \quad \mathsf{symb}_2(\Gamma|_2) \cap \mathsf{symb}_2(\varphi|_2) \neq \emptyset$$
$$\text{iff}$$
$$\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \neq \emptyset.$$

**Proof:** Assume with no loss of generality that $c_1 \in \mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1)$. Then, $\lceil c_1 c_2 \rceil \in \mathsf{symb}_1(\Gamma)$ and $\lceil c_1 c_2' \rceil \in \mathsf{symb}_1(\varphi)$ for some $c_2, c_2'$ in $\Sigma_2$ of arity 0. Hence, $c_1 \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$.

Conversely, assume that $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \neq \emptyset$ and let $c \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$. Then, we can consider the following cases:
(a) $c = \lceil c_1 c_2 \rceil$ where $c_1, c_2$ are propositional symbols. Then, $c_k \in \mathsf{symb}_k(\Gamma|_k) \cap \mathsf{symb}_k(\varphi|_k)$ for $k = 1, 2$.

(b) $c = \lceil c_1 \mathsf{tt}_2^{(0)} \rceil$. Then, either $\lceil c_1 c_2 \rceil$ or $\lceil c_1 \mathsf{tt}_2^{(0)} \rceil$ occurs in $\Gamma$ and either $\lceil c_1 c_2' \rceil$ or $\lceil c_1 \mathsf{tt}_2^{(0)} \rceil$ occurs in $\varphi$. Hence, $c_1$ occurs in $\Gamma|_1$ and $c_1$ occurs in $\varphi|_1$. Then, $c_1 \in \mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1)$.
(c) $c = \lceil \mathsf{tt}_1^{(0)} c_2 \rceil$. Similar to case (b). QED

With the results above at hand, we are ready to establish the key result concerning the interpolation property in meet-combinations.

**Theorem 4.4 (Preservation of interpolation)**
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be suitable matrix logics with identity and enjoying the Craig interpolation property. Then, their meet-combination $\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil$ enjoys the relaxed Craig interpolation property.

**Proof:** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be logics over sets $Q_1$ and $Q_2$ of propositional symbols, respectively, enjoying Craig interpolation, and $\Gamma \cup \{\varphi\} \subseteq L_{\lceil 12 \rceil}$. Assume that $\Gamma \vdash_{\lceil 12 \rceil} \varphi$. We need to consider two scenarios:

(A) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) = \emptyset$:

Then, $\mathsf{symb}_k(\Gamma|_k) \cap \mathsf{symb}_k(\varphi|_k) = \emptyset$ for $k = 1, 2$. Hence, for each $k = 1, 2$, either $\Gamma|_k \vdash_k \mathsf{ff}_k$ or $\vdash_k \varphi|_k$. Thus, either $\Gamma \vdash_{\lceil 12 \rceil} \lceil \mathsf{ff}_1 \mathsf{ff}_2 \rceil$ or $\vdash_{\lceil 12 \rceil} \varphi$. Indeed:

(1) If $\vdash_1 \varphi|_1$ and $\vdash_2 \varphi|_2$, then, by Proposition 2.1, $\vdash_{\lceil 12 \rceil} \varphi|_k$ for $k = 1, 2$. So, by LFT, we have $\vdash_{\lceil 12 \rceil} \varphi$.

(2) Otherwise, $\Gamma|_1 \vdash_1 \mathsf{ff}_1$ or $\Gamma|_2 \vdash_2 \mathsf{ff}_2$. Assume, without loss of generality, that $\Gamma|_1 \vdash_1 \mathsf{ff}_1$. Observe that by cLFT, $\Gamma \vdash_{\lceil 12 \rceil} \Gamma|_1$ and, by Proposition 2.1, $\Gamma|_1 \vdash_{\lceil 12 \rceil} \mathsf{ff}_1$. Thus, $\Gamma \vdash_{\lceil 12 \rceil} \mathsf{ff}_1$ and so, by Proposition 2.2, $\Gamma \vdash_{\lceil 12 \rceil} \lceil \mathsf{ff}_1 \mathsf{ff}_2 \rceil$.

(B) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \neq \emptyset$:

Assume that $d$ is a derivation sequence of $\varphi$ from $\Gamma$. We consider two cases:

(1) $d$ uses an FX rule. Thus, $\Gamma \vdash_{\lceil 12 \rceil} \mathsf{ff}_1$. Then, $\{\mathsf{ff}_1\}$ is an interpolant for $\Gamma \vdash_{\lceil 12 \rceil} \varphi$. Indeed:

(i) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\{\mathsf{ff}_1\}) = \emptyset \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$. (ii) $\Gamma \vdash_{\lceil 12 \rceil} \mathsf{ff}_1$ and (iii) $\mathsf{ff}_1 \vdash_{\lceil 12 \rceil} \varphi$ using Proposition 2.2.

(2) $d$ does not use FX rules. Then,

$$\Gamma|_1 \vdash_1 \varphi|_1 \quad \text{and} \quad \Gamma|_2 \vdash_2 \varphi|_2$$

by Proposition 2.4. Moreover, by Proposition 4.3,

$$\mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1) \neq \emptyset \quad \text{or} \quad \mathsf{symb}_2(\Gamma|_2) \cap \mathsf{symb}_2(\varphi|_2) \neq \emptyset.$$

Before proceeding, observe that there is a right 1-inverse $f_1$ such that, for each $c_1 \in \mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1)$:

- $f_1(c_1) \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$;
- if there is $c'_2 \neq \mathtt{tt}_2$ with $\lceil c_1 c'_2 \rceil \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$ then $f_1(c_1)|_2 \neq \mathtt{tt}_2$.

Similarly, there is a right 2-inverse $f_2$ such that, for each $c_2 \in \mathsf{symb}_2(\Gamma|_2) \cap \mathsf{symb}_2(\varphi|_2)$:

- $f_2(c_2) \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$;
- if there is $c'_1 \neq \mathtt{tt}_1$ with $\lceil c'_1 c_2 \rceil \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$ then $f_2(c_2)|_1 \neq \mathtt{tt}_1$.

We consider two subcases:
(a) Both $\mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1)$ and $\mathsf{symb}_2(\Gamma|_2) \cap \mathsf{symb}_2(\varphi|_2)$ are non empty. Then, for each $k = 1, 2$, there is $\Theta_k \subseteq L_k$ such that

$$(\dagger)\ \mathsf{symb}_k(\Theta_k) \subseteq \mathsf{symb}_k(\Gamma|_k) \cap \mathsf{symb}_k(\varphi|_k);$$
$$(\ddagger)\ \Gamma|_k \vdash_k \theta_k \text{ for each } \theta_k \in \Theta_k, \quad \text{and} \quad \Theta_k \vdash_k \varphi|_k.$$

Take $f_1$ and $f_2$ fulfilling the conditions above and let

$$\Theta = \tau_1(f_1(\Theta_1)) \cup \tau_2(f_2(\Theta_2)).$$

Then, $\Theta$ is an interpolant for $\Gamma \vdash_{\lceil 12 \rceil} \varphi$. Indeed:
(i) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta) \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$.
Let $c \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta)$. Assume that $c \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\tau_1(f_1(\Theta_1)))$. Then,

$$c \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(f_1(\Theta_1)).$$

Thus, $c = f_1(c_1)$ and, so, $c \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$. The other case follows similarly.
(ii) $\Gamma \vdash_{\lceil 12 \rceil} \theta$ for each $\theta \in \Theta$. Assume, without loss of generality, that $\theta \in \tau_1(f_1(\Theta_1))$. Let $\theta = \tau_1(f_1(\theta_1))$ where $f_1(\theta_1)$ is obtained from $\theta_1 \in \Theta_1$ by replacing each propositional symbol $c_1$ by $f_1(c_1)$. Observe that $\Gamma \vdash_{\lceil 12 \rceil} \Gamma|_1$ by cLFT. Hence, by $(\ddagger)$,

$$\Gamma \vdash_{\lceil 12 \rceil} \theta_1.$$

Therefore, by Proposition 4.1,

$$\Gamma \vdash_{\lceil 12 \rceil} \theta|_1.$$

On the other hand, $\vdash_{\lceil 12 \rceil} \theta|_2$ by Proposition 4.2. Thus, the result follows by LFT.

(iii) $\Theta \vdash_{\lceil 12 \rceil} \varphi$. Assume, without loss of generality, that the main constructor of $\varphi$ is in $\Sigma_1$. Observe, by cLFT, that

$$\Theta \vdash_{\lceil 12 \rceil} \Theta|_1.$$

Hence, by Proposition 4.1

$$\Theta \vdash_{\lceil 12 \rceil} \Theta_1$$

and so, by (‡), $\Theta \vdash_{\lceil 12 \rceil} \varphi|_1$. On the other hand, $\Theta \vdash_{\lceil 12 \rceil} \varphi|_2$ since the main constructor of $\varphi|_2$ is $\mathtt{tt}_2^{(n)}$ for some $n$. The result follows by LFT.

(b) Otherwise, without loss of generality, let $\mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1) \neq \emptyset$ and $\mathsf{symb}_2(\Gamma|_2) \cap \mathsf{symb}_2(\varphi|_2) = \emptyset$. Then, there is $\Theta_1 \subseteq L_1$ such that

(††) $\mathsf{symb}_1(\Theta_1) \subseteq \mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1)$;

(‡‡) $\Gamma|_1 \vdash_1 \theta_1$ for each $\theta_1 \in \Theta_1$, and $\Theta_1 \vdash_1 \varphi|_1$.

In this situation, $f_1$ as used in (a) does not help since $f_1(\Theta_1) = \Theta_1$. Accordingly, let

$$\Theta = \tau_1(\Theta_1).$$

Then, $\Theta$ is an interpolant for $\Gamma \vdash_{\lceil 12 \rceil} \varphi$. Indeed:

(i) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta) \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$.

Let $c \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta)$. Since $\Theta_1 \subseteq L_1$ then,

$$\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta) = \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta_1) = \mathsf{symb}_1(\Theta_1)$$

and, therefore, $c = \lceil c_1 \mathtt{tt}_2^{(0)} \rceil = c_1$. Since $\mathsf{symb}_2(\Gamma|_2) \cap \mathsf{symb}_2(\varphi|_2) = \emptyset$ and $c \in \mathsf{symb}_1(\Gamma|_1) \cap \mathsf{symb}_1(\varphi|_1)$ we can consider two cases:

- either $\lceil c_1 c_2 \rceil$ or $c_1$ occurs in $\Gamma$, $\lceil c_1 c_2 \rceil$ does not occur in $\varphi$ and $c_1$ occurs in $\varphi$.

- either $\lceil c_1 c_2 \rceil$ or $c_1$ occurs in $\varphi$, $\lceil c_1 c_2 \rceil$ does not occur in $\Gamma$ and $c_1$ occurs in $\Gamma$.

Then, in both cases, $c_1 \in \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$.

(ii) $\Gamma \vdash_{\lceil 12 \rceil} \theta$ for each $\theta \in \Theta$. Let $\theta = \tau_1(\theta_1)$. Observe that $\Gamma \vdash_{\lceil 12 \rceil} \Gamma|_1$ by cLFT. So, by (‡‡), $\Gamma \vdash_{\lceil 12 \rceil} \theta_1$ and, thus, by Proposition 4.1,

$$\Gamma \vdash_{\lceil 12 \rceil} \theta|_1.$$

Moreover, since $f_1(\theta_1) = \theta_1$, by Proposition 4.2, $\vdash_{\lceil 12 \rceil} \theta|_2$ and, so,

$$\Gamma \vdash_{\lceil 12 \rceil} \theta|_2.$$

Therefore, the result follows by LFT.

(iii) $\Theta \vdash_{\lceil 12 \rceil} \varphi$. Observe, by cLFT, that $\Theta \vdash_{\lceil 12 \rceil} \Theta|_1$. Hence, by Proposition 4.1, $\Theta \vdash_{\lceil 12 \rceil} \Theta_1$ and, so, by (‡‡),

$$\Theta \vdash_{\lceil 12 \rceil} \varphi|_1.$$

Moreover, $\Gamma|_2 \vdash_{\lceil 12 \rceil} \varphi|_2$ by Propositions 2.4 and 2.1. So, $\vdash_2 \varphi|_2$ because $\mathcal{L}_2$ enjoys the Craig interpolation property, $\mathsf{symb}_2(\Gamma|_2) \cap \mathsf{symb}_2(\varphi|_2) = \emptyset$ and $\Gamma|_2 \not\vdash_2 \mathsf{ff}_2$. Hence, by Proposition 2.1 $\vdash_{\lceil 12 \rceil} \varphi|_2$ and, so,

$$\Theta \vdash_{\lceil 12 \rceil} \varphi|_2.$$

The result follows by LFT. QED

Therefore, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are suitable matrix logics with identity, with componentship relation coinciding with the diagonal and enjoying the relaxed Craig interpolation property, then, their meet-combination $\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil$ also enjoys the relaxed Craig interpolation property.

Furthermore, as an immediate corollary of Theorem 4.4, we obtain: *given two axiomatized suitable matrix logics with identity and enjoying the Craig interpolation property, their product enjoys the relaxed Craig interpolation property.*

Observe also that these results are easily extended to the meet-combination and product of $n$ matrix logics. Therefore, one can use them for establishing the relaxed Craig interpolation property of the combination of combinations of logics by flattening.

Finally, note that the relaxed interpolant obtained in the proof of Theorem 4.4 coincides with the usual notion of interpolant when

$$(\mathsf{symb}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}_{\lceil 12 \rceil}(\varphi))^{\sqsubseteq} = \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi).$$

Indeed in this case, if $\Gamma \vdash_{\lceil 12 \rceil} \varphi$ and

$$\mathsf{symb}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}_{\lceil 12 \rceil}(\varphi) \neq \emptyset$$

then

$$\mathsf{symb}_{\lceil 12\rceil}(\Theta) \subseteq \mathsf{symb}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}_{\lceil 12\rceil}(\varphi)$$

where $\Theta$ is the interpolant obtained following the proof of the theorem.

We now investigate preservation of interpolation in the presence of interaction axioms. We start by defining two relevant notions. Given a set $\Gamma \cup \{\varphi, \psi\} \subseteq L_{\lceil 12\rceil}$, the formula $\psi$ is *separable* for $\Gamma$ and $\varphi$ whenever:

(SH) either $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\psi) \subseteq (Q_{\lceil 12\rceil} \setminus \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma)) \cup (\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi))$;

(SC) or $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\psi) \subseteq (Q_{\lceil 12\rceil} \setminus \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi)) \cup (\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi))$.

Moreover, the set $\Psi \subseteq L_{\lceil 12\rceil}$ is *separable* for $\Gamma$ and $\varphi$ whenever:

- each $\psi \in \Psi$ is separable for $\Gamma$ and $\varphi$;
- given $\psi', \psi'' \in \Psi$ such that $\psi'$ satisfies (SC) and $\psi''$ satisfies (SH) then $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\psi') \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\psi'') \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi)$.

**Theorem 4.5 (Preservation of interpolation with interaction)**
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be suitable matrix logics with identity and enjoying the Craig interpolation property, the MTD and the MP with respect to $\supset_1$ and $\supset_2$, respectively. Let Ax a set of interaction axioms. Assume that $d$ is a derivation sequence for

$$\Gamma \vdash_{\lceil 12\rceil + \mathrm{Ax}} \varphi$$

where the set of instances of axioms in Ax used in $d$ is separable for $\Gamma$ and $\varphi$. Then, $\Gamma \vdash_{\lceil 12\rceil + \mathrm{Ax}} \varphi$ has a relaxed Craig interpolant.

**Proof:** We consider two cases.

(A) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi) = \emptyset$.

Then, $\mathsf{symb}_k(\Gamma|_k) \cap \mathsf{symb}_k(\varphi|_k) = \emptyset$ for $k = 1, 2$. Hence, for each $k = 1, 2$, either $\Gamma|_k \vdash_k \mathsf{ff}_k$ or $\vdash_k \varphi|_k$. Therefore, either $\Gamma \vdash_{\lceil 12\rceil + \mathrm{Ax}} \lceil \mathsf{ff}_1 \mathsf{ff}_2 \rceil$ or $\vdash_{\lceil 12\rceil + \mathrm{Ax}} \varphi$ since either $\Gamma \vdash_{\lceil 12\rceil} \lceil \mathsf{ff}_1 \mathsf{ff}_2 \rceil$ or $\vdash_{\lceil 12\rceil} \varphi$ with a proof similar to case (A) in the proof of Theorem 4.4.

(B) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi) \neq \emptyset$.

(1) $d$ uses an FX rule. Then, $\{\mathsf{ff}_1\}$ is an interpolant for $\Gamma \vdash_{\lceil 12\rceil + \mathrm{Ax}} \varphi$. Indeed:

(i) $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\{\mathsf{ff}_1\}) = \emptyset \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi)$; (ii) $\Gamma \vdash_{\lceil 12\rceil + \mathrm{Ax}} \mathsf{ff}_1$ and (iii) $\{\mathsf{ff}_1\} \vdash_{\lceil 12\rceil + \mathrm{Ax}} \varphi$.

(2) No FX rules were used in $d$. Let $\Psi' = \{\psi'_1, \ldots, \psi'_{n'}\}$ be the set of instances

of axioms in Ax occurring in $d$ satisfying (SC) and $\Psi'' = \{\psi''_1, \ldots, \psi''_{n''}\}$ be the set of instances of axioms in Ax occurring in $d$ satisfying (SH). Then

$$\Gamma, \psi'_1, \ldots, \psi'_{n'}, \psi''_1, \ldots, \psi''_{n''} \vdash_{\lceil 12 \rceil} \varphi.$$

Hence, using $m''$-times the MTD, see Theorem 2.5,

$$(\dagger)\ \Gamma, \psi'_1, \ldots, \psi'_{n'} \vdash_{\lceil 12 \rceil} \psi''_1 \lceil \supset_1 \supset_2 \rceil (\ldots (\psi''_{n''} \lceil \supset_1 \supset_2 \rceil \varphi) \ldots).$$

By Theorem 4.4, let $\Theta$ be the interpolant for $(\dagger)$. Then

$$\Gamma \vdash_{\lceil 12 \rceil + \mathrm{Ax}} \Theta$$

and

$$\Theta \vdash_{\lceil 12 \rceil + \mathrm{Ax}} \varphi$$

since

$$\Theta, \psi''_1, \ldots, \psi''_{n''} \vdash_{\lceil 12 \rceil} \varphi$$

using $n''$-times the MP (see Theorem 2.6). It remains to show $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta) \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$. Indeed,

- $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Theta) \subseteq (\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cup \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Psi')) \cap (\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \cup \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Psi''))$;
- $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Psi') \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$;
- $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Psi'') \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$;
- $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Psi') \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Psi'') \subseteq \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi)$.

So $\Theta$ is also a relaxed Craig interpolant for $\Gamma \vdash_{\lceil 12 \rceil + \mathrm{Ax}} \varphi$. QED

We now present an example of non preservation of interpolation in the presence of interaction when the conditions of Theorem 4.5 are not fullfilled. Let CPL be classical propositional logic and $\lceil \mathsf{CPLCPL} \rceil$ the meet combination of CPL with CPL. Assume that Ax is a singleton with the axiom

$$\lceil \mathtt{tt}\, p_1 \rceil \lceil \supset \supset \rceil \lceil p_2\, \mathtt{tt} \rceil .$$

Then

$$(*) \quad \lceil \mathtt{tt}\, p_1 \rceil \vdash_{\lceil 12 \rceil + \mathrm{Ax}} \lceil p_2\, \mathtt{tt} \rceil$$

and there is no relaxed Craig interpolant for the obvious derivation sequence of (*) since

- $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\lceil \mathsf{tt} p_1\rceil) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\lceil p_2 \mathsf{tt}\rceil) = \emptyset$;
- $\lceil \mathsf{tt} p_1\rceil \not\vdash_{\lceil 12\rceil + \mathrm{Ax}} \mathsf{ff}$;
- $\not\vdash_{\lceil 12\rceil + \mathrm{Ax}} \lceil p_2 \mathsf{tt}\rceil$.

Observe that Theorem 4.5 is not applicable since $\lceil \mathsf{tt} p_1\rceil \lceil \supset\supset\rceil \lceil p_2 \mathsf{tt}\rceil$ is not separable for $\lceil \mathsf{tt} p_1\rceil$ and $\lceil p_2 \mathsf{tt}\rceil$.

As an immediate consequence of Theorem 4.5 we get the following result that guarantees the existence of interpolant independently of the derivation sequence at hand when the additional axioms introduce interaction in a very restricted way.

**Corollary 4.6**
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be suitable matrix logics with identity and enjoying the Craig interpolation property, the MTD and the MP with respect to $\supset_1$ and $\supset_2$, respectively. Let $\lceil c_1 c_2\rceil$ be a propositional symbol of $\lceil \mathcal{L}_1 \mathcal{L}_2\rceil$ and Ax a set of interaction axioms each with no schema variables and with no propositional symbols besides $\lceil c_1 c_2\rceil$. Then, $\lceil \mathcal{L}_1 \mathcal{L}_2\rceil_{\mathrm{Ax}}$ has relaxed Craig interpolation.

# 5 Worked examples

Taking advantage of the constructive nature of the proofs of Theorem 4.4 and Theorem 4.5, we proceed to illustrate the computation of interpolants in three representative scenarios, one of them with interaction.

### Matrix product of classical and intuitionistic logics

We start by considering the classical propositional logic $\mathsf{CPL} = (\Sigma_{\mathsf{C}}, \Delta_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}})$ over the set $\{q_{\mathsf{C}j} : j \in \mathbb{N}\}$ of propositional symbols and the intuitionistic propositional logic $\mathsf{IPL} = (\Sigma_{\mathsf{I}}, \Delta_{\mathsf{I}}, \mathcal{M}_{\mathsf{I}})$ over the set $\{q_{\mathsf{I}j} : j \in \mathbb{N}\}$ of propositional symbols, as defined in [20]. Clearly, these two logics are both suitable and with identity ($\mathsf{id}_{\mathsf{C}}\varphi$ defined as $\mathsf{tt}_{\mathsf{C}} \supset_{\mathsf{C}} \varphi$ and $\mathsf{id}_{\mathsf{I}}\varphi$ as $\mathsf{tt}_{\mathsf{I}} \supset_{\mathsf{I}} \varphi$). Accordingly,

- $\Sigma_{\mathsf{C}0} = \{q_{\mathsf{C}j} : j \in \mathbb{N}\} \cup \{\mathsf{tt}_{\mathsf{C}}, \mathsf{ff}_{\mathsf{C}}\}$;
  $\Sigma_{\mathsf{C}1} = \{\neg_{\mathsf{C}}, \mathsf{id}_{\mathsf{C}}, \mathsf{tt}^{(1)}_{\mathsf{C}}\}$;
  $\Sigma_{\mathsf{C}2} = \{\supset_{\mathsf{C}}, \wedge_{\mathsf{C}}, \vee_{\mathsf{C}}, \mathsf{tt}^{(2)}_{\mathsf{C}}\}$;
  $\Sigma_{\mathsf{C}n} = \{\mathsf{tt}^{(n)}_{\mathsf{C}}\}$ for $n \geq 3$.

- $\Sigma_{\mathsf{I}0} = \{q_{\mathsf{I}j} : j \in \mathbb{N}\} \cup \{\mathsf{tt}_{\mathsf{I}}, \mathsf{ff}_{\mathsf{I}}\}$;
  $\Sigma_{\mathsf{I}1} = \{\neg_{\mathsf{I}}, \mathsf{id}_{\mathsf{I}}, \mathsf{tt}_{\mathsf{I}}^{(1)}\}$;
  $\Sigma_{\mathsf{I}2} = \{\supset_{\mathsf{I}}, \wedge_{\mathsf{I}}, \vee_{\mathsf{I}}, \mathsf{tt}_{\mathsf{I}}^{(2)}\}$;
  $\Sigma_{\mathsf{I}n} = \{\mathsf{tt}_{\mathsf{I}}^{(n)}\}$ for $n \geq 3$.

Let CIPL be $\lceil \mathsf{CPL}\,\mathsf{IPL} \rceil$. Observe that, by Theorem 4.4, CIPL has the relaxed Craig interpolation since:

- CPL and IPL enjoy theoremhood Craig interpolation (see [11]);
- and, so, CPL and IPL enjoy (turnstile) Craig interpolation, since they have the metatheorems of deduction and conjunction.

In the sequel, we denote by $\neg_{\mathsf{CI}}$, $\wedge_{\mathsf{CI}}$ and $\vee_{\mathsf{CI}}$ the meet-combined constructors $\lceil \neg_{\mathsf{C}} \neg_{\mathsf{I}} \rceil$, $\lceil \wedge_{\mathsf{C}} \wedge_{\mathsf{I}} \rceil$ and $\lceil \vee_{\mathsf{C}} \vee_{\mathsf{I}} \rceil$, respectively. Moreover, we denote by $q_{\mathsf{CI}j}$ the meet-combined constructor $\lceil q_{\mathsf{C}j} q_{\mathsf{I}j} \rceil$ for $j \in \mathbb{N}$.

We now illustrate the interpolant construction for

$$(\dagger) \quad q_{\mathsf{CI}1} \wedge_{\mathsf{CI}} (\neg_{\mathsf{CI}}\, q_{\mathsf{CI}2}) \vdash_{\mathsf{CI}} (\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{CI}}\, q_{\mathsf{CI}2}) \vee_{\mathsf{CI}} q_{\mathsf{CI}3}.$$

following the proof of Theorem 4.4. Observe that there is a derivation sequence for (†) not using the FX rules. Then, following that proof we observe that:

(a) $\mathsf{symb}^{\sqsubseteq}_{\mathsf{CI}}(q_{\mathsf{CI}1} \wedge_{\mathsf{CI}} (\neg_{\mathsf{CI}}\, q_{\mathsf{CI}2})) \cap \mathsf{symb}^{\sqsubseteq}_{\mathsf{CI}}((\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{CI}}\, q_{\mathsf{CI}2}) \vee_{\mathsf{CI}} q_{\mathsf{CI}3}) = \{q_{\mathsf{C}2}, q_{\mathsf{I}2}, q_{\mathsf{CI}2}\}$.

(b) $\mathsf{symb}_{\mathsf{C}}(q_{\mathsf{C}1} \wedge_{\mathsf{C}} (\neg_{\mathsf{C}}\, q_{\mathsf{C}2})) \cap \mathsf{symb}_{\mathsf{C}}((\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{C}}\, q_{\mathsf{C}2}) \vee_{\mathsf{C}} q_{\mathsf{C}3}) = \{q_{\mathsf{C}2}\}$.

(c) $\mathsf{symb}_{\mathsf{I}}(q_{\mathsf{I}1} \wedge_{\mathsf{I}} (\neg_{\mathsf{I}}\, q_{\mathsf{I}2})) \cap \mathsf{symb}_{\mathsf{I}}((\mathsf{tt}_{\mathsf{I}}^{(1)}(\mathsf{tt}_{\mathsf{I}}^{(1)}(\neg_{\mathsf{I}}(q_{\mathsf{I}2})))) \vee_{\mathsf{I}} q_{\mathsf{I}3}) = \{q_{\mathsf{I}2}\}$.

(d) $q_{\mathsf{C}1} \wedge_{\mathsf{C}} (\neg_{\mathsf{C}}\, q_{\mathsf{C}2}) \vdash_{\mathsf{C}} (\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{C}}\, q_{\mathsf{C}2}) \vee_{\mathsf{C}} q_{\mathsf{C}3}$.

(e) $q_{\mathsf{I}1} \wedge_{\mathsf{I}} (\neg_{\mathsf{I}}\, q_{\mathsf{I}2}) \vdash (\mathsf{tt}_{\mathsf{I}}^{(1)}(\mathsf{tt}_{\mathsf{I}}^{(1)}(\neg_{\mathsf{I}}(q_{\mathsf{I}2})))) \vee_{\mathsf{I}} q_{\mathsf{I}3}$.

(f) $\Theta_{\mathsf{C}} = \{\neg_{\mathsf{C}}\, q_{\mathsf{C}2}\}$ is an interpolant for (d).

(g) $\Theta_{\mathsf{I}} = \{\neg_{\mathsf{I}}\, q_{\mathsf{I}2}\}$ is an interpolant for (e).

(h) $\Theta_{\mathsf{CI}} = \{\neg_{\mathsf{C}}\, q_{\mathsf{CI}2}, \neg_{\mathsf{I}}\, q_{\mathsf{CI}2}\}$ is an interpolant for (†). Indeed:

($\mathrm{h}_1$) $q_{\mathsf{CI}1} \wedge_{\mathsf{CI}} (\neg_{\mathsf{CI}}\, q_{\mathsf{CI}2}) \vdash_{\mathsf{CI}} \Theta_{\mathsf{CI}}$ since

| | | |
|---|---|---|
| 1 | $q_{\mathsf{CI}1} \wedge_{\mathsf{CI}} (\neg_{\mathsf{CI}}\, q_{\mathsf{CI}2})$ | HYP |
| 2 | $q_{\mathsf{C}1} \wedge_{\mathsf{C}} (\neg_{\mathsf{C}}\, q_{\mathsf{C}2})$ | cLFT 1 |
| 3 | $\neg_{\mathsf{C}}\, q_{\mathsf{C}2}$ | TAUT 2 |
| 4 | $\mathsf{tt}_{\mathsf{I}}^{(1)}(q_{\mathsf{I}2})$ | $\mathsf{tt}_{\mathsf{I}}^{(1)}$ |
| 5 | $\neg_{\mathsf{C}}\, q_{\mathsf{CI}2}$ | LFT 3, 4 |

and similarly for $\neg_{\mathsf{I}}\, q_{\mathsf{CI}2}$.

(h$_2$) $\Theta_{\mathsf{CI}} \vdash_{\mathsf{CI}} (\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{CI}}\, q_{\mathsf{CI}2}) \vee_{\mathsf{CI}} q_{\mathsf{CI}3}$ since

| | | |
|---|---|---|
| 1 | $\neg_{\mathsf{C}}\, q_{\mathsf{CI}2}$ | HYP |
| 2 | $\neg_{\mathsf{I}}\, q_{\mathsf{CI}2}$ | HYP |
| 3 | $\neg_{\mathsf{C}}\, q_{\mathsf{C}2}$ | cLFT 1 |
| 4 | $(\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{C}}\, q_{\mathsf{C}2}) \vee_{\mathsf{C}} q_{\mathsf{C}3}$ | TAUT 3 |
| 5 | $\mathsf{tt}_{\mathsf{I}}^{(1)}(\mathsf{tt}_{\mathsf{I}}^{(1)}(\neg_{\mathsf{I}}\, q_{\mathsf{I}2}))$ | $\mathsf{tt}_{\mathsf{I}}^{(1)}$ |
| 6 | $(\mathsf{tt}_{\mathsf{I}}^{(1)}(\mathsf{tt}_{\mathsf{I}}^{(1)}(\neg_{\mathsf{I}}\, q_{\mathsf{I}2}))) \vee_{\mathsf{I}} q_{\mathsf{I}3}$ | TAUT 5 |
| 7 | $(\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{CI}}\, q_{\mathsf{CI}2}) \vee_{\mathsf{CI}} q_{\mathsf{CI}3}$ | LFT 4, 6 |

(h$_3$) $\mathsf{symb}_{\mathsf{CI}}^{\sqsubseteq}(\Theta_{\mathsf{CI}}) \subseteq \mathsf{symb}_{\mathsf{CI}}^{\sqsubseteq}(q_{\mathsf{CI}1} \wedge_{\mathsf{CI}} (\neg_{\mathsf{CI}}\, q_{\mathsf{CI}2})) \cap \mathsf{symb}_{\mathsf{CI}}^{\sqsubseteq}((\neg_{\mathsf{C}} \neg_{\mathsf{C}} \neg_{\mathsf{CI}}\, q_{\mathsf{CI}2}) \vee_{\mathsf{CI}} q_{\mathsf{CI}3})$. Immediate since $\mathsf{symb}_{\mathsf{CI}}^{\sqsubseteq}(\Theta_{\mathsf{CI}}) = \{q_{\mathsf{C}2}, q_{\mathsf{I}2}, q_{\mathsf{CI}2}\}$.

## Matrix product of modal logics

Consider the S4 modal logic $\mathsf{MSPL} = (\Sigma_{\mathsf{S}}, \Delta_{\mathsf{S}}, \mathcal{M}_{\mathsf{S}})$ over the set $\{q_{\mathsf{S}j} : j \in \mathbb{N}\}$ of propositional symbols, as defined in [20]. Let $\mathsf{INTL} = (\Sigma_{\mathsf{L}}, \Delta_{\mathsf{L}}, \mathcal{M}_{\mathsf{L}})$ over the set $\{q_{\mathsf{L}j} : j \in \mathbb{N}\}$ of propositional symbols be the propositional interpretability logic presented in [1]. Clearly, these logics are both suitable and with identity (introduced as in CPL). Accordingly:

- $\Sigma_{\mathsf{S}0} = \{q_{\mathsf{S}j} : j \in \mathbb{N}\} \cup \{\mathsf{tt}_{\mathsf{S}}, \mathsf{ff}_{\mathsf{S}}\}$;
  $\Sigma_{\mathsf{S}1} = \{\neg_{\mathsf{S}}, \Box_{\mathsf{S}}, \mathsf{id}_{\mathsf{S}}, \mathsf{tt}_{\mathsf{S}}^{(1)}\}$;
  $\Sigma_{\mathsf{S}2} = \{\supset_{\mathsf{S}}, \wedge_{\mathsf{S}}, \vee_{\mathsf{S}}, \mathsf{tt}_{\mathsf{S}}^{(2)}\}$;
  $\Sigma_{\mathsf{S}n} = \{\mathsf{tt}_{\mathsf{S}}^{(n)}\}$ for $n \geq 3$.
- $\Sigma_{\mathsf{L}0} = \{q_{\mathsf{L}j} : j \in \mathbb{N}\} \cup \{\mathsf{tt}_{\mathsf{L}}, \mathsf{ff}_{\mathsf{L}}\}$;
  $\Sigma_{\mathsf{L}1} = \{\neg_{\mathsf{L}}, \Box_{\mathsf{L}}, \mathsf{id}_{\mathsf{L}}, \mathsf{tt}_{\mathsf{L}}^{(1)}\}$;
  $\Sigma_{\mathsf{L}2} = \{\supset_{\mathsf{L}}, \wedge_{\mathsf{L}}, \vee_{\mathsf{L}}, \rhd_{\mathsf{L}}, \mathsf{tt}_{\mathsf{L}}^{(2)}\}$;
  $\Sigma_{\mathsf{L}n} = \{\mathsf{tt}_{\mathsf{L}}^{(n)}\}$ for $n \geq 3$

Let SL be $\lceil \mathsf{MSPL}\ \mathsf{INTL} \rceil$. Then, by Theorem 4.4, SL has the relaxed Craig interpolation since:

- MSPL has theoremhood Craig interpolation (see [10, 7]).
- MSPL enjoys (turnstile) Craig interpolation. Indeed assume that $\varphi \vdash_{\mathsf{MSPL}} \psi$ and $\mathsf{symb}_{\mathsf{S}}(\varphi) \cap \mathsf{symb}_{\mathsf{S}}(\psi) \neq \emptyset$. Then, by the metatheorem of deduction,
$$\vdash_{\mathsf{MSPL}} (\Box_{\mathsf{S}}\, \varphi) \supset_{\mathsf{S}} \psi.$$

Using the theoremhood Craig interpolation, there is a formula $\theta$ such that $\mathsf{symb}_{\mathsf{S}}(\theta) \subseteq \mathsf{symb}_{\mathsf{S}}(\Box_{\mathsf{S}}\,\varphi) \cap \mathsf{symb}_{\mathsf{S}}(\psi)$, $\vdash_{\mathsf{MSPL}} (\Box_{\mathsf{S}}\,\varphi) \supset \theta$ and $\vdash_{\mathsf{MSPL}} \theta \supset_{\mathsf{S}} \psi$. We now prove that $\theta$ is also the interpolant for $\varphi \vdash_{\mathsf{MSPL}} \psi$. Indeed:
(a) $\varphi \vdash_{\mathsf{MSPL}} \theta$. A derivation sequence is easily built using necessitation and tautological reasoning;
(b) $\theta \vdash_{\mathsf{MSPL}} \varphi$ by MP;
(c) $\mathsf{symb}_{\mathsf{S}}(\theta) \subseteq \mathsf{symb}_{\mathsf{S}}(\varphi) \cap \mathsf{symb}_{\mathsf{S}}(\psi)$ since $\mathsf{symb}_{\mathsf{S}}(\varphi) = \mathsf{symb}_{\mathsf{S}}(\Box_{\mathsf{S}}\,\varphi)$.

- $\mathsf{INTL}$ enjoys (turnstile) Craig interpolation (see [1]).

In the sequel, we denote by $\neg_{\mathsf{SL}}$, $\Box_{\mathsf{SL}}$, $\wedge_{\mathsf{SL}}$ and $\vee_{\mathsf{SL}}$ the meet-combined constructors $\lceil \neg_{\mathsf{S}}\, \neg_{\mathsf{L}} \rceil$, $\lceil \Box_{\mathsf{S}}\, \Box_{\mathsf{L}} \rceil$, $\lceil \wedge_{\mathsf{S}} \wedge_{\mathsf{L}} \rceil$ and $\lceil \vee_{\mathsf{S}} \vee_{\mathsf{L}} \rceil$, respectively. Moreover, we denote by $q_{\mathsf{SL}j}$ the meet-combined constructor $\lceil q_{\mathsf{S}j} q_{\mathsf{L}j} \rceil$ for $j \in \mathbb{N}$.

We now illustrate the interpolant construction for

$$(+) \quad q_{\mathsf{SL}1} \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}}\, q_{\mathsf{SL}2}) \vdash_{\mathsf{SL}} q_{\mathsf{SL}1} \lceil \wedge_{\mathsf{S}} \vee_{\mathsf{L}} \rceil\, q_{\mathsf{SL}2}.$$

This case will illustrate the role of the identity constructor. A derivation sequence for (+) is as follows:

| | | |
|---|---|---|
| 1 | $q_{\mathsf{SL}1} \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}}\, q_{\mathsf{SL}2})$ | HYP |
| 2 | $q_{\mathsf{S}1} \wedge_{\mathsf{S}} (\Box_{\mathsf{S}}\, q_{\mathsf{S}2})$ | cLFT 1 |
| 3 | $q_{\mathsf{S}1}$ | $\mathrm{TAUT}_{\mathsf{S}}$ 2 |
| 4 | $\Box_{\mathsf{S}}\, q_{\mathsf{S}2}$ | $\mathrm{TAUT}_{\mathsf{S}}$ 2 |
| 5 | $(\Box_{\mathsf{S}}\, q_{\mathsf{S}2}) \supset_{\mathsf{S}} q_{\mathsf{S}2}$ | $\mathrm{T}_{\mathsf{S}}$ |
| 6 | $q_{\mathsf{S}2}$ | $\mathrm{MP}_{\mathsf{S}}$ $4,5$ |
| 7 | $q_{\mathsf{S}1} \wedge_{\mathsf{S}} q_{\mathsf{S}2}$ | $\mathrm{TAUT}_{\mathsf{S}}$ $3,6$ |
| 8 | $q_{\mathsf{L}1} \wedge_{\mathsf{L}} \mathtt{tt}^{(1)}_{\mathsf{L}}(q_{\mathsf{L}2})$ | cLFT 1 |
| 9 | $q_{\mathsf{L}1}$ | $\mathrm{TAUT}_{\mathsf{L}}$ 8 |
| 10 | $q_{\mathsf{L}1} \vee_{\mathsf{L}} q_{\mathsf{L}2}$ | $\mathrm{TAUT}_{\mathsf{L}}$ 9 |
| 11 | $q_{\mathsf{SL}1} \lceil \wedge_{\mathsf{S}} \vee_{\mathsf{L}} \rceil\, q_{\mathsf{SL}2}$ | LFT $7,10$. |

Then, following the proof of Theorem 4.4, we have:
(a) $\mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(q_{\mathsf{SL}1} \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}}\, q_{\mathsf{SL}2})) \cap \mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(q_{\mathsf{SL}1} \lceil \wedge_{\mathsf{S}} \vee_{\mathsf{L}} \rceil\, q_{\mathsf{SL}2}) = \{q_{\mathsf{SL}1}, q_{\mathsf{SL}2}, q_{\mathsf{S}1}, q_{\mathsf{L}1}, q_{\mathsf{S}2}, q_{\mathsf{L}2}\}$.
(b) $\mathsf{symb}_{\mathsf{S}}(q_{\mathsf{S}1} \wedge_{\mathsf{S}} (\Box_{\mathsf{S}}\, q_{\mathsf{S}2})) \cap \mathsf{symb}_{\mathsf{S}}(q_{\mathsf{S}1} \wedge_{\mathsf{S}} q_{\mathsf{S}2}) = \{q_{\mathsf{S}1}, q_{\mathsf{S}2}\}$.
(c) $\mathsf{symb}_{\mathsf{L}}(q_{\mathsf{L}1} \wedge_{\mathsf{L}} \mathtt{tt}^{(1)}_{\mathsf{L}}(q_{\mathsf{L}2})) \cap \mathsf{symb}_{\mathsf{L}}(q_{\mathsf{L}1} \vee_{\mathsf{L}} q_{\mathsf{L}2}) = \{q_{\mathsf{L}1}, q_{\mathsf{L}2}\}$.
(d) $q_{\mathsf{S}1} \wedge_{\mathsf{S}} (\Box_{\mathsf{S}}\, q_{\mathsf{S}2}) \vdash_{\mathsf{L}} q_{\mathsf{S}1} \wedge_{\mathsf{S}} q_{\mathsf{S}2}$.

(e) $q_{\mathsf{L}1} \wedge_{\mathsf{L}} \mathsf{tt}^{(1)}_{\mathsf{L}}(q_{\mathsf{L}2}) \vdash_{\mathsf{S}} q_{\mathsf{L}1} \vee_{\mathsf{L}} q_{\mathsf{L}2}$.

(f) $\Theta_{\mathsf{S}} = \{q_{\mathsf{S}1}, q_{\mathsf{S}2}\}$ is an interpolant for (d).

(g) $\Theta_{\mathsf{L}} = \{q_{\mathsf{L}1}\}$ is an interpolant for (e).

(h) $\Theta_{\mathsf{SL}} = \{\lceil \mathsf{id}_{\mathsf{S}} \mathsf{tt}^{(1)}_{\mathsf{L}} \rceil(q_{\mathsf{SL}1}), \lceil \mathsf{id}_{\mathsf{S}} \mathsf{tt}^{(1)}_{\mathsf{L}} \rceil(q_{\mathsf{SL}2}), \lceil \mathsf{tt}^{(1)}_{\mathsf{S}} \mathsf{id}_{\mathsf{L}} \rceil(q_{\mathsf{SL}1})\}$ is an interpolant for (+). Indeed:

($h_1$) $q_{\mathsf{SL}1} \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}}\, q_{\mathsf{SL}2}) \vdash_{\mathsf{SL}} \Theta_{\mathsf{SL}}$.

($h_2$) $\Theta_{\mathsf{SL}} \vdash_{\mathsf{SL}} q_{\mathsf{SL}1} \lceil \wedge_{\mathsf{S}} \vee_{\mathsf{L}} \rceil\, q_{\mathsf{SL}2}$.

($h_3$) $\mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(\Theta_{\mathsf{SL}}) \subseteq \mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(q_{\mathsf{SL}1} \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}}\, q_{\mathsf{SL}2})) \cap \mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(q_{\mathsf{SL}1} \lceil \wedge_{\mathsf{S}} \vee_{\mathsf{L}} \rceil\, q_{\mathsf{SL}2})$.

We now illustrate the interpolant construction for

$$(\ddagger) \quad ((\Box_{\mathsf{L}}\, q_{\mathsf{SL}1}) \supset_{\mathsf{L}} q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{SL}1} \wedge_{\mathsf{S}} q_{\mathsf{L}2})) \vdash_{\mathsf{SL}} (\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} ((\Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}) \supset_{\mathsf{SL}} (\Box_{\mathsf{SL}} \Box_{\mathsf{SL}}\, q_{\mathsf{SL}2})).$$

For a better understanding we build a derivation sequence for (‡) in two parts. Let $d_1$ be the derivation sequence

| | | |
|---|---|---|
| 1 | $((\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \supset_{\mathsf{L}} q_{\mathsf{L}1}) \wedge_{\mathsf{L}} (\mathsf{tt}^{(1)}_{\mathsf{L}}(\mathsf{tt}^{(1)}_{\mathsf{L}}(\mathsf{tt}^{(2)}_{\mathsf{L}}(q_{\mathsf{L}1}, q_{\mathsf{L}2}))))$ | HYP |
| 2 | $(\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \supset_{\mathsf{L}} q_{\mathsf{L}1}$ | $\mathrm{TAUT}_{\mathsf{L}}$ 1 |
| 3 | $\Box_{\mathsf{L}}((\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \supset_{\mathsf{L}} q_{\mathsf{L}1})$ | $\mathrm{NEC}_{\mathsf{L}}$ 2 |
| 4 | $\Box_{\mathsf{L}}\, q_{\mathsf{L}1}$ | $\mathrm{Löb}_{\mathsf{L}} + \mathrm{MP}_{\mathsf{L}}$ 3 |
| 5 | $(\Box_{\mathsf{L}}\, q_{\mathsf{L}2}) \supset_{\mathsf{L}} (\Box_{\mathsf{L}} \Box_{\mathsf{L}}\, q_{\mathsf{L}2})$ | $4_{\mathsf{L}}$ |
| 6 | $(\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \wedge_{\mathsf{L}} ((\Box_{\mathsf{L}}\, q_{\mathsf{L}2}) \supset_{\mathsf{L}} (\Box_{\mathsf{L}} \Box_{\mathsf{L}}\, q_{\mathsf{L}2}))$ | $\mathrm{TAUT}_{\mathsf{L}}$ 4, 5 |

and $d_2$ the derivation sequence

| | | |
|---|---|---|
| 1 | $(\mathsf{tt}^{(2)}_{\mathsf{S}}(\mathsf{tt}^{(1)}_{\mathsf{S}}(q_{\mathsf{S}1}), q_{\mathsf{S}1})) \wedge_{\mathsf{S}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{S}1} \wedge_{\mathsf{S}} \mathsf{tt}^{(0)}_{\mathsf{S}}))$ | HYP |
| 2 | $\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{S}1} \wedge_{\mathsf{S}} \mathsf{tt}^{(0)}_{\mathsf{S}})$ | $\mathrm{TAUT}_{\mathsf{S}}$ 1 |
| 3 | $\Box_{\mathsf{S}}(q_{\mathsf{S}1} \wedge_{\mathsf{S}} \mathsf{tt}^{(0)}_{\mathsf{S}})$ | $\mathrm{T}_{\mathsf{S}} + \mathrm{MP}_{\mathsf{S}}$ 2 |
| 4 | $(\Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \wedge_{\mathsf{S}} (\Box_{\mathsf{S}}\, \mathsf{tt}^{(0)}_{\mathsf{S}})$ | $\mathrm{K}_{\mathsf{S}} + \mathrm{MP}_{\mathsf{S}}$ 3 |
| 5 | $\Box_{\mathsf{S}}\, q_{\mathsf{S}1}$ | $\mathrm{TAUT}_{\mathsf{S}}$ 4 |
| 6 | $(\Box_{\mathsf{S}}\, q_{\mathsf{S}2}) \supset_{\mathsf{S}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}\, q_{\mathsf{S}2})$ | $4_{\mathsf{S}}$ |
| 7 | $(\Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \wedge_{\mathsf{S}} ((\Box_{\mathsf{S}}\, q_{\mathsf{S}2}) \supset_{\mathsf{S}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}\, q_{\mathsf{S}2}))$ | $\mathrm{TAUT}_{\mathsf{S}}$ 5, 6 |

Hence, a derivation sequence for (‡) is as follows:

| | | |
|---|---|---|
| 1 | $((\Box_{\mathsf{L}}\, q_{\mathsf{SL}1}) \supset_{\mathsf{L}} q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{SL}1} \wedge_{\mathsf{S}} q_{\mathsf{L}2}))$ | HYP |
| 2 … 6 | $d_1$ | |
| 7 … 13 | $d_2$ | |
| 14 | $(\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} ((\Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}) \supset_{\mathsf{SL}} (\Box_{\mathsf{SL}} \Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}))$ | LFT 6, 13 |

where the justification of steps 2 and 7 is cLFT 1. Again, we follow the construction of the interpolant in the proof of Theorem 4.4:

(a) $\mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(((\Box_{\mathsf{L}}\, q_{\mathsf{SL}1}) \supset_{\mathsf{L}} q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{SL}1} \wedge_{\mathsf{S}} q_{\mathsf{L}2})))$
$\cap\, \mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} ((\Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}) \supset_{\mathsf{SL}} (\Box_{\mathsf{SL}} \Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}))$
$= \{q_{\mathsf{S}1}, q_{\mathsf{L}1}, q_{\mathsf{SL}1}, q_{\mathsf{L}2}\}$;

(b) $\mathsf{symb}_{\mathsf{L}}(((\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \supset_{\mathsf{L}} q_{\mathsf{L}1}) \wedge_{\mathsf{L}} (\mathtt{tt}^{(1)}_{\mathsf{L}}(\mathtt{tt}^{(1)}_{\mathsf{L}}(\mathtt{tt}^{(2)}_{\mathsf{L}}(q_{\mathsf{L}1}, q_{\mathsf{L}2})))))$
$\cap\, \mathsf{symb}_{\mathsf{L}}((\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \wedge_{\mathsf{L}} ((\Box_{\mathsf{L}}\, q_{\mathsf{L}2}) \supset_{\mathsf{L}} (\Box_{\mathsf{L}} \Box_{\mathsf{L}}\, q_{\mathsf{L}2}))) = \{q_{\mathsf{L}1}, q_{\mathsf{L}2}\}$;

(c) $\mathsf{symb}_{\mathsf{S}}((\mathtt{tt}^{(2)}_{\mathsf{S}}(\mathtt{tt}^{(1)}_{\mathsf{S}}(q_{\mathsf{S}1}), q_{\mathsf{S}1})) \wedge_{\mathsf{S}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{S}1} \wedge_{\mathsf{S}} \mathtt{tt}^{(0)}_{\mathsf{S}})))$
$\cap\, \mathsf{symb}_{\mathsf{S}}((\Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \wedge_{\mathsf{S}} ((\Box_{\mathsf{S}}\, q_{\mathsf{S}2}) \supset_{\mathsf{S}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}\, q_{\mathsf{S}2}))) = \{q_{\mathsf{S}1}\}$;

(d) $((\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \supset_{\mathsf{L}} q_{\mathsf{L}1}) \wedge_{\mathsf{L}} (\mathtt{tt}^{(1)}_{\mathsf{L}}(\mathtt{tt}^{(1)}_{\mathsf{L}}(\mathtt{tt}^{(2)}_{\mathsf{L}}(q_{\mathsf{L}1}, q_{\mathsf{L}2}))))$
$\vdash_{\mathsf{L}} (\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \wedge_{\mathsf{L}} ((\Box_{\mathsf{L}}\, q_{\mathsf{L}2}) \supset_{\mathsf{L}} (\Box_{\mathsf{L}} \Box_{\mathsf{L}}\, q_{\mathsf{L}2}))$ (see $d_1$ above);

(e) $(\mathtt{tt}^{(2)}_{\mathsf{S}}(\mathtt{tt}^{(1)}_{\mathsf{S}}(q_{\mathsf{S}1}), q_{\mathsf{S}1})) \wedge_{\mathsf{S}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{S}1} \wedge_{\mathsf{S}} \mathtt{tt}^{(0)}_{\mathsf{S}}))$
$\vdash_{\mathsf{S}} (\Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \wedge_{\mathsf{S}} ((\Box_{\mathsf{S}}\, q_{\mathsf{S}2}) \supset_{\mathsf{S}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}\, q_{\mathsf{S}2}))$ (see $d_2$ above);

(f) $\Theta_{\mathsf{L}} = \{(\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \wedge_{\mathsf{L}} ((\Box_{\mathsf{L}}\, q_{\mathsf{L}2}) \supset_{\mathsf{L}} (\Box_{\mathsf{L}} \Box_{\mathsf{L}}\, q_{\mathsf{L}2}))\}$ is an interpolant for (d);

(g) $\Theta_{\mathsf{S}} = \{\Box_{\mathsf{S}}\, q_{\mathsf{S}1}\}$ is an interpolant for (e);

(h) $\Theta_{\mathsf{SL}} = \{(\Box_{\mathsf{L}}\, q_{\mathsf{SL}1}) \wedge_{\mathsf{L}} ((\Box_{\mathsf{L}}\, q_{\mathsf{L}2}) \supset_{\mathsf{L}} (\Box_{\mathsf{L}} \Box_{\mathsf{L}}\, q_{\mathsf{L}2})), \Box_{\mathsf{S}}\, q_{\mathsf{SL}1}\}$ is an interpolant for (‡). Inded:

($h_1$) $((\Box_{\mathsf{L}}\, q_{\mathsf{SL}1}) \supset_{\mathsf{L}} q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{SL}1} \wedge_{\mathsf{S}} q_{\mathsf{L}2})) \vdash_{\mathsf{SL}} \Theta_{\mathsf{SL}}$;

($h_2$) $\Theta_{\mathsf{SL}} \vdash_{\mathsf{SL}} (\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} ((\Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}) \supset_{\mathsf{SL}} (\Box_{\mathsf{SL}} \Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}))$;

($h_3$) $\mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(\Theta_{\mathsf{SL}}) \subseteq \mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}((\Box_{\mathsf{L}}\, q_{\mathsf{SL}1}) \supset_{\mathsf{L}} q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}(q_{\mathsf{SL}1} \wedge_{\mathsf{S}} q_{\mathsf{L}2}))$
$\cap\, \mathsf{symb}^{\sqsubseteq}_{\mathsf{SL}}(\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}) \wedge_{\mathsf{SL}} ((\Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}) \supset_{\mathsf{SL}} (\Box_{\mathsf{SL}} \Box_{\mathsf{SL}}\, q_{\mathsf{SL}2}))$.

We now exemplify a case where there are no common propositional symbols between the formulas obtained by projection of the hypothesis and the formula obtained by projection of the conclusion to one of the component logics of the combination. Consider the following assertion:

$$(*) \quad \Box_{\mathsf{SL}}\, q_{\mathsf{SL}1} \vdash_{\mathsf{SL}} (\Box_{\mathsf{S}} \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{SL}} q_{\mathsf{L}3}.$$

A derivation sequence for (*) is as follows:

| | | |
|---|---|---|
| 1 | $\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}$ | HYP |
| 2 | $\Box_{\mathsf{S}}\, q_{\mathsf{S}1}$ | cLFT 1 |
| 3 | $(\Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \supset_{\mathsf{S}} (\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1})$ | $4_{\mathsf{S}}$ |
| 4 | $\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}$ | $\mathrm{MP}_{\mathsf{S}}\ 2, 3$ |
| 5 | $(\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{S}} \mathsf{tt}_{\mathsf{S}}^{(0)}$ | $\mathrm{TAUT}_{\mathsf{S}}\ 4$ |
| 6 | $\mathsf{tt}_{\mathsf{L}}^{(1)}(\mathsf{tt}_{\mathsf{L}}^{(1)}(\mathsf{tt}_{\mathsf{L}}^{(0)}))$ | $\mathsf{tt}_{\mathsf{L}}$ |
| 7 | $(\mathsf{tt}_{\mathsf{L}}^{(1)}(\mathsf{tt}_{\mathsf{L}}^{(1)}(\mathsf{tt}_{\mathsf{L}}^{(0)}))) \vee_{\mathsf{L}} q_{\mathsf{L}3}$ | $\mathrm{TAUT}_{\mathsf{L}}\ 6$ |
| 8 | $(\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{SL}} q_{\mathsf{L}3}$ | LFT 5, 7. |

Then, following the construction of the interpolant in the proof of Theorem 4.4:

(a) $\mathsf{symb}_{\mathsf{SL}}^{\sqsubseteq}(\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}) \cap \mathsf{symb}_{\mathsf{SL}}^{\sqsubseteq}((\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{SL}} q_{\mathsf{L}3}) = \{q_{\mathsf{S}1}\}$;

(b) $\mathsf{symb}_{\mathsf{S}}(\Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \cap \mathsf{symb}_{\mathsf{S}}((\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{S}} \mathsf{tt}_{\mathsf{S}}^{(0)}) = \{q_{\mathsf{S}1}\}$.

(c) $\mathsf{symb}_{\mathsf{L}}(\Box_{\mathsf{L}}\, q_{\mathsf{L}1}) \cap \mathsf{symb}_{\mathsf{L}}((\mathsf{tt}_{\mathsf{L}}^{(1)}(\mathsf{tt}_{\mathsf{L}}^{(1)}(\mathsf{tt}_{\mathsf{L}}^{(0)}))) \vee_{\mathsf{L}} q_{\mathsf{L}3}) = \emptyset$.

(d) $\Box_{\mathsf{S}}\, q_{\mathsf{S}1} \vdash_{\mathsf{S}} (\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{S}} \mathsf{tt}_{\mathsf{S}}^{(0)}$.

(e) $\Theta_{\mathsf{S}} = \{\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}\}$ is an interpolant for (d).

(f) $\tau_{\mathsf{S}}(\Theta_{\mathsf{S}}) = \{\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}\}$.

(g) $\Theta_{\mathsf{SL}} = \{\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}\}$ is an interpolant for $(*)$. Indeed:

($g_1$) $\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1} \vdash_{\mathsf{SL}} \Theta_{\mathsf{SL}}$.

($g_2$) $\Theta_{\mathsf{SL}} \vdash_{\mathsf{SL}} (\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{SL}} q_{\mathsf{L}3}$.

($g_3$) $\mathsf{symb}_{\mathsf{SL}}^{\sqsubseteq}(\Theta_{\mathsf{SL}}) \subseteq \mathsf{symb}_{\mathsf{SL}}^{\sqsubseteq}(\Box_{\mathsf{SL}}\, q_{\mathsf{SL}1}) \cap \mathsf{symb}_{\mathsf{SL}}^{\sqsubseteq}((\Box_{\mathsf{S}}\, \Box_{\mathsf{S}}\, q_{\mathsf{S}1}) \vee_{\mathsf{SL}} q_{\mathsf{L}3})$.

### Matrix product of modal logics with interaction

Consider the K4 modal logic $\mathsf{M4PL} = (\Sigma_4, \Delta_4, \mathcal{M}_4)$ over the set $\{q_{4j} : j \in \mathbb{N}\}$ of propositional symbols and the T modal logic $\mathsf{MTPL} = (\Sigma_{\mathsf{T}}, \Delta_{\mathsf{T}}, \mathcal{M}_{\mathsf{T}})$ over the set $\{q_{\mathsf{T}j} : j \in \mathbb{N}\}$ of propositional symbols. Clearly, these logics are both suitable and with identity (introduced as in $\mathsf{CPL}$). Accordingly:

- $\Sigma_{40} = \{q_{4j} : j \in \mathbb{N}\} \cup \{\mathsf{tt}_4, \mathsf{ff}_4\}$;
  $\Sigma_{41} = \{\neg_4, \Box_4, \mathsf{id}_4, \mathsf{tt}_4^{(1)}\}$;
  $\Sigma_{42} = \{\supset_4, \wedge_4, \vee_4, \mathsf{tt}_4^{(2)}\}$;
  $\Sigma_{4n} = \{\mathsf{tt}_4^{(n)}\}$ for $n \geq 3$.

- $\Sigma_{\mathsf{T}0} = \{q_{\mathsf{T}j} : j \in \mathbb{N}\} \cup \{\mathtt{tt}_{\mathsf{T}}, \mathtt{ff}_{\mathsf{T}}\}$;
  $\Sigma_{\mathsf{T}1} = \{\neg_{\mathsf{T}}, \Box_{\mathsf{T}}, \mathsf{id}_{\mathsf{T}}, \mathtt{tt}^{(1)}_{\mathsf{T}}\}$;
  $\Sigma_{\mathsf{T}2} = \{\supset_{\mathsf{T}}, \wedge_{\mathsf{T}}, \vee_{\mathsf{T}}, \mathtt{tt}^{(2)}_{\mathsf{T}}\}$;
  $\Sigma_{\mathsf{T}n} = \{\mathtt{tt}^{(n)}_{\mathsf{T}}\}$ for $n \geq 3$.

We assume that both logics are endowed with deductive systems for local derivability (necessitation only applies to theorems). Observe that both $\mathsf{M4PL}$ and $\mathsf{MTPL}$ enjoy modus ponens and the metatheorems of deduction with respect to $\supset_{\mathsf{4}}$ and $\supset_{\mathsf{T}}$, respectively.

Let $\mathsf{4TL}$ be $\lceil \mathsf{M4PL}\,\mathsf{MTPL} \rceil$. Then, by Theorem 4.4, $\mathsf{4TL}$ has the relaxed Craig interpolation (with respect to local derivability) since $\mathsf{M4PL}$ and $\mathsf{MTPL}$ have (turnstile) Craig interpolation (with respect to local derivability) because they have theoremhood Craig interpolation (see [7]).

In the sequel, we denote by $\neg_{\mathsf{4TL}}$, $\Box_{\mathsf{4TL}}$, $\supset_{\mathsf{4TL}}$, $\wedge_{\mathsf{4TL}}$ and $\vee_{\mathsf{4TL}}$ the meet-combined constructors $\lceil \neg_{\mathsf{4}}\, \neg_{\mathsf{T}} \rceil$, $\lceil \Box_{\mathsf{4}}\, \Box_{\mathsf{T}} \rceil$, $\lceil \supset_{\mathsf{4}} \supset_{\mathsf{T}} \rceil$, $\lceil \wedge_{\mathsf{4}} \wedge_{\mathsf{T}} \rceil$ and $\lceil \vee_{\mathsf{4}} \vee_{\mathsf{T}} \rceil$, respectively. Moreover, for $j \in \mathbb{N}$, we denote by $q_{\mathsf{4TL}j}$ the meet-combined constructor $\lceil q_{\mathsf{4}j} q_{\mathsf{T}j} \rceil$.

Let Ax be the singleton set composed by the axiom:

$$\Box_{\mathsf{4}}(\xi_1 \wedge_{\mathsf{4}} \xi_2) \supset_{\mathsf{4TL}} \Box_{\mathsf{4TL}}(\xi_1 \vee_{\mathsf{4TL}} \xi_3)$$

We now illustrate the interpolant construction for

$$(*) \quad (\Box_{\mathsf{4}}\, q_{\mathsf{4TL}1}) \wedge_{\mathsf{4TL}} q_{\mathsf{4TL}2} \vdash_{\mathsf{4TL}+\mathrm{Ax}} \Box_{\mathsf{4TL}}(q_{\mathsf{4TL}1} \vee_{\mathsf{4TL}} q_{\mathsf{4TL}3}).$$

based on the following derivation sequence:

| | | |
|---|---|---|
| 1 | $(\Box_{\mathsf{4}}\, q_{\mathsf{4TL}1}) \wedge_{\mathsf{4TL}} q_{\mathsf{4TL}2}$ | HYP |
| 2 | $(\Box_{\mathsf{4}}\, q_{41}) \wedge_{\mathsf{4}} q_{42}$ | cLFT 1 |
| 3 | $\Box_{\mathsf{4}}\, q_{41}$ | $\mathrm{TAUT}_{\mathsf{4}}$ 2 |
| 4 | $\Box_{\mathsf{4}}(q_{41} \wedge_{\mathsf{4}} q_{41})$ | $\mathsf{M4PL}$ 3 |
| 5 | $(\Box_{\mathsf{4}}(q_{\mathsf{4TL}1} \wedge_{\mathsf{4}} q_{\mathsf{4TL}1})) \supset_{\mathsf{4TL}} \Box_{\mathsf{4TL}}(q_{\mathsf{4TL}1} \vee_{\mathsf{4TL}} q_{\mathsf{4TL}3})$ | Ax |
| 6 | $(\Box_{\mathsf{4}}(q_{41} \wedge_{\mathsf{4}} q_{41})) \supset_{\mathsf{4}} \Box_{\mathsf{4}}(q_{41} \vee_{\mathsf{4}} q_{43})$ | cLFT 5 |
| 7 | $\Box_{\mathsf{4}}(q_{41} \vee_{\mathsf{4}} q_{43})$ | $\mathrm{MP}_{\mathsf{4}}$ 4, 6 |
| 8 | $(\mathtt{tt}^{(1)}_{\mathsf{T}}(\mathtt{tt}^{(2)}_{\mathsf{T}}(q_{\mathsf{T}1}, q_{\mathsf{T}1}))) \supset_{\mathsf{T}} \Box_{\mathsf{T}}(q_{\mathsf{T}1} \vee_{\mathsf{T}} q_{\mathsf{T}3})$ | cLFT 5 |
| 9 | $\mathtt{tt}^{(1)}_{\mathsf{T}}(\mathtt{tt}^{(2)}_{\mathsf{T}}(q_{\mathsf{T}1}, q_{\mathsf{T}1}))$ | $\mathtt{tt}_{\mathsf{T}}$ |
| 10 | $\Box_{\mathsf{T}}(q_{\mathsf{T}1} \vee_{\mathsf{T}} q_{\mathsf{T}3})$ | $\mathrm{MP}_{\mathsf{T}}$ 8, 9 |
| 11 | $\Box_{\mathsf{4TL}}(q_{\mathsf{4TL}1} \vee_{\mathsf{4TL}} q_{\mathsf{4TL}3})$ | LFT 7, 10. |

Then, following the construction of the interpolant in the proof of Theorem 4.5:

(a) $\mathsf{symb}^{\sqsubseteq}_{\mathsf{4TL}}((\Box_{\mathsf{4}}\, q_{\mathsf{4TL}1}) \wedge_{\mathsf{4TL}} q_{\mathsf{4TL}2}) \cap\ \mathsf{symb}^{\sqsubseteq}_{\mathsf{4TL}}(\Box_{\mathsf{4TL}}(q_{\mathsf{4TL}1} \vee_{\mathsf{4TL}} q_{\mathsf{4TL}3}))$
$= \{q_{\mathsf{4TL}1}, q_{\mathsf{4}1}, q_{\mathsf{L}1}\}$.

(b) $(\Box_{\mathsf{4}}(q_{\mathsf{4TL}1} \wedge_{\mathsf{4}} q_{\mathsf{4TL}1})) \supset_{\mathsf{4TL}} \Box_{\mathsf{4TL}}(q_{\mathsf{4TL}1} \vee_{\mathsf{4TL}} q_{\mathsf{4TL}3})$ is the instance of Ax used in the derivation sequence.

(c) $\mathsf{symb}^{\sqsubseteq}_{\mathsf{4TL}}((\Box_{\mathsf{4}}(q_{\mathsf{4TL}1} \wedge_{\mathsf{4}} q_{\mathsf{4TL}1})) \supset_{\mathsf{4TL}} \Box_{\mathsf{4TL}}(q_{\mathsf{4TL}1} \vee_{\mathsf{4TL}} q_{\mathsf{4TL}3})) =$
$\{q_{\mathsf{4TL}1}, q_{\mathsf{4}1}, q_{\mathsf{L}1}, q_{\mathsf{4TL}3}, q_{\mathsf{4}3}, q_{\mathsf{L}3}\} \subseteq \mathsf{symb}^{\sqsubseteq}_{\mathsf{4TL}}(\Box_{\mathsf{4TL}}(q_{\mathsf{4TL}1} \vee_{\mathsf{4TL}} q_{\mathsf{4TL}3}))$.

(d1) $(\Box_{\mathsf{4}}\, q_{\mathsf{4}1}) \wedge_{\mathsf{4}} q_{\mathsf{4}2} \vdash_{\mathsf{4}}$
$((\Box_{\mathsf{4}}(q_{\mathsf{4}1} \wedge_{\mathsf{4}} q_{\mathsf{4}1})) \supset_{\mathsf{4}} \Box_{\mathsf{4}}(q_{\mathsf{4}1} \vee_{\mathsf{4}} q_{\mathsf{4}3})) \supset_{\mathsf{4}} (\Box_{\mathsf{4}}(q_{\mathsf{4}1} \vee_{\mathsf{4}} q_{\mathsf{4}3}))$.

(d2) $(\mathtt{tt}^{(1)}_{\mathsf{T}}(q_{\mathsf{T}1})) \wedge_{\mathsf{T}} q_{\mathsf{T}2} \vdash_{\mathsf{T}}$
$((\mathtt{tt}^{(1)}_{\mathsf{T}}(\mathtt{tt}^{(2)}_{\mathsf{T}}(q_{\mathsf{T}1}, q_{\mathsf{T}1}))) \supset_{\mathsf{T}} \Box_{\mathsf{T}}(q_{\mathsf{T}1} \vee_{\mathsf{T}} q_{\mathsf{T}3})) \supset_{\mathsf{T}} (\Box_{\mathsf{T}}(q_{\mathsf{T}1} \vee_{\mathsf{T}} q_{\mathsf{T}3}))$.

(e1) $\Theta_{\mathsf{4}} = \{\Box_{\mathsf{4}}\, q_{\mathsf{4}1}\}$ is an interpolant for (d1).

(e2) $\Theta_{\mathsf{T}} = \{\mathtt{tt}^{(1)}_{\mathsf{T}}(q_{\mathsf{T}1})\}$ is an interpolant for (d2).

(f) $\Theta_{\mathsf{4TL}} = \{\Box_{\mathsf{4}}\, q_{\mathsf{4TL}1}, \mathtt{tt}^{(1)}_{\mathsf{T}}(q_{\mathsf{4TL}1})\}$ is an interpolant for $(*)$.

# 6 Interpolation algorithm and its complexity

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be suitable logics with Craig interpolation and identity. The objective is to extract, from the proof of Theorem 4.4, an algorithm for computing interpolants in $\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil$. We assume that an algorithm for finding interpolants in each of the component logics is available.

The envisaged interpolation algorithm for $\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil$ is required to produce an interpolant only for any given derivation sequence of $\varphi$ from $\Gamma$ fulfilling the following requirements:

- $\Gamma$ is finite and consistent;
- $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \neq \emptyset$.

More concretely, for each $k = 1, 2$, let $\mathsf{IAlg}_{\mathcal{L}_k}$ be an algorithm for $\mathcal{L}_k$ that, given a finite set of formulas $\Gamma_k$ and a formula $\varphi_k$ such that $\Gamma_k \vdash_k \varphi_k$ and $\mathsf{symb}_k(\Gamma_k) \cap \mathsf{symb}_k(\varphi_k) \neq \emptyset$, returns an interpolant $\Theta_k$.

Consider the algorithm $\mathsf{IAlg}_{\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil}$ presented in Figure 1 where the auxiliary algorithms, possibly extended as expected to finite sets of formulas, are as follows:

- $\mathsf{symb}_k$ receives a formula in $L_k$ and returns the set of propositional symbols in $Q_k$ occurring in the given formula, for $k = 1, 2$;

- $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}$ receives a formula in $L_{\lceil 12\rceil}$ and returns the set of propositional symbols in $Q_{\lceil 12\rceil}$ occurring in the given formula together with their component propositional symbols;

- $\cdot|_k$ receives a formula in $L_{\lceil 12\rceil}$ and returns a formula in $L_k$ with the same structure where each constructor $\lceil c_1 c_2\rceil$ is replaced by $c_k$, for $k = 1, 2$;

- $\tau_1$ receives a formula $\psi$ in $L_{\lceil 12\rceil}$ and returns the same formula if it is not a 0-ary constructor, otherwise $\tau_1$ returns

$$\lceil \mathsf{id}_1 \mathsf{tt}_2^{(1)}\rceil\, \psi$$

  (similarly for $\tau_2$);

- $f_1$ receives a finite set of formulas $\Theta$ contained in $L_1$, a finite set $\Gamma$ and a formula $\varphi$ both in $L_{\lceil 12\rceil}$ and returns a finite set of formulas $\Theta'$ in $L_{\lceil 12\rceil}$ such that each $\theta' \in \Theta'$ is obtained from a formula $\theta \in \Theta$ by replacing each propositional symbol $c_1$ by $\lceil c_1 c_2\rceil$ in $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi)$ such that if there is

$$c_2' \neq \mathsf{tt}_2^{(0)}$$

  with $\lceil c_1 c_2'\rceil$ in $\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil}(\varphi)$ then

$$c_2 \neq \mathsf{tt}_2^{(0)}$$

  (similarly for $f_2$).

Observe that the algorithm $\mathsf{IAlg}_{\lceil \mathcal{L}_1\mathcal{L}_2\rceil}$ in Figure 1 follows closely the steps in the proof of Theorem 4.4 and so its correctness comes directly from that proof.

We now analyze the time complexity in the worst case of algorithm $\mathsf{IAlg}_{\lceil \mathcal{L}_1\mathcal{L}_2\rceil}$, that is, the time complexity class of the running time of $\mathsf{IAlg}_{\lceil \mathcal{L}_1\mathcal{L}_2\rceil}$ in the worst case. In the sequel, we denote by $\mathsf{RT}(A)$ the running time of algorithm $A$ (which is a function of the total length of the arguments of $A$). Furthermore, as usual, we denote by $\mathsf{Pol}$ the class of all polynomials on a single variable.

We start by investigating the time complexity of the auxiliary algorithms. We assume an appropriate representation of formulas (using, for example, a prefix notation). Then

- $\mathsf{RT}(\mathsf{symb}_1)$, $\mathsf{RT}(\mathsf{symb}_2)$ and $\mathsf{RT}(\mathsf{symb}^{\sqsubseteq}_{\lceil 12\rceil})$ are in $\mathsf{Pol}$;

```
IAlg_{⌈L1L2⌉}(Γ, φ):
    if symb^⊑_{⌈12⌉}(Γ) ∩ symb^⊑_{⌈12⌉}(φ) = ∅ then
        return
        "Hypotheses and conclusion do not share propositional symbols"
    fi;
    Γ1 = Γ|1; φ1 = φ|1;
    Γ2 = Γ|2; φ2 = φ|2;
    if symb2(Γ2) ∩ symb2(φ2) = ∅ then
        Θ1 = IAlg_{L1}(Γ1, φ1);
        return τ1(Θ1)
    fi;
    if symb1(Γ1) ∩ symb1(φ1) = ∅ then
        Θ2 = IAlg_{L2}(Γ2, φ2);
        return τ2(Θ2)
    fi;
    Θ1 = IAlg_{L1}(Γ1, φ1); Θ'1 = f1(Θ1, Γ, φ);
    Θ2 = IAlg_{L2}(Γ2, φ2); Θ'2 = f2(Θ2, Γ, φ);
    return τ1(Θ'1) ∪ τ2(Θ'2)
```

Figure 1: Interpolation algorithm $\mathsf{IAlg}_{\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil}$.

- $\mathsf{RT}(\cdot\,|_1)$ and $\mathsf{RT}(\cdot\,|_2)$ are in $\mathsf{Pol}$;
- $\mathsf{RT}(\tau_1)$ and $\mathsf{RT}(\tau_2)$ are in $\mathsf{Pol}$;
- $\mathsf{RT}(f_1)$ and $\mathsf{RT}(f_2)$ are in $\mathsf{Pol}$.

We denote by $\mathcal{C}_k$ the time complexity class of $\mathsf{RT}(\mathsf{IAlg}_{\mathcal{L}_k})$ for $k = 1, 2$. Note that the worst case of the running time of algorithm $\mathsf{IAlg}_{\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil}$ for arguments $\Gamma$ and $\varphi$ is when $\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\Gamma) \cap \mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil}(\varphi) \neq \emptyset$ and $\mathsf{symb}_k(\Gamma|_k) \cap \mathsf{symb}_k(\varphi|_k) \neq \emptyset$ for $k = 1, 2$. Hence, modulo the cost of basic effective operations (like assignments to variables), we have

$$
\begin{aligned}
\mathsf{RT}(\mathsf{IAlg}_{\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil})(|\Gamma| + |\varphi|) \quad \leq \quad & \mathsf{RT}(\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil})(|\Gamma|) + \mathsf{RT}(\mathsf{symb}^{\sqsubseteq}_{\lceil 12 \rceil})(|\varphi|) + \\
& \mathsf{RT}(\cdot\,|_1)(|\Gamma|) + \mathsf{RT}(\cdot\,|_1)(|\varphi|) + \\
& \mathsf{RT}(\cdot\,|_2)(|\Gamma|) + \mathsf{RT}(\cdot\,|_2)(|\varphi|) + \\
& \mathsf{RT}(\mathsf{symb}_1)(\mathsf{RT}(\cdot\,|_1)(|\Gamma|)) + \\
& \mathsf{RT}(\mathsf{symb}_1)(\mathsf{RT}(\cdot\,|_1)(|\varphi|)) + \\
& \mathsf{RT}(\mathsf{symb}_2)(\mathsf{RT}(\cdot\,|_2)(|\Gamma|)) + \\
& \mathsf{RT}(\mathsf{symb}_2)(\mathsf{RT}(\cdot\,|_2)(|\varphi|)) + \\
& n_1 + n_2 + m_1 + m_2 + \\
& \mathsf{RT}(\tau_1)(m_1) + \mathsf{RT}(\tau_2)(m_2)
\end{aligned}
$$

where, for $k = 1, 2$,

- $n_k = \mathsf{RT}(\mathsf{IAlg}_{\mathcal{L}_k})(\mathsf{RT}(\cdot\,|_k)(|\Gamma|) + \mathsf{RT}(\cdot\,|_k)(|\varphi|))$;
- $m_k = \mathsf{RT}(f_k)(n_k + |\Gamma| + |\varphi|)$.

Since $\mathsf{RT}(\cdot\,|_k)$ and $\mathsf{RT}(f_k)$ are in $\mathsf{Pol}$, if $\mathsf{Pol} \subseteq \mathcal{C}_k$ for $k = 1, 2$ then

$$
\begin{cases}
\mathsf{RT}(\mathsf{IAlg}_{\mathcal{L}_k}) \circ \mathsf{RT}(\cdot\,|_k) \in \mathcal{C}_k; \\
\mathsf{RT}(f_k) \circ \mathsf{RT}(\mathsf{IAlg}_{\mathcal{L}_k}) \in \mathcal{C}_k.
\end{cases}
$$

So, we have the following result:

**Theorem 6.1 (Complexity of the interpolation algorithm)**
For each $k = 1, 2$, assume that $\mathcal{L}_k$ is a suitable matrix logic with identity and enjoying the Craig Interpolation property. Furthermore, for each $k = 1, 2$, assume that $\mathsf{IAlg}_{\mathcal{L}_k}$ is an algorithm, with time complexity $\mathcal{C}_k \supseteq \mathsf{Pol}$, for computing interpolants within $\mathcal{L}_k$. Then, the time complexity of $\mathsf{IAlg}_{\lceil \mathcal{L}_1 \mathcal{L}_2 \rceil}$ is $\mathsf{max}(\mathcal{C}_1, \mathcal{C}_2)$.

Observe that, barring specially designed logics, the time complexity of the interpolation algorithm is expected to be greater than polynomial. Indeed, it was proved in [18] that for classical propositional logic the existence of a polynomial-time interpolation algorithm implies that $\mathsf{P} = \mathsf{NP}$ or $\mathsf{NP} \neq \mathsf{CoNP}$.

## 7 Outlook

Capitalizing on the axiomatization of the product of two matrix logics provided by their meet-combination, we were able to establish by proof-theoretic means that such a product preserves a variant of the Craig interpolation property. The proposed variant seems to be quite natural within the relevant setting of product of signatures. We also prove weaker results for the preservation of interpolation in the presence of interaction axioms, taking advantage of the preservation of the metatheorem of deduction by meet-combination.

The computation of the interpolant in the product of two matrix logics was shown to have only a polynomial penalty over the computation in the two given logics.

Concerning further work, it seems worthwhile to investigate the preservation of alternative interpolation notions, like extension interpolation [6] and Maehara interpolation [14]. Moreover, we intend to investigate preservation of interpolation from a semantic point of view motivated by the recent results and techniques in [9].

In another front, it seems promising to apply the algorithm proposed in this paper for computing the interpolant to the field of model checking when dealing with logics that can be obtained as products of simpler matrix logics.

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## References

[1] C. Areces, E. Hoogland, and D. de Jongh. Interpolation, definability and fixed points in interpretability logics. In *Advances in Modal Logic*, volume 119 of *CSLI Lecture Notes*, pages 35–58. CSLI Publications, 2001.

[2] J. Bicarregui, T. Dimitrakos, D. Gabbay, and T. Maibaum. Interpolation in practical formal development. *Logic Journal of the IGPL*, 9(2):231–243, 2001.

[3] A. Carbone. The Craig interpolation theorem for schematic systems. In *Collegium logicum, Vol. 2*, volume 2 of *Colloquium Logicum Ann. Kurt-Gödel-Society*, pages 87–100. Springer, 1996.

[4] W. A. Carnielli, J. Rasga, and C. Sernadas. Preservation of interpolation features by fibring. *Journal of Logic and Computation*, 18(1):123–151, 2008.

[5] W. Craig. Linear reasoning. A new form of the Herbrand-Gentzen theorem. *The Journal of Symbolic Logic*, 22:250–268, 1957.

[6] J. Czelakowski and D. Pigozzi. Amalgamation and interpolation in abstract algebraic logic. In *Models, Algebras, and Proofs*, volume 203 of *Lecture Notes in Pure and Applied Mathematics*, pages 187–265. Dekker, 1999.

[7] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*, volume 169 of *Synthese Library*. D. Reidel Publishing Co., 1983.

[8] J. M. Font, R. Jansana, and D. Pigozzi. Fully adequate Gentzen systems and the deduction theorem. *Reports on Mathematical Logic*, (35):115–165, 2001.

[9] D. Gabbay and K. Schlechta. *Conditionals and Modularity in General Logics*. Springer, 2011.

[10] D. M. Gabbay. Craig's interpolation theorem for modal logics. In *Conference in Mathematical Logic*, pages 111–127. Lecture Notes in Mathematics, Vol. 255. Springer, 1972.

[11] D. M. Gabbay and L. Maksimova. *Interpolation and Definability*, volume 46 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, 2005. Modal and Intuitionistic Logics.

[12] R. Jhala and K. L. McMillan. Interpolant-based transition relation approximation. *Logical Methods in Computer Science*, 3(4):4:1, 17 pp. (electronic), 2007.

[13] M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *The Journal of Symbolic Logic*, 56(4):1469–1485, 1991.

[14] S. Maehara. On the interpolation theorem of Craig. *Mathematical Society of Japan. Sugaku (Mathematics)*, 12:235–237, 1960/1961.

[15] M. Marx and C. Areces. Failure of interpolation in combined modal logics. *Notre Dame Journal of Formal Logic*, 39(2):253–273, 1998.

[16] K. L. McMillan. An interpolating theorem prover. *Theoretical Computer Science*, 345(1):101–121, 2005.

[17] C. Miller, S. Kupferschmid, M. Lewis, and B. Becker. Encoding techniques, Craig interpolants and bounded model checking for incomplete designs. In *Theory and Applications of Satisfiability Testing—SAT*, volume 6175 of *Lecture Notes in Computer Science*, pages 194–208. Springer, 2010.

[18] D. Mundici. NP and Craig's interpolation theorem. In *Logic colloquium '82*, volume 112, pages 345–358. North-Holland, 1984.

[19] G. F. Schumm. Some failures of interpolation in modal logic. *Notre Dame Journal of Formal Logic*, 27(1):108–110, 1986.

[20] A. Sernadas, C. Sernadas, and J. Rasga. On meet-combination of logics. *Journal of Logic and Computation*, 22(6):1453–1470, 2012.

[21] A. Urquhart. Failure of interpolation in relevant logics. *Journal of Philosophical Logic*, 22(5):449–479, 1993.