Preservation of interpolation features by fibring

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Abstract

Fibring is a metalogical constructor that permits to combine different logics by operating on their deductive systems under certain natural restrictions, as for example that the two given logics are presented by deductive systems of the same type. Under such circumstances, fibring will produce a new deductive system by means of the free use of inference rules from both deductive systems, provided the rules are schematic, in the sense of using variables that are open for application to formulas with new linguistic symbols (from the point of view of each logic component). Fibring is a generalization of fusion, a less general but wider developed mechanism which permits results of the following kind: if each logic component is decidable (or sound, or complete with respect to a certain semantics) then the resulting logic heirs such a property. The interest for such preservation results for combining logics is evident, and they have been achieved in the more general setting of fibring in several cases. The Craig interpolation property and the Maehara interpolation have a special significance when combining logics, being related to certain problems of complexity theory, some properties of model theory and to the usual (global) metatheorem of deduction. When the peculiarities of the distinction between local and global deduction interfere, justifying what we call *careful reasoning*, the question of preservation of interpolation becomes more subtle and other forms of interpolation can be distinguished. These questions are investigated and several (global and local) preservation results for interpolation are obtained for fibring logics that fulfill mild requirements.

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1 Introduction

Among the methods for combining logics, fusion [39] is the best understood, mainly in what concerns preservation of properties as soundness, weak completeness, uniform Craig interpolation (for theoremhood) and decidability (see [41, 25]).

Further research has been directed at fibring, a more general combination mechanism proposed by Gabbay [18, 19], including fusion as a special case. Although already well understood at the proof-theoretic level, fibring raises some difficulties at the semantic level [36]. The general quest for *preservation* (in the sense of characterizing which logical properties like decidability, interpolation, completeness, finite algebraizability and so on are preserved through the operation of fibring logics) represents one of the main research trends in fibring.

Although preservation of soundness and completeness has been already investigated in the context of propositional-based logics [42, 38, 7], first-order quantification [37], higher-order quantification [13], non truth-functional semantics [6], sequent and other deductive systems [22, 33], other forms of preservation are still to be fully understood.

Herein we concentrate on preservation of several forms of interpolation and on the preservation of related properties, including metatheorems of deduction and derivation with different sets of variables. We restrict our attention in this paper to the context of propositional-based logics endowed with a Hilbert calculus coping with global and local derivability consequences.

What is now generally known as Craig interpolation is a heritage of the seminal results proved by W. Craig [14] in a proof-theoretic context for first-order logic. Several abstractions have been considered either in proof-theoretical vein (e.g. [9, 8]) or in (non-constructive) model-theoretical style (e.g. for modal and positive logics as in [28, 29], for intuitionistic logic as in [17] and for hybrid logics as in [1, 2]). The importance of Craig interpolation for some fundamental problems of complexity theory as analyzed in [31] and further developed in [32], permits to associate the rate of growth of the interpolant and measures of complexity. Interpolation has recently acquired practical relevance in engineering applications namely when formality and modularity are invoked [4], in software model-checking as in [23] and SAT-based methods of unbounded symbolic model-checking as in [30].

Interpolation properties can be regarded as a kind of density in topological terms and are known to be related with properties of model theory as exemplified by the correspondence between Craig interpolation and joint consistency properties for classical propositional logic. This correspondence is mediated in the classical case by finite algebraizability in the sense of [5] and by the familiar (global) metatheorem of deduction. However, in the general case of deducibility relations, specially in those where the peculiarities of local and global deduction interfere, this correspondence opens difficult and challenging problems. We refer here to *careful reasoning* when the distinction of global and local deduction is relevant: careful reasoning may lead to other forms of interpolation even at propositional level. Typical case of this distinction occurs in modal logic when reasoning based upon a single world versus reasoning with all the worlds.

The concept of amalgamation seems to be the correct semantic setting for relating interpolation and model-theoretic properties in general. In [15] it is proven that amalgamation and interpolation-type properties are related in several guises. Semantic proofs of interpolation are usually not constructive. This is one of the main reasons why in this paper, we investigate preservation of interpolation in the context of Hilbert calculi. The advantages of this approach are twofold: First dealing with proof systems we do not need to make any previous commitments to semantical notions (which tend to be different from logic to logic) and second constructive proofs in Hilbert systems can be shown in several cases to be preserved under the combination mechanisms. Complexity results can then be obtained.

The main contributions of the paper are presented in Sections 3, 4 and 5. In Section 2, we present the basic concepts about deductive systems distinguishing between local and global reasoning. Moreover, we discuss the metatheorem of deduction as a key ingredient for the rest of the paper. We conclude the section with the presentation of (unconstrained and constrained) fibring of deductive systems. Several examples are given for the different concepts.

Section 3 is dedicated to interpolation. Three forms of interpolation are considered: extension, Craig and Maehara interpolations. We prove that for deductive systems enjoying what we call careful-reasoning-by-cases local interpolation implies global interpolation. We also prove results showing that all forms of interpolations have a formulation in terms of finite sets thus showing that interpolation has an inherent character of compactness. Finally, we stress the importance of a general form of metatheorem of deduction by proving that Craig interpolation implies another form of interpolation proposed by S. Maehara [27] (in the context of intuitionistic logic), thus showing that the mediation of metatheorem of deduction plays a central role.

Section 4 is the preparation for preservation of interpolation. The starting point is the proof that the metatheorem of deduction is preserved. The preservation of the metatheorem of deduction is needed for proving the preservation of the Maehara interpolation. The main achievements of this section are the technical results related to the translation of derivations from the fibring deductive system $\mathcal{D}$ to the component deductive systems $\mathcal{D}'$ and $\mathcal{D}''$. We manage to do so by enriching the deductive system $\mathcal{D}'$ with "ghost" variables that represent in $\mathcal{D}'$ formulas from the deductive system $\mathcal{D}''$ and vice-versa.

In Section 5, we investigate preservation of interpolation in the context of Hilbert systems in a much broader sense for a wide-scoped fibring combinations covering global and local reasoning for several logics. Preservation of careful-reasoning-by-cases is proved without further assumptions. After that we establish sufficient conditions for preservation of interpolation by fibring. Preservation of Craig interpolation either global or local depends on the existence of a bridge to one of the component deductive systems. The extension interpolation property holds in the fibring depending on the preservation of metatheorems of modus ponens and deduction. Finally the preservation of the Maehara interpolation property involves either the presence of a bridge in the fibring as well as the metatheorems of deduction and modus ponens.

Along the paper we give several examples concerning interpolation. General techniques for obtaining interpolation are not known: Craig interpolation fails unexpectedly, for example, in all Łukasiewicz logics L_n with n finite or infinite see [26], and also in all Gödel logics G_n for $n \geq 4$, see [3]. Understanding the reasons behind the failure and developing constructive proofs of interpolation are still hard problems. In Section 6, we obtain a constructive method of Craig interpolation for special logics as it is the case of some many-valued logics and logics of formal inconsistency (as studied in [10]). Open issues are discussed in Section 7, namely related to the possibility of extending the results to first-order based logics and of investigating semantic characterizations of interpolation in a general setting.

2 Preliminaries

In this section we introduce the basic relevant concepts about deductive systems namely global and local derivations, a general version of the metatheorem of deduction and several forms of interpolation. We also define fibring of deductive systems.

2.1 Deductive systems

A *signature* C is a family of sets indexed by natural numbers. The elements of each C_k are called *constructors* of arity k. Let $L(C, \Xi)$ be the free algebra over C generated by Ξ (a denumerable set of variables). Fixed Ξ, we denote by $L(C, \Upsilon)$ the subset of $L(C, \Xi)$ including all formulas with variables in $\Upsilon \subseteq \Xi$, $\mathrm{var}(\varphi)$ the set of elements of Ξ occurring in a formula φ and $\mathrm{var}(\Gamma) = \cup_{\gamma\in\Gamma}\mathrm{var}(\gamma)$ the set of variables occurring in a set of formulae Γ.

A *substitution* is any map $\sigma: \Xi \to L(C, \Xi)$. Substitutions can be inductively extended to formulas: $\sigma(\gamma)$ is the formula where each $\xi \in \Xi$ is replaced by $\sigma(\xi)$ and also to sets: $\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\}$. When $\mathrm{var}(\varphi) = \{\xi_1, \ldots, \xi_n\}$ and $\sigma(\xi_i) = \psi_i$ for $i = 1, \ldots, n$, we use $\varphi(\psi_1, \ldots, \psi_n)$ to denote $\sigma(\varphi)$. Furthermore, we may also extend this notation to sets of formulas when all the formulas in the set have the same set of variables.

A *rule* over C is a pair $r = \langle \Theta, \eta \rangle$ where $\Theta \cup \{\eta\} \subseteq L(C, \Xi)$. As usual the elements of Θ are the *premises* and η is the *conclusion* of the rule. We shall work as usual with finitary rules, that is, we assume that the set Θ of premises is finite. A *careful deductive system* is a triple

$$\mathcal{D} = \langle C, R_{\rm l}, R_{\rm g} \rangle$$

where C is a signature and both $R_{\rm l}$ and $R_{\rm g}$ are sets of rules over C such that $R_{\rm l} \subseteq R_{\rm g}$. For reasons that are clear in the example below, for modal logic, the rules in $R_{\rm l}$ are called *local rules* and those in $R_{\rm g}$ are called *global rules.* The distinction between local and global rules is imparted in the concept of careful-reasoning and is crucial when investigating metatheoretical properties and their preservation.

In the sequel we omit sometimes the adjective *careful* when referring to this kind of twofold deductive system.

A *global derivation* of $\varphi \in L(C, \Xi)$ from $\Gamma \subseteq L(C, \Xi)$, indicated

$$\text{either by } \Gamma \vdash^{\rm g}_{\mathcal{D},\Xi} \varphi \;\text{ or by }\; \varphi \in \Gamma^{\vdash^{\rm g}_{\mathcal{D},\Xi}},$$

is a sequence $\psi_1 \dots \psi_n$ such that ψ_n is φ and each ψ_i is either an element of Γ or there are a rule $r = \langle \{\theta_1, \dots, \theta_m\}, \eta \rangle \in R_{\rm g}$ and a substitution σ such that ψ_i is $\sigma(\eta)$ and $\sigma(\theta_j)$ appears among $\psi_1 \dots \psi_{i-1}$ for every $j = 1, \dots, m$.

A *local derivation* of $\varphi \in L(C, \Xi)$ from $\Gamma \subseteq L(C, \Xi)$, indicated

$$\text{either by } \Gamma \vdash^{\rm l}_{\mathcal{D},\Xi} \varphi \;\text{ or by }\; \varphi \in \Gamma^{\vdash^{\rm l}_{\mathcal{D},\Xi}},$$

is a sequence $\psi_1 \dots \psi_n$ such that ψ_n is φ and each ψ_i is either an element of Γ, or is globally derivable from the empty set or there are a rule $r = \langle \{\theta_1, \dots, \theta_m\}, \eta \rangle \in R_{\rm l}$ and a substitution σ such that ψ_i is $\sigma(\eta)$ and $\sigma(\theta_j)$ appears among $\psi_1 \dots \psi_{i-1}$ for every $j = 1, \dots, m$. We use the notation $\Gamma \vdash^{d}_{\mathcal{D},\Xi} \varphi$ when stating properties that hold either for global derivations when d is g or for local derivations when d is l. We extend the derivations to sets. For instance $\Gamma \vdash^{\rm g}_{\mathcal{D},\Xi} \Psi$ with $\Psi \subseteq L(C, \Xi)$ iff $\Gamma \vdash^{\rm g}_{\mathcal{D},\Xi} \psi$ for every $\psi \in \Psi$. Moreover we may write $\gamma_1, \dots, \gamma_n \vdash^{d}_{\mathcal{D},\Xi} \varphi$ to denote $\{\gamma_1, \dots, \gamma_n\} \vdash^{d}_{\mathcal{D},\Xi} \varphi$. Axioms can be seen as rules with no premises.

Example 2.1 The normal modal deductive system K is defined as follows:

- $C_0 = \{\mathbf{t}, \mathbf{f}\}$; $C_1 = \{\neg, \Box\}$; $C_2 = \{\Rightarrow\}$; $C_k = \emptyset$ for $k \geq 3$;
- $R_{\rm l}$ consists of the following rules:
  $\langle \emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)) \rangle$;
  $\langle \emptyset, ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_2 \Rightarrow \xi_3)))) \rangle$;
  $\langle \emptyset, (((\neg\, \xi_1) \Rightarrow (\neg\, \xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))) \rangle$;
  $\langle \emptyset, ((\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2)))) \rangle$ K axiom;
  $\langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2 \rangle$;
- $R_{\rm g} = R_{\rm l} \cup \langle \{\xi_1\}, (\Box\xi_1) \rangle$ necessitation rule.

To illustrate the distinction between local and global derivability, observe that $\{(\xi_1 \Rightarrow \xi_2)\} \vdash^{\rm g}_{K,\Xi} ((\Box\xi_1) \Rightarrow (\Box\xi_2))$ but $\{(\xi_1 \Rightarrow \xi_2)\} \not\vdash^{\rm l}_{K,\Xi} ((\Box\xi_1) \Rightarrow (\Box\xi_2))$.
The classical propositional deductive system can be obtained by deleting: $\Box$ from the unary connectives, the K axiom from the local rules and the necessitation rule from the global rules. $\triangleleft$

Example 2.2 The Gödel's G3 (three valued intermediate) deductive system is defined as follows:

- $C_0 = \{\mathbf{t}, \mathbf{f}\}$; $C_1 = \{\neg\}$; $C_2 = \{\wedge, \vee, \Rightarrow\}$; $C_k = \emptyset$ for $k \geq 3$;

- R_{l} includes $\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle$ plus:
  - the axiom schemata of propositional intuitionistic logic;
  - the axiom schema $(((\neg\,\xi_1) \Rightarrow \xi_2) \Rightarrow (((\xi_2 \Rightarrow \xi_1) \Rightarrow \xi_2) \Rightarrow \xi_2))$;
- $R_{\mathrm{g}} = R_{\mathrm{l}}$. ◁

Note that, as a consequence of the finite character of provability, the definition of derivation implies immediately compactness that is, for d equals l or g:

$$\text{if } \Gamma \vdash^d_{\mathcal{D},\Xi} \varphi \text{ then there is a finite } \Phi \subseteq \Gamma \text{ such that } \Phi \vdash^d_{\mathcal{D},\Xi} \varphi.$$

Moreover derivations are structural, that is closed under substitution:

$$\text{if } \Gamma \vdash^d_{\mathcal{D},\Xi} \varphi \text{ then } \sigma(\Gamma) \vdash^d_{\mathcal{D},\Xi} \sigma(\varphi).$$

Observe also that $\Gamma \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$ always implies $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$ and $\emptyset \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$ iff $\emptyset \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$. So each careful deductive system $\mathcal{D}$ induces two familiar Tarskian consequence systems $\langle L(C,\Xi), \vdash^{\mathrm{l}}_{\mathcal{D},\Xi}\rangle$ and $\langle L(C,\Xi), \vdash^{\mathrm{g}}_{\mathcal{D},\Xi}\rangle$ such that the latter extends the former.

Derivations in a deductive system with respect to different sets of variables can be related.

Proposition 2.3 Assume that Γ' is finite, $\delta_1, \ldots, \delta_n$ is a derivation of $\Gamma' \vdash^d_{\mathcal{D},\Xi\cup\Xi'} \varphi'$ where Ξ' is disjoint of Ξ and Υ' is the set of variables in Ξ' occurring in the derivation. Let Υ be a set of variables in Ξ not occurring in the derivation such that $|\Upsilon'| = |\Upsilon|$ and μ a bijection from Υ' to Υ. Consider a substitution $\rho : \Xi \cup \Xi' \to L(C,\Xi)$ such that:

- $\rho(\xi) = \xi$ for $\xi \in \Xi$;
- $\rho(\xi') = \mu(\xi')$ for $\xi' \in \Upsilon'$.

Then $\rho(\Gamma') \vdash^d_{\mathcal{D},\Xi} \rho(\varphi')$.

Proof: The sequence $\rho(\delta_1) \ldots \rho(\delta_n)$ is a derivation of $\rho(\varphi')$ from $\rho(\Gamma')$ using variables in Ξ. ◇

Proposition 2.4 Assume that Γ is finite, $\delta_1, \ldots, \delta_n$ is a derivation of $\Gamma \vdash^d_{\mathcal{D},\Xi} \varphi$, Υ is a subset of the set of variables in Ξ occurring in the derivation, and Ξ' is a set of variables disjoint of Ξ with cardinality greater than Υ. Let $\Upsilon' \subseteq \Xi'$ be such that $|\Upsilon'| = |\Upsilon|$ and μ be a bijection from Υ to Υ'. Consider a substitution $\sigma : \Xi \to L(C, \Xi \cup \Xi')$ such that:

- $\sigma(\xi) = \xi$ for $\xi \in \Xi \setminus \Upsilon$;
- $\sigma(\xi) = \mu(\xi)$ for $\xi \in \Upsilon$.

Then $\sigma(\Gamma) \vdash^d_{\mathcal{D}',\Xi\cup\Xi'} \sigma(\varphi)$.

Proof: The sequence $\sigma(\delta_1) \ldots \sigma(\delta_n)$ is a derivation of $\sigma(\varphi)$ from $\sigma(\Gamma)$ using variables in $\Xi \cup \Xi'$. ◇

Several distinct deduction metatheorems can be considered as indicated in [16]: they generalize the usual deduction metatheorems that require the existence of a deductive implication in the signature.

Herein, we consider extended versions of deduction metatheorems taking into account global and local reasoning as follows: A deduction system $\mathcal{D}$ has the *d-metatheorem of deduction* (d-MTD) if there is a finite set of formulas $\Delta \subseteq L(C, \{\xi_1, \xi_2\})$ such that:

$$\text{if } \Gamma, \varphi_1 \vdash^d_{\mathcal{D},\Xi} \varphi_2 \text{ then } \Gamma \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \varphi_2)$$

where $\Delta(\varphi_1, \varphi_2)$ is obtained from Δ by substituting ξ_i by φ_i for $i = 1, 2$. And it has the *d-metatheorem of modus ponens* (d-MTMP) if there is a finite set of formulas $\Delta \subseteq L(C, \{\xi_1, \xi_2\})$ such that the converse holds. We may refer to Δ as the *base set*.

Example 2.5 For instance:

- Classical propositional logic has g-MTD, g-MTMP, l-MTD and l-MTMP taking $\Delta = \{(\xi_1 \Rightarrow \xi_2)\}$.
- Modal and intuitionistic logics have l-MTD and l-MTMP with base set $\Delta = \{(\xi_1 \Rightarrow \xi_2)\}$.
- Modal logic in general does not have g-MTD.
- Modal logic K4 has g-MTD and g-MTMP taking $\Delta = \{(((\Box\xi_1) \wedge \xi_1) \Rightarrow \xi_2)\}$.
- Modal logic S4 has g-MTD and g-MTMP taking $\Delta = \{((\Box\xi_1) \Rightarrow \xi_2)\}$.
- Gödel logic G3 has g-MTD, g-MTMP, l-MTD and l-MTMP taking $\Delta = \{(\xi_1 \Rightarrow \xi_2)\}$.
- Łukasiewicz logic L3 has g-MTD and g-MTMP taking $\Delta = \{(\xi_1 \Rightarrow (\xi_1 \Rightarrow \xi_2))\}$.
- Similarly Łukasiewicz logic Ln, for each $n \geq 4$, also has g-MTD and g-MTMP $\Delta = \{(\xi_1^{n-1} \Rightarrow \xi_2)\}$ where $(\xi_1^{n-1} \Rightarrow \xi_2)$ is $(\xi_1 \Rightarrow (\xi_1^{n-2} \Rightarrow \xi_2))$. ◁

2.2 Fibring

Given two deductive systems $\mathcal{D}'$ and $\mathcal{D}''$, their fibring is the deductive system $\mathcal{D} = \langle C, R_\text{l}, R_\text{g}\rangle$ defined as follows:

- $C_k = C'_k \cup C''_k$ for every $k \in \mathbb{N}$;
- $R_\text{l} = {R_\text{l}}' \cup {R_\text{l}}''$;
- $R_\text{g} = {R_\text{g}}' \cup {R_\text{g}}''$.

Observe that the deductive system induced by $\mathcal{D}$ is not the union (in the sense of [40]) of consequence systems induced by $\mathcal{D}'$ and $\mathcal{D}''$ neither for local nor for global derivation. Moreover taking $\Gamma' \subseteq L(C', \Xi)$ and $\Gamma'' \subseteq L(C'', \Xi)$ in general we obtain that $(\Gamma')^{\vdash'^d_{\mathcal{D}',\Xi}} \subset (\Gamma')^{\vdash^d_{\mathcal{D},\Xi}}$ and $(\Gamma'')^{\vdash''^d_{\mathcal{D}'',\Xi}} \subset (\Gamma'')^{\vdash^d_{\mathcal{D},\Xi}}$. Usually in the fibred deductive system we have a much richer notion of derivation.

Fibring can be defined in a categorial setting, considering the category whose objects are deductive systems and where a *morphism*

$$h : \mathcal{D} \to \mathcal{D}'$$

is a signature morphism $h : C \to C'$, that is a family $h_k : C_k \to C'_k$ of functions for every $k \in \mathbb{N}$, such that $h(R_\text{l}) \subseteq {R_\text{l}}'$ and $h(R_\text{g}) \subseteq {R_\text{g}}'$. Observe that

$$\text{if } \ \Gamma \vdash^d_{\mathcal{D},\Xi} \varphi \text{ then } h(\Gamma) \vdash^d_{\mathcal{D}',\Xi} h(\varphi)$$

that is, morphisms preserve local and global derivations. A signature morphism $h : C \to C'$ can be extended to $\overline{h} : L(C, \Xi) \to L(C', \Xi)$ as follows: (i) $\overline{h}(c) = h(c)$, $c \in C_0$; (ii) $\overline{h}(\xi) = \xi$; (iii) $\overline{h}(c)(\varphi_1, \ldots, \varphi_k) = h(c)(\overline{h}(\varphi_1), \ldots, \overline{h}(\varphi_k))$. We will denote $\overline{h}(\varphi)$ by $h(\varphi)$.

Unconstrained fibring is a coproduct in the category of deductive systems when no sharing of constructors is allowed ($C' \cap C'' = \emptyset$). *Constrained fibring*, (when $C' \cap C'' \neq \emptyset$) is a pushout of $h' : \mathcal{D}^0 \to \mathcal{D}'$ and $h'' : \mathcal{D}^0 \to \mathcal{D}''$ where the constructors and the rules of $\mathcal{D}^0$ are shared. In any case, we denote by $i' : \mathcal{D}' \to \mathcal{D}$ and $i'' : \mathcal{D}'' \to \mathcal{D}$ the morphisms from the components to the fibring. These morphisms preserve local and global derivations. That is, everything that we derive for the components is also derived in the fibring. Hence fibring is a conservative extension of its components. However, usually, for fibring much more can be proved.

Example 2.6 Fibring S4 and K4 deductive systems
Let $\mathcal{D}^0$ be a propositional deductive system defined as follows:

- $C^0_0 = \{\mathbf{t}, \mathbf{f}\}$, $C^0_1 = \{\neg\}$, $C^0_2 = \{\Rightarrow\}$ and $C^0_k = \emptyset$ for every $k \geq 3$;
- ${R_\mathrm{l}}^0$ consists of the following rules:
  $\langle \emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)) \rangle$;
  $\langle \emptyset, ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_2 \Rightarrow \xi_3))) \rangle$;
  $\langle \emptyset, ((\neg\, \xi_1 \Rightarrow (\neg\, \xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1)) \rangle$;
  $\langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2 \rangle$;

Let $\mathcal{D}'$ be a S4 modal deductive system and $\mathcal{D}''$ a K4 modal deductive system such that:

- $C'_0 = C''_0 = C^0_0$, $C'_1 = C^0_1 \cup \{\Box'\}$, $C''_1 = C^0_1 \cup \{\Box''\}$, $C'_2 = C''_2 = C^0_2$, $C'_k = C''_k = \emptyset$ for $k \geq 3$:
- ${R_\mathrm{l}}'$ is ${R_\mathrm{l}}^0$ plus
  $\langle \emptyset, ((\Box'(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box'\xi_1) \Rightarrow (\Box'\xi_2))) \rangle$;
  $\langle \emptyset, ((\Box'\xi_1) \Rightarrow \xi_1) \rangle$;
  $\langle \emptyset, ((\Box'\xi_1) \Rightarrow (\Box'(\Box'\xi_1))) \rangle$;
- ${R_\mathrm{l}}''$ is ${R_\mathrm{l}}^0$ plus
  $\langle \emptyset, ((\Box''(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box''\xi_1) \Rightarrow (\Box''\xi_2))) \rangle$;
  $\langle \emptyset, ((\Box''\xi_1) \Rightarrow (\Box''(\Box''\xi_1))) \rangle$;
- ${R_\mathrm{g}}' = {R_\mathrm{l}}' \cup \{\langle \xi_1, (\Box'\xi_1) \rangle\}$;
- ${R_\mathrm{g}}'' = {R_\mathrm{l}}'' \cup \{\langle \xi_1, (\Box''\xi_1) \rangle\}$.

Then the constrained fibring of $\mathcal{D}'$ and $\mathcal{D}''$ sharing $\mathcal{D}^0$ is the deductive system $\langle C, R_l, R_g \rangle$ with

- $C_0 = \{\mathbf{t}, \mathbf{f}\}$, $C_1 = \{\neg, \Box', \Box''\}$, $C_2 = \{\Rightarrow\}$, $C_k = \emptyset$ for $k \geq 3$:
- $R_\mathrm{l} = {R_\mathrm{l}}' \cup {R_\mathrm{l}}''$;
- $R_\mathrm{g} = {R_\mathrm{g}}' \cup {R_\mathrm{g}}''$.

Hence $\mathcal{D}$ is a bimodal logic with two unary modal operators: a S4 $\Box'$ and a $K4$ $\Box''$ having two necessitations and two K axioms. The morphisms involved are in this case inclusions. ◁

Example 2.7 Fibring super-intuitionistic deductive systems
Let $\mathcal{D}^0$ be the propositional intuitionistic deductive system defined as follows:

- $C_0^0 = \{\mathbf{t}, \mathbf{f}\}$, $C_1^0 = \{\neg\}$, $C_2^0 = \{\Rightarrow, \wedge, \vee\}$ and $C_k^0 = \emptyset$ for every $k \geq 3$;
- ${R_\mathrm{l}}^0 = {R_\mathrm{g}}^0$ consists of the following rules:
  $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$;
  $((\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow (\xi_1 \Rightarrow \xi_3)))$;
  $(\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)))$;
  $((\xi_1 \wedge \xi_2) \Rightarrow \xi_1)$;
  $((\xi_1 \wedge \xi_2) \Rightarrow \xi_2)$;
  $(\xi_1 \Rightarrow (\xi_1 \vee \xi_2))$;
  $(\xi_2 \Rightarrow (\xi_1 \vee \xi_2))$;
  $((\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3)))$
  $((\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow (\neg\xi_2)) \Rightarrow (\neg\xi_1)))$;
  $(\xi_1 \Rightarrow ((\neg\xi_1) \Rightarrow \xi_2))$;
  $\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle$.

Let $\mathcal{D}'$ and $\mathcal{D}''$ be super-intuitionistic deductive systems such that:

- $C' = C'' = C^0$:
- ${R_\mathrm{l}}'$ is ${R_\mathrm{l}}^0$ plus
  $\langle\emptyset, ((\neg\,\xi_1) \vee (\neg(\neg\,\xi_1)))\rangle$;
- ${R_\mathrm{l}}''$ is ${R_\mathrm{l}}^0$ plus
  $\langle\emptyset, (\xi_1 \vee (\xi_1 \Rightarrow (\xi_2 \vee (\neg\,\xi_2))))\rangle$;
- ${R_\mathrm{g}}' = {R_\mathrm{l}}'$ and ${R_\mathrm{g}}'' = {R_\mathrm{l}}''$.

Then the constrained fibring of $\mathcal{D}'$ and $\mathcal{D}''$ sharing $\mathcal{D}^0$ is the deductive system

- $C = C' = C'' = C^0$;
- $R_\mathrm{l} = {R_\mathrm{l}}' \cup {R_\mathrm{l}}''$;
- $R_\mathrm{g} = {R_\mathrm{g}}' \cup {R_\mathrm{g}}''$.

Hence $\mathcal{D}$ is the super-intuitionistic system $H5$ in the terminology of [20]. ◁

Example 2.8 Fibring Gödel and classical propositional deductive systems
The unconstrained fibring of the propositional deductive system $\mathcal{D}'$ and the Gödel G3 deductive system $\mathcal{D}''$ is the deductive system $\mathcal{D}$ such that:

- $C_0 = \{\mathbf{t}', \mathbf{f}'\}$, $C_1 = \{\neg', \neg''\}$, $C_2 = \{\Rightarrow', \Rightarrow'', \wedge'', \vee''\}$, $C_k = \emptyset$ for $k \geq 3$:
- R_l and R_g are the same and include all local and global rules for the connectives of both deductive systems.

For instance, two versions $\langle\{\xi_1, (\xi_1 \Rightarrow' \xi_2)\}, \xi_2\rangle$ and $\langle\{\xi_1, (\xi_1 \Rightarrow'' \xi_2)\}, \xi_2\rangle$ of the modus ponens for the propositional and the Gödel implications are included in R_l. In this case, if the negation is shared then the fibring colapses to the propositional deductive system. For more details about collapses and ways to solve them see [38]. ◁

3 Interpolation in several guises

We recast here some forms of interpolation taking into account the distinction between local and global deduction.

3.1 Extension interpolation property

A deductive system has the *d-extension interpolation property* (d-EIP) with respect to Ξ whenever:

$$\text{if } \Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi \text{ then there is } \Gamma' \subseteq L(C, \text{var}(\Psi) \cup \text{var}(\varphi))$$
$$\text{such that } \Gamma \vdash^d_{\mathcal{D},\Xi} \Gamma' \text{ and } \Gamma', \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$$

for every $\Gamma, \Psi \subseteq L(C, \Xi)$ and $\varphi \in L(C, \Xi)$. The set Γ' is said to be an *extension interpolant* for $\Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$. Extension interpolation can be defined in terms of finite sets as the following result shows.

Proposition 3.1 A deductive system $\mathcal{D}$ has d-extension interpolation iff for every $\Psi_1 \cup \Psi_2 \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ_1, Ψ_2 finite, there is a finite extension interpolant whenever $\Psi_1, \Psi_2 \vdash^d_{\mathcal{D},\Xi} \eta$.

Proof: Assume that $\mathcal{D}$ has d-extension interpolation and that $\Theta_1, \Theta_2 \vdash^d_{\mathcal{D},\Xi} \varphi$ where Θ_1, Θ_2 are finite sets. Then there is $\Theta' \subseteq L(C, \Xi)$ such that $\text{var}(\Theta') \subseteq \text{var}(\Theta_2) \cup \text{var}(\varphi)$, $\Theta_1 \vdash^d_{\mathcal{D},\Xi} \Theta'$ and $\Theta', \Theta_2 \vdash^d_{\mathcal{D},\Xi} \varphi$. Using the fact that $\vdash^d_{\mathcal{D},\Xi}$ is finitary and $\Theta', \Theta_2 \vdash^d_{\mathcal{D},\Xi} \varphi$, there is a finite set Φ such that $\Phi \subseteq \Theta'$ and $\Phi, \Theta_2 \vdash^d_{\mathcal{D},\Xi} \varphi$. Moreover $\Theta_1 \vdash^d_{\mathcal{D},\Xi} \Phi$ and $\text{var}(\Phi) \subseteq \text{var}(\Theta_2) \cup \text{var}(\varphi)$. Hence, Φ is an extension interpolant for $\Theta_1, \Theta_2 \vdash^d_{\mathcal{D},\Xi} \varphi$.
Assume now that for every $\Psi_1 \cup \Psi_2 \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ_1, Ψ_2 finite, there is a finite extension interpolant whenever $\Psi_1, \Psi_2 \vdash^d_{\mathcal{D},\Xi} \eta$. Furthermore, assume $\Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$. Since $\vdash^d_{\mathcal{D},\Xi}$ is finitary, there are finite sets $\Theta_1 \subseteq \Gamma$ and $\Theta_2 \subseteq \Psi$ such that $\Theta_1, \Theta_2 \vdash^d_{\mathcal{D},\Xi} \varphi$. Then, by hypothesis, there is a finite extension interpolant Φ for $\Theta_1, \Theta_2 \vdash^d_{\mathcal{D},\Xi} \varphi$ which is also an extension interpolant for $\Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$. $\diamond$

An interesting relationship can be established between the d-extension interpolation property and d-metatheorems of deduction and modus ponens.

Proposition 3.2 d-extension interpolation holds in a deductive system with d-metatheorem of modus ponens and deduction with the same base set.

Proof: Let $\mathcal{D}$ be a deductive system with d-deductive conjunction $\wedge$ and with d-metatheorem of modus ponens and deduction with the same base set Δ. Let $\Gamma, \Psi \subseteq L(C, \Xi)$ and $\varphi \in L(C, \Xi)$ and assume that $\Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$. For the sake of simplicity assume that $\Psi = \{\psi_1, \psi_2\}$. Then $\Gamma \vdash^d_{\mathcal{D},\Xi} \Upsilon$ where Υ is $\cup_{\delta \in \Delta(\psi_2, \varphi)} \Delta(\psi_1, \delta)$ using twice d-MTD. Observe that, for every $\mu \in \Delta(\psi_1, \delta)$ and $\delta \in \Delta(\psi_2, \varphi)$, $\text{var}(\mu) = \text{var}(\psi_1) \cup \text{var}(\delta)$, $\text{var}(\delta) = \text{var}(\psi_2) \cup \text{var}(\varphi)$ and so $\text{var}(\Upsilon) = \text{var}(\Psi) \cup \text{var}(\varphi)$. Note that $\Upsilon, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$ using twice the d-metatheorem of modus ponens. Hence Υ is a d-extension interpolant for $\Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$. $\diamond$

Example 3.3 As an illustration we prove that Łukasiewicz logic L_n with $n \geq 3$ has global extension interpolation property (although not the Craig interpolation property as defined below).
For simplicity we consider the case where $n = 3$ and $\Psi = \{\psi\}$ (since this logic has conjunction with the usual properties and is compact there is no loss of generality in considering the set of hypothesis Ψ as a singleton). Assume that $\Gamma, \psi \vdash^{\text{g}}_{\mathcal{D},\Xi} \varphi$ then $\Gamma \vdash^{\text{g}}_{\mathcal{D},\Xi} (\psi \Rightarrow (\psi \Rightarrow \varphi))$ using g-MTD. Taking $\Gamma' = \{(\psi \Rightarrow (\psi \Rightarrow \varphi))\}$ we obtain $\Gamma \vdash^{\text{g}}_{\mathcal{D},\Xi} \Gamma'$ and also $\Gamma', \psi \vdash^{\text{g}}_{\mathcal{D},\Xi} \varphi$ using modus ponens, satisfying the definition of global extension interpolation property above. $\triangleleft$

3.2 Craig interpolation property

A deductive system has the *d-Craig interpolation property* (d-CIP) with respect to Ξ if:

$$\begin{aligned}&\text{if } \Gamma \vdash^d_{\mathcal{D},\Xi} \varphi \text{ and } \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset \text{ then}\\ &\quad \text{there is } \Gamma' \subseteq L(C, \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi))\\ &\quad \text{such that } \Gamma \vdash^d_{\mathcal{D},\Xi} \Gamma' \text{ and } \Gamma' \vdash^d_{\mathcal{D},\Xi} \varphi\end{aligned}$$

for every $\Gamma \subseteq L(C, \Xi)$ and $\varphi \in L(C, \Xi)$. The set Γ' is said to be a *Craig interpolant* for $\Gamma \vdash^d_{\mathcal{D},\Xi} \varphi$. Again Craig interpolation can be stated in terms of finite sets.

Proposition 3.4 A deductive system $\mathcal{D}$ has d-Craig interpolation iff for every $\Psi \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ finite and $\mathrm{var}(\Psi) \cap \mathrm{var}(\eta) \neq \emptyset$, there is a finite Craig interpolant whenever $\Psi \vdash^d_{\mathcal{D},\Xi} \eta$.

Proof: The proof from left to right is similar to the one of Proposition 3.1.

For the other implication, assume that for every $\Psi \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ finite, there is a finite Craig interpolant whenever $\Psi \vdash^d_{\mathcal{D},\Xi} \eta$. Furthermore, assume that $\Gamma \vdash^d_{\mathcal{D},\Xi} \varphi$ and $\mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$. Then, since $\vdash^d_{\mathcal{D},\Xi}$ is finitary, there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash^d_{\mathcal{D},\Xi} \varphi$. We consider two cases. (i) $\mathrm{var}(\Gamma') \cap \mathrm{var}(\varphi) \neq \emptyset$. Then the proof is once again similar to the one of Proposition 3.1. (ii) $\mathrm{var}(\Gamma') \cap \mathrm{var}(\varphi) = \emptyset$. Let $\gamma \in \Gamma$ be such that there is $\xi \in \mathrm{var}(\gamma) \cap \mathrm{var}(\varphi)$. Then $\Gamma', \gamma \vdash^d_{\mathcal{D},\Xi} \varphi$ with $\Gamma' \cup \{\gamma\}$ is finite and $\mathrm{var}(\Gamma' \cup \{\gamma\}) \cap \mathrm{var}(\varphi) \neq \emptyset$ and so by hypothesis there is a finite Craig interpolant Φ for $\Gamma', \gamma \vdash^d_{\mathcal{D},\Xi} \varphi$. Moreover Φ is also a d-Craig interpolant for $\Gamma \vdash^d_{\mathcal{D},\Xi} \varphi$. $\diamond$

Craig interpolation is preserved when enriching the set of variables with another disjoint denumerable set of variables.

Proposition 3.5 Given a denumerable set Ξ' of variables disjoint from Ξ, a deductive system has d-Craig interpolation with respect to Ξ iff it has d-Craig interpolation with respect to $\Xi \cup \Xi'$.

Proof: Let Ξ' be a denumerable set of variables disjoint from Ξ and $\mathcal{D}$ a deductive system with d-Craig interpolation with respect to Ξ. Let $\Gamma' \subseteq L(C, \Xi \cup \Xi')$ and $\varphi' \in L(C, \Xi \cup \Xi')$ such that $\Gamma' \vdash^d_{\mathcal{D},\Xi\cup\Xi'} \varphi'$ and $\mathrm{var}(\Gamma') \cap \mathrm{var}(\varphi') \neq \emptyset$. Let ρ and μ be as defined in Proposition 2.3. Then $\rho(\Gamma') \vdash^d_{\mathcal{D},\Xi} \rho(\varphi')$ Observe that $\mathrm{var}(\rho(\Gamma')) \cap \mathrm{var}(\rho(\varphi')) \neq \emptyset$. Then, using the fact that $\mathcal{D}$ has d-Craig interpolation with respect to Ξ there is a finite set $\Psi \subseteq L(C, \Xi)$ such that:

- $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\rho(\Gamma')) \cap \mathrm{var}(\rho(\varphi'))$
- $\rho(\Gamma') \vdash^d_{\mathcal{D},\Xi} \Psi$
- $\Psi \vdash^d_{\mathcal{D},\Xi} \rho(\varphi')$

Since Ξ' is a denumerable set of variables, using Proposition 2.4 for μ^{-1} and ρ^{-1} there is a finite set $\rho^{-1}(\Psi) \subseteq L(C, \Xi \cup \Xi')$ such that:

- $\mathrm{var}(\rho^{-1}(\Psi)) \subseteq \mathrm{var}(\Gamma') \cap \mathrm{var}(\varphi')$
- $\Gamma' \vdash^d_{\mathcal{D},\Xi\cup\Xi'} \rho^{-1}(\Psi)$
- $\rho^{-1}(\Psi) \vdash^d_{\mathcal{D},\Xi\cup\Xi'} \varphi'$

showing that $\rho^{-1}(\Psi)$ is a d-Craig interpolant for $\Gamma' \vdash^{d}_{\mathcal{D},\Xi\cup\Xi'} \varphi'$.
For the other implication let $\Gamma \subseteq L(C,\Xi)$ and $\varphi \in L(C,\Xi)$ such that $\Gamma \vdash^{d}_{\mathcal{D},\Xi} \varphi$ and $\mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$. Then $\Gamma \vdash^{d}_{\mathcal{D},\Xi\cup\Xi'} \varphi$ and using the fact that $\mathcal{D}$ has d-Craig interpolation with respect to $\Xi \cup \Xi'$ there is a finite set $\Psi \subseteq L(C, \Xi \cup \Xi')$ such that:

- $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi)$
- $\Gamma \vdash^{d}_{\mathcal{D},\Xi\cup\Xi'} \Psi$
- $\Psi \vdash^{d}_{\mathcal{D},\Xi\cup\Xi'} \varphi$

Then using Proposition 2.3

- $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi)$
- $\Gamma \vdash^{d}_{\mathcal{D},\Xi} \Psi$
- $\Psi \vdash^{d}_{\mathcal{D},\Xi} \varphi$

showing that Ψ is a d-Craig interpolant for $\Gamma \vdash^{d}_{\mathcal{D},\Xi} \varphi$. $\diamond$

In most of the cases Craig interpolation has been investigated in deductive systems with implication and when no hypotheses are needed in the derivation. A deductive system has a *d-deductive implication* $\Rightarrow$ if $\Rightarrow \in C_2$ and the d-MTD and the d-MTMP hold with $\Delta = \{(\xi_1 \Rightarrow \xi_2)\}$. A deductive system with d-deductive implication has *d-theoremhood-Craig interpolation* if:

$$\text{if } \vdash^{d}_{\mathcal{D},\Xi} (\varphi_1 \Rightarrow \varphi_2) \text{ and } \mathrm{var}(\varphi_1) \cap \mathrm{var}(\varphi_2) \neq \emptyset \text{ then there is } \psi$$

$$\text{such that } \mathrm{var}(\psi) \subseteq \mathrm{var}(\varphi_1) \cap \mathrm{var}(\varphi_2),\ \vdash^{d}_{\mathcal{D},\Xi} (\varphi_1 \Rightarrow \psi) \text{ and } \vdash^{d}_{\mathcal{D},\Xi} (\psi \Rightarrow \varphi_2)$$

for every $\varphi_1, \varphi_2 \in L(C,\Xi)$.
The relevance of careful reasoning (see Subsection 2.1) is measured by the fact that in some cases it is also possible to relate local and global CIP. That is the case of deductive systems which share with modal and first-order logics the important property that we call *careful-reasoning-by-cases*. The property is present when there is a procedure which permits that hypotheses in global reasoning can be modified so as to transform a global derivation into a local derivation.

Example 3.6 As an illustration, we observe that the global derivation of

$$\{(\xi_1 \Rightarrow \xi_2)\} \vdash^{\mathrm{g}}_{K,\Xi} ((\Box\xi_1) \Rightarrow (\Box\xi_2))$$

in the normal modal system K can be transformed into a local derivation of

$$\{(\Box(\xi_1 \Rightarrow \xi_2))\} \vdash^{\mathrm{l}}_{K,\Xi} ((\Box\xi_1) \Rightarrow (\Box\xi_2))$$

where $((\Box\xi_1) \Rightarrow (\Box\xi_2)) \in \{(\xi_1 \Rightarrow \xi_2)\}^{\vdash^{\mathrm{g}}_{K,\Xi}}$. A similar procedure can be used in first-order logic by means of universal closure. $\triangleleft$

A deduction system $\mathcal{D}$ is said to enjoy *careful-reasoning-by-cases* with respect to Ξ if:

$$\text{if } \Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi, \text{ then there is } \Psi \subseteq \Gamma^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$$

$$\text{such that } \mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma) \text{ and } \Psi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$$

where $\Gamma \cup \{\varphi\} \subseteq L(C,\Xi)$.

Theorem 3.7 A deductive system enjoying careful-reasoning-by-cases has global Craig interpolation whenever it has local Craig interpolation property.

Proof: Assume that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$ and $\mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$. Then, since $\mathcal{D}$ enjoys careful-reasoning-by-cases, there is $\Psi \subseteq L(C, \Xi)$ such that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \Psi$, $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma)$ and $\Psi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$. There are two cases. (1) $\mathrm{var}(\Psi) \cap \mathrm{var}(\varphi) \neq \emptyset$. Since $\mathcal{D}$ has l-Craig interpolation there is $\Gamma' \subseteq L(C, \mathrm{var}(\Psi) \cap \mathrm{var}(\varphi))$ such that $\Psi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \Gamma'$ and $\Gamma' \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$. Therefore, there is $\Gamma' \subseteq L(C, \mathrm{var}(\Psi) \cap \mathrm{var}(\varphi))$ such that $\Psi \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \Gamma'$ and $\Gamma' \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$ and so, by transitivity of $\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}$, there is $\Gamma' \subseteq L(C, \mathrm{var}(\Psi) \cap \mathrm{var}(\varphi))$ such that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \Gamma'$ and $\Gamma' \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$. Since $\mathrm{var}(\Psi) \cap \mathrm{var}(\varphi)) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi)$ then there is $\Gamma' \subseteq L(C, \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi))$ such that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \Gamma'$ and $\Gamma' \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$. (2) $\mathrm{var}(\Psi) \cap \mathrm{var}(\varphi) = \emptyset$. Take $\gamma \in \Gamma$ such that $\mathrm{var}(\gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$. Then $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \Psi \cup \{\gamma\}$, $\mathrm{var}(\Psi) \cup \mathrm{var}(\gamma) \subseteq \mathrm{var}(\Gamma)$ and $\Psi \cup \{\gamma\} \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$. And we can now proceed in a similar way to case (1). $\diamond$

Careful-reasoning-by-cases can be expressed in terms of finite sets as indicated in the following result.

Proposition 3.8 A deductive system $\mathcal{D}$ enjoys careful-reasoning-by-cases iff for every $\Psi \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ finite such that $\Psi \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \eta$, there is a finite Ψ' such that $\mathrm{var}(\Psi') \subseteq \mathrm{var}(\Psi)$, $\Psi' \subseteq \Psi^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$ and $\Psi' \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \eta$.

Proof: Assume that $\mathcal{D}$ enjoys careful-reasoning-by-cases and let $\Psi \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ finite, be such that $\Psi \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \eta$. Then there is $\Upsilon \subseteq \Psi^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$ such that $\mathrm{var}(\Upsilon) \subseteq \mathrm{var}(\Psi)$ and $\Upsilon \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \eta$. Since $\vdash^{\mathrm{l}}_{\mathcal{D},\Xi}$ is finitary, there is a finite set $\Psi' \subseteq \Upsilon$, and so with $\Psi' \subseteq \Psi^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$ and $\mathrm{var}(\Psi') \subseteq \mathrm{var}(\Psi)$, with $\Psi' \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \eta$.
Assume now that for every $\Psi \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ finite such that $\Psi \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \eta$, there is a finite Ψ' such that $\mathrm{var}(\Psi') \subseteq \mathrm{var}(\Psi)$, $\Psi' \subseteq \Psi^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$ and $\Psi' \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \eta$. Let $\Gamma \cup \{\varphi\} \subseteq L(C, \Xi)$ with $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$. Since $\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}$ is finitary, there is a finite set Υ with $\Upsilon \subseteq \Gamma$ and $\Upsilon \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$. So there is a finite set Ψ such that $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Upsilon)$, $\Psi \subseteq \Upsilon^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$, and so with $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma)$ and $\Psi \subseteq \Gamma^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$, and $\Psi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$. $\diamond$

Example 3.9 Some illustrations can be given of Craig interpolation:

- In [20] it is shown that some modal deductive systems have theoremhood-Craig interpolation.
- We can conclude that the modal deductive systems referred to above have local Craig interpolation:
  Assume that $\{\gamma_1, \ldots, \gamma_n\} \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$ and $\mathrm{var}(\{\gamma_1, \ldots, \gamma_n\}) \cap \mathrm{var}(\varphi) \neq \emptyset$. Then, by l-MTD, $\vdash^{\mathrm{l}}_{\mathcal{D},\Xi} ((\gamma_1 \wedge \ldots \wedge \gamma_n) \Rightarrow \varphi)$ and so by theoremhood Craig interpolation, there is a formula ψ such that $\mathrm{var}(\psi) \subseteq \mathrm{var}(\{\gamma_1, \ldots, \gamma_n\}) \cap \mathrm{var}(\varphi)$, $\vdash^{\mathrm{l}}_{\mathcal{D},\Xi} ((\gamma_1 \wedge \ldots \wedge \gamma_n) \Rightarrow \psi)$ and $\vdash^{\mathrm{l}}_{\mathcal{D},\Xi} (\psi \Rightarrow \varphi)$. Using l-MTMP there is a finite set $\{\psi\}$ such that $\mathrm{var}(\{\psi\}) \subseteq \mathrm{var}(\{\gamma_1, \ldots, \gamma_n\}) \cap \mathrm{var}(\varphi)$, $\{\gamma_1, \ldots, \gamma_n\} \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \{\psi\}$ and $\{\psi\} \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$.
- Since modal deductive systems enjoy careful-reasoning-by-cases, then the modal deductive systems referred to above have global Craig interpolation.
- Intuitionistic logic [17] has theoremhood Craig interpolation and so has l-Craig interpolation using a reasoning similar to the one proved above for modal systems.

In general, proofs of theoremhood Craig interpolation are not constructive. In Section 6, we show that under certain natural conditions some classes of deductive systems can be shown to have constructive interpolation. In particular, deductive systems that can be axiomatized using the Rosser-Turquette method are included in this class. ◁

3.3 Maehara interpolation property

A deductive system has the d-*Maehara interpolation property* (d-MIP) whenever:

$$\text{if } \Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi \text{ and } \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi)) \neq \emptyset \text{ then}$$
$$\text{there is } \Gamma' \subseteq \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi))$$
$$\text{such that } \Gamma \vdash^d_{\mathcal{D},\Xi} \Gamma' \text{ and } \Gamma', \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$$

for every $\Gamma, \Psi \subseteq L(C, \Xi)$ and $\varphi \in L(C, \Xi)$. We say then that Γ' is a *Maehara interpolant*. As happened before, Maehara interpolation can be stated for finite sets.

Proposition 3.10 A deductive system $\mathcal{D}$ has d-Maehara interpolation iff for every $\Psi_1 \cup \Psi_2 \cup \{\eta\} \subseteq L(C, \Xi)$ with Ψ_1, Ψ_2 finite sets there is a finite Maehara interpolant whenever $\Psi_1, \Psi_2 \vdash^d_{\mathcal{D},\Xi} \eta$.

Proof: This result is a direct corollary of Proposition 3.1 and Proposition 3.4. ◇

It is proved in [15], that a deductive system has global MIP iff it has global EIP and global CIP. Our notion of careful reasoning allows the following improvement.

Theorem 3.11 A deductive system with d-Craig interpolation, d-metatheorem of modus ponens and d-metatheorem of deduction over the same base set has d-Maehara interpolation.

Proof: Suppose that $\Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$ and $\mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi)) \neq \emptyset$ where Γ and Ψ are finite sets. For the sake of simplicity assume that $\Psi = \{\psi_1, \psi_2\}$. Then $\Gamma \vdash^d_{\mathcal{D},\Xi} \mu$ for every $\mu \in \Delta(\psi_1, \delta)$ and $\delta \in \Delta(\psi_2, \varphi)$ using twice d-MTD. Observe that, for every $\mu \in \Delta(\psi_1, \delta)$ and $\delta \in \Delta(\psi_2, \varphi)$, $\mathrm{var}(\mu) = \mathrm{var}(\psi_1) \cup \mathrm{var}(\delta)$, $\mathrm{var}(\delta) = \mathrm{var}(\psi_2) \cup \mathrm{var}(\varphi)$ and so $\mathrm{var}(\mu) = \mathrm{var}(\psi_1) \cup \mathrm{var}(\psi_2) \cup \mathrm{var}(\varphi)$. Hence there is $\Gamma^{\mu,\delta}$ such that $\mathrm{var}(\Gamma^{\mu,\delta}) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\mu)$, $\Gamma \vdash^d_{\mathcal{D},\Xi} \Gamma^{\mu,\delta}$ and $\Gamma^{\mu,\delta} \vdash^d_{\mathcal{D},\Xi} \mu$ for every $\mu \in \Delta(\psi_1, \delta)$ and $\delta \in \Delta(\psi_2, \varphi)$ using d-CIP. The set

$$\Phi = \bigcup_{\mu \in \Delta(\psi_1, \delta)} \bigcup_{\delta \in \Delta(\psi_2, \varphi)} \Gamma^{\mu,\delta}$$

is a Maehara interpolant for $\Gamma, \Psi \vdash^d_{\mathcal{D},\Xi} \varphi$: (i) $\mathrm{var}(\Phi) \subseteq \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi))$; (ii) $\Gamma \vdash^d_{\mathcal{D},\Xi} \Phi$; (iii) $\Phi \vdash^d_{\mathcal{D},\Xi} \varphi$: $\Phi \vdash^d_{\mathcal{D},\Xi} \Delta(\psi_1, \delta)$ for every $\delta \in \Delta(\psi_2, \varphi)$, hence $\Phi, \psi_1 \vdash^d_{\mathcal{D},\Xi} \delta$ for every $\delta \in \Delta(\psi_2, \varphi)$ using d-MTMP, so $\Phi, \psi_1 \vdash^d_{\mathcal{D},\Xi} \Delta(\psi_2, \varphi)$ and finally, by d-MTMP $\Phi, \psi_1, \psi_2 \vdash^d_{\mathcal{D},\Xi} \varphi$. ◇

The previous theorem along with the result in [15] shows that d-MTD and d-EIP are provable from each other. Of course, it is easier to prove that a deduction system has the metatheorem of deduction than that it has the extension interpolation. See also Section 4 where a necessary and sufficient condition for the existence of the MTD is presented.

Example 3.12 For instance K4 modal logic has g-metatheorem of modus ponens, g-metatheorem of deduction and g-Craig interpolation. Therefore by Theorem 3.11 it also has g-Maehara interpolation. ◁

4 Preserving metatheoretical character

We start by studying the preservation of the metatheorems of modus ponens and deduction. Afterwards we analyze preservation of derivations when adding new variables and also when translating derivations from the fibring to the component deductive systems (as observed before every derivation in the components is a derivation in the fibring).

4.1 Preserving metatheorems

In order to analyze the preservation of MTD it is easier to provide an alternative characterization involving derivations in the object logic. We start with a lemma about a characterization of the metatheorem of modus ponens.

Lemma 4.1 A deductive system has d-metatheorem of modus ponens iff there is a finite set $\Delta \subseteq L(C, \{\xi_1, \xi_2\})$ of formulas such that $\Delta, \xi_1 \vdash^d_{\mathcal{D},\Xi} \xi_2$.

Proof: Assume that $\mathcal{D}$ has the d-MTMP. Since $\Delta \vdash^d_{\mathcal{D},\Xi} \Delta$ then $\Delta, \xi_1 \vdash^d_{\mathcal{D},\Xi} \xi_2$ using the hypothesis.

Assume that there is a finite set $\Delta \subseteq L(C, \{\xi_1, \xi_2\})$ of formulas such that $\Delta, \xi_1 \vdash^d_{\mathcal{D},\Xi}$ ξ_2. Suppose that $\Gamma \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \varphi_2)$. Using the preservation of derivations by substitution we have $\Delta(\varphi_1, \varphi_2), \varphi_1 \vdash^d_{\mathcal{D},\Xi} \varphi_2$. Hence $\Gamma, \varphi_1 \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \varphi_2) \cup \{\varphi_1\}$ and so $\Gamma, \varphi_1 \vdash^d_{\mathcal{D},\Xi} \varphi_2$. ⋄

Lemma 4.2 A deductive system with d-metatheorem of modus ponens has d-metatheorem of deduction with the same base set Δ iff:

1. $\{\xi_1\} \vdash^d_{\mathcal{D},\Xi} \Delta(\xi_2, \xi_1)$;
2. $\vdash^d_{\mathcal{D},\Xi} \Delta(\xi_1, \xi_1)$;
3. $\Delta(\xi_1, \theta_1) \cup \cdots \cup \Delta(\xi_1, \theta_m) \vdash^d_{\mathcal{D},\Xi} \Delta(\xi_1, \eta)$
   for each rule $r = \langle\{\theta_1, \ldots, \theta_m\}, \eta\rangle \in R_d$.

Proof: Let $\mathcal{D}$ be a deductive system with d-MTMP over the base set Δ. Assume that $\mathcal{D}$ has d-MTD for Δ.
1. $\{\xi_1, \xi_2\} \vdash^d_{\mathcal{D},\Xi} \xi_1$ and so $\{\xi_1\} \vdash^d_{\mathcal{D},\Xi} \Delta(\xi_2, \xi_1)$ by d-MTD.
2. $\{\xi_1\} \vdash^d_{\mathcal{D},\Xi} \xi_1$ and so $\vdash^d_{\mathcal{D},\Xi} \Delta(\xi_1, \xi_1)$ by d-MTD.
3. $\Delta(\xi_1, \theta_1) \cup \cdots \cup \Delta(\xi_1, \theta_m) \vdash^d_{\mathcal{D},\Xi} \Delta(\xi_1, \theta_i)$ for $i = 1, \ldots, m$, so by d-MTMP we have $\Delta(\xi_1, \theta_1) \cup \cdots \cup \Delta(\xi_1, \theta_m), \xi_1 \vdash^d_{\mathcal{D},\Xi} \theta_i$ for $i = 1, \ldots, m$, therefore, since $\{\theta_1, \ldots, \theta_m\} \vdash^d_{\mathcal{D},\Xi} \eta$, then $\Delta(\xi_1, \theta_1) \cup \cdots \cup \Delta(\xi_1, \theta_m), \xi_1 \vdash^d_{\mathcal{D},\Xi} \eta$ and finally, by d-MTD, $\Delta(\xi_1, \theta_1) \cup \cdots \cup \Delta(\xi_1, \theta_m) \vdash^d_{\mathcal{D},\Xi} \Delta(\xi_1, \eta)$.

Assume that 1, 2 and 3 hold. We show the d-MTD by induction on the length of a derivation $\psi_1 \ldots \psi_n$ of $\Gamma, \varphi_1 \vdash^d_{\mathcal{D},\Xi} \varphi_2$.
Base (i) φ_2 is either in Γ or is an instance of a rule with no premises or is in $\emptyset^{\vdash^{\rm g}}$ in the case of $\vdash^{\rm l}$. By 1, $\{\varphi_2\} \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \varphi_2)$, also $\vdash^d_{\mathcal{D},\Xi} \varphi_2$ hence $\vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \varphi_2)$ and so $\Gamma \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \varphi_2)$. (ii) φ_2 is φ_1. Then by 2 and monotonicity, $\Gamma \vdash^d_{\mathcal{D},\Xi}$ $\Delta(\varphi_1, \varphi_1)$. (iii) Let $r = \langle\{\theta_1, \ldots, \theta_m\}, \eta\rangle \in R_d$ and assume that there is σ such that $\sigma(\eta) = \psi_n$ and there are $n_1, \ldots, n_m$ such that $\sigma(\theta_j) = \psi_{n_j}$ for $j = 1, \ldots, m$. Hence $\Gamma, \varphi_1 \vdash^d_{\mathcal{D},\Xi} \psi_{n_j}$, hence, by induction hypothesis, $\Gamma \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \psi_{n_j})$, so $\Gamma \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \psi_{n_1}) \cup \cdots \cup \Delta(\varphi_1, \psi_{n_m})$ and so by 3, $\Gamma \vdash^d_{\mathcal{D},\Xi} \Delta(\varphi_1, \psi_n)$. ⋄

Theorem 4.3 d-metatheorem of modus ponens holds in the fibring when at least one of the components has d-metatheorem of modus ponens.

Proof: Assume that $\mathcal{D}$ is the fibring of $\mathcal{D}'$ and $\mathcal{D}''$ and that $\mathcal{D}'$ has d-MTMP. Take $\Delta = \Delta'$. Then $\Delta, \xi_1 \vdash^d_{\mathcal{D},\Xi} \xi_2$. The result follows by Lemma 4.1. $\diamond$

Theorem 4.4 d-metatheorem of deduction is preserved by fibring deductive systems when at least one of the components has d-metatheorem of modus ponens and all the base sets coincide in the fibring deductive system.

Proof: The proof is similar to the one above but we need both systems to have d-MTD and that the base sets coincide in the deductive system resulting of the fibring because of condition 3 in Lemma 4.2. $\diamond$

4.2 Derivation with different variables

In order to investigate the preservation of interpolation, we must be able to *transform* derivations in the fibring into derivations in the components (the other way around we already know how to do). For this purpose, we start by translating formulas from a deductive system to another in the presence of a deductive system morphism. Assume that $h : \mathcal{D} \to \mathcal{D}'$ is a deductive system morphism. Take

$$\Xi^\bullet = \Xi \cup \{\xi_{c'(\varphi_1,\dots,\varphi_k)} : c'(\varphi_1,\dots,\varphi_k) \in L(C,\Xi), c' \in C'_k \setminus h(C_k)\}$$

as a new set of variables. Take Ξ' as $\Xi^\bullet \setminus \Xi$ and assume without loss of generality that Ξ' is a denumerable set. Each $\xi_{c'(\varphi_1,\dots,\varphi_k)}$ is a *ghost* of $c'(\varphi_1,\dots,\varphi_k)$ in $\mathcal{D}$ and will only have an auxiliary role. Observe that the set of ghosts can be dealt with both in $\mathcal{D}$ and in $\mathcal{D}'$. The introduction of ghosts is similar to the introduction of surrogates used in [41] for proving preservation of properties in the fusion of modal logics sharing the propositional connectives.

The translation

$$\tau : L(C',\Xi) \to L(C,\Xi^\bullet)$$

is a map defined inductively as follows:

- $\tau(\xi) = \xi$ for $\xi \in \Xi$;
- $\tau(h(c)) = c$ for $c \in C_0$;
- $\tau(c') = \xi_{c'}$ for $c' \in C'_0 \setminus h(C_0)$;
- $\tau(h(c)(\gamma'_1,\dots,\gamma'_k)) = c(\tau(\gamma'_1),\dots,\tau(\gamma'_k))$ for $c \in C_k$ and $\gamma'_1,\dots,\gamma'_k \in L(C',\Xi)$;
- $\tau(c'(\gamma'_1,\dots,\gamma'_k)) = \xi_{c'(\gamma'_1,\dots,\gamma'_k)}$ for $c' \in C'_k \setminus h(C_k)$ and $\gamma'_1,\dots,\gamma'_k \in L(C',\Xi)$.

On the other hand, let $\tau^{-1} : \Xi^\bullet \to L(C',\Xi)$ be the following substitution:

- $\tau^{-1}(\xi) = \xi$ for $\xi \in \Xi$;
- $\tau^{-1}(\xi_{c'(\gamma'_1,\dots,\gamma'_k)}) = c'(\gamma'_1,\dots,\gamma'_k)$
  for $c'(\gamma'_1,\dots,\gamma'_k) \in L(C',\Xi)$ and $c' \in C'_k \setminus h(C_k)$.

The following are technical lemmas that will be needed to relate derivations in $\mathcal{D}'$ with derivations in $\mathcal{D}$. The first one relates substitutions in $\mathcal{D}$ and in $\mathcal{D}'$. The second one states the invertible character of translation τ, morphism h and assignment ρ^{-1}.

Lemma 4.5 Let $h : C \to C'$ be a signature morphism, $\rho' : \Xi \to L(C', \Xi)$ and $\rho : \Xi^\bullet \to L(C, \Xi^\bullet)$ substitutions such that $\rho(\xi) = \tau(\rho'(\xi))$ for every $\xi \in \Xi$. Then $\rho(\gamma) = \tau(\rho'(h(\gamma)))$ for every $\gamma \in L(C, \Xi)$.

Proof: By induction on the structure of γ.
Base: (1) γ is ξ: direct from the definition of ρ. (2) γ is $c \in C_0$: $\rho(c) = c = \tau(h(c)) = \tau(\rho'(h(c)))$. Step: γ is $c(\gamma_1, \ldots, \gamma_k)$: hence $\rho(c(\gamma_1, \ldots, \gamma_k)) = c(\rho(\gamma_1), \ldots, \rho(\gamma_k))$; on the other hand, $\tau(\rho'(h(c(\gamma_1, \ldots, \gamma_k)))) = \tau(h(c)(\rho'(h(\gamma_1)), \ldots, \rho'(h(\gamma_k))))$, which is, by definition of τ equal to $c(\tau(\rho'(h(\gamma_1))), \ldots, \tau(\rho'(h(\gamma_k))))$ which is, using the induction hypothesis, $c(\rho(\gamma_1), \ldots, \rho(\gamma_k))$. $\diamond$

The following lemma is also proved using a straightforward induction.

Lemma 4.6 If $h : C \to C'$ is a signature morphism, then $\tau^{-1} \circ h \circ \tau = \mathrm{id}$.

Proof: By induction on the structure of γ in $L(C', \Xi)$.
Base: γ is ξ: direct from the definition of τ, h and τ^{-1}.
Step:
(1) γ is $h(c)(\gamma'_1, \ldots, \gamma'_k)$ for $c \in C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C', \Xi)$: note that $\tau^{-1}(h(\tau(\gamma)))$ is $\tau^{-1}(h(c(\tau(\gamma'_1), \ldots, \tau(\gamma'_k))))$ which is $\tau^{-1}(h(c)(h(\tau(\gamma'_1)), \ldots, h(\tau(\gamma'_k))))$ that is, by definition of τ^{-1}, $h(c)(\tau^{-1}(h(\tau(\gamma'_1))), \ldots, \tau^{-1}(h(\tau(\gamma'_k)))))$, which, by induction hypothesis is γ.
(2) γ is $c'(\gamma'_1, \ldots, \gamma'_k)$ for $c' \in C'_k \setminus h(C_k)$ and $\gamma'_1, \ldots, \gamma'_k \in L(C', \Xi)$: note that $\tau^{-1}(h(\tau(\gamma)))$ is $\tau^{-1}(h(\xi_\gamma))$ which is $\tau^{-1}(\xi_\gamma)$ which by definition of τ^{-1} is γ. $\diamond$

We are now ready to relate global derivations in $h(\mathcal{D})$ with global derivations in $\mathcal{D}$ where:

$$h(\mathcal{D}) = \langle C', h(R_\mathrm{l}), h(R_\mathrm{g}) \rangle.$$

Of course in $\mathcal{D}'$ we can prove more things than in $h(\mathcal{D})$ since in $h(\mathcal{D})$ no rules that are in ${R_\mathrm{g}}' \setminus h(R_\mathrm{g})$ can be used.

Lemma 4.7 Let $h : \mathcal{D} \to \mathcal{D}'$ be a deductive system morphism. Then

$$\Gamma' \vdash^d_{h(\mathcal{D}),\Xi} \psi' \text{ iff } \tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \tau(\psi')$$

for every $\Gamma' \cup \{\psi'\} \subseteq L(C', \Xi)$.

Proof:
(i) Assume that $\Gamma' \vdash^d_{h(\mathcal{D}),\Xi} \psi'$. We prove that $\tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \tau(\psi')$ by induction on the length n of a proof of ψ' from Γ'.
Base: a) ψ' is an instance of the axiom $\langle \emptyset, h(\eta) \rangle$ in $h(\mathcal{D})$ with substitution $\rho' : \Xi \to L(C', \Xi)$. Then $\emptyset \vdash^d_{h(\mathcal{D}),\Xi} \rho'(h(\eta))$. Let $\rho : \Xi^\bullet \to L(C, \Xi^\bullet)$ be a substitution such that $\rho(\xi) = \tau(\rho'(\xi))$ for every $\xi \in \Xi$. Hence $\emptyset \vdash^d_{\mathcal{D},\Xi^\bullet} \rho(\eta)$, by monotonicity $\tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \rho(\eta)$ and since $\rho'(h(\eta)) = \psi'$ we get $\tau(\rho'(h(\eta)) = \tau(\psi')$ and so using Lemma 4.5 $\rho(\eta) = \tau(\psi')$. b) Straightforward when ψ' is an hypothesis.
Step: Assume that ψ' is an instance of $h(\eta)$ in the proof rule $\langle \{\omega_1, \ldots, \omega_k\}, \eta \rangle$ in $\mathcal{D}$ with substitution $\rho' : \Xi \to L(C', \Xi)$. Then $\Gamma' \vdash^d_{h(\mathcal{D}),\Xi} \rho'(h(\omega_i))$ for $i = 1, \ldots, k$ and so by the induction hypothesis $\tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \tau(\rho'(h(\omega_i)))$ for $i = 1, \ldots, k$. Taking substitution ρ such that $\rho(\xi) = \tau(\rho'(\xi))$ for every $\xi \in \Xi$ then $\tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \rho(\omega_i)$ for $i = 1, \ldots, k$, hence $\tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \rho(\eta)$ and so $\tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \tau(\psi')$.
(ii) Suppose that $\tau(\Gamma') \vdash^d_{\mathcal{D},\Xi^\bullet} \tau(\psi')$. Then, since deductive system morphisms preserve derivation, $h(\tau(\Gamma')) \vdash^d_{h(\mathcal{D}),\Xi^\bullet} h(\tau(\psi'))$, so $\tau^{-1}(h(\tau(\Gamma'))) \vdash^d_{h(\mathcal{D}),\Xi} \tau^{-1}(h(\tau(\psi')))$ since derivation is closed for substitution and by Lemma 4.6 $\Gamma' \vdash^d_{h(\mathcal{D}),\Xi} \psi'$. $\diamond$

Lemma 4.7 states the relationship between derivations in $\mathcal{D}$ (over $\Xi \cup \Xi'$) with (parts of) derivations in $\mathcal{D}'$ (over Ξ) using only rules in $\mathcal{D}$.

5 Preserving interpolation properties

We are ready to investigate preservation of different kinds of interpolation by fibring. In the presence of fibring we have to deal with the ghost variables for each component deductive system as well as two translations. We start by investigating preservation of careful-reasoning-by-cases.

5.1 Preserving careful-reasoning-by-cases

Before analyzing preservation by fibring of careful-reasoning-by-cases we prove a technical lemma about preservation of careful-reasoning-by-cases when changing the set of variables. This is the situation which occurs when a morphism $h : \mathcal{D} \to \mathcal{D}'$ is present and we want to transfer derivations from $\mathcal{D}'$ to $\mathcal{D}$.

Lemma 5.1 Let $\mathcal{D}$ be a deductive system with careful-reasoning-by-cases with respect to Ξ and $h : \mathcal{D} \to \mathcal{D}'$ a deductive system morphism. Then $\mathcal{D}$ also has careful-reasoning-by-cases with respect to $\Xi^\bullet$.

Proof: Assume that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi^\bullet} \varphi$ where Γ is finite. Pick up a derivation of φ from Γ. Let Υ' be the set of variables in Ξ' appearing in the derivation, μ a bijection to a set Υ of variables in Ξ not occurring in that derivation and ρ an assignment as required in Proposition 2.3. Then, by the same lemma, $\rho(\Gamma) \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \rho(\varphi)$. Since $\mathcal{D}$ has careful-reasoning-by-cases with respect to Ξ, using Proposition 3.8, there is a finite set $\Phi \subseteq L(C, \Xi)$ such that:

- (1) $\mathrm{var}(\Phi) \subseteq \mathrm{var}(\rho(\Gamma))$;
- (2) $\Phi \subseteq \rho(\Gamma)^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$;
- (3) $\Phi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \rho(\varphi)$.

Consider substitution ρ^{-1}. Then, there is a finite set $\rho^{-1}(\Phi) \subseteq L(C, \Xi^\bullet)$ such that:

- $\mathrm{var}(\rho^{-1}(\Phi)) \subseteq \mathrm{var}(\Gamma)$;
- $\rho^{-1}(\Phi) \subseteq \Gamma^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi^\bullet}}$
  Assume that $\varphi \in \Phi$. Then $\rho(\Gamma) \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$
  hence, by Proposition 2.4, $\rho^{-1}(\rho(\Gamma)) \vdash^{\mathrm{g}}_{\mathcal{D},\Xi^\bullet} \rho^{-1}(\varphi)$ and so $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi^\bullet} \rho^{-1}(\varphi)$;
- $\rho^{-1}(\Phi) \vdash^{\mathrm{l}}_{\mathcal{D},\Xi^\bullet} \varphi$.

Therefore, $\mathcal{D}$ has careful-reasoning-by-cases with respect to $\Xi^\bullet$. $\diamond$

Before analyzing the preservation of careful-reasoning-by-cases by a morphism we need two more lemmas.

Lemma 5.2 Let $h : \mathcal{D} \to \mathcal{D}'$ be a deductive system morphism and $\Psi \vdash^{d}_{\mathcal{D},\Xi^\bullet} \varphi$ where Ψ is finite. Let $\Upsilon \subseteq \Xi'$ be the set of variables used in a derivation of φ from Ψ. Choose μ and ρ as in Proposition 2.3 and take also μ^{-1} and ρ^{-1}. Then for every $\gamma \in L(C, \Upsilon)$, we have

$$\rho^{-1}(h(\rho(\gamma))) = h(\gamma).$$

Proof: The proof follows by induction on the structure of γ. We only consider as an illustration the case of $\gamma \in \Upsilon$: $\rho^{-1}(h(\rho(\gamma))) = \rho^{-1}(h(\mu(\gamma))) = \rho^{-1}(\mu(\gamma)) = \mu^{-1}(\mu(\gamma)) = \gamma = h(\gamma)$. $\diamond$

We can now relate derivation in $\mathcal{D}$ and $h(\mathcal{D})$ over the set of variables $\Xi^\bullet$.

Lemma 5.3 Let $h : \mathcal{D} \to \mathcal{D}'$ be a deductive system morphism. Then

$$h(\Gamma) \vdash^d_{h(\mathcal{D}),\Xi^\bullet} h(\varphi) \text{ whenever } \Gamma \vdash^d_{\mathcal{D},\Xi^\bullet} \varphi.$$

Proof: Assume that $\Gamma \vdash^d_{\mathcal{D},\Xi^\bullet} \varphi$. Then by Proposition 2.3, choosing μ and ρ as indicated there, $\rho(\Gamma) \vdash^d_{\mathcal{D},\Xi} \rho(\varphi)$ and, since morphisms preserve derivations over variables in Ξ, $h(\rho(\Gamma)) \vdash^d_{h(\mathcal{D}),\Xi} h(\rho(\varphi))$. Using Proposition 2.4, choosing μ^{-1} and ρ^{-1}, $\rho^{-1}(h(\rho(\Gamma))) \vdash^d_{h(\mathcal{D}),\Xi^\bullet} \rho^{-1}(h(\rho(\varphi)))$ and so, by Lemma 5.2, $h(\Gamma) \vdash^d_{h(\mathcal{D}),\Xi^\bullet} h(\varphi)$. $\diamond$

Proposition 5.4 If $\mathcal{D}$ is a deductive system enjoying careful-reasoning-by-cases with respect to Ξ then $h(\mathcal{D})$ also enjoys careful-reasoning-by-cases with respect to Ξ.

Proof: Assume that $\mathcal{D}$ enjoys careful-reasoning-by-cases with respect to Ξ and $\Gamma' \vdash^{\mathrm{g}}_{h(\mathcal{D}),\Xi} \psi'$. Then $\tau(\Gamma') \vdash^{\mathrm{g}}_{\mathcal{D},\Xi^\bullet} \tau(\psi')$ by Proposition 4.7. Since, by Lemma 5.1, $\mathcal{D}$ enjoys careful-reasoning-by-cases with respect to $\Xi^\bullet$, there is a finite set $\Phi \subseteq L(C, \Xi^\bullet)$ such that:

- $\mathrm{var}(\Phi) \subseteq \mathrm{var}(\tau(\Gamma'))$;
- $\Phi \subseteq \tau(\Gamma')^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi^\bullet}}$;
- $\Phi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi^\bullet} \tau(\psi')$.

Hence there is a finite set $h(\Phi) \subseteq L(C', \Xi^\bullet)$ such that:

- $\mathrm{var}(h(\Phi)) \subseteq \mathrm{var}(h(\tau(\Gamma')))$;
- $h(\Phi) \subseteq h(\tau(\Gamma'))^{\vdash^{\mathrm{g}}_{h(\mathcal{D}),\Xi^\bullet}}$ using Lemma 5.3;
- $h(\Phi) \vdash^{\mathrm{l}}_{h(\mathcal{D}),\Xi^\bullet} h(\tau(\psi'))$ using Lemma 5.3.

Moreover, there is a finite set $\tau^{-1}(h(\Phi)) \subseteq L(C', \Xi)$ such that:

- $\mathrm{var}(\tau^{-1}(h(\Phi))) \subseteq \mathrm{var}(\tau^{-1}(h(\tau(\Gamma'))))$;
- $\tau^{-1}(h(\Phi)) \subseteq \tau^{-1}(h(\tau(\Gamma')))^{\vdash^{\mathrm{g}}_{h(\mathcal{D}),\Xi}}$, using closure for substitution;
- $\tau^{-1}(h(\Phi)) \vdash^{\mathrm{l}}_{h(\mathcal{D}),\Xi} \tau^{-1}(h(\tau(\psi')))$, using closure for substitution.

Hence, by Lemma 4.6, there is a finite set $\tau^{-1}(h(\Phi)) \subseteq L(C', \Xi)$ such that:

- $\mathrm{var}(\tau^{-1}(h(\Phi))) \subseteq \mathrm{var}(\Gamma')$;
- $\tau^{-1}(h(\Phi)) \subseteq \Gamma'^{\vdash^{\mathrm{g}}_{h(\mathcal{D}),\Xi}}$;
- $\tau^{-1}(h(\Phi)) \vdash^{\mathrm{l}}_{h(\mathcal{D}),\Xi} \psi'$.

Therefore, $h(\mathcal{D})$ enjoys careful-reasoning-by-cases with respect to Ξ. $\diamond$

We want to investigate the preservation of careful-reasoning-by-cases by the fibring. We start by setting-up the ghost variables and the translations. Let $\mathcal{D}$ be the fibring of $\mathcal{D}'$ and $\mathcal{D}''$. Then we need to work with:

- $\Xi'' = \{\xi_{c''(\varphi_1,\ldots,\varphi_k)} : i''(c'')(\varphi_1, \ldots, \varphi_k) \in L(C, \Xi), c'' \in C''_k\}$;

- $\Xi' = \{\xi_{c'(\varphi_1,\dots,\varphi_k)} : i'(c')(\varphi_1,\dots,\varphi_k) \in L(C,\Xi), c' \in C'_k\}$;

as the ghosts of $\mathcal{D}''$ in $\mathcal{D}'$ and of $\mathcal{D}'$ in $\mathcal{D}''$, respectively. Let

- $\tau' : L(C,\Xi) \to L(C',\Xi \cup \Xi'')$;
- $\tau'' : L(C,\Xi) \to L(C'',\Xi \cup \Xi')$;

be the translations and τ'^{-1} and τ''^{-1} assignments as defined in the beginning of Section 4.2. Recall that

$$i'(\mathcal{D}') = \langle C, i'(R_{\mathrm{l}}{}'), i'(R_{\mathrm{g}}{}')\rangle \text{ and } i''(\mathcal{D}'') = \langle C, i''(R_{\mathrm{l}}{}''), i''(R_{\mathrm{g}}{}'')\rangle$$

are deductive systems with the same connectives as the fibring but where only the rules from $\mathcal{D}'$ and $\mathcal{D}''$ can be used, respectively.

Theorem 5.5 Careful-reasoning-by-cases is preserved by fibring deductive systems.

Proof: Let $\mathcal{D}$ be the fibring of two deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ both enjoying careful-reasoning-by-cases with respect to Ξ. Assume that Γ is finite and that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi$ with a proof

$$\underbrace{\delta_1 \dots \delta_k}_{i''(\mathcal{D}'')\text{ rules}} \qquad \underbrace{\delta_{k+1} \dots \delta_n}_{i'(\mathcal{D}')\text{ rules}}$$

such that $\delta_1 \dots \delta_k$, $\delta_{k+1} \dots \delta_n$ were justified by rules in $i''(\mathcal{D}'')$ and $i'(\mathcal{D}')$, respectively and $\Gamma_1 \subseteq \Gamma$ is the part of Γ used in the derivation until step k and $\Gamma_2 \subseteq \Gamma$ is the part of Γ used from $k+1$ onwards. Then

$$\Gamma_1 \vdash^{\mathrm{g}}_{i''(\mathcal{D}''),\Xi} \delta_k.$$

Assume also that, as a simplification, only δ_k is used as premise of a rule applied in $\delta_{k+1} \dots \delta_n$. Since $\mathcal{D}''$ enjoys careful-reasoning-by-cases with respect to Ξ then, by Proposition 5.4, $i''(\mathcal{D}'')$ enjoys careful-reasoning-by-cases with respect to Ξ and so there is a finite set $\Phi \subseteq L(C,\Xi)$ such that:

- $\mathrm{var}(\Phi) \subseteq \mathrm{var}(\Gamma_1)$;
- $\Phi \subseteq {\Gamma_1}^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$;
- $\Phi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \delta_k$.

Hence

$$\Gamma_2, \Phi \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} \varphi.$$

Since $\mathcal{D}'$ enjoys careful-reasoning-by-cases with respect to Ξ then, by Proposition 5.4, $i'(\mathcal{D}')$ enjoys careful-reasoning-by-cases with respect to Ξ and so there is a finite set $\Psi \subseteq L(C,\Xi)$ such that:

- $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma_2) \cup \mathrm{var}(\Phi)$;
- $\Psi \subseteq (\Gamma_2 \cup \Phi)^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$;
- $\Psi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$.

Therefore, there is a finite set $\Psi \subseteq L(C,\Xi)$ such that:

- $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma)$;

- $\Psi \subseteq \Gamma^{\vdash^{\mathrm{g}}_{\mathcal{D},\Xi}}$;
- $\Psi \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} \varphi$.

Hence $\mathcal{D}$ enjoys careful-reasoning-by-cases with respect to Ξ. The case where in $\delta_{k+1}\dots\delta_n$ more than one element of $\delta_1\dots\delta_k$ is used is proved in a similar way. The same applies to the case where more than two blocks of rules from $\mathcal{D}''$ and $\mathcal{D}'$ are applied. $\diamond$

Example 5.6 We provide an illustration of careful-reasoning-by-cases in the context of modal logics. Let $\mathcal{D}$ be the fibring of two modal deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ that share the propositional part, in particular $\Rightarrow$ and $\vee$ but have two different modalities $\Box'$ and $\Box''$ (recall Example 2.6).

Consider the following global derivation of

$$\{(\Box''(\varphi' \Rightarrow \varphi'')), (\Box'(\Box''\varphi'))\} \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} ((\Box'(\Box''\varphi'')) \vee \psi')$$

(note the use of the necessitation rule Nec$'$) using variables in Ξ:

1	$(\Box''(\varphi' \Rightarrow \varphi''))$	hyp
2	$((\Box''(\varphi' \Rightarrow \varphi'')) \Rightarrow ((\Box''\varphi') \Rightarrow (\Box''\varphi'')))$	K$''$
3	$((\Box''\varphi') \Rightarrow (\Box''\varphi''))$	MP 1,2
4	$(\Box'((\Box''\varphi') \Rightarrow (\Box''\varphi'')))$	Nec$'$ 3
5	$((\Box'((\Box''\varphi') \Rightarrow (\Box''\varphi''))) \Rightarrow ((\Box'(\Box''\varphi')) \Rightarrow (\Box'(\Box''\varphi''))))$	K'
6	$((\Box'(\Box''\varphi')) \Rightarrow (\Box'(\Box''\varphi'')))$	MP 4,5
7	$(\Box'(\Box''\varphi'))$	hyp
8	$(\Box'(\Box''\varphi''))$	MP 6,7
9	$((\Box'(\Box''\varphi'')) \vee \psi')$	$\vee$ I 8

Observe that steps 1 and 2 are justified by rules in $\mathcal{D}''$ and that all the other steps are justified by rules in $\mathcal{D}'$ (since the implication is shared step 3 can be seen as an hypothesis in $\mathcal{D}'$).

Hence, from the derivation, we extract a global derivation in $\mathcal{D}'$ of

$$\{(\xi_{(\Box''\varphi')} \Rightarrow \xi_{(\Box''\varphi'')}), (\Box'\xi_{(\Box''\varphi')})\} \vdash^{\mathrm{g}}_{\mathcal{D}'} ((\Box'\xi_{(\Box''\varphi'')}) \vee \psi')$$

using variables in Ξ but also ghost variables in Ξ'':

1	$(\xi_{(\Box''\varphi')} \Rightarrow \xi_{(\Box''\varphi'')})$	hyp
2	$(\Box'(\xi_{(\Box''\varphi')} \Rightarrow \xi_{(\Box''\varphi'')}))$	Nec$'$ 1
3	$((\Box'(\xi_{((\Box''\varphi')} \Rightarrow \xi_{(\Box''\varphi'')})) \Rightarrow ((\Box'\xi_{(\Box''\varphi')}) \Rightarrow (\Box'\xi_{(\Box''\varphi'')})))$	K'
4	$((\Box'\xi_{(\Box''\varphi')}) \Rightarrow (\Box'\xi_{(\Box''\varphi'')}))$	MP 2,3
5	$(\Box'\xi_{(\Box''\varphi')})$	hyp
6	$(\Box'\xi_{(\Box''\varphi'')})$	MP 5,4
7	$((\Box'\xi_{(\Box''\varphi'')}) \vee \psi')$	$\vee$ I 6

Using the fact that $\mathcal{D}'$ enjoys careful-reasoning-by-cases with respect to Ξ by Lemma 5.1 also enjoys careful-reasoning-by-cases with respect to $\Xi^\bullet$, and taking

$$\Psi' = \{(\Box'(\xi_{(\Box''\varphi')} \Rightarrow \xi_{(\Box''\varphi'')})), (\Box'\xi_{(\Box''\varphi')})\}$$

we have

$$\Psi' \vdash^{\mathrm{l}}_{\mathcal{D}'} ((\Box'\xi_{(\Box''\varphi'')}) \vee \psi') \text{ and } \Psi' \subseteq \{(\xi_{(\Box''\varphi')} \Rightarrow \xi_{(\Box''\varphi'')}), (\Box'\xi_{(\Box''\varphi')})\}^{\vdash^{\mathrm{g}}_{\mathcal{D}'}}.$$

Hence $\{(\Box'((\Box''\varphi') \Rightarrow (\Box''\varphi''))), (\Box'(\Box''\varphi'))\}$ is such that:

- $\{(\Box'((\Box''\varphi') \Rightarrow (\Box''\varphi''))), (\Box'(\Box''\varphi'))\} \vdash^{\mathrm{l}}_{\mathcal{D},\Xi} ((\Box'(\Box''\varphi'')) \vee \psi')$;
- $\{(\Box''(\varphi' \Rightarrow \varphi'')), (\Box'(\Box''\varphi'))\} \vdash^{\mathrm{g}}_{\mathcal{D},\Xi} (\Box'((\Box''\varphi') \Rightarrow (\Box''\varphi'')))$.

Therefore careful-reasoning-by-cases still holds in the bi-modal logic with the modalities $\Box'$ and $\Box''$. $\triangleleft$

5.2 Preserving Craig interpolation

The goal is to show that Craig interpolation is preserved by fibring under mild conditions. That conditions involve the notion of bridge.

Definition 5.7 A *d-bridge to* the deductive system $\mathcal{D}'$ in the fibring $\mathcal{D}$ of deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ sharing $\mathcal{D}^0$ is a pair $\langle h_1, h_2\rangle$ of maps from $L(C,\Xi)$ to $L(C',\Xi)$ such that

- $h_1(\Psi) \vdash^d_{i'(\mathcal{D}'),\Xi} h_2(\varphi)$ whenever $\Psi \vdash^d_{\mathcal{D},\Xi} \varphi$
- $\mathrm{var}(h_i(\varphi)) = \mathrm{var}(\varphi)$
- $\gamma \vdash^d_{\mathcal{D},\Xi} h_1(\gamma)$ and $h_2(\delta) \vdash^d_{\mathcal{D},\Xi} \delta$ for any γ and δ in $L(C,\Xi)$

where i' is the morphism from $\mathcal{D}'$ to $\mathcal{D}$, $\Psi \subseteq L(C,\Xi)$ and $\varphi \in L(C,\Xi)$.

We now present an example of a bridge involving intuitionistic and classical propositional deductive systems.

Example 5.8 Consider the fibring $\mathcal{D}$ of the deductive system $\mathcal{D}_c$ for classical logic where

- C_{c0} contains $\perp_c$ and a denumerable set of propositional symbols
- $C_{c2} = \{\rightarrow_c\}$

and the deductive system $\mathcal{D}_i$ for intuitionistic logic where

- C_{i0} contains $\perp_i$ and a denumerable set of variables containing the classical propositional symbols
- $C_{i2} = \{\wedge_i, \vee_i, \rightarrow_i\}$

sharing a deductive system $\mathcal{D}^0$ where ${C^0}_0$ is the set of classical propositional symbols and the other components are empty. Consider the map h_1 from $L(C,\Xi)$ to $L(C_i,\Xi)$ inductively defined as follows

- $h_1(\varphi) = \varphi$ whenever φ is in Ξ
- $h_1(\perp_c) = \perp_i$
- $h_1(\varphi_1 \rightarrow_c \varphi_2) = (\neg_i\neg_i h_1(\varphi_1)) \rightarrow_i (\neg_i\neg_i h_1(\varphi_2))$
- $h_1(c_i(\varphi_1,\ldots,\varphi_n)) = c_i(\neg_i\neg_i h_1(\varphi_1),\ldots,\neg_i\neg_i h_1(\varphi_n))$ if c_i is in C_{in}, $n \geq 0$.

Then the pair

$$\langle h_1, h_2\rangle$$

where h_2 is $\lambda\varphi.\neg_i\neg_i h_1(\varphi)$ constitute a bridge to $\mathcal{D}_i$ in $\mathcal{D}$ since considering h_c a map from $L(C,\Xi)$ to $L(C,\Xi)$ inductively defined as follows

- $h_c(\varphi) = \varphi$ whenever φ is either $\perp_i$ or $\perp_c$
- $h_c(\varphi) = \neg_c\neg_c\varphi$ whenever φ is in $\Xi \cup C_{i0}$ and is neither $\perp_i$ nor $\perp_c$
- $h_c(\varphi_1 \rightarrow_c \varphi_2) = (\neg_c\neg_c h_c(\varphi_1)) \rightarrow_c (\neg_c\neg_c h_c(\varphi_2))$
- $h_c(c_i(\varphi_1,\ldots,\varphi_n)) = c_i(\neg_c\neg_c h_c(\varphi_1),\ldots,\neg_c\neg_c h_c(\varphi_n))$ if c_i is in C_{in} and $n > 0$.

it happens that

1. $h_1(\Psi) \vdash^d_{i_i(\mathcal{D}_i),\Xi} h_2(\varphi)$ whenever $\Psi \vdash^d_{\mathcal{D},\Xi} \varphi$

2. $h_2(\varphi) \dashv\vdash^d_{\mathcal{D},\Xi} h_c(\varphi)$

3. $h_1(\varphi) \dashv\vdash^d_{\mathcal{D},\Xi} h_c(\varphi)$

4. $h_c(\varphi) \dashv\vdash^d_{\mathcal{D},\Xi} \varphi$.

We now show that Craig interpolation can be preserved by constrained or unconstrained fibring whenever there is a bridge in the fibring.

Theorem 5.9 d-Craig interpolation holds in the deductive system resulting from constrained or unconstrained fibring provided that one of the component deductive systems has d-Craig interpolation and there is a d-bridge to that deductive system in the fibring.

Proof: Let $\mathcal{D}$ be the fibring of deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ sharing $\mathcal{D}^0$. Note that $\mathcal{D}$ can be the unconstrained fibring of $\mathcal{D}'$ and $\mathcal{D}''$. Assume without loss of generality that $\mathcal{D}'$ has d-Craig interpolation and that there is a d-bridge $\langle h_1, h_2 \rangle$ to $\mathcal{D}'$ in $\mathcal{D}$. Let $\Gamma \subseteq L(C, \Xi)$ be finite, $\varphi \in L(C, \Xi)$ and assume that $\Gamma \vdash^d_{\mathcal{D},\Xi} \varphi$ and that $\text{var}(\Gamma) \cap \text{var}(\varphi) \neq \emptyset$. Then

$$h_1(\Gamma) \vdash^d_{i'(\mathcal{D}'),\Xi} h_2(\varphi)$$

and $\text{var}(h_1(\Gamma)) \cap \text{var}(h_2(\varphi)) \neq \emptyset$ since h_1 and h_2 constitute a d-bridge, see Definition 5.7. Taking into account Lemma 4.7 then

$$\tau'(h_1(\Gamma)) \vdash^d_{\mathcal{D}',\Xi\cup\Xi''} \tau'(h_2(\varphi))$$

and since $\mathcal{D}'$ has d-Craig interpolation with respect to $\Xi \cup \Xi''$ by Proposition 3.5, there is a finite set $\Psi' \subseteq L(C', \Xi \cup \Xi'')$ such that

- $\text{var}(\Psi') \subseteq \text{var}(\tau'(\Gamma)) \cap \text{var}(\tau'(\varphi))$
- $\tau'(h_1(\Gamma)) \vdash^d_{\mathcal{D}',\Xi\cup\Xi''} \Psi'$;
- $\Psi' \vdash^d_{\mathcal{D}',\Xi\cup\Xi''} \tau'(h_2(\varphi))$;

therefore

- $\text{var}(i'(\Psi')) \subseteq \text{var}(i'(\tau'(\Gamma))) \cap \text{var}(i'(\tau'(\varphi)))$;
- $i'(\tau'(h_1(\Gamma))) \vdash^d_{\mathcal{D},\Xi\cup\Xi''} i'(\Psi')$;
- $i'(\Psi') \vdash^d_{\mathcal{D},\Xi\cup\Xi''} i'(\tau'(h_2(\varphi)))$.

and so, by Lemma 4.6,

- $\text{var}(\tau'^{-1}(i'(\Psi'))) \subseteq \text{var}(\Gamma) \cap \text{var}(\varphi)$;
- $h_1(\Gamma) \vdash^d_{\mathcal{D},\Xi} \tau'^{-1}(i'(\Psi'))$
- $\tau'^{-1}(i'(\Psi')) \vdash^d_{\mathcal{D},\Xi} h_2(\varphi)$.

Henceforth $\tau'^{-1}(i'(\Psi'))$ is a d-Craig interpolant for $\Gamma \vdash^d_{\mathcal{D},\Xi} \varphi$ since $\Gamma \vdash^d_{\mathcal{D},\Xi} h_1(\Gamma)$ and $h_2(\varphi) \vdash^d_{\mathcal{D},\Xi} \varphi$. ◇

Note that in the proof of Theorem 5.9 it is not required the preservation of the metatheorems of modus ponens and deduction.

Example 5.10 The fibring of the propositional intuitionistic deductive system and the propositional classical deductive system sharing only the classical propositional symbols has local Craig interpolation. Indeed Theorem 5.9 can be applied, see Example 3.9 and Example 5.8. $\triangleleft$

Interpolation in the presence of deductive implication is discussed in the next theorem.

Theorem 5.11 d-Theoremhood-Craig interpolation holds in the deductive system resulting from constrained or unconstrained fibring of deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ provided that either $\mathcal{D}'$ or $\mathcal{D}''$ has d-deductive implication, one of the component deductive systems has d-Craig interpolation, there is a d-bridge to that deductive system in the fibring, and either $\mathcal{D}'$ or $\mathcal{D}''$ has d-deductive conjunction.

Proof: Let $\mathcal{D}$ be the fibring of deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ sharing $\mathcal{D}^0$ and that one of the component deductive systems has d-Craig interpolation and there is a d-bridge to that deductive system in the fibring. Note that $\mathcal{D}$ can be the unconstrained fibring of $\mathcal{D}'$ and $\mathcal{D}''$. Assume without loss of generality that $\mathcal{D}'$ has d-deductive implication and that $\mathcal{D}''$ has d-deductive conjunction. Suppose that $\vdash^d_{\mathcal{D},\Xi} (\varphi_1 \Rightarrow' \varphi_2)$ and $\mathrm{var}(\varphi_1) \cap \mathrm{var}(\varphi_2) \neq \emptyset$. Then by d-MTMP, $\varphi_1 \vdash^d_{\mathcal{D},\Xi} \varphi_2$ and $\mathrm{var}(\varphi_1) \cap \mathrm{var}(\varphi_2) \neq \emptyset$ and so by Theorem 5.9 there is a finite set $\Psi \subseteq L(C, \Xi)$ such that $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\varphi_1) \cap \mathrm{var}(\varphi_2)$, $\varphi_1 \vdash^d_{\mathcal{D},\Xi} \Psi$ and $\Psi \vdash^d_{\mathcal{D},\Xi} \varphi_2$. The result follows by taking $(\bigwedge''_{\psi \in \Psi} \psi)$ as interpolant and using d-MTD. $\diamond$

Example 5.12 The fibring of the propositional intuitionistic deductive system and the propositional classical deductive system sharing only classical propositional symbols has local theoremhood-Craig interpolation. $\triangleleft$

5.2.1 Complexity of the Craig interpolation procedure

Craig interpolation is constructive in the deductive system resulting from the unconstrained or constrained fibring of deductive systems where there is a bridge to one of the components having also that property whenever 1) Craig interpolation is constructive in that deductive system, and 2) it is constructive the procedure of obtaining a deduction in the deductive system with Craig interpolation for each deduction in the fibring.

We analyze the time complexity of the algorithm I to obtain the interpolant of derivations in the fibring, described in the proof of Theorem 5.9, assuming that 1) there is an algorithm I° to obtain the interpolant in the component deductive system with Craig interpolation and 2) there is an algorithm $I_{\langle h_1,h_2 \rangle}$ that given a deduction in the fibring and the bridge $\langle h_1, h_2 \rangle$ returns a similar deduction in the deductive system enjoying Craig interpolation. In order to obtain a time complexity result it is important to consider the size (in bits) of the derivation. For derivation $\varphi_1 \dots \varphi_k$, the size is

$$||\varphi_1 \dots \varphi_k|| = \sum_{i=1,\dots,k} ||\varphi_i||$$

where $||\varphi_i||$ is the number of bits required to represent (efficiently) the formula φ_i for $i = 1, \dots, n$. We denote by $\mathrm{Time}(I^\circ, D)$ the cost in time of applying algorithm I° to deduction D, and similarly for the algorithms I and $I_{\langle h_1,h_2 \rangle}$.

Proposition 5.13 Let $\mathcal{D}$ be the fibring of deductive systems such that there is a bridge $\langle h_1, h_2 \rangle$ to a component deductive system $\mathcal{D}^\circ$ with Craig interpolation. Assume that

- $\mathrm{Time}(I^\circ, D^\circ) \in \mathrm{O}(f^\circ(||D^\circ||))$ for each derivation D° in $\mathcal{D}^\circ$
- $\mathrm{Time}(I_{\langle h_1,h_2\rangle}, D) \in \mathrm{O}(f_{\langle h_1,h_2\rangle}(||D||))$ for each derivation D in $\mathcal{D}$

then

$$\mathrm{Time}(I, D) \in \mathrm{O}(f_{\langle h_1,h_2\rangle}(||D||) + f^\circ(||I_{\langle h_1,h_2\rangle}(D)||))$$

for each derivation D of $\Gamma \vdash_{\mathcal{D},\Xi} \varphi$ in $\mathcal{D}$.

We omit the proof of the proposition since it follows straightforwardly. Observe that if I° and $I_{\langle h_1,h_2\rangle}$ take polynomial time so does I. Of course if, for instance, I° takes exponential time and $I_{\langle h_1,h_2\rangle}$ takes polynomial time then I also takes, in the worst case, exponential time.

5.3 Preserving extension interpolation

Capitalizing on the relationship of the extension interpolation property with the metatheorems of modus ponens and deduction, as established in Proposition 3.2, it is possible to obtain the following preservation result.

Theorem 5.14 d-extension interpolation holds in the fibring of deductive systems with d-metatheorem of deduction where one of them also has d-metatheorem of modus ponens and where the base sets coincide in fibring.

Proof: The result follows by Proposition 3.2 since by Theorem 4.3 and Theorem 4.4 the deductive system resulting of the fibring has d-MTMP and d-MTD with the same base set. ◇

Example 5.15 Deductive systems resulting from fibring that enjoy d-extension interpolation can be obtained just by looking to Example 2.5 and considering pairs of deductive systems satisfying the conditions of Theorem 5.14. Implication should be shared in the fibring. We present now a small list of these combinations:

- the fibring of classical propositional logic with Łukasiewicz logic Ln for $n \geq 3$ enjoy global extension interpolation property;
- the fibring of modal logics with Gödel logic G3 enjoy local extension interpolation property;
- the fibring of Gödel logic G3 with Łukasiewicz logic Ln for $n \geq 3$ enjoy global extension interpolation property;
- the fibring of intuitionistic logic with Gödel logic G3 enjoy local extension interpolation property. ◁

5.4 Preserving Maehara interpolation

We conclude by investigating preservation of Maehara interpolation.

Theorem 5.16 d-Maehara interpolation holds in the deductive system resulting from constrained or unconstrained fibring of deductive systems provided that one of the component systems has d-metatheorem of modus ponens and both have d-metatheorem of deduction over base sets that coincide in the fibring, and moreover one of the component deductive systems has d-Craig interpolation and there is a d-bridge to that deductive system in the fibring.

Proof: The result follows by Theorem 3.11 since under the conditions of the theorem the deductive system resulting from the fibring has Craig interpolation by Theorem 5.9, has d-MTMP by Theorem 4.3 and d-MTD by Theorem 4.4. $\diamond$

Next example capitalizes on the bridge to intuitionistic logic on the fibring of intuitionistic and classical logic.

Example 5.17 We observe that local Maehara interpolation is a property enjoyed by the fibring of classical propositional logic and intuitionistic propositional logic when the implication and the classical propositional symbols are shared, see Example 2.5, Example 3.9 and Example 5.8. $\triangleleft$

6 Constructive interpolation and examples

Constructive proofs of Craig interpolation can be given for deductive systems that enjoy certain properties. For this purpose we need a few semantic notions. A *matrix* is a triple $\langle B, \cdot, D\rangle$ where $m = \langle B, \cdot\rangle$ is an algebra over C (of truth-values) and $D \subseteq B$ (D is the set of distinguished values). A *valuation* is any map from Ξ to B. The denotation of a formula $[\![\varphi]\!]^m_v$ is defined inductively in the expected way. A formula φ is a *global semantic consequence* of Γ, written $\Gamma \vDash^{\mathrm{g}} \varphi$ if $[\![\varphi]\!]^m_v \in D$ whenever $[\![\gamma]\!]^m_v \in D$ for every $\gamma \in \Gamma$. When we need semantic considerations we restrict ourselves to logics that are characterized by a unique matrix up to isomorphism. Then we can use $v(\varphi)$ to refer to $[\![\varphi]\!]^m_v$. Constructive forms of Craig interpolation can now be given for deductive systems allowing the possibility of expressing all truth values at the syntactical level. The so called Rosser-Turquette deductive systems [34] do have this property including as particular cases the Post systems [21].

A deductive system is *syntactically faithful* if it has a deductive disjunction $\vee$ and there are $\beta_1, \dots, \beta_n$ depending at most upon the variables $\xi_1, \dots, \xi_n$ such that $v(\beta_i) = b_i$ for every valuation v and for every truth value $b_i \in B$.

Several finite-valued and non-truth functional logics share this property. Besides Post logics and Rosser-Turquette systems referred to above, several logics of formal inconsistency such as **mbC**, **bC**, **Ci** and da Costa's $\mathcal{C}_n$ for $n \in \mathbb{N}$ are also syntactically faithful (see [12]) and thus enjoy Craig interpolation.

Proposition 6.1 Every syntactically faithful logic has g-Craig interpolation.

Proof: The following procedure indeed constructs an interpolant whenever $\varphi \vdash^{\mathrm{g}} \psi$ such that $\mathrm{var}(\varphi) \cap \mathrm{var}(\psi) \neq \emptyset$. Let $\mathrm{var}(\varphi) \cap \mathrm{var}(\psi) = \{\theta_1, \dots, \theta_r\}$ and $\mathrm{var}(\varphi) \setminus \mathrm{var}(\psi) = \{\sigma_1, \dots, \sigma_s\}$. Consider the formula

$$\rho = \bigvee_{i=1}^{n^s} \varphi^{\sigma_i}_{\beta_1,\dots,\beta_n}$$

where $\varphi^{\sigma_i}_{\beta_1,\dots,\beta_n}$ is the deductive disjunction obtained by substituting σ_i by each β_j. It is easy to see that that $\varphi \vdash^{\mathrm{g}} \rho$. On the other hand by the construction itself $\rho \vdash^{\mathrm{g}} \varphi$ and since by hypothesis $\varphi \vdash^{\mathrm{g}} \psi$ then $\rho \vdash^{\mathrm{g}} \psi$. $\diamond$

Example 6.2 Rosser-Turquette logics have global interpolation as we explain now. Consider an n-valued Rosser-Turquette logic with the connectives $C_1 = \{\neg\} \cup \{J_k : 0 \le k \le n-1\}$ and $C_2 = \{\wedge, \vee\}$, with truth-values $V = \{0, 1, \dots, n-1\}$ and $D = \{s, \dots, n-1\} \subseteq V$ the set of designated values. Fixing an assignment $\rho : \Xi \to V$, the denotation of formulas is inductively defined as follows:

- $[\![\xi]\!]_\rho = \rho(\xi)$;
- $[\![(\varphi_1 \vee \varphi_2)]\!]_\rho = \max([\![\varphi_1]\!]_\rho, [\![\varphi_2]\!]_\rho)$;
- $[\![(\varphi_1 \wedge \varphi_2)]\!]_\rho = \min([\![\varphi_1]\!]_\rho, [\![\varphi_2]\!]_\rho)$;
- $[\![J_k(\varphi)]\!]_\rho = \begin{cases} n-1 & \text{if } [\![\varphi]\!]_\rho = k \\ 0 & \text{otherwise} \end{cases}$;
- $[\![(\neg\,\varphi)]\!]_\rho = [\![(J_0(\varphi) \vee \ldots \vee J_{s-1}(\varphi))]\!]_\rho = \begin{cases} n-1 & 0 \leq [\![\varphi]\!]_\rho \leq s-1 \\ 0 & \text{otherwise} \end{cases}$

Some immediate properties of such logics are the following:

- $[\![(\varphi \vee (\neg\,\varphi)]\!]_\rho = \begin{cases} n-1 & \text{if } 0 \leq [\![\varphi]\!]_\rho \leq s-1 \\ k & \text{if } [\![\varphi]\!]_\rho = k \geq s \end{cases}$
  (hence a tautology);
- $[\![(\varphi \wedge (\neg\,\varphi)]\!]_\rho = \begin{cases} k & \text{if } 0 \leq [\![\varphi]\!]_\rho = k \leq s-1 \\ 0 & \text{if } [\![\varphi]\!]_\rho = k \geq s \end{cases}$
  (hence a contradiction);
- $[\![(J_0(\varphi) \vee \ldots \vee J_{n-1}(\varphi)]\!]_\rho) = n-1$;
- $[\![(J_0(\varphi) \wedge \ldots \wedge J_{n-1}(\varphi))]\!]_\rho = 0$.

We can apply directly the algorithm of Proposition 6.1 and obtain the interpolation formula. $\triangleleft$

Theorem 6.3 The fibring of syntactically faithful logics is syntactically faithful providing that deductive disjunction is shared. Therefore the fibring of such logics has Craig interpolation.

Proof: Deductive disjunction is easily seen to be preserved by fibring and if $\mathcal{L}'$ and $\mathcal{L}''$ have $\beta'_1, \ldots, \beta'_{n'}$ and $\beta''_1, \ldots, \beta''_{n'}$ depending at most upon the variables $\xi'_1, \ldots, \xi'_{n'}$ and $\xi''_1, \ldots, \xi''_{n''}$ such that $v'(\beta'_i) = b'_i$ for every valuation v' and for every truth value $b'_i \in B'$ and $v''(\beta''_i) = b''_i$ for every valuation v'' and for every truth value $b''_i \in B''$ then $\beta'_1, \ldots, \beta'_{n'}, \beta''_1, \ldots, \beta''_{n'}$ depending at most upon the variables $\xi'_1, \ldots, \xi'_{n'}, \xi''_1, \ldots, \xi''_{n''}$ and $v(\beta_i) = b_i$. $\diamond$

Example 6.4 Fibring of syntactically faithful logics
The fibring of logics of formal inconsistency and/or Rosser Turquette logics enjoys Craig interpolation. $\triangleleft$

7 Concluding remarks

This paper has dealt with investigating, based upon Hilbert-style deductive systems, the basic question of preservation of metatheoretical properties via the operation of (constrained and unconstrained) fibring. Special emphasis was placed on interpolation and on the analysis of the effects of the distinction between local and global deducibility.

The obvious generalization of the results in this paper are related to considering logics with a first-order or even a higher-order basis and also exploring the subtleties of local and global reasoning. In this case, several possibilities can be considered in what concerns symbols to be allowed in the common signature. The usual difficulties with quantifiers and variables are expected but not substantially more. Namely one has to deal with provisos in inference rules which always make the mathematical machinery heavier.

Semantic issues related to interpolation were left out, namely semantic characterizations of interpolation in a very generic context that can include several logics. Among them, the relationship between Craig interpolation, definability and amalgamation is worthwhile to explore. A starting point for the basic setting was done in [24]. The relationship between amalgamation and (global) Craig interpolation is related to finitely algebraizable logics endowed with a matrix semantics. It is worthwhile to look into the preservation of the results obtained in [15]. Moreover it is worthwhile to extend the results to logics with other semantic structures. More challenging are the problems related to the distinction between local and global reasoning. Of course a first option is to choose the semantic domain that can encompass several logics. We believe that the simple algebraic setting in [35] can be a starting point. A more sophisticated level is to employ topos semantics as in [13].

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