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# FIBRING OF LOGICS AS A UNIVERSAL CONSTRUCTION

## 1 PROLEGOMENON TO FIBRING

It is a task of philosophy to explain the sense in which contemporary science uses the label "logics", specially through "logics in" (natural language, program verification, machine learning, knowledge representation, abductive and inductive reasoning, etc.) as well as "logics for" (hybrid reasoning systems, ontology, engineering, reasoning about cryptographic construction, defeasible argumentation, reasoning with uncertainty, reasoning under contradiction, reasoning about action, agents with bounded rationality, and so on) and even "logics that" (that characterize classes of finite structures as in finite model theory, that characterize formal grammars, that characterize processes, etc.)

The Greek term *logos* (and *ratio* in Latin) from which "logic" and "reason" derive, with its original meaning of "to put together", and later "to speak about" is suggestive: it may be relevant for such domains to start by collecting peculiar concepts and thoughts, and then recompiling them in an orderly way using logical tools so that talking and reasoning about the resulting concepts becomes something practical and effective.

Whether or not such usage favours logical pluralism (in the sense that there is more than one "real logic"') or just reflects isolated parts of the conception of reason as cosmic ordering, is also a matter for philosophy, as it is also to reconcile this practice with logic regarded as an epistemological enterprise or to Kant's transcendental deduction. But what is more: the contemporary usage of the term logic specializes from the formal logic (in the sense of abiding to the criteria of concept, judgment, and inference) not only towards using symbolic logic (i.e., a development of formal logic by means of mathematical concepts), but also by means of mechanized, computer-based concepts, or in other words, by means of the algorithmic side of logic.

It is natural to think that the intense use of "logics in", "logics that" and even "logics" with no specifications can be combined again by mathematical methods, realizing a certain philosophers and logicians dream to building mechanisms where several different logics could interact and cooperate, instead of clashing. In this sense the project of reducing reasoning to symbolic computation is an old one. The philosopher and mathematician Bernard Bolzano born in Prague, Bohemia was not far from proposing the general idea of combining notions of consequence in his monumental *Wissenschaftslehre* already in 1837. Some ideas of Bolzano are considered by some authors to be philosophically related to the thought and conceptions of Gottfried Wilhelm Leibniz, well known in his concerns about the possibilities of a universal language and an all-purposes *calculus ratiocinator*.

The pragmatical significance of the problem of combining logics is widely recognized, and its philosophical interpretations are just emerging. The uses we mentioned can now be categorized in the main areas of knowledge representation (within artificial intelligence), formal grammars and structures (within formal linguistics) and in formal specification and verification of algorithms and protocols (within software engineering and security). In these fields, the need for working simultaneously with several calculi is the rule rather than the exception. For instance, in a knowledge representation problem it may be necessary to deal with temporal, spatial, deontic, linguistic and probabilistic aspects (e.g., for reasoning with mixed assertions like "with probability greater than 0.99, in the near future smoking will be forbidden almost everywhere"). As another important example, in a security protocol specification it may be necessary to combine temporal, equational, epistemic and dynamic logic features. We thus need to study general methods for combining different logic systems and to gain control on the complex resulting theories, understanding their expressive power and their mathematical and computational aspects in general. In other words, we need a kind of *ars combinatoria*, as already proposed by Leibniz, and much before by Raimundus Lullus (Ramon Lull, the *doctor illuminatus*, in the 13th century) to express the several deduction formalisms.

Not only the interest in the pragmatical side of combination of logic systems has recently been growing (as reflected in the series [de Rijke and Blackburn, 1996; Gabbay and Pirri, 1997; Baader and Schulz, 1996; Gabbay and de M. Rijke, 2000; Kirchner and Ringeissen, 2000; Armando, 2002]), but the topic is also interesting on purely theoretical grounds. It might be illuminating, for instance, to look at predicate temporal logic as resulting from the combination of first-order logic and propositional temporal logic. However, this approach will be significant as much as general results can be obtained about the *preservation properties* of the combination mechanism at hand. For example, suppose that it has been established that completeness is preserved by a certain combination mechanism $\bullet$, and it is known that a logic system $\mathcal{L}$ can be obtained by $\mathcal{L}' \bullet \mathcal{L}''$; in this case, if we had preservation of completeness, the completeness of $\mathcal{L}$ would follow from the completeness of $\mathcal{L}'$ and $\mathcal{L}''$. A similar phenomenon would occur if we could establish that the combination mechanism $\bullet$ preserves meta-logical features as interpolation, cut-elimination, decidability and so on. It is then understandable that theoretical impetus has been directed to establishing preservation results in general, and in finding limits for the preservation in the different combination mechanisms. For an early overview of the practical and theoretical issues see [Blackburn and de Rijke, 1997].

Several forms of combination have been studied, like product [Marx, 1999; Gabbay and Shehtman, 1998; Gabbay and Shehtman, 2000; Gabbay and Shehtman, 2002], fusion [Thomason, 1984; Kracht and Wolter, 1991; Kracht and Wolter, 1997; Wolter, 1998; Gabbay *et al.*, 2003], temporalization [Finger and Gabbay, 1992; Finger and Gabbay, 1996; Wolter and Zakharyaschev, 2000; Finger and Weiss, 2002], parameterization [Caleiro *et al.*, 1999], synchronization [Sernadas *et al.*, 1997] and fibring [Gabbay, 1996a; Gabbay, 1996b; Beckert and Gabbay, 1998; Gabbay, 1999; Sernadas *et al.*, 1999; Zanardo *et al.*, 2001]. Fusion is the simplest, and the best understood combination mechanism. In short, the fusion of two modal systems leads to a bimodal system including the two original modal operators and common propositional connectives. Several interesting properties of logic systems (like soundness, weak completeness, Craig interpolation property and decidability) were shown to be preserved when fusing modal systems (see [Kracht and Wolter, 1991; Kracht, 1999]).

Among such diverse possibilities of procedures for combining logics, fibring occupies a central place. Fibring has to do with joining two or more inference mechanisms by careful genetic manipulation of their formulas and their inferences rules. The resulting fibred system has the capability of express reasoning not only in both ways but also in combined ways. An essential ingredient is to use meta-variables for allowing the instantiation of rules of one logic with formulas from the other logic. Fibring is in a certain sense a metamathematical construction that can be manipulated at the object level. This permits to distinguish between *constrained fibring by sharing* (when the logics are allowed to share constructors in their languages), and *unconstrained* otherwise.

The theoretical significance of fibring results from the fact that it is more easily accessible to results of meta-theoretical preservation in the sense that, in many cases, a certain property of a fibred logic can be obtained by preserving that property from the fibring components. For example preserving completeness is a recurrent issue in fibring logics.

The fact that preservation results have been obtained in the scope of higher-order, modal, relevance and non-truth-functional logics, and that refinements on the notion of fibring as the modulated fibring have proved to be keen tools to solve some collapsing problems within the combination of logics justify the interest on fibring. A broader research scope has been devoted to integrating, comparing and fostering other forms of composing and decomposing logics (see [Carnielli *et al.*, 2004]), such as fusion, splicing, splitting, synchronization and temporalization. Applications of the amply ideal of combining rationalities deeply influences the area of software specification, knowledge representation, architectures for intelligent computing and applications to security protocols and authentication, secure computation and zero-knowledge proof protocols and even quantum computation.

The dominant Kantian tradition deeply influenced the 20th Century logicians, and it is still not clear if we are talking about the same thing when referring to logic as formal (in the sense of being topic neutral) and to logic as symbolic (in the sense of providing norms of calculation). It is not an easy task to offer a prefatory introduction on how logics can be combined, if we do not have a universal agreement about what logic is, and even less whether logic and rationality coincide.

Kant's *Prolegomena to any Future Metaphysics* was published in Latvia in 1783, two years after the *Critique of Pure Reason*, a book to which it was meant to serve as an introduction.[1] Apart any discussion whether it was really helpful for understanding the difficult parts of the *Critique*, the illustrious example of an introduction appearing after what it should have introduced encourages us to postpone, any value judgment about the real meaning of combination of logics for the whole discipline of logic, and about the role of fibring therein. Our more unpretentious aims are to guide the reader on what has been done, and to motivate what could have been done.

Our aim in this paper is to bring together the rich variety of results, problems and perspectives involving fibring, making clear the role of the underlying constructions as universal arguments in the categorial sense. We depart from a basic universe of logic systems encompassing only propositional-based systems endowed with Hilbert calculi and ordered algebraic semantics. We shall see that this universe is already rich enough to illustrate interesting features of fibring and to provide the basis for understanding the trade of combining systems varying from intuitionistic to many-valued logics (including modal systems as special cases). We also explain fibring in a bolder perspective namely encompassing non truth-functional semantics and first-order quantification. Those interested in additional topics like, fibring non Hilbert calculi and higher-order based logics should consult [Coniglio *et al.*, 2003; Caleiro *et al.*, 2003a; Governatori *et al.*, 2002; Rasga *et al.*, 2002].

With this in mind, in Section 2 we offer a general description of propositional fibring, its scope and methods; Section 3 treats a sharp variant called modulated fibring, which was tailored to solve certain problems of collapsing: when two logics are combined, in some cases one of them eclipses the other, and the combination mechanisms must be redefined to keep a finer control on the procedure. Modulated fibring is able to do this, and we shall see how this sharper version of fibring can be very naturally described in categorial terms. Section 4 shows how to extend fibring to non-truth-functional logics, a quite important improvement since several of the new logics subject to fibring operations are not truth-functional; Section 5 is devoted to first-order fibring, and to discussing the expected complications and problems it

[1] Immanuel Kant, *Prolegomena zu einer jeden künftigen Metaphysik die als Wissenschaft wird auftreten können*, Riga, 1783, and *Kritik der reinen Vernunft*, Riga, 1781; revised edition in 1787. Several English translations are available.

poses. Section 6 discusses trends, missing links and tendencies. Finally we include an Appendix which is structured as such, not for being marginal to the subject of fibring logics, neither too technical in its content, but rather because of its historical nature. It focused on bridging the gap that separates the initial intuitions and ideas on fibring of logics (cf. [Gabbay, 1996a; Gabbay, 1996b; Gabbay, 1999]), to the abstract, point free, perspective whose most relevant aspects are told in this Chapter. In all sections we try, as much as possible, to offer an implicit or explicit categorial perspective to the constructions we are dealing with. The motivation for using tools of category theory is to make clear which are the minimal assumptions behind such constructions and the common way of reasoning with them. The reader not so fond with category theory can almost always skip such details.

## 2 PROPOSITIONAL FIBRING

For the sake of simplicity, we start by adopting a basic universe of logic systems encompassing propositional-based systems endowed with Hilbert calculi and ordered algebraic semantics (based on [Sernadas and Sernadas, 2003]). We show with all details how to define fibring in this basic universe. This allows us to introduce many of the definitions, concepts and notations that shall be needed throughout the chapter. Fibring is first defined in a proof-theoretical level. Afterwards, we concentrate on model-theoretic fibring. To what concerns preservation results, we concentrate on illustrating preservation of (global) completeness.

### *2.1 Deductive systems*

When defining deductive systems the first thing to consider is how to present them. We adopt the *homogeneous scenario*, that is, assume that all the deductive systems are presented in the same way. We use Hilbert-style (familiar method of axioms and rules), due to its simplicity, allowing to concentrate on fibring instead of dispersing the reader's attention in other details.

DEFINITION 1. A *signature* $C$ is a family of countable sets $C_k$ where $k \in \mathbb{N}$. The elements of each $C_k$ are called *constructors* or *connectives* of arity $k$.

As usual, a $C$-algebra $\mathcal{B}$ consists of a non-empty *carrier* set $B$ together with a *denotation function* $\nu_k(c) : B^k \to B$ for each $c \in C_k$ and $k \in \mathbb{N}$. A *free algebra* $\mathcal{B}$ over $C$, or a *free $C$-algebra*, where $C$ is a signature, is a $C$-algebra whose carrier $B$ is inductively defined as follows: $c \in B$ whenever $c \in C_0$ and, for every $k \in \mathbb{N}$, $c(b_1, \ldots, b_k) \in B$ whenever $c \in C_k$ and $b_1, \ldots, b_k \in B$. A *free algebra* over $C$ *generated* by a set $A$ is the free algebra over $C'$ where $C'_0 = C_0 \cup A$ and $C'_k = C_k$ for $k$ greater than 0.

The *language* $L(C)$ induced by a signature $C$ is the carrier of the free $C$-algebra. The elements of $L(C)$ are called *$C$-formulas*, or simply *formulas* when the signature $C$ is in the context. We consider different signatures and in order to express schematic inference rules we assume fixed once and for all a denumerable set $\Xi = \{\xi_n : n \in \mathbb{N}\}$ of *schema variables*. The *schema language* $sL(C)$ induced by $C$ is the carrier of the free $C$-algebra generated by $\Xi$. The elements of $sL(C)$ are called *$C$-schema formulas*, or simply *schema formulas*. Of course, $L(C)$ is contained in $sL(C)$ since formulas are precisely the schema formulas where schema variables do not occur.

EXAMPLE 2. Taking a denumerable set $\Pi$ of propositional symbols, we can consider the following signatures:

- Propositional logic: $C^p_0 = \Pi \cup \{\mathbf{t}\}$, $C^p_1 = \{\neg\}$, $C^p_2 = \{\Rightarrow\}$, $C^p_n = \emptyset$ for $n > 2$;
- Intuitionistic logic: $C^i_0 = \Pi \cup \{\mathbf{t}\}$, $C^i_1 = \{\neg\}$, $C^i_2 = \{\Rightarrow, \wedge, \vee\}$, $C^i_n = \emptyset$ for $n > 2$;
- Modal logic: $C^m_0 = \Pi \cup \{\mathbf{t}\}$, $C^m_1 = \{\neg, \Box\}$, $C^m_2 = \{\Rightarrow\}$, $C^m_n = \emptyset$ for $n > 2$;
- Paraconsistent logic: $C^{\mathcal{C}_1}_0 = \Pi \cup \{\mathbf{t}, \mathbf{f}\}$, $C^{\mathcal{C}_1}_1 = \{\neg\}$, $C^{\mathcal{C}_1}_2 = \{\Rightarrow, \wedge, \vee\}$, $C^{\mathcal{C}_1}_n = \emptyset$ for $n > 2$. △

In the context of the modal signature introduced in the example above a schema formula is for instance $\neg(\xi_1 \wedge p)$ and a formula is $\neg(p \wedge q)$, where $p$ and $q$ are in $\Pi$ and $\xi_1$ is in $\Xi$.

A *substitution* on $sL(C)$ is a map $\sigma : \Xi \to sL(C)$. The *instance* of a schema formula $\gamma$ by a substitution $\sigma$, denoted by $\gamma\sigma$, is the schema formula obtained from $\gamma$ by simultaneously replacing each occurrence of $\xi$ in $\gamma$ by $\sigma(\xi)$ for every $\xi \in \Xi$. Instantiation by $\sigma$ thus corresponds to the free extension of $\sigma$ to schema formulas. We also extend the notion of instantiation to sets of schema formulas: $\Gamma\sigma$ denotes the set $\{\gamma\sigma : \gamma \in \Gamma\}$.

We now introduce the notion of Hilbert-style deductive system as an abstraction capturing the proof-theoretic aspects of a logic at the level of detail that we need: language constructors plus inference rules. Inference rules are seen as schemas that can be instantiated by replacing the occurring schema variables with concrete formulas, this is the whole idea of introducing the schema language.

DEFINITION 3. An *inference rule* over $C$ is a pair $r = \langle \Gamma, \delta \rangle$ where $\Gamma \cup \{\delta\} \subseteq sL(C)$ and $\Gamma$ is finite.

We use $Prem(r)$ to denote the *set of premises* $\Gamma$ of $r$, and $Conc(r)$ to denote the *conclusion* $\delta$. We denote by $R(C)$ the set of all inference rules over $C$. A rule with an empty set of premises is called *axiomatic*, and its conclusion an *axiom schema*. It is convenient to distinguish among local rules for deriving consequences from a given set of hypotheses, and global rules used for proving theorems.

DEFINITION 4. A *deductive system* is a triple $\mathcal{D} = \langle C, R_\ell, R_{\mathrm{g}}\rangle$ where $C$ is a signature and $R_\ell \subseteq R_{\mathrm{g}} \subseteq R(C)$.

The distinct roles played by the sets $R_\ell$ and $R_{\mathrm{g}}$ of *local* and *global* rules is made clear in the last of the following deductive systems. The distinction between local and global deduction appeared in the context of modal logic: local means carried out at a single world and global refers to reasoning about all worlds, and thus any correct local deduction is also global but not necessarily vice-versa. This distinction can also be useful in other contexts.[2]

EXAMPLE 5. Taking into account the corresponding signatures as introduced in Example 2 the following deductive systems can be characterized:

Propositional logic

$\mathcal{D}^p = \langle C^p, R_\ell, R_{\mathrm{g}}\rangle$, where $R_\ell = R_{\mathrm{g}}$ contains the axiom schemas:

- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$
- $((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))$
- $(((\neg\, \xi_1) \Rightarrow (\neg\, \xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))$
- $(\mathbf{t} \Leftrightarrow (\xi_1 \Rightarrow \xi_1))$

plus one inference rule:

- $\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle$,

where $(\gamma_1 \Leftrightarrow \gamma_2)$ is an abbreviation of $((\gamma_1 \Rightarrow \gamma_2) \wedge (\gamma_2 \Rightarrow \gamma_1))$. We will re-use this abbreviation in the other examples.

Intuitionistic logic

$\mathcal{D}^i = \langle C^i, R_\ell, R_{\mathrm{g}}\rangle$, where $R_\ell = R_{\mathrm{g}}$ contains the axiom schemas:

- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$
- $((\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow (\xi_1 \Rightarrow \xi_3)))$
- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)))$
- $((\xi_1 \wedge \xi_2) \Rightarrow \xi_1)$
- $((\xi_1 \wedge \xi_2) \Rightarrow \xi_2)$
- $(\xi_1 \Rightarrow (\xi_1 \vee \xi_2))$
- $(\xi_2 \Rightarrow (\xi_1 \vee \xi_2))$

[2]This distinction can be sharpened towards the notions of *local* and *global reasoning*, as explained in [Carnielli and Sernadas, 2004].

- $((\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3)))$
- $((\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow (\neg \xi_2)) \Rightarrow (\neg \xi_1)))$
- $(\xi_1 \Rightarrow ((\neg \xi_1) \Rightarrow \xi_2))$
- $(\mathbf{t} \Leftrightarrow (\xi_1 \Rightarrow \xi_1))$

plus one inference rule:

- $\langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2 \rangle$.

Modal logic K

$\mathcal{D}^{\mathrm{K}} = \langle C^m, R_\ell, R_{\mathrm{g}} \rangle$ where $R_\ell$ contains the axiom schemas:

- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$
- $((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))$
- $(((\neg \xi_1) \Rightarrow (\neg \xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))$
- $(\mathbf{t} \Leftrightarrow (\xi_1 \Rightarrow \xi_1))$
- $((\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box \xi_1) \Rightarrow (\Box \xi_2)))$

plus one inference rule:

- $\langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2 \rangle$,

and $R_{\mathrm{g}}$ further contains the inference rule:

- $\langle \{\xi_1\}, (\Box \xi_1) \rangle$.

Modal logic S4

$\mathcal{D}^{\mathrm{S4}} = \langle C^m, R_\ell, R_{\mathrm{g}} \rangle$ where $R_\ell$ is obtained from the local rules for modal logic K by adding the axiom schemas:

- $((\Box \xi_1) \Rightarrow \xi_1)$
- $((\Box \xi_1) \Rightarrow (\Box(\Box \xi_1)))$.

Modal logic D

$\mathcal{D}^{\mathrm{D}} = \langle C^m, R_\ell, R_{\mathrm{g}} \rangle$ where $R_\ell$ is obtained from the local rules for modal logic K by adding the axiom schema:

- $((\Box \xi_1) \Rightarrow (\Diamond \xi_1))$,

where $(\Diamond \gamma)$ is the usual abbreviation of $(\neg(\Box(\neg \gamma)))$.

Paraconsistent logic $\mathcal{C}_1$

$\mathcal{D}^{\mathcal{C}_1} = \langle C^{\mathcal{C}_1}, R_\ell, R_{\mathrm{g}} \rangle$, where $R_\ell = R_{\mathrm{g}}$ contains the axiom schemas:

- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$
- $((\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow (\xi_1 \Rightarrow \xi_3)))$
- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)))$
- $((\xi_1 \wedge \xi_2) \Rightarrow \xi_1)$
- $((\xi_1 \wedge \xi_2) \Rightarrow \xi_2)$
- $(\xi_1 \Rightarrow (\xi_1 \vee \xi_2))$
- $(\xi_2 \Rightarrow (\xi_1 \vee \xi_2))$
- $((\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3)))$
- $((\neg(\neg\, \xi_1)) \Rightarrow \xi_1)$
- $(\xi_1 \vee (\neg\, \xi_1))$
- $(\xi_1^\circ \Rightarrow (\xi_1 \Rightarrow ((\neg\, \xi_1) \Rightarrow \xi_2)))$
- $((\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ)$
- $((\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ)$
- $((\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ)$
- $(\mathbf{t} \Leftrightarrow (\xi_1 \Rightarrow \xi_1))$
- $(\mathbf{f} \Leftrightarrow (\xi_1^\circ \wedge (\xi_1 \wedge (\neg\, \xi_1))))$

plus one inference rule:

- $\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle$,

where $\gamma^\circ$ is an abbreviation of $(\neg(\gamma \wedge (\neg\, \gamma)))$. △

To build deductions in a given deductive system, we can obviously freely instantiate the schema variables appearing in the rules.

In the sequel, unless otherwise stated, we assume fixed a deductive system $\langle C, R_\ell, R_{\rm g}\rangle$ denoted by $\mathcal{D}$.

DEFINITION 6. We say that $\delta$ is a *global deduction* of $\Gamma$ in $\mathcal{D}$, and write $\Gamma \vdash^{\rm g}_{\mathcal{D}} \delta$ if there is a sequence $\gamma_1 \ldots \gamma_m \in sL(C)^+$ such that:

- $\gamma_m$ is $\delta$;
- each $\gamma_i$ is either an element of $\Gamma$, or there exist $r \in R_{\rm g}$ and a substitution $\sigma_i$ such that $\gamma_i = Conc(r)\sigma_i$ and $Prem(r)\sigma_i \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

When $\Gamma = \emptyset$ we say that $\delta$ is a *theorem schema* and just write $\vdash^{\rm g}_{\mathcal{D}} \delta$. Note that we do not allow substitutions on hypotheses (the elements of $\Gamma$). Indeed, such substitutions do not make sense. For instance, from $\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}$ we want to be able to prove $\xi_2$, but not every formula as it would be possible by substitution on $\xi_1$.

DEFINITION 7. We say that $\delta$ is a *local deduction* of $\Gamma$ in $\mathcal{D}$, and write $\Gamma \vdash^{\ell}_{\mathcal{D}} \delta$ if there is a sequence $\gamma_1 \ldots \gamma_m \in sL(C)^+$ such that:

- $\gamma_m$ is $\delta$;
- each $\gamma_i$ is either an element of $\Gamma$, or a theorem schema, or there exist $r \in R_\ell$ and a substitution $\sigma_i$ with $\gamma_i = Conc(r)\sigma_i$ and $Prem(r)\sigma_i \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

If $\Gamma = \emptyset$, again we just write $\vdash^\ell_\mathcal{D} \delta$. If $\mathcal{D}$ is clear from the context, we simplify the notation and simply write $\vdash^{\mathrm{g}}$ and $\vdash^\ell$.

EXAMPLE 8. Observe that in the deductive system presented in Example 5 for modal logic K we have

$$\{(\xi_1 \Rightarrow \xi_2)\} \vdash^{\mathrm{g}}_{\mathcal{D}^{\mathrm{K}}} ((\Box\xi_1) \Rightarrow (\Box\xi_2))$$

but

$$\{(\xi_1 \Rightarrow \xi_2)\} \nvdash^{\ell}_{\mathcal{D}^{\mathrm{K}}} ((\Box\xi_1) \Rightarrow (\Box\xi_2)).$$

△

Note that our definition of deduction immediately implies compactness (that is, if $\Gamma \vdash^d_\mathcal{D} \delta$ then there is a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vdash^d_\mathcal{D} \delta$, for any $d$ equal to g or $\ell$).

REMARK 9. Our presentation could be situated at the level of a general theory of consequence relations, within what is known as general abstract logics, if we were not concerned with fibring. Usual consequence systems are not concerned with the structure of formulas, and so are not adequate as a starting point for fibring. In order to understand why, we now briefly introduce the theory of consequence relations. Let $\wp(X)$ be the powerset of a set $X$. As usual, given a set $L$ of formulas, we say that $\triangleright \subseteq \wp(L) \times L$ defines a *(Tarskian) consequence relation* on $L$ if the following clauses hold, for any formulas $\alpha$ and $\beta$, and subsets $\Gamma$ and $\Delta$ of $L$ (formulas and commas at the left-hand side of $\triangleright$ denote, as usual, sets and unions of sets of formulas):

- $\alpha \in \Gamma$ implies $\Gamma \triangleright \alpha$ (reflexivity);
- $(\Delta \triangleright \alpha$ and $\Delta \subseteq \Gamma)$ implies $\Gamma \triangleright \alpha$ (monotonicity);
- $(\Delta \triangleright \alpha$ and $\Gamma, \alpha \triangleright \beta)$ implies $\Gamma, \Delta \triangleright \beta$ (transitivity);
- $(\Delta \triangleright \alpha$ and $\rho$ is a substitution) implies $\rho(\Delta) \triangleright \rho(\alpha)$ (structurality).

So, a logic could be seen as a structure of the form $\langle L, \triangleright\rangle$, containing a set of formulas and a consequence relation defined on this set. This structure will be called a *consequence system*.

A *consequence system morphism* $h : \langle L, \triangleright\rangle \to \langle L', \triangleright'\rangle$ is a map $h : L \to L'$ such that if $\Gamma \triangleright \varphi$ then $h(\Gamma) \triangleright' h(\varphi)$[3]. Then union of consequence systems

[3]Consequence system morphisms are also called translations, see e.g. [Coniglio and Carnielli, 2002] and references therein.

$\langle L', \triangleright' \rangle$ and $\langle L'', \triangleright'' \rangle$ is the consequence system $\langle L, \triangleright \rangle$ where $L = L' \cup L''$ and $\triangleright = \triangleright' \cup \triangleright''$.

Consequence systems are too poor for fibring, and not adequate, as mentioned before, since the key point in fibring is to be able to write formulas where connectives can be intertwined. For instance if we have two modal logics with $\Box'$ and $\Box''$ we want to be able in the fibring to write formulas like $((\Box'(\Box''\delta)) \Rightarrow \gamma)$ which does not belong to the union of the consequence systems associated with both logics. △

It is straightforward to verify that both local and global deduction fulfill Tarski's axioms, and so that both the global and the local deduction relations are *structural*, in the sense that $\Gamma \vdash^d \delta$ implies $\Gamma\sigma \vdash^d \delta\sigma$ for every substitution $\sigma$ and any $d$ equal to g or $\ell$.

PROPOSITION 10. *Every deductive system $\mathcal{D}$ induces two consequence systems $\langle sL(C) \vdash^{\mathrm{g}}_{\mathcal{D}} \rangle$ and $\langle sL(C), \vdash^{\ell}_{\mathcal{D}} \rangle$, and $\vdash^{\mathrm{g}}_{\mathcal{I}}$ extends $\vdash^{\ell}_{\mathcal{I}}$.*

Nevertheless, note that if $\vdash^{\mathrm{g}} \delta$ then also $\vdash^{\ell} \delta$. Although both deduction relations are defined over schema formulas, they restrict to just formulas in a natural way.

We can finally define the fibring of deductive systems in formal terms.

DEFINITION 11. The *fibring of deductive systems* $\mathcal{D}'$ and $\mathcal{D}''$ denoted by

$$\mathcal{D}' + \mathcal{D}''$$

is the deductive system $\langle C, R_\ell, R_{\mathrm{g}} \rangle$ where $C_k = C'_k \cup C''_k$ for each $k \in \mathbb{N}$, and $R_\ell = {R_\ell}' \cup {R_\ell}''$ and $R_{\mathrm{g}} = {R_{\mathrm{g}}}' \cup {R_{\mathrm{g}}}''$.

Clearly, it makes sense to combine the signatures $C'$ and $C''$ into a larger signature $C' \cup C''$, where all shared constructors appear in their common subsignature $C' \cap C''$. Indeed, we say that the fibring is *constrained* precisely if there are shared constructors. Otherwise, we say that the fibring is *unconstrained*. Then, by also putting together the rules of both systems we obtain the deductive system over the combined language. Note that the richness of the combination lies on the schematicity of the rules of each of the calculi. In the fibred system the rules are considered in the context of a richer language, and so their schema variables can now be instantiated with mixed formulas built using constructors from both signatures.

EXAMPLE 12. Consider the deductive systems for modal logics D and S4 of Example 5 where the $\Box$ connective was renamed to $\Box'$ and $\Box''$ respectively. Then the deductive system resulting from the fibring

$$\mathcal{D}^{\mathrm{D}} + \mathcal{D}^{\mathrm{S4}}$$

is a deductive system for a modal logic with two modalities: a deontic modality $\Box'$ and an S4 modality $\Box''$. △

*Categorial perspective*

Fibring of deductive systems corresponds to a universal construction in the category of deductive systems. The categorial approach is important because it requires that the objects of study and their interrelationships are made completely precise. Moreover, in category theory, a universal construction plays the role of an abstract definition, large enough to accommodate common constructions and at the same time restricted enough so as to guarantee certain uniqueness conditions. The notion of fibring can be recast as a coproduct or a pushout (universal constructions) in the category of deductive systems or other appropriate categories. We refer to the introductory chapters of [Mac Lane, 1998] for the few basic notions involved in the categorial presentation of fibring.

In order to present that construction we need to define the category of signatures.

DEFINITION 13. A *signature morphism* $h : C \to C'$ is a family of functions $h_k : C_k \to C'_k$ where $k \in \mathbb{N}$.

Naturally, each signature morphism $h : C \to C'$ freely extends to a *language translation map* $h^* : sL(C) \to sL(C')$, by defining $h^*(\xi) = \xi$ for every $\xi \in \Xi$. For ease of notation we use $h$ for this extension.

Signatures and their morphisms constitute a category **Sig**, with identity and composition of functions defined on each arity. Clearly **Sig** is (small) cocomplete, that is, it is closed under coproducts and pushouts. We are now ready to define the notion of deductive system morphism.

DEFINITION 14. A *deductive system morphism*

$$h : \mathcal{D} \to \mathcal{D}'$$

is a signature morphism $h : C \to C'$ such that $h(\Gamma) \vdash^{\ell}_{\mathcal{D}'} h(\delta)$ for every rule $\langle \Gamma, \delta \rangle \in R_\ell$, and $h(\Gamma) \vdash^{\mathrm{g}}_{\mathcal{D}'} h(\delta)$ for every rule $\langle \Gamma, \delta \rangle \in R_{\mathrm{g}}$.

A morphism of deductive systems is thus a signature morphism that preserves the inference rules. Note that for every rule $r$ in $R_\ell$, it is sufficient to prove that $h(Conc(r)) \vdash^{\ell}_{\mathcal{D}'} h(Prem(r))$ since this also implies that $h(Conc(r)) \vdash^{\mathrm{g}}_{\mathcal{D}'} h(Prem(r))$. It is straightforward to show that deductive system morphisms preserve consequence.

Deductive systems and their morphisms constitute a category **Ded**, with identity and composition borrowed from **Sig**. Indeed, **Ded** is concrete over **Sig** via the obvious forgetful functor. Using this fact it is easy to show that also **Ded** is (small) cocomplete.

PROPOSITION 15. *Given a deductive system morphism* $h : \mathcal{D} \to \mathcal{D}'$ *if* $\Gamma \vdash^d_{\mathcal{D}} \delta$ *then* $h(\Gamma) \vdash^d_{\mathcal{D}'} h(\delta)$*, that is,* $h$ *induces a consequence system morphism from* $\langle sL(C), \vdash^d_{\mathcal{D}} \rangle$ *to* $\langle sL(C'), \vdash^d_{\mathcal{D}'} \rangle$*, for* $d$ *equal to* g *or* $\ell$*.*

We are now ready to characterize fibring as a universal construction.

DEFINITION 16. Let $\mathcal{D}'$ and $\mathcal{D}''$ be deductive systems. Their *unconstrained fibring* is a coproduct $\mathcal{D}$ in the category **Ded**.

That means that there are deductive system morphisms $i' : \mathcal{D}' \to \mathcal{D}$ and $i'' : \mathcal{D}'' \to \mathcal{D}$ such that whenever there are morphisms $h' : \mathcal{D}' \to \mathcal{D}'''$ and $h'' : \mathcal{D}'' \to \mathcal{D}'''$ there is a unique morphism $h : \mathcal{D} \to \mathcal{D}'''$ such that $h \circ i' = h'$ and $h \circ i'' = h''$.

DEFINITION 17. Let $f' : \mathcal{D}^0 \to \mathcal{D}'$ and $f'' : \mathcal{D}^0 \to \mathcal{D}''$ be deductive system morphisms, where $\mathcal{D}^0$ is $\langle C' \cap C'', \emptyset, \emptyset \rangle$. The *constrained fibring* of $\mathcal{D}'$ and $\mathcal{D}''$ sharing $\mathcal{D}^0$ is a pushout in the category **Ded**.

That means that there is a triple $\langle \mathcal{D}, i', i'' \rangle$ where $\mathcal{D}$ is a deductive system, $i' : \mathcal{D}' \to \mathcal{D}$ and $i'' : \mathcal{D}'' \to \mathcal{D}$ are deductive system morphisms such that $i' \circ f' = i'' \circ f''$ and moreover for every triple $\langle \mathcal{D}''', h', h'' \rangle$ where $\mathcal{D}'''$ is a deductive system, $h' : \mathcal{D}' \to \mathcal{D}'''$ and $h'' : \mathcal{D}'' \to \mathcal{D}'''$ are deductive system morphisms such that $h' \circ f' = h'' \circ f''$, there is a unique $h : \mathcal{D} \to \mathcal{D}'''$ such that $h \circ i' = h'$ and $h \circ i'' = h''$.

## *2.2 Interpretation systems*

We adopt as the basic semantic unit a simple algebraic structure. This departs from the point basic semantics of fibring as originally proposed in [Gabbay, 1996a]. We have nevertheless some good reasons to use a more abstract approach based on ordered algebras instead of the rather narrow Kripke-style interpretation semantics; this is explained in detail in the Appendix. So, according to the perspective used here, it is required that a given logic can be semantically presented using models endowed with an ordered algebra.

DEFINITION 18. An *interpretation structure* $\mathcal{B}$ over the signature $C$ is a tuple $\langle B, \leq, \nu, \top \rangle$ where $\langle B, \leq, \top \rangle$ is a partial order with a top, and $\langle B, \nu \rangle$ is a $C$-algebra.

The set $B$ is the set of *truth values* and $\top$ is the *designated value* whose intended purpose is to state when a formula is true in a structure. The relation $\leq$ allows the comparison between truth values. We denote by $Str(C)$ the class of all interpretation structures over $C$.

Formulas are to be evaluated over interpretation structures. We will use *assignments* over an interpretation structure $\mathcal{B}$, that is, maps $\alpha : \Xi \to B$.

DEFINITION 19. The *denotation* of a schema formula over $\mathcal{B}$ and $\alpha$ is inductively defined as follows:

- $[\![\xi]\!]^\alpha_{\mathcal{B}} = \alpha(\xi)$;
- $[\![c(\gamma_1, \ldots, \gamma_k)]\!]^\alpha_{\mathcal{B}} = \nu_k(c)([\![\gamma_1]\!]^\alpha_{\mathcal{B}}, \ldots, [\![\gamma_k]\!]^\alpha_{\mathcal{B}})$.

Now we can introduce the concept of interpretation system.

DEFINITION 20. An *interpretation system* $\mathcal{I}$ is a pair $\langle C, \mathcal{A} \rangle$ where $C$ is a signature and $\mathcal{A} \subseteq Str(C)$.

We present some examples of interpretation systems in the sequel.

EXAMPLE 21. Taking into account the signatures introduced in Example 2, we can consider the following interpretation systems:

Propositional logic

$\mathcal{I}^p = \langle C^p, \mathcal{A} \rangle$, where $\mathcal{A}$ is the class of all interpretation structures $\mathcal{B} = \langle B, \leq, \nu, \top \rangle$ built from a Boolean algebra $\langle A, \sqcap, \sqcup, -, \top, \bot \rangle$ and a valuation $v : \Pi \to A$, as follows:

- $x \leq y$ if and only if $x \sqcap y = x$;
- $\nu_0(p) = v(p)$ for $p \in \Pi$;
- $\nu_0(\mathbf{t}) = \top$;
- $\nu_1(\neg)(x) = -x$;
- $\nu_2(\Rightarrow)(x, y) = (-x) \sqcup y$.

Intuitionistic logic

$\mathcal{I}^i = \langle C^i, \mathcal{A} \rangle$, where $\mathcal{A}$ is the class of all interpretation structures $\mathcal{B} = \langle B, \leq, \nu, \top \rangle$ built from an Heyting algebra $\langle A, \sqcap, \sqcup, \rightarrow, \top, \bot \rangle$ and a valuation $v : \Pi \to A$, as follows:

- $x \leq y$ if and only if $x \rightarrow y = \top$;
- $\nu_0(p) = v(p)$ for $p \in \Pi$;
- $\nu_0(\mathbf{t}) = \top$;
- $\nu_1(\neg)(x) = x \rightarrow \bot$;
- $\nu_2(\wedge) = \sqcap$;
- $\nu_2(\vee) = \sqcup$;
- $\nu_2(\Rightarrow)(x, y) = x \rightarrow y$.

Modal logic K

$\mathcal{I}^{\mathrm{K}} = \langle C^m, \mathcal{A} \rangle$, where $\mathcal{A}$ is the class of all interpretation structures $\mathcal{B} = \langle \wp(W), \subseteq, \nu, W \rangle$ built from a Kripke frame $\langle W, R \rangle$ and a valuation $v : \Pi \to \wp(W)$, as follows:

- $\nu_0(p) = v(p)$ for $p \in \Pi$;
- $\nu_0(\mathbf{t}) = W$;
- $\nu_1(\sim)(X) = W \setminus X$;
- $\nu_1(\Box)(X) = \{w \in W : wRw' \text{ implies } w' \in X\}$;

- $\nu_2(\Rightarrow)(X, Y) = (W \setminus X) \cup Y$.

Modal logic S4

$\mathcal{I}^{\mathrm{S4}} = \langle C^m, \mathcal{A}\rangle$, where $\mathcal{A}$ is the class of all interpretation structures built from a Kripke frame $\langle W, R\rangle$ where $R$ is reflexive and transitive, and from a valuation $v : \Pi \to \wp(W)$, as for modal logic K.

Modal logic D

$\mathcal{I}^{\mathrm{D}} = \langle C^m, \mathcal{A}\rangle$, where $\mathcal{A}$ is the class of all interpretation structures built from a Kripke frame $\langle W, R\rangle$ where $R$ is serial, and from a valuation $v : \Pi \to \wp(W)$, as for modal logic K.

Paraconsistent logic $\mathcal{C}_1$

As shown in [Mortensen, 1980; Lewin *et al.*, 1991], there is no way of presenting $\mathcal{C}_1$ as an interpretation system with a meaningful algebraic truth-functional semantics. To treat this case, we shall have to enlarge the scope of our definitions and methods to *non-truth-functional* logics, as explained in Section 4. △

Given an interpretation system $\mathcal{I}$ we define the notions of global and local semantic consequences.

DEFINITION 22. We say that $\delta$ is *globally entailed* from $\Gamma$ with respect to an interpretation system $\mathcal{I}$, written

$$\Gamma \vDash^{\mathrm{g}}_{\mathcal{I}} \delta$$

if, for every $\mathcal{B}$ in $\mathcal{I}$ and assignment $\alpha$ over $\mathcal{B}$, if $[\![\gamma]\!]^{\alpha}_{\mathcal{B}} = \top$ for each $\gamma \in \Gamma$ then $[\![\delta]\!]^{\alpha}_{\mathcal{B}} = \top$.

DEFINITION 23. We say that $\delta$ is *locally entailed* from $\Gamma$ with respect to an interpretation system $\mathcal{I}$, written

$$\Gamma \vDash^{\ell}_{\mathcal{I}} \delta$$

if, for every $\mathcal{B}$ in $\mathcal{I}$, assignment $\alpha$ over $\mathcal{B}$ and $b \in B$, if $b \leq [\![\gamma]\!]^{\alpha}_{\mathcal{B}}$ for each $\gamma \in \Gamma$ then $b \leq [\![\delta]\!]^{\alpha}_{\mathcal{B}}$.

As is evident from the definition, global entailment requires less structure at the semantic level than local entailment. So, if we were only interested in global entailment it would not be necessary to consider structures with a partial order.

As in the deductive setting, both local and global entailment relations fulfill Tarski's axioms, and so they are structural.

PROPOSITION 24. *Given an interpretation system $\mathcal{I}$, the pair $\langle sL(C), \vDash^{\mathrm{g}}_{\mathcal{I}}\rangle$ and the pair $\langle sL(C), \vDash^{\ell}_{\mathcal{I}}\rangle$ are consequence systems, and $\vDash^{\mathrm{g}}_{\mathcal{I}}$ extends $\vDash^{\ell}_{\mathcal{I}}$.*

We now introduce the concept of reduct of an ordered algebra over a signature that will be used in the definition of interpretation system morphism and in the definition of fibring.

DEFINITION 25. Given signatures $C$ and $C'$ and an interpretation structure $\mathcal{B}'$ over $C'$ such that $C \subseteq C'$, the reduct of $\mathcal{B}'$ to $C$ is the interpretation structure over $C$

$$\mathcal{B}'|_C = \langle B', \leq', \nu'_{|_C}, \top' \rangle$$

where $\nu'_{|_C k}(c) = \nu'_k(c)$ for every $c \in C_k$.

We can now define the fibring of interpretation systems.

DEFINITION 26. The *fibring of interpretation systems* $\mathcal{I}'$ and $\mathcal{I}''$ denoted by

$$\mathcal{I}' + \mathcal{I}''$$

is the interpretation system $\langle C' \cup C'', \mathcal{A} \rangle$ where $\mathcal{A}$ is the class of all interpretation structures $\mathcal{B}$ over $C' \cup C''$ such that $\mathcal{B}|_{C'} \in \mathcal{A}'$ and $\mathcal{B}|_{C''} \in \mathcal{A}''$.

As in the case of deductive systems, the fibring of interpretation systems is constrained or unconstrained depending on whether the shared subsignature $C' \cap C''$ is non-empty. As should be clear from the definition, shared constructors are forced to have the same interpretation in fibred models. Then, by considering in $\mathcal{I}' + \mathcal{I}''$ all possible interpretation structures over the combined signature $C' \cup C''$ that simultaneously extend some interpretation structure of each of the given interpretation systems, we achieve the desired degree of generality. Still, it should be clear that each pair of interpretation structures can only be combined if there is a full agreement on the ordered set of truth values and on the interpretation of shared constructors.

EXAMPLE 27. Consider the interpretation systems of Example 21 for classical and intuitionistic logics with connectives for negation and implication respectively denoted as $\neg_c$ and $\neg_i$, and $\Rightarrow_c$ and $\Rightarrow_i$. Let $\mathcal{I}$ be the interpretation system resulting from their fibring. So $\mathcal{B}|_{C^p}$ and $\mathcal{B}|_{C^i}$ are respectively a Boolean algebra and a Heyting algebra, for each semantic structure $\mathcal{B}$ in $\mathcal{I}$. As the carrier set of both reducts coincide, $\mathcal{B}|_{C^i}$ will also be a Boolean algebra and so we obtain the phenomenon of fibring collapsing described in [del Cerro and Herzig, 1996] and [Gabbay, 1996b]. Section 3 introduces the modulated fibring as a solution to this collapsing phenomenon. $\triangle$

*Categorial perspective*

We start by defining the notion of reduct in a categorial perspective.

DEFINITION 28. Given a signature morphism $h : C \to C'$ and an interpretation structure $\mathcal{B}'$ over $C'$, the *h-reduct* of $\mathcal{B}'$ is the interpretation structure $\mathcal{B}'|_C = \langle B', \leq', \nu' \circ h, \top' \rangle$ over $C$.

This construction induces a map $\cdot|_h : Str(C') \to Str(C)$ that is reminiscent of the corresponding operation on the underlying algebras.

We are now ready to set-up the category of interpretation systems.

DEFINITION 29. An *interpretation system morphism* $h : \langle C, \mathcal{A}\rangle \to \langle C', \mathcal{A}'\rangle$ is a signature morphism $h : C \to C'$ such that:

- $\mathcal{B}'|_h \in \mathcal{A}$ for every $\mathcal{B}' \in \mathcal{A}'$.

A morphism of interpretation systems is simply a signature morphism that additionally reduces models to models in the opposite direction. It is well known from reducts of algebras that $[\![\gamma]\!]^{\alpha}_{\mathcal{B}|_h} = [\![h(\gamma)]\!]^{\alpha}_{\mathcal{B}}$ for every assignment $\alpha$ and every $\gamma \in sL(C)$.

Using this fact, it is straightforward to show that morphisms of interpretation systems preserve entailment.

PROPOSITION 30. *Given an interpretation system morphism $h : \mathcal{I} \to \mathcal{I}'$, if $\Gamma \vDash^d_{\mathcal{I}} \delta$ then $h(\Gamma) \vDash^d_{\mathcal{I}'} h(\delta)$, that is, $h$ induces a consequence system morphism from $\langle sL(C), \vDash^d_{\mathcal{I}}\rangle$ to $\langle sL(C'), \vDash^d_{\mathcal{I}'}\rangle$, for $d$ equal to* g *or $\ell$.*

Interpretation systems and their morphisms constitute a category **Int**, with identity and composition borrowed from **Sig**. Indeed, **Int** is concrete over **Sig** via the obvious forgetful functor. Using this fact it is easy to show that also **Int** is (small) cocomplete.

We are now ready to characterize semantic fibring as a universal construction, as done for deductive systems at part 2.1 of this section.

DEFINITION 31. Let $\mathcal{I}'$ and $\mathcal{I}''$ be interpretation systems. Their *unconstrained fibring* is a coproduct $\mathcal{I}$ in the category **Int**.

That means that there are interpretation system morphisms $i' : \mathcal{I}' \to \mathcal{I}$ and $i'' : \mathcal{I}'' \to \mathcal{I}$ such that whenever there are morphisms $h' : \mathcal{I}' \to \mathcal{I}'''$ and $h'' : \mathcal{I}'' \to \mathcal{I}'''$ there is a unique morphism $h : \mathcal{I} \to \mathcal{I}'''$ such that $h \circ i' = h'$ and $h \circ i'' = h''$.

DEFINITION 32. Let $f' : \mathcal{I}^0 \to \mathcal{I}'$ and $f'' : \mathcal{I}^0 \to \mathcal{I}''$ be interpretation system morphisms, where $\mathcal{I}^0$ is $\langle C' \cap C'', Str(C' \cap C'')\rangle$. The *constrained fibring* of $\mathcal{I}'$ and $\mathcal{I}''$ sharing $\mathcal{I}^0$ is a pushout in the category **Int**.

That means that there is a triple $\langle \mathcal{I}, i', i''\rangle$ where $\mathcal{I}$ is an interpretation system, $i' : \mathcal{I}' \to \mathcal{I}$ and $i'' : \mathcal{I}'' \to \mathcal{I}$ are interpretation system morphisms such that $i' \circ f' = i'' \circ f''$ and moreover for every triple $\langle \mathcal{I}''', h', h''\rangle$ where $\mathcal{I}'''$ is an interpretation system, $h' : \mathcal{I}' \to \mathcal{I}'''$ and $h'' : \mathcal{I}'' \to \mathcal{I}'''$ are interpretation system morphisms such that $h' \circ f' = h'' \circ f''$ there is a unique $h : \mathcal{I} \to \mathcal{I}'''$ such that $h \circ i' = h'$ and $h \circ i'' = h''$.

### *2.3 Preservation results*

One of the important aspects of fibring is the possibility of obtaining transference results from the logics to be fibred to the logic resulting from the fibring. In this subsection we study the preservation by fibring of soundness and of strong global completeness. For preservation of strong local completeness we recommend [Zanardo *et al.*, 2001].

We start by introducing logic systems as systems obtained by putting together deduction and semantics and so offering the right context to speak about soundness and completeness. A *logic system* $\mathcal{L}$ is a tuple $\langle C, \mathcal{A}, R_\ell, R_\mathrm{g}\rangle$ where $\langle C, \mathcal{A}\rangle$ is an interpretation system, denoted by $\mathcal{I}(\mathcal{L})$, and $\langle C, R_\ell, R_\mathrm{g}\rangle$ is a deductive system, denoted by $\mathcal{D}(\mathcal{L})$. In the following when there is no ambiguity we assume that a logic system $\mathcal{L}$ is $\langle C, \mathcal{A}, R_\ell, R_\mathrm{g}\rangle$ and we denote its deductive part by $\mathcal{D}$ and its interpretation part by $\mathcal{I}$. Moreover given $\mathcal{L}$ we may denote the consequence relation $\vdash^d_\mathcal{D}$ by $\vdash^d_\mathcal{L}$ or simply by $\vdash^d$, and similarly for the entailment relations, for $d$ equal to g or $\ell$.

DEFINITION 33. A logic system $\mathcal{L}$ is said to be *globally sound* if $\Gamma \vDash^\mathrm{g}_\mathcal{L} \delta$ whenever $\Gamma \vdash^\mathrm{g}_\mathcal{L} \delta$, for every $\Gamma$ and $\varphi$ in $sL(C)$. And it is said to be *globally complete* if $\Gamma \vdash^\mathrm{g} \delta$ whenever $\Gamma \vDash^\mathrm{g} \delta$, for every $\Gamma$ and $\delta$ in $sL(C)$. If we consider $\Gamma = \emptyset$ we get the corresponding weak notions. The local versions are defined *mutatis mutandis*.

#### *Preservation of soundness*

Preservation of soundness follows straightforwardly by exploiting the fact that interpretation system morphisms preserve entailment.

THEOREM 34. *Soundness is preserved by fibring.*

It is straightforward to prove that (strong and weak, global and local) soundness is unconditionally preserved by fibring in the basic universe of logic systems considered here. However, in larger universes things can be more complicated. As it is shown in Section 5, when fibring logic systems with quantifiers and using rules with side provisos (such as "provided that term $\theta$ is free for variable $x$ in formula $\xi$"), soundness is not always preserved. See also [Sernadas *et al.*, 2002a; Coniglio *et al.*, 2003].

#### *Preservation of completeness*

We now turn our attention to the preservation by fibring of strong global completeness. To this end we establish sufficient properties for a logic system to be strongly global complete and then show that fibring preserves these properties.

We start by defining important concepts needed along this subsection. Given an inference rule $\langle\Gamma, \delta\rangle$, denoted by $r$, an interpretation structure $\mathcal{B}$ over $C$ *locally satisfies* $r$ whenever for every assignment $\alpha$ over $\mathcal{B}$ and $b \in B$, if $b \leq [\![\gamma]\!]^{\alpha}_{\mathcal{B}}$ for each $\gamma \in \Gamma$ then $b \leq [\![\delta]\!]^{\alpha}_{\mathcal{B}}$, and *globally satisfies* $r$ whenever for every assignment $\alpha$ over $\mathcal{B}$, if $[\![\gamma]\!]^{\alpha}_{\mathcal{B}} = \top$ for each $\gamma \in \Gamma$ then $[\![\delta]\!]^{\alpha}_{\mathcal{B}} = \top$.

DEFINITION 35. A logic system $\langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}} \rangle$ is said to be *full* when $\langle C, \mathcal{A} \rangle$ contains all interpretation structures over $C$ that locally satisfy the rules in $R_\ell$ and that globally satisfy the rules in $R_{\mathrm{g}}$.

DEFINITION 36. A logic system $\mathcal{L}$ is said to be *congruent* when for every $\Gamma$ closed under global deduction, $c \in C_k$ and $\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_k \in sL(C)$:

$$\frac{\begin{array}{c}\Gamma, \gamma_i \vdash^{\ell} \delta_i \\ \Gamma, \delta_i \vdash^{\ell} \gamma_i\end{array} \quad i = 1, \ldots, k}{\Gamma, c(\gamma_1, \ldots, \gamma_k) \vdash^{\ell} c(\delta_1, \ldots, \delta_k)},$$

and $\mathcal{L}$ has *verum* if its language contains a theorem that denotes $\top$ in every interpretation structure.

We are now ready to state a completeness theorem for global reasoning, which can be proved using a common Lindenbaum-Tarski construction.

THEOREM 37. *Every full and congruent logic system with verum is globally complete.*

Observe that the requirements of congruence and verum are quite weak and usually fulfilled by commonly used logic systems (including those mentioned above as examples). Furthermore, any complete logic system can be made full without changing its entailment. And if *verum* is not present, it can be conservatively added in congruent logic systems. But if the system at hand is not congruent, there is nothing we can do within the scope of the basic theory of fibring outlined here.

Note also that through a mild strengthening of the requirements of the theorem we can ensure finitary strong local completeness (see for instance [Sernadas *et al.*, 2002b]). A similar strong (local and global) completeness theorem is obtained in [Zanardo *et al.*, 2001] without extra requirements for local reasoning but assuming a more complex semantics and using a Henkin construction.

Herein we examine in detail the question of preserving strong global completeness when fibring basic logic systems (as defined above). Note that weak completeness is not always preserved as shown in [Zanardo *et al.*, 2001]. Preservation of strong global completeness follows by adapting the technique originally proposed in [Zanardo *et al.*, 2001], and capitalizing on the completeness theorem stated above about such logic systems. That is, when fibring two given logic systems that are full, congruent and with *verum* (and, therefore, strongly globally complete) we shall try to obtain the strong global completeness of the result by identifying the conditions under which fullness, congruence and *verum* are preserved by fibring.

LEMMA 38. *Fullness is preserved by fibring.*

LEMMA 39. *The logic system resulting from fibring has verum provided that at least one of the given logic systems has verum.*

However, congruence is not always preserved by fibring. Consider the fibring of two logic systems $\mathcal{L}', \mathcal{L}''$ with the following signatures and rules:

$$C_0' = \{\pi_0, \pi_1, \pi_2\} \quad C_1' = \{c\} \quad C_k' = \emptyset \text{ for } k > 1$$
$${R_\ell}' = \emptyset \quad {R_{\mathrm{g}}}' = \{\langle \{\xi\}, c(\xi)\rangle\}$$

$$C_0'' = \{\pi_0, \pi_1, \pi_2\} \quad C_k'' = \emptyset \text{ for } k > 0$$
$${R_\ell}'' = {R_{\mathrm{g}}}'' = \{\langle \{\pi_0, \pi_1\}, \pi_2\rangle, \langle \{\pi_0, \pi_2\}, \pi_1\rangle\}$$

Clearly, both $\mathcal{L}'$ and $\mathcal{L}''$ are congruent, but their fibring $\mathcal{L} = \mathcal{L}' + \mathcal{L}''$ is not congruent. Indeed, consider $\Gamma = \{\pi_0\}^{\vdash^{\mathrm{g}}_{\mathcal{L}}} = \{c^n(\pi_0) : n \geq 0\}$. So, from $\Gamma$, $\pi_1$ and $\pi_2$ are locally interderivable in $\mathcal{L}$ but, $c(\pi_1)$ and $c(\pi_2)$ are not.

Fortunately, it is possible to establish a useful sufficient condition for the preservation of congruence by fibring. In order to define that condition we need first to say when a deductive system has implication and equivalence. A logic system $\mathcal{L}$ has *implication* if there is a binary connective $\Rightarrow$ fulfilling the following Metatheorem of Modus Ponens (MTMP)

$$\frac{\Gamma \vdash^{\ell}_{\mathcal{L}} (\delta_1 \Rightarrow \delta_2)}{\Gamma, \delta_1 \vdash^{\ell}_{\mathcal{L}} \delta_2}$$

where $\Gamma$ is a set of schema formulas, and the following Metatheorem of Deduction (MTD)

$$\frac{\Gamma, \delta_1 \vdash^{\ell}_{\mathcal{L}} \delta_2}{\Gamma \vdash^{\ell}_{\mathcal{L}} (\delta_1 \Rightarrow \delta_2)}$$

where $\Gamma$ is a globally closed set of schema formulas. Moreover a logic system is said to have *equivalence* if it has implication and its signature contains a binary connective $\Leftrightarrow$ fulfilling the two Metatheorems of Biconditionality (relating implication with equivalence)

$$\frac{\Gamma \vdash^{\ell} (\delta_1 \Rightarrow \delta_2) \qquad \Gamma \vdash^{\ell} (\delta_2 \Rightarrow \delta_1)}{\Gamma \vdash^{\ell} (\delta_1 \Leftrightarrow \delta_2)} \qquad \frac{\Gamma \vdash^{\ell} (\delta_1 \Leftrightarrow \delta_2)}{\Gamma \vdash^{\ell} (\delta_1 \Rightarrow \delta_2) \qquad \Gamma \vdash^{\ell} (\delta_2 \Rightarrow \delta_1)}$$

for every globally closed set $\Gamma$ contained in $sL(C)$ and $\delta_1$ and $\delta_2$ in $sL(C)$, and the Metatheorem of Substitution of Equivalents (MTSE)

$$\frac{\Gamma \vdash^{\ell} (\delta_1 \Leftrightarrow \delta_2)}{\Gamma \vdash^{\ell} (\epsilon \Leftrightarrow \epsilon')}$$

where $\epsilon'$ is obtained from $\epsilon$ by replacing one or more occurrences of $\delta_1$ by $\delta_2$. In order to show that a logic system with equivalence is congruent we state first that the MTD holds in a logic system iff a couple of deductions hold.

LEMMA 40. *The MTD holds in a logic system $\mathcal{L}$ iff:*

- $\vdash^{\ell}_{\mathcal{L}} (\xi \Rightarrow \xi)$*;*
- $\{\xi_1\}^{\vdash^{g}_{\mathcal{L}}} \vdash^{\ell}_{\mathcal{L}} (\xi_2 \Rightarrow \xi_1)$*; and*
- $\{(\xi \Rightarrow \gamma_1), \ldots, (\xi \Rightarrow \gamma_k)\}^{\vdash^{g}_{\mathcal{L}}} \vdash^{\ell}_{\mathcal{L}} (\xi \Rightarrow \gamma)$ *for each local rule* $\langle \{\gamma_1, \ldots, \gamma_k\}, \gamma \rangle$ *where* $\xi$ *does not occur in the rule.*

The proof of this lemma and of the next proposition can be found in [Zanardo *et al.*, 2001] so we omit them. We can now establish that equivalence is a sufficient condition for congruence.

PROPOSITION 41. *Any logic system with equivalence is congruent.*

Note that the fibring of logic systems with implication while sharing the implication symbol, also has implication.

PROPOSITION 42. *The logic system resulting from the fibring has MTMP provided that at least one of the given logic systems has MTMP and the implication symbol is shared.*

PROPOSITION 43. *The logic system resulting from the fibring has MTD provided that both given logic systems have MTD and the implication symbol is shared.*

Moreover the fibring of two logic systems with equivalence while sharing the implication symbol as well as the equivalence symbol is a logic system with equivalence. So we can establish sufficient conditions for fibring to preserve strong global completeness.

THEOREM 44. *The fibring while sharing implication and equivalence of full logic systems with equivalence and verum is strongly globally complete.*

This preservation result is quite useful because many widely used logic systems, as modal logic, classical propositional logic, intuitionistic logic, do have equivalence in the sense above.

## 3 MODULATED FIBRING

Albeit the great significance of fibring as a conceptual tool, it simply collapses into one of the components when applied to certain logics, even when no symbols are shared. Recall on this issue the case of the collapsing of fibring propositional and intuitionistic logics, as explained in Example 27. To solve this problem we have defined a variation of fibring called *modulated fibring* (for a detailed exposition see [Sernadas *et al.*, 2002b]). The main goal of modulated fibring is to achieve a mechanism for combining logics both at the semantic and at the deductive levels but avoiding the collapsing phenomenon. By its own nature, modulated fibring is more akin to categorial presentation *ab initio*, so this section has an underlying categorial perspective of its own.

### *3.1 Deductive systems*

The notion of signature must be enriched. In order to do that it is necessary to define what is a pre-modulated signature.

DEFINITION 45. A *pre-modulated signature* is a triple $\langle C, \&, \Xi \rangle$ where $C$ is a signature, & is a symbol, and $\Xi$ is a set.

The role of the symbol & will become clear when giving the semantics.

DEFINITION 46. A *pre-modulated signature morphism* $h : \langle C, \&, \Xi \rangle \to \langle C', \&', \Xi' \rangle$ is a pair $\langle h_1, h_2 \rangle$ such that $h_1 : C \to C'$ is a signature morphism and $h_2 : \Xi \to \Xi'$ is a map.

Pre-modulated signatures and their morphisms constitute the category **pSig**. This category has finite colimits.

DEFINITION 47. A *modulated signature* $\Sigma$ is a co-cone in **pSig**, that is $\langle C, \&, \Xi, S \rangle$, where $S$ is a set of **pSig** morphisms with codomain in the pre-modulated signature $\langle C, \&, \Xi \rangle$.

The set $S$ contains the "safe-relevant" pre-modulated signature morphisms whose destination is $\langle C, \&, \Xi \rangle$. Safety will play an important role in the definition of the entailments by constraining the admissible assignments to meta-variables in the range of safe-relevant pre-modulated signature morphisms. This is also the reason why the meta-variables are local to signatures which was not true in the case of fibring.

DEFINITION 48. A *modulated signature morphism* $h : \Sigma \to \Sigma'$ is a co-cone pre-modulated signature morphism, that is, $h$ is a pre-modulated signature morphism such that $h \circ f \in S'$ whenever $f \in S$.

Modulated signatures and their modulated signature morphisms constitute the category **mSig**. Again this category has finite colimits, in particular pushouts.

The set $sL(\Sigma)$ of $\Sigma$-*formulas* is the free algebra over $C$ and $\Xi$. We denote by $L(C, \&)$ the subset of $sL(\Sigma)$ composed by *ground formulas*, that is formulas without meta-variables. Given $s : \breve{\Sigma} \to \Sigma$ we denote by $sL(\Sigma, s)$ the set of formulas in $sL(\Sigma)$ whose main constructor is from $s(\breve{C})$ and by $L(C, \&, s)$ the subset of $sL(\Sigma, s)$ composed by ground formulas whose main constructor is from $s(\breve{C})$. A *substitution* over $\Sigma$ is defined as in the previous section.

DEFINITION 49. A substitution $\sigma$ is *safe* for a set of formulas $\Gamma \subseteq sL(\Sigma)$ if $\sigma(s(\breve{\xi})) \in sL(\Sigma, s)$ for every $s : \breve{\Sigma} \to \Sigma$ in $S$ and $s(\breve{\xi}) \in \Gamma$.

Therefore we should be careful whenever we have in a set of schema formulas images by safe-relevant signature morphisms of meta-variables that come from another signature. They have to be substituted by schema formulas whose main constructor belongs to that signature.

Inference rules are defined as in the previous section so we omit its definition.

DEFINITION 50. A *pre-modulated deductive system* over $\Sigma$ is a triple $\langle\Sigma, R_\ell, R_{\mathrm{g}}\rangle$ where $\Sigma$ is a modulated signature and $R_\ell$ and $R_{\mathrm{g}}$ are sets of inference rules with $R_\ell \subseteq R_{\mathrm{g}}$.

The notion of *local and global deduction* in the context of a pre-modulated deductive system is the same as in Section 2 so we omit its explicit definition.

DEFINITION 51. A *modulated deductive system* $\mathcal{D}$ over $\Sigma$ is a pre-modulated deductive system where

1. $\{\delta_1\&\delta_2\} \vdash^\ell \delta_i$ with $i$ equal to 1 or 2 (& elimination), and
2. $\{\delta_1, \delta_2\} \vdash^\ell \delta_1\&\delta_2$ (& introduction) for every formulas $\delta_1$ and $\delta_2$.

We denote by $\gamma_1 \cong_\Gamma \gamma_2$ the fact that $\Gamma, \gamma_1 \vdash^\ell \gamma_2$ and $\Gamma, \gamma_2 \vdash^\ell \gamma_1$. When $\Gamma = \emptyset$ then we will omit the reference to the set.

To illustrate modulated deductive systems we present a modulated deductive system for Gödel logics. Gödel logics were introduced as approximations to intuitionistic logic, and extended the propositional intuitionistic calculus.

EXAMPLE 52. *(3-valued) Gödel modulated deductive system.* We adapt from the axiomatic system in [Hähnle, 2001]. The signature $\langle C, \&, \Xi, S\rangle$ is such that: $\mathbf{t}, \mathbf{f} \in C_0$, $C_1 = \{\neg\}$, $C_2 = \{\wedge, \vee, \Rightarrow\}$, $C_k = \emptyset$ for all $k \geq 3$, & is $\wedge$, and $\Xi = \{\xi_i : i \in \mathbb{N}\}$,

- $R_\ell = \{\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\}$;
- $R_{\mathrm{g}}$ includes $R_\ell$ plus:
  - the axiom schemas of propositional intuitionistic logic, see Example 5;
  - the axiom schema $(((\neg\,\xi_1) \Rightarrow \xi_2) \Rightarrow (((\xi_2 \Rightarrow \xi_1) \Rightarrow \xi_2) \Rightarrow \xi_2))$. $\triangle$

We now introduce the notion of modulated deductive system morphism as a pair. The first component of the pair is a modulated signature morphism.

DEFINITION 53. A *modulated deductive system morphism* from $\mathcal{D}$ to $\mathcal{D}'$ is a pair $\langle\hat{h}, \check{h}\rangle$ such that $\hat{h} : \Sigma \to \Sigma'$ is a modulated signature morphism and $\check{h} : L(C') \to L(C)$ is a monotonic map with:

1. $\hat{h}(r) \in R'_{\mathrm{g}}$ for every $r \in R_{\mathrm{g}}$;

2. $\hat{h}(r) \in R'_\ell$ for every $r \in R_\ell$;

3. $\check{h}$ is left adjoint of $\hat{h}$;

4. $\hat{h}(c(\check{h}(\vec{\varphi}'))) \vdash'^{\ell} \hat{h}(c)(\vec{\varphi}')$.

The more complex notion of modulated deductive system morphism is the adequate one for fulfilling the requirements that are necessary for preserving congruence by modulated fibring. The contravariant map $\check{h}$ can be seen as a map relating truth values (formulas) in the Lindendaum-Tarski algebras. Differently from Section 2 where preservation of congruence was obtained by sharing implication and equivalence, here this may not be the best solution because sharing of implication and equivalence leads in some cases to collapse.

Observe that $\check{h}(\Phi') \vdash^{\mathrm{g}} \check{h}(\varphi')$ whenever $\Phi' \vdash'^{\mathrm{g}} \varphi'$ and $\check{h}(\Phi') \vdash^{\ell} \check{h}(\varphi')$ whenever $\Phi' \vdash'^{\ell} \varphi'$ for every $\Phi'$ and $\varphi'$ in $L(C', \&')$.

PROPOSITION 54. *Modulated deductive systems and their morphisms constitute a category named **mDed**.*

We now show that modulated deductive system morphisms do preserve global and local deduction.

PROPOSITION 55. *Let $h : \mathcal{D} \to \mathcal{D}'$ be a modulated deductive system morphism such that $\hat{h}$ is injective for $\Xi$ and $\hat{h}(C) \subseteq \hat{s}'(C_{\hat{s}'})$ whenever $\hat{h}(\Xi) \cap \hat{s}'(\Xi_{\hat{s}'}) \neq \emptyset$ for every $\hat{s}' \in S'$. Thus $\hat{h}(\Gamma) \vdash'^{\mathrm{g}} \hat{h}(\varphi)$ whenever $\Gamma \vdash^{\mathrm{g}} \varphi$. A similar result holds for local deduction.*

We now define what is a bridge for modulated deductive systems. The bridge allows a mild relationship between the formulas in the modulated deductive systems that we want to combine as well as between their consequence relations. Again modulated fibring appears as a pushout in the category of modulated deductive systems.

DEFINITION 56. A *bridge* between modulated deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ is a diagram $\beta = \langle f' : \breve{\mathcal{D}} \to \mathcal{D}', f'' : \breve{\mathcal{D}} \to \mathcal{D}'' \rangle$ in **mDed** such that $\hat{f}'$, $\hat{f}''$ are injective and $\check{f}'$ and $\check{f}''$ are surjective.

DEFINITION 57. The *modulated fibring* of deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ by a bridge $\beta$ is a pushout of $\beta$ in **mDed**.

We now give an example of modulated fibring illustrating non-collapsing situations.

EXAMPLE 58. *Modulated fibring of propositional and Gödel logics.* Let $\mathcal{D}'$ be the modulated deductive system for propositional logic, $\mathcal{D}''$ the modulated deductive system for 3-valued Gödel logic and $\beta$ a bridge such that $\breve{C}_0 = \{\breve{\mathbf{t}}\}$, $\breve{C}_k = \emptyset$ for all $k \neq 0$, $\breve{\Xi} = \emptyset$, $\breve{S} = \emptyset$, $\breve{R}_{\mathrm{g}} = \breve{R}_\ell$ include $\{\langle \emptyset, \breve{\mathbf{t}} \rangle\}$ and the rules for $\breve{\&}$ elimination and introduction, $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$,

$\hat{f}'(\breve{\mathbf{t}}) = \mathbf{t}'$, $\hat{f}''(\breve{\mathbf{t}}) = \mathbf{t}''$, $\check{f}'(\varphi') = \breve{\mathbf{t}}$ for every $\varphi'$ and $\check{f}''(\varphi'') = \breve{\mathbf{t}}$ for every $\varphi''$. Then the modulated fibring of $\mathcal{D}'$ and $\mathcal{D}''$ by $\beta$ does not collapse. △

We now analyze an example of modulated fibring of deductive systems sharing the negation constructor.

EXAMPLE 59. *Modulated fibring of propositional and Gödel logics sharing negation.* Let $\mathcal{D}'$ be the modulated deductive system for 3-valued Gödel logic (see Example 52) and $\mathcal{D}''$ be the modulated deductive system for propositional logic. Consider a bridge $\beta$ such that:

- $\breve{C}_0 = \{\breve{\mathbf{f}}, \breve{\mathbf{t}}\}$, $\breve{C}_1 = \{\breve{\neg}\}$, $\breve{C}_k = \emptyset$ for $k \geq 2$, $\breve{\Xi} = \emptyset$, $\breve{S} = \emptyset$;
- $S' = \{id_{\Sigma'}\}$ and $S'' = \{id_{\Sigma''}\}$;
- $\hat{f}'(\breve{\mathbf{f}}) = \mathbf{f}'$, $\hat{f}'(\breve{\mathbf{t}}) = \mathbf{t}'$ and $\hat{f}'(\breve{\neg}) = \neg'$;
- $\check{f}'(\varphi') = \begin{cases} \breve{\varphi} & \text{p.t. } \varphi' \text{ is } \hat{f}'(\breve{\varphi}) \\ \breve{\mathbf{f}} & \text{p.t. } \varphi' \vdash^{\ell} \mathbf{f}' \\ \breve{\mathbf{t}} & \text{otherwise} \end{cases}$;
- $\hat{f}''$ and $\check{f}''$ defined in a similar way;
- $\breve{R}_{\mathrm{g}}$ and $\breve{R}_\ell$ are the translations of the ground instances of $R'_{\mathrm{g}}$, $R''_\ell$, $R'_\ell$, $R''_\ell$ by $\check{f}'$ and $\check{f}''$ plus the rules $\breve{\&}$ elimination and introduction.

Note that the pair $\langle \hat{f}', \check{f}' \rangle$ is a morphism. In the modulated fibring $C_k = \hat{g}'(C'_k) \cup \hat{g}''(C''_k)$ and $\Xi = \hat{g}'(\Xi') \cup \hat{g}''(\Xi'')$, $R_{\mathrm{g}} = \hat{g}'(R'_{\mathrm{g}}) \cup \hat{g}''(R''_{\mathrm{g}}) \cup R_\ell$ and $R_\ell$ includes $\hat{g}'(R'_\ell) \cup \hat{g}''(R''_\ell)$, the rules for & elimination and introduction, and the rules for the modulated fibring. △

### *3.2 Interpretation systems*

In this section we investigate modulated fibring from a semantic point of view. The basic semantic unit is the structure for a modulated signature.

DEFINITION 60. A *modulated structure* $\mathcal{B} = \langle B, \leq, \nu \rangle$ over $\Sigma$ is a pre-ordered algebra over $C$ and & with finite meets[4] such that

1. $\nu_2(\&)(b_1, b_2) = b_1 \sqcap b_2$;
2. $\nu_k(c)(b_1, \ldots, b_k) \cong \nu_k(c)(d_1, \ldots, d_k)$ whenever $b_i \cong d_i$ for $i = 1, \ldots, k$.

The symbol & is the syntactical counterpart of 2-ary meets. Note that constraint 2 is a congruence requirement: denotations of a constructor on "equivalent" truth values should be "equivalent". We omit the reference to the arity of the constructors and the subscripts in signature morphisms in

[4] In a pre-order, by $b_1 \cong b_2$ it is meant $b_1 \leq b_2$ and $b_2 \leq b_1$.

order to make the notation lighter. Sometimes we also use $\vec{b}$ as a short hand for $b_1, \ldots, b_k$. Moreover, as is more convenient, we will refer to a structure over a signature $\Sigma$ as a $\Sigma$-structure.

DEFINITION 61. A *modulated interpretation system* is a tuple $\mathcal{I} = \langle \Sigma, M, A \rangle$ where $\Sigma$ is a modulated signature, $M$ is a class (of models), $A$ is a map associating to each $m$ in $M$ a modulated structure $\mathcal{B}_m$ over $\Sigma$.

The modulated interpretation system could be a pair $\langle \Sigma, \mathbb{B} \rangle$ where $\mathbb{B}$ is a class of modulated structures. We include $M$ because one can take the models of the logic at hand and use $A$ to extract the underlying algebras. In this sense, $A(M)$ can be understood as the class of interpretation structures.

Some of the examples we consider are many-valued logics. For more details about these logics see [Gottwald, 2001; Carnielli, 1987; Carnielli and Marcos, 1999; Carnielli and Marcos, 2001; Hähnle, 2001; Carnielli and Marcos, 2002].

EXAMPLE 62. Taking into account the modulated signature for intuitionistic logic based on the signature presented in Example 2 and the modulated signature for (3-valued) Gödel logic in Example 52, we define the following modulated interpretation systems:

Intuitionistic interpretation system

- $M$ is the class of all pairs $m = \langle H, v \rangle$ where $H = \langle B, \sqcap, \sqcup, \sqsupset, \bot, \top \rangle$ is a Heyting algebra and $v : C_0 \to B$ such that $v(\mathbf{t}) = \top$;
- $A(m) = \langle B, \leq, \nu \rangle$ where
  - $b_1 \leq b_2$ iff $b_1 \sqcap b_2 = b_1$;
  - $\nu_0(c) = v(c)$, $\nu_2(\wedge) = \sqcap$ and $\nu_2(\vee) = \sqcup$;
  - $\nu_2(\Rightarrow) = \sqsupset$;
  - $\nu_1(\neg) = \lambda b.b \sqsupset \bot$.

(3-valued) Gödel interpretation system

- $M$ is the class of all pairs $m = \langle G, v \rangle$ where $G = \langle B, \sqcap, \sqcup, \sqsupset, \ominus, \bot, \top \rangle$ is a 3-valued Gödel algebra[5] and $v : C_0 \to B$ such that $v(\mathbf{t}) = \top$;
- $A(m) = \langle B, \leq, \nu \rangle$ where
  - $b_1 \leq b_2$ iff $b_1 \sqcap b_2 = b_1$;
  - $\nu_0(c) = v(c)$, $\nu_2(\wedge) = \sqcap$, $\nu_2(\vee) = \sqcup$ and $\nu_1(\neg) = \ominus$;
  - $\nu(\Rightarrow) = \sqsupset$. △

[5]Recall that the typical 3-valued Gödel algebra has $B = \{\bot, 1/2, \top\}$ and operations $\ominus$ and $\sqsupset$ are defined as follows: $\ominus b = 1$ whenever $b = 0$ and 0 otherwise, and $b_1 \sqsupset b_2$ is $\top$ if $b_1 \leq b_2$ and $b_2$ otherwise.

The objective now is to introduce the notion of entailment. As in section 2 we have two entailments: global entailment and local entailment.

We need the notion of assignment for defining the denotation of formulas and entailments. Assignments that give values to schema variables that come from safe-relevant morphisms are referred to as safe.

Let $s : \breve{\Sigma} \to \Sigma$ be a modulated signature morphism and $\mathcal{B}$ a modulated structure over $\Sigma$. Then, $\mathcal{B}(s)$ is the smallest subalgebra of $\mathcal{B}$ for signature $s(\breve{\Sigma})$. The notion of assignment is defined as in the previous section.

DEFINITION 63. Given a modulated interpretation system we say that an assignment is *over a model* $m$ if $\alpha : \Xi \to B_m$. The assignment $\alpha$ is said to be *safe* for a set of formulas $\Gamma \subseteq sL(\Sigma)$ if $\alpha(s(\breve{\xi})) \in B(s)$ for every $s : \breve{\Sigma} \to \Sigma$ in $S$ and $s(\breve{\xi}) \in \Gamma$.

Safe assignments show the relevance of having the component $S$ in signatures and will be necessary when defining the entailment. Interpretation of formulas in an algebra is defined as usual.

DEFINITION 64. Given a $\Sigma$-structure $\mathcal{B}$ and an assignment $\alpha$, a formula $\gamma$ is *globally satisfied* by $\mathcal{B}$ and a safe assignment $\alpha$ for $\gamma$, written $\mathcal{B}\alpha \Vdash \gamma$, if $[\![\gamma]\!]^{\alpha}_{\mathcal{B}} \cong \top$. A formula $\gamma$ is *locally satisfied* by $\mathcal{B}$, a safe assignment $\alpha$ for $\gamma$ and $b \in B$, written $\mathcal{B}\alpha b \Vdash \gamma$, if $b \leq [\![\gamma]\!]^{\alpha}_{\mathcal{B}}$.

In the context of a modulated interpretation system, we may use $[\![\gamma]\!]^{\alpha}_{m}$ instead of $[\![\gamma]\!]^{\alpha}_{A(m)}$. Moreover, we write $m\alpha \Vdash \gamma$ and $m\alpha b \Vdash \gamma$ whenever $\mathcal{B}_m\alpha \Vdash \gamma$ and $\mathcal{B}_m\alpha b \Vdash \gamma$, respectively. Observe that local satisfaction of a formula at a truth value $b$ indicates that a formula is at least as true as $b$.

DEFINITION 65. A formula $\delta$ is *globally entailed* from a finite set of formulas $\Gamma$, written $\Gamma \vDash^{\mathrm{g}} \delta$, if, for every model $m$ and safe assignment $\alpha$ for $\Gamma \cup \{\delta\}$, $m\alpha \Vdash \delta$ whenever $m\alpha \Vdash \gamma$ for every $\gamma \in \Gamma$. A formula $\delta$ is *globally entailed* from a set of formulas $\Delta$, written $\Delta \vDash^{\mathrm{g}} \delta$, if there is a finite set $\Gamma$ contained in $\Delta$ such that $\Gamma \vDash^{\mathrm{g}} \delta$.

DEFINITION 66. A formula $\delta$ is a *locally entailed* from a finite set of formulas $\Gamma$, written $\Gamma \vDash^{\ell} \delta$, if $m\alpha b \Vdash \delta$ whenever $m\alpha b \Vdash \gamma$ for every $\gamma \in \Gamma$, $m \in M$, safe assignment $\alpha$ over $m$ for $\Gamma \cup \{\delta\}$ and $b \in B_m$. A formula $\delta$ is a *locally entailed* from a set of formulas $\Delta$, written $\Delta \vDash^{\ell} \delta$, if there is a finite set $\Gamma$ contained in $\Delta$ such that $\Gamma \vDash^{\ell} \delta$.

DEFINITION 67. A *modulated interpretation system morphism* $h : \mathcal{I} \to \mathcal{I}'$ is a tuple $\langle \hat{h}, \underline{h}, \dot{h}, \ddot{h} \rangle$ where:

- $\hat{h} : \Sigma \to \Sigma'$ is a morphism in **mSig**;
- $\underline{h} : M' \to M$ is a map;
- $\dot{h} = \{\dot{h}_{m'}\}_{m' \in M'}$ where $\dot{h}_{m'} : \langle B_{\underline{h}(m')}, \leq_{\underline{h}(m')} \rangle \to \langle B'_{m'}, \leq'_{m'} \rangle$ is a monotonic map;

- $\ddot{h} = \{\ddot{h}_{m'}\}_{m' \in M'}$ where $\ddot{h}_{m'} : \langle B'_{m'}, \leq'_{m'}\rangle \to \langle B_{\underline{h}(m')}, \leq_{\underline{h}(m')}\rangle$ is a monotonic map preserving preserving finite meets;

such that for every $m' \in M'$, $\vec{b} \in B^k_{\underline{h}(m')}$ and $\vec{b}' \in B^k_{m'}$:

1. $\ddot{h}_{m'}$ is left adjoint of $\dot{h}_{m'}$;
2. $\nu'_{m'}(\hat{h}(c))(\vec{b}') \cong_{m'} \dot{h}_{m'}(\nu_{\underline{h}(m')}(c)(\ddot{h}_{m'}(\vec{b}')))$ for every $c \in C_k$. △

The map $\underline{h}$ is contravariant as expected. The family of maps $\dot{h}_{m'}$ and $\ddot{h}_{m'}$ indicate that we need to represent the truth values of $B_{\underline{h}(m')}$ in the truth values of $B'_{m'}$ and vice versa. Clause 1. establishes constraints that the maps should fulfill. Clause 2. indicates that denotations of constructors from $C$ in a model $m'$ can be given for any truth values in $B'_{m'}$ by using the two maps.

The morphism between interpretation systems presented in Section 2 is a particular case of the one in Definition 67 with $\dot{h}_{m'} = \mathrm{id}_{B'_{m'}}$, $\ddot{h}_{m'} = \mathrm{id}_{B_{\underline{h}(m')}}$ and hence, $B_{\underline{h}(m')} = B'_{m'}$, etc.

PROPOSITION 68. *Modulated interpretation systems and their morphisms constitute a category, named* ***mInt****.*

A signature morphism $\hat{h}$ can be extended to a map $\hat{h}^*$ between formulas: $\hat{h}^*(c) = \hat{h}(c)$ for $c \in C_0$, $\hat{h}^*(\xi) = \hat{h}(\xi)$, and $\hat{h}^*(c(\gamma_1, \dots, \gamma_k)) = \hat{h}(c)(\hat{h}^*(\gamma_1), \dots, \hat{h}^*(\gamma_k))$ for a $k$-ary connective $c$. Below, $\hat{h}$ is used for the map $\hat{h}^*$. We show below that global and local semantic entailments are preserved by some kind of morphisms.

PROPOSITION 69. *Let $h : \mathcal{I} \to \mathcal{I}'$ be an interpretation system morphism such that $\ddot{h}_{m'}$ is surjective for every $m'$ in $M'$ and $\hat{h} \in S'$ whenever $\Gamma \cup \{\delta\}$ has meta-variables. Then (1) $\hat{h}(\Gamma) \vDash'^{\mathrm{g}} \hat{h}(\delta)$ whenever $\Gamma \vDash^{\mathrm{g}} \delta$ and (2) $\hat{h}(\Gamma) \vDash'^{\ell} \hat{h}(\delta)$ whenever $\Gamma \vDash^{\ell} \delta$.*

As we shall see in the modulated fibring the morphisms that relate interpretation systems do have the required properties. The underlying intuition is that each model in the modulated fibring of $\mathcal{I}'$ and $\mathcal{I}''$ will be a pair $\langle m', m''\rangle$ where $m'$ is a model of $\mathcal{I}'$ and $m''$ is a model of $\mathcal{I}''$. Moreover the truth values in the algebra of $\langle m', m''\rangle$ should be the union of the truth values in the algebras of $m'$ and $m''$. However, for denotations of formulas we need some relationship between the truth values of $m'$ and $m''$ for every $m'$ and $m''$. Such a relationship is established by the notion of bridge.

DEFINITION 70. A *bridge* between modulated interpretation systems $\mathcal{I}'$ and $\mathcal{I}''$ is a diagram $\beta = \langle f' : \breve{\mathcal{I}} \to \mathcal{I}', f'' : \breve{\mathcal{I}} \to \mathcal{I}''\rangle$ in **mInt** such that $\hat{f}'$, $\hat{f}''$, $\dot{f}'_{m'}$ and $\dot{f}''_{m''}$ are injective maps and $\ddot{f}'_{m'}$ and $\ddot{f}''_{m''}$ are surjective maps for every $m' \in M'$ and $m'' \in M''$, respectively.

The category **mInt** has pushouts. The proof of this result is intricate and we invite the reader to follow the steps of the proof in [Sernadas *et al.*, 2002b].

DEFINITION 71. The *modulated fibring* of interpretation systems $\mathcal{I}'$ and $\mathcal{I}''$ by a bridge $\beta$ is a pushout of $\beta$ in **mInt**.

*Examples and the collapsing problem*

We give some examples of modulated fibring showing how the collapse can be avoided. To see a description of the most common collapse we refer the reader to [Sernadas *et al.*, 2002b]. We now define a specific bridge that leads to a non-collapsing situation whenever there is no sharing of constructors.

EXAMPLE 72. *Modulated fibring of propositional and intuitionistic logics.* Let $\mathcal{I}'$ be a propositional modulated interpretation system, $\mathcal{I}''$ an intuitionistic modulated interpretation system and $\beta$ a bridge such that $\breve{C}_0 = \{\breve{\mathbf{t}}\}$, $\breve{C}_k = \emptyset$ for all $k \neq 0$, $\breve{\Xi} = \emptyset$, $\breve{S} = \emptyset$ $\breve{M} = \{\breve{m}\}$, $B_{\breve{m}} = \{\breve{\top}\}$, $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$, $\underline{f}'(m') = \underline{f}''(m'') = \breve{m}$ and $\ddot{f}'_{m'}(b') = \ddot{f}''_{m''}(b'') = \breve{\top}$ for every $m' \in M'$, $m'' \in M''$, $b' \in B'_{m'}$ and $b'' \in B''_{m''}$. Then the modulated fibring of $\mathcal{I}'$ and $\mathcal{I}''$ does not collapse. Intuitionistic logic collapses into propositional logic when the formula $((\neg(\neg\varphi))\Leftrightarrow\varphi)$ becomes valid which is not the case. Observe that in the modulated fibring, $\dot{g}'(B'_{m'})$ is a Boolean algebra "equivalent" to $B'_{m'}$ and $\dot{g}''(B''_{m''})$ is a Heyting algebra "equivalent" to $B''_{m''}$. △

Similarly to Fariñas del Cerro and Herzig's C+J logic as presented in [del Cerro and Herzig, 1996], in the modulated fibring of propositional logic $\mathcal{I}'$ and intuitionistic logic $\mathcal{I}''$ considered above, we have also no problems with the validity of the formula $\hat{g}'(\varphi' \Rightarrow' (\psi' \Rightarrow' \varphi'))$ since, according to our semantics, the formula is only valid for "intuitionistic values". Propositional values are converted to the intuitionistic value "**t**".

The following example illustrates several possible combinations of propositional logic and Gödel logic through different bridges.

EXAMPLE 73. *Modulated fibring of propositional and Gödel logics.* Let $\mathcal{I}'$ and $\mathcal{I}''$ be the modulated interpretation systems for 3-valued Gödel logic and propositional logic (see Example 62). For propositional logic only 2-valued algebras are included. Consider the fibring of propositional and Gödel logics modulated by three different bridges $\beta = \langle f' : \breve{\mathcal{I}} \to \mathcal{I}', f'' : \breve{\mathcal{I}} \to \mathcal{I}'' \rangle$ as follows:

Bridge 1:

- $\breve{\mathcal{I}}$ is such that
  - $\breve{M} = \{\breve{m}\}$;
  - $\breve{A}(\breve{m}) = \langle \{\breve{\top}\}, \{\langle \breve{\top}, \breve{\top} \rangle\}, \breve{\nu} \rangle$;

- $f'$ and $f''$ are such that
  - $\underline{f}'(m') = \underline{f}''(m'') = \breve{m}$;
  - $\dot{f}'_{m'}(\breve{\top}) = \top'_{m'}$ and $\dot{f}''_{m''}(\breve{\top}) = \top''_{m''}$;
  - $\ddot{f}'_{m'}(b') = \breve{\top}$ and $\ddot{f}''_{m''}(b'') = \breve{\top}$ for every $b' \in B'_{m'}$, $b'' \in B''_{m''}$;

Bridge 2:

- $\breve{\mathcal{I}}$ is such that
  - $\breve{M} = \{\breve{m}\}$;
  - $\breve{A}(\breve{m}) = \langle \{\breve{\bot}, \breve{\top}\}, \{\langle \breve{\bot}, \breve{\bot}\rangle, \langle \breve{\bot}, \breve{\top}\rangle, \langle \breve{\top}, \breve{\top}\rangle\}, \breve{\nu}\rangle$;
- $f'$ and $f''$ are such that
  - $\underline{f}'(m') = \underline{f}''(m'') = \breve{m}$;
  - $\dot{f}'_{m'}(\breve{\bot}_{\breve{m}}) = \bot'_{m'}$, $\dot{f}'_{m'}(\breve{\top}_{\breve{m}}) = \top'_{m'}$;
  - $\dot{f}''_{m''}(\breve{\bot}_{\breve{m}}) = \bot''_{m''}$ and $\dot{f}''_{m''}(\breve{\top}_{\breve{m}}) = \top''_{m''}$;
  - $\ddot{f}'_{m'}(\bot'_{m'}) = \breve{\bot}_{\breve{m}}$ and $\ddot{f}'_{m'}(b') = \breve{\top}_{\breve{m}}$ for every $b' \neq \bot'_{m'}$;
  - $\ddot{f}''_{m''}(\bot''_{m''}) = \breve{\bot}_{\breve{m}}$ and $\ddot{f}''_{m''}(b'') = \breve{\top}_{\breve{m}}$ for every $b'' \neq \bot''_{m''}$;

Bridge 3:

- $\breve{\mathcal{I}}$ is such that
  - $\breve{M} = A'(M')|_{\breve{C}} \cup A''(M'')|_{\breve{C}}$;
  - $\breve{A}$ is the identity map;
- $f'$ and $f''$ are such that
  - $\underline{f}'(m') = A'(m')|_{\breve{C}}$ and $\underline{f}''(m'') = A''(m'')|_{\breve{C}}$;
  - $\dot{f}'_{m'} = \mathrm{id}_{B'_{m'}}$ and $\dot{f}''_{m''} = \mathrm{id}_{B''_{m''}}$;
  - $\ddot{f}'_{m'} = \mathrm{id}_{\breve{B}_{\underline{f}'(m')}}$ and $\ddot{f}''_{m''} = \mathrm{id}_{\breve{B}_{\underline{f}''(m'')}}$.

Bridges 1, 2 and 3 can be used to modulate the fibring when $\breve{C}_0 = \{\breve{\mathbf{t}}\}$ and $\breve{C}_k = \emptyset$, $\breve{\Xi} = \emptyset$ and $\breve{S} = \emptyset$. Then $\breve{\nu}$ is a family of empty maps except for $\breve{\nu}_0$ and $\hat{f}'$ and $\hat{f}''$ are also empty maps except for $k = 0$. Bridges 2 and 3 can be used to modulate the fibring when $\breve{C}_0 = \{\breve{\mathbf{f}}, \breve{\mathbf{t}}\}$, $\breve{C}_1 = \{\breve{\neg}\}$, $\breve{C}_k = \emptyset$ for every $k \geq 2$, $\breve{\Xi} = \emptyset$, $\breve{S} = \emptyset$, $\breve{\nu}(\breve{\neg})(\breve{\bot}) = \breve{\top}$, $\breve{\nu}(\breve{\neg})(\breve{\top}) = \breve{\bot}$ and $\hat{f}'$ and $\hat{f}''$ are such that $\hat{f}'(\breve{\neg}) = \neg'$ and $\hat{f}''(\breve{\neg}) = \neg''$. Bridge 3 can be used to modulate the fibring when $\breve{C} = C' = C''$, $\breve{\Xi} = \emptyset$, $\breve{S} = \emptyset$ and $\hat{f}'$ and $\hat{f}''$ are such that $\hat{f}'(\breve{\neg}) = \neg'$, $\hat{f}'(\breve{\wedge}) = \wedge'$, $\hat{f}''(\breve{\neg}) = \neg''$ and $\hat{f}''(\breve{\wedge}) = \wedge''$ (corresponding to the collapse of Gödel logics into propositional logics since in the fibring we will only have Boolean algebras). △

### *3.3 Preservation results*

We start by putting together interpretation systems and deductive systems in modulated logic systems. Then we define modulated fibring as a pushout in the category of modulated logic systems. Finally we give new examples of modulated fibring and investigate preservation of soundness and completeness.

DEFINITION 74. A *modulated logic system* $\mathcal{L}$ is a tuple $\langle \Sigma, M, A, R_{\mathrm{g}}, R_{\ell} \rangle$ such that $\langle \Sigma, M, A \rangle$ is a modulated interpretation system and $\langle \Sigma, R_{\mathrm{g}}, R_{\ell} \rangle$ is a modulated deductive system.

To simplify we follow the conventions for presenting modulated logic systems introduced in Section 2. So given a modulated logic system $\mathcal{L}$ we denote its deductive part by $\mathcal{D}(\mathcal{L})$ and its interpretation part by $\mathcal{I}(\mathcal{L})$. Moreover we assume that $\mathcal{L}$ is $\langle \Sigma, M, A, R_{\mathrm{g}}, R_{\ell} \rangle$ and we omit its reference in the consequence relations when there is no ambiguity.

DEFINITION 75. A modulated logic system is *globally sound* if $\Gamma \vDash^{\mathrm{g}} \delta$ whenever $\Gamma \vdash^{\mathrm{g}} \delta$ for every $\Gamma$ and $\delta$ in $L(C)$. A logic system is *globally complete* if $\Gamma \vdash^{\mathrm{g}} \delta$ whenever $\Gamma \vDash^{\mathrm{g}} \delta$, for every $\Gamma$ and $\delta$ in $L(C)$. Analogously for the local notions. When $\Gamma$ is $\emptyset$ we obtain weak completeness and weak soundness.

Note that for modulated fibring we define soundness and completeness over non-schematic formulas.

*Preservation of soundness*

We now concentrate our attention on soundness. The main objective is to obtain a result stating that if we start with sound modulated logic systems then the logic system obtained by modulated fibring is again sound.

DEFINITION 76. Given a modulated logic system $\mathcal{L}$, a model $m$ in $M$ is a *model* for $\mathcal{D}(\mathcal{L})$ if for every rule $\langle \Gamma, \delta \rangle \in R_{\mathrm{g}}$, $m\alpha \Vdash \delta$ whenever $m\alpha \Vdash \gamma$ for every $\gamma \in \Gamma$ and safe assignment $\alpha$ for $\Gamma \cup \{\delta\}$ and for every rule $\langle \Gamma, \delta \rangle \in R_{\ell}$, $m\alpha b \Vdash \delta$ whenever $m\alpha b \Vdash \gamma$ for every $\gamma \in \Gamma$, safe assignment $\alpha$ for $\Gamma \cup \{\delta\}$ and $b \in B_m$.

The next result establishes sufficient conditions for a modulated logic system to be sound.

PROPOSITION 77. *Let $\mathcal{L}$ be a logic system such that each $m$ in $M$ is a model for $\mathcal{D}(\mathcal{L})$. Then $\mathcal{L}$ is globally and locally sound.*

We conclude with the main result on preservation of soundness.

THEOREM 78. *The modulated fibring $\langle g' : \mathcal{L}' \to \mathcal{L}, g'' : \mathcal{L}'' \to \mathcal{L} \rangle$ of $\mathcal{L}'$ and $\mathcal{L}''$ by a bridge $\beta$ is sound, provided that $\mathcal{L}'$ and $\mathcal{L}''$ are sound, $id_{\Sigma'} \in S'$ and $id_{\Sigma''} \in S''$.*

*Preservation of completeness*

Herein we revisit completeness with the objective of obtaining preservation results. We restrict our results to strong global completeness. For local completeness see [Sernadas *et al.*, 2002b]. The first main result is Theorem 80 which establishes a sufficient condition for global completeness of a modulated logic system. The second main result is Theorem 83 that provides sufficient conditions for preservation of global completeness.

Observe that & is congruent: assume that $\Gamma, \gamma_i \vdash_d \delta_i$, for $i$ equal to 1 or 2. Note that $\Gamma, (\gamma_1 \& \gamma_2) \vdash_\ell \gamma_i$ with $i$ equal to 1 or 2. Hence $\Gamma, (\gamma_1 \& \gamma_2) \vdash_\ell \delta_i$ for $i$ equal to 1 or 2 and so $\Gamma, (\gamma_1 \& \gamma_2) \vdash_\ell (\delta_1 \& \delta_2)$.

Another restriction is to be assumed: we will work with modulated logic systems that have a special *verum* constructor $\mathbf{t}$ of 0-arity.

Completeness is obtained based on the fact that for each globally closed set of formulas $\Gamma$, we have a model whose underlying structure is the Lindenbaum-Tarski algebra for $\Gamma$. See [Sernadas *et al.*, 2002b] to check the details of the definition of Lindenbaum-Tarski algebra for a set over a modulated deductive system. As shown in that work the Lindenbaum-Tarski algebra validates the rules in the modulated deductive system at hand. Recall the notions of congruence and with *verum* in Definition 36. In the context of modulated logic systems, fullness has a slightly different formulation.

DEFINITION 79. A modulated logic system $\mathcal{L}$ with congruence and *verum* is *full* if, for every set of formulas $\Gamma$ globally closed, there is a model $m_\Gamma$ such that $A(m_\Gamma)$ is isomorphic to the Lindenbaum-Tarski algebra for $\Gamma$.

It is possible to enrich the class of models of a modulated interpretation system with one extra model corresponding to the Lindenbaum-Tarski algebra for $\Gamma$ for each globally closed set $\Gamma$. Now we can state the main result of this section.

THEOREM 80. *Every full logic system $\mathcal{L}$ with congruence and* verum *is global strong complete.*

The main goal is to establish preservation of strong global completeness by modulated fibring under reasonable conditions. According to Theorem 80 we can conclude that a modulated logic system is complete provided that it is full and with congruence and *verum*. Therefore we prove that congruence and *verum* are preserved by modulated fibring. Moreover we also prove that fullness is preserved by modulated fibring provided that the bridge has additional properties.

THEOREM 81. *The modulated fibring $\langle g' : \mathcal{L}' \to \mathcal{L}, g'' : \mathcal{L}'' \to \mathcal{L} \rangle$ of logic systems $\mathcal{L}'$ and $\mathcal{L}''$ with congruence and* verum *by a bridge $\beta$ is with congruence and* verum.

Observe that the more complex notion of modulated deductive system morphism was essential for the preservation of congruence without the requirement of sharing implication and equivalence (as in Section 2 leading to the unwanted collapse). For the preservation of fullness by modulated fibring we need further constraints on the bridge.

DEFINITION 82. A bridge $\langle f' : \check{\mathcal{L}} \to \mathcal{L}', f'' : \check{\mathcal{L}} \to \mathcal{L}'' \rangle$ is *adequate* whenever $\mathcal{L}'$, $\mathcal{L}''$, $\check{\mathcal{L}}$ are full, with congruence and *verum* and $\underline{f}'(m'_{\Gamma'}) = m_{\check{f}(\Gamma')}$ and $\underline{f}''(m''_{\Gamma''}) = m_{\check{f}(\Gamma'')}$ for every globally closed sets of ground formulas $\Gamma'$ and $\Gamma''$.

So by using Theorem 80 we can conclude that the modulated fibring of full logic systems with congruence and *verum* by an adequate bridge is strong global complete. The proof of this result and of the preservation of fullness can be checked in [Sernadas *et al.*, 2002b].

THEOREM 83. *The modulated fibring* $\langle g' : \mathcal{L}' \to \mathcal{L}, g'' : \mathcal{L}'' \to \mathcal{L} \rangle$ *of logic systems* $\mathcal{L}'$ *and* $\mathcal{L}''$ *by an adequate bridge* $\beta$ *is strong global complete.*

EXAMPLE 84. The following modulated fibrings are strong global complete:

- Unconstrained modulated fibring of full logic systems with congruence and *verum* by an adequate bridge. In particular, the unconstrained modulated fibring of full propositional and intuitionistic logics is strong global complete. The same holds for the unconstrained modulated fibring of full propositional and Łukasiewicz logics.

- The modulated fibring of full propositional logic and Gödel logic sharing negation is strong global complete.

- The modulated fibring of full Gödel logic and Łukasiewicz logic sharing conjunction and disjunction is strong global complete. △

## 4 FIBRING OF NON-TRUTH FUNCTIONAL LOGICS

We now present an extension of fibring towards *non-truth-functional logics* (based on [Caleiro *et al.*, 2003a]). In fact, the question is primarily related to the kind of structure that we have used to present the semantics of a logic. Although quite general, there are very interesting logics that fail to have a meaningful semantics presented as in Section 2. Paradigmatic examples of this phenomenon are, for instance, the paraconsistent systems $\mathcal{C}_n$ of da Costa [da Costa, 1963], subsystems of propositional classical logic in which the principle of *Pseudo Scotus* $\gamma, (\neg\gamma) \vdash \delta$ does not hold. It is well known that, in all the $\mathcal{C}_n$ systems, negation cannot be given a truth-functional semantics [Mortensen, 1980; Lewin *et al.*, 1991]. That is, homomorphic interpretation of formulas on a truth-value algebra over the syntactical signature of the logic simply does not work. In order to be able to also deal with such logics, we need to consider a more general notion of semantic interpretation structure and to redefine fibring in a way that is consistent with the previous characterization. The main ingredient will be the use of a suitable auxiliary logic, that we call the meta-logic, where the (possibly) non-truth-functional valuations are specified.

## *4.1 Deductive systems*

Deductively, we adopt without any changes the basic setting introduced in Section 2. We shall just present our running example for this section. In [da Costa and Carnielli, 1988], a paraconsistent deontic logic called $\mathcal{C}_1^{\mathrm{D}}$ is introduced including the paraconsistent system $\mathcal{C}_1$ and the modal system D (interpreting the modal operator $\Box$ as "obligatory"). Our aim will be to analyze $\mathcal{C}_1^{\mathrm{D}}$ at the light of fibring the paraconsistent logic $\mathcal{C}_1$ of da Costa [da Costa, 1963] with the deontic modal logic D.

EXAMPLE 85. (*Deontic paraconsistent logic*) We consider the deductive systems $\mathcal{D}^{\mathrm{D}}$ and $\mathcal{D}^{\mathcal{C}_1}$ of modal logic D and of paraconsistent logic, respectively, as defined in Example 5. We shall consider their fibring by sharing the propositional symbols, conjunction, disjunction and implication. For that purpose we shall rename their negation connectives to $\neg_{\mathrm{D}}$ and $\neg_{\mathcal{C}_1}$, respectively. The fibred deductive system $\mathcal{D}^{\mathrm{D}}+\mathcal{D}^{\mathcal{C}_1}$ puts together all the local and global rules of each of the calculi, with their corresponding negations now indexed as explained above. In order to get the deontic paraconsistent system $\mathcal{C}_1^D$ of [da Costa and Carnielli, 1988], at the proof-theoretic level, we need to introduce the following axiom schema:

- $\xi_1^\circ \Rightarrow (\Box\xi_1)^\circ$.

This interaction axiom could never be obtained using the basic fibring operation since it makes full use of the mixed language. Just note that $\gamma^\circ$ is now an abbreviation of $(\neg_{\mathcal{C}_1}(\gamma \wedge (\neg_{\mathcal{C}_1} \gamma)))$. $\triangle$

## *4.2 Interpretation system presentations*

As hinted above, the big difference with respect to Section 2 concerns the semantic aspects of logic. Observe that, when setting-up an algebraic semantics for a truth-functional logic, we endow it with models that are algebras (of truth-values) over the signature of the logic and evaluate formulas homomorphically. This approach fails when the logic is not truth-functional. But yet within the spirit of "algebraic semantics", there is a solution: work instead with two-sorted algebras of formulas and truth-values and include the valuation map as an operation between the two sorts. This approach, stemming from [Caleiro *et al.*, 2003a], captures, as a special case, truth-functional logics by imposing the homomorphism conditions on the valuation map, which can be done with equations. Looking at examples of non-truth-functional logics we find that the envisaged requirements on the valuation map could also be imposed by, albeit conditional, equations. Therefore, we are led to the following algebraic notion of possibly non-truth-functional semantics: each model is a two-sorted algebra (of formulas and truth-values) including a valuation operation that satisfies some requirements written in a suitable conditional equational meta-logic. Since it is enough for the present purposes, we choose conditional equational logic (CEQ, [Goguen and Meseguer, 1985; Meseguer, 1998]) as the meta-logic.

DEFINITION 86. Given a signature $C$, the induced two-sorted *meta-signature* $\Sigma(C, \Xi)$, with sort $\phi$ (for formulas) and sort $\tau$ (for truth-values), has the following operations:

- $O_{\epsilon\,\phi} = C_0 \cup \Xi$;
- $O_{\phi^k\,\phi} = C_k$ for $k > 0$;
- $O_{\phi\,\tau} = \{v\}$;
- $O_{\epsilon\,\tau} = \{\top, \bot\}$;
- $O_{\tau\,\tau} = \{-\}$;
- $O_{\tau\tau\,\tau} = \{\sqcap, \sqcup, \sqsupset\}$; and
- $O_{\omega\,s} = \emptyset$ in the other cases.

We shall use $\Sigma(C)$ to denote the subsignature $\Sigma(C, \emptyset)$, that is, where $O_{\epsilon\,\phi} = C_0$. The operations $\top, \bot, -, \sqcap, \sqcup, \sqsupset$ are used as generators of truth-values. The symbol $v$ is interpreted as a valuation map.

We consider the following sets of variables for $\Sigma(C, \Xi)$ and $\Sigma(C)$: $X_\phi = \{y_1, y_2, \dots\}$ and $X_\tau = \{x_1, x_2, \dots\}$. For ease of notation we simply use $X$ to denote the two-sorted family $\{X_\phi, X_\tau\}$. Recall that a term $t$ is called a ground term if it does not contain variables, and that a substitution $\theta$ is said to be ground if it replaces every variable by a ground term.

We want to write valuation specifications (within the adopted meta-logic CEQ) over $\Sigma(C)$ and $X$. Recall that a CEQ-specification is composed of conditional equations of the general form:

$$(\mathrm{eq}_1 \;\&\; \dots \;\&\; \mathrm{eq}_n \;\to\; \mathrm{eq})$$

with $n \geq 0$. Each equation is of the form $t = t'$ where $t, t'$ are terms of the same sort built over $\Sigma(C)$ and $X$. The sort of each equation is defined to be the sort of its terms. A conditional equation that only involves equations of a given sort is said to be a conditional equation of that sort. Conditional equations are universally quantified, although, for the sake of simplicity, we omit the quantifier, contrarily to the notation used in [Meseguer, 1998]. For example, $( \to v(y_1 \wedge y_2) = v(y_1) \sqcap v(y_2))$ is a conditional equation of sort $\tau$, supposing that $\wedge \in C_2^\phi$. In the sequel we shall only need to consider specifications containing exclusively conditional equations (or *meta-axioms*) of sort $\tau$. Such specifications are called $\tau$-specifications in the sequel. The deductive system of CEQ [Meseguer, 1998] is a system for deriving equations from a given specification of conditional equations. It consists of the usual rules for reflexivity, symmetry, transitivity and congruence of equality, plus a form of *Modus Ponens* that allows us to obtain an equation $\mathrm{eq}\,\theta$ from already obtained equations $\mathrm{eq}_1\theta, \ldots, \mathrm{eq}_n\theta$, given a conditional equation $(\mathrm{eq}_1 \;\&\; \ldots \;\&\; \mathrm{eq}_n \;\to\; \mathrm{eq})$ in the specification and a substitution $\theta$. In the sequel, we use $\vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)}$ to denote the corresponding consequence relation.

DEFINITION 87. An *interpretation system presentation* is a pair $\mathcal{S} = \langle C, S\rangle$ where $C$ is a signature and $S$ is a $\tau$-specification over $\Sigma(C)$.

As a minimal requirement regarding the compatibility with our previous notion of interpretation structure, we shall assume that the truth-values constitute a Heyting algebra. We denote by $S^\bullet$ the specification composed of the meta-axioms in $S$ plus equations specifying precisely the class of Heyting algebras.

DEFINITION 88. Given an interpretation system presentation $\mathcal{S}$, the class $Str(\mathcal{S})$ of *interpretation structures* presented by $\mathcal{S}$ is the class of all algebras over $\Sigma(C, \Xi)$ satisfying $S^\bullet$.

Note that, being a two-sorted algebra over $\Sigma(C, \Xi)$, an interpretation structure $\mathcal{B}$ can be seen as a tuple $\langle B_\phi, B_\tau, \nu, v_\mathcal{B}, \top_\mathcal{B}, \bot_\mathcal{B}, -_\mathcal{B}, \sqcap_\mathcal{B}, \sqcup_\mathcal{B}, \sqsupset_\mathcal{B}\rangle$ such that $\langle B_\phi, \nu\rangle$ is a $C$-algebra, $v_\mathcal{B} : B_\phi \to B_\tau$ is a function, and moreover $\langle B_\tau, \top_\mathcal{B}, \bot_\mathcal{B}, -_\mathcal{B}, \sqcap_\mathcal{B}, \sqcup_\mathcal{B}, \sqsupset_\mathcal{B}\rangle$ is a Heyting algebra. In the sequel, we need to refer to the denotation $[\![t]\!]^\rho_\mathcal{B}$ of a meta-term $t$ given an assignment $\rho$ over an interpretation structure $\mathcal{B}$. As expected, an assignment maps each variable to an element in the carrier set of the sort of the variable. In the case of a ground term $t$, as usual, we just write $[\![t]\!]_\mathcal{B}$ for its denotation in $\mathcal{B}$. For the sake of economy of presentation, we introduce the following abbreviations: $x_1 \leq x_2$ for $x_1 \sqcap x_2 = x_1$, and $x_1 \equiv x_2$ for $(x_1 \sqsupset x_2) \sqcap (x_2 \sqsupset x_1)$. The relation symbol $\leq_\mathcal{B}$ denotes the partial order on truth-values in a given interpretation structure $\mathcal{B}$. Of course $B_\tau$ is also a bounded lattice with meet $\sqcap_\mathcal{B}$, join $\sqcup_\mathcal{B}$, top $\top_\mathcal{B}$ and bottom $\bot_\mathcal{B}$ (cf. [Birkhoff, 1967]). As expected, $b_1 \leq_\mathcal{B} b_2$ and $b_1 \equiv_\mathcal{B} b_2$ are abbreviations of $b_1 \sqcap_\mathcal{B} b_2 = b_1$ and $(b_1 \sqsupset_\mathcal{B} b_2) \sqcap_\mathcal{B} (b_2 \sqsupset_\mathcal{B} b_1)$, respectively. It is also well known that the Heyting algebra axioms further imply that $[\![t_1]\!]^\rho_\mathcal{B} \leq_\mathcal{B} [\![t_2]\!]^\rho_\mathcal{B}$ if and only if $[\![t_1 \sqsupset t_2]\!]^\rho_\mathcal{B} = \top_\mathcal{B}$, and $[\![t_1]\!]^\rho_\mathcal{B} = [\![t_2]\!]^\rho_\mathcal{B}$ if and only if $[\![t_1 \equiv t_2]\!]^\rho_\mathcal{B} = \top_\mathcal{B}$.

EXAMPLE 89. Taking into account the signatures introduced in Example 2, we can consider the following interpretation system presentations:

Paraconsistent logic $\mathcal{C}_1$

$\mathcal{S}^{\mathcal{C}_1} = \langle C^{\mathcal{C}_1}, S\rangle$, where $S$ contains one further meta-axiom in order to obtain a specification of the class of all Boolean algebras; e.g.

- $( \ \to -(-x_1) = x_1)$.

plus the valuation axioms

- $( \ \to v(\mathbf{t}) = \top)$;
- $( \ \to v(\mathbf{f}) = \bot)$;
- $( \ \to v(y_1 \wedge y_2) = v(y_1) \sqcap v(y_2))$;
- $( \ \to v(y_1 \vee y_2) = v(y_1) \sqcup v(y_2))$;
- $( \ \to v(y_1 \Rightarrow y_2) = v(y_1) \sqsupset v(y_2))$;
- $( \ \to -v(y_1) \leq v(\neg\, y_1))$;
- $( \ \to v(\neg(\neg\, y_1)) \leq v(y_1))$;
- $( \ \to v(y_1^\circ) \sqcap v(y_1) \sqcap v(\neg\, y_1) = \bot)$;
- $( \ \to v(y_1^\circ) \sqcap v(y_2^\circ) \leq v((y_1 \wedge y_2)^\circ))$;
- $( \ \to v(y_1^\circ) \sqcap v(y_2^\circ) \leq v((y_1 \vee y_2)^\circ))$;
- $( \ \to v(y_1^\circ) \sqcap v(y_2^\circ) \leq v((y_1 \Rightarrow y_2)^\circ))$.

The reader should be warned that we are using Boolean algebras here as a metamathematical environment sufficient to carry out the computations of truth-values for the formulas in $\mathcal{C}_1$. Specifically we are not introducing any unary operator in the Boolean algebras corresponding to paraconsistent negation, but we are computing the values of formulas of the form $(\neg\,\gamma)$ by means of conditional equations in the algebras. In other words, $\neg$ does not correspond to the Boolean algebra complement $-$. It is straightforward to verify that every paraconsistent bivaluation introduced in [da Costa and Alves, 1977] has a counterpart in $Str(\mathcal{S}^{\mathcal{C}_1})$. Furthermore, the additional interpretation structures do not change the semantic entailment (as defined below). Note that it is easy to adapt this example in order to set up the interpretation system presentations for the whole hierarchy $\mathcal{C}_n$ by specifying the paraconsistent $n$-valuations introduced in [Loparić and Alves, 1980].

Modal logic D

$\mathcal{S}^{\mathrm{D}} = \langle C^{\mathrm{D}}, S\rangle$, where $S$ contains a meta-axiom in order to obtain a specification of the class of all Boolean algebras, as above, plus the valuation axioms

- $( \;\to v(\mathbf{t}) = \top)$;
- $( \;\to v(\neg\, y_1) = -v(y_1))$;
- $( \;\to v(y_1 \wedge y_2) = v(y_1) \sqcap v(y_2))$;
- $( \;\to v(y_1 \vee y_2) = v(y_1) \sqcup v(y_2))$;
- $( \;\to v(y_1 \Rightarrow y_2) = v(y_1) \sqsupset v(y_2))$;
- $( \;\to v(\Box \mathbf{t}) = \top)$;
- $( \;\to v(\Box(y_1 \wedge y_2)) = v(\Box y_1) \sqcap v(\Box y_2))$;
- $( \;\to v(\Box y_1) \sqcap v(\Diamond y_1) = v(\Box y_1))$;
- $(v(y_1) = v(y_2) \to v(\Box y_1) = v(\Box y_2))$.

It is straightforward to verify that every Kripke model has a counterpart in $Str(\mathcal{S}^{\mathrm{D}})$: consider the algebra of truth-values given by the power set of the set of worlds. Furthermore, every general model in [Zanardo *et al.*, 2001] also has a counterpart in $Str(\mathcal{S}^{\mathrm{D}})$: take $\langle \mathcal{B}, \nu \rangle$ as the algebra of the truth-values. Again, the extra interpretation structures do not change the semantic entailment. △

After this example, we can now clarify the meaning of non-truth-functional semantics. To be as general as possible we shall not only consider primitive connectives (as given by the object signature) but also derived ones. As usual, a *derived connective* of arity $k$ is a $\lambda$-term $\lambda y_1 \dots y_k \,.\, \delta$, where the variables occurring in the schema formula $\delta$ are taken from $y_1, \dots, y_k$. Of course, if $c \in C_k$ is a primitive connective it can also be considered as the derived connective $\lambda y_1 \dots y_k \,.\, c(y_1, \dots, y_k)$.

DEFINITION 90. A derived connective $\lambda y_1 \dots y_k \,.\, \delta$ is said to be *truth-functional* in a given interpretation system presentation $\mathcal{S}$ if

$$S^{\bullet} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\delta) = t\, \theta^{v(y)}_{x}$$

for some $\tau$-term $t$ written only on the variables $x_1, ..., x_k$, where $\theta^{v(y)}_{x}$ is the substitution such that $\theta^{v(y)}_{x}(x_n) = v(y_n)$ for every $n \geq 1$.

If it is not possible to fulfill the above requirement, the connective is said to be *non-truth-functional* in $\mathcal{S}$.

Showing that a certain connective is non-truth-functional can be a very hard task. In $\mathcal{C}_1$, classical negation $\sim := \lambda y_1 \,.\, ((\neg\, y_1) \wedge y_1^{\circ})$ (take $t$ as $-x_1$) and equivalence $\Leftrightarrow\; := \lambda y_1 y_2 \,.\, (y_1 \Leftrightarrow y_2)$ (take $t$ as $x_1 \equiv x_2$) are both truth-functional. And, of course, so are the primitive conjunction $\lambda y_1 y_2 \,.\, (y_1 \wedge y_2)$, disjunction $\lambda y_1 y_2 \,.\, (y_1 \vee y_2)$, and implication $\lambda y_1 y_2 \,.\, (y_1 \Rightarrow y_2)$. On the other hand, paraconsistent negation $\lambda y_1 \,.\, (\neg\, y_1)$ is known to be non-truth-functional. We refer the reader to [Mortensen, 1980] for a proof of this fact.

In the modal interpretation system presentation D above, all derived connectives are truth-functional, but the modality $\lambda y_1 \,.\, (\Box\, y_1)$ would require in $\Sigma(C)$ the extra generator $D$ in $O_{\tau\,\tau}$ satisfying:

- $( \;\to D(\top) = \top)$;
- $( \;\to D(x_1 \sqcap x_2) = D(x_1) \sqcap D(x_2))$;
- $( \;\to D(x_1) \sqcap -D(-x_1) = D(x_1))$;
- $( \;\to v(\Box\, y_1) = D(v(y_1)))$.

Note that these axioms on $D$ are very closely related to the last four valuation axioms of $\mathcal{S}^{\mathrm{D}}$, which allowed us to specify the intended modal algebras and still avoid the use of $D$. Although such an operation $D$ could be easily defined over the set of truth-values according to the axioms above, our definition does not comply with its inclusion in the signature $\Sigma(C)$.

We are now ready to define the (global and local) semantic entailments.

DEFINITION 91. We say that $\delta$ is *globally entailed* from $\Gamma$ in an interpretation system presentation $\mathcal{S}$, written

$$\Gamma \vDash^{\mathrm{g}}_{\mathcal{S}} \delta$$

when, for every $\mathcal{B} \in Str(\mathcal{S})$, if $v_{\mathcal{B}}([\![\gamma]\!]_{\mathcal{B}}) = \top_{\mathcal{B}}$ for each $\gamma \in \Gamma$ then $v_{\mathcal{B}}([\![\delta]\!]_{\mathcal{B}}) = \top_{\mathcal{B}}$.

DEFINITION 92. We say that $\delta$ is *locally entailed* from $\Gamma$ in an interpretation system presentation $\mathcal{S}$, written

$$\Gamma \vDash^{\ell}_{\mathcal{S}} \delta$$

when, for every $\mathcal{B} \in Str(\mathcal{S})$ and every $b \in \mathcal{B}_{\phi}$, if $v_{\mathcal{B}}(b) \leq_{\mathcal{B}} v_{\mathcal{B}}([\![\gamma]\!]_{\mathcal{B}})$ for each $\gamma \in \Gamma$ then $v_{\mathcal{B}}(b) \leq_{\mathcal{B}} v_{\mathcal{B}}([\![\delta]\!]_{\mathcal{B}})$.

We now need to define the fibring of interpretation system presentations in a way that is consistent with the previous characterization. Here we face the novel problem of defining fibring as an operation on logics endowed with non-truth-functional semantics as defined above. In the fibring, like in the truth-functional case, we still expect to find two-sorted algebras over the new signature whose reducts are models of the logics being fibred. Therefore, when fibring two interpretation system presentations, we expect to put together the signatures and the requirements on the valuation map.

DEFINITION 93. The *fibring of interpretation system presentations* $\mathcal{S}'$ and $\mathcal{S}''$ denoted by

$$\mathcal{S}' + \mathcal{S}''$$

is the interpretation system presentation $\langle C' \cup C'', S' \cup S'' \rangle$.

Given signatures $C$ and $C'$ such that $C \subseteq C'$ and an interpretation structure $\mathcal{B}' = \langle B'_\phi, B'_\tau, \nu', v_\mathcal{B}, \top_\mathcal{B}, \bot_\mathcal{B}, -_\mathcal{B}, \sqcap_\mathcal{B}, \sqcup_\mathcal{B}, \sqsupset_\mathcal{B} \rangle$ over $\Sigma(C', \Xi)$, the reduct of $\mathcal{B}'$ to $\Sigma(C, \Xi)$ is $\mathcal{B}'|_C = \langle B'_\phi, B'_\tau, \nu'|_C, v_\mathcal{B}, \top_\mathcal{B}, \bot_\mathcal{B}, -_\mathcal{B}, \sqcap_\mathcal{B}, \sqcup_\mathcal{B}, \sqsupset_\mathcal{B} \rangle$. The following result confirms the intuitions that guided the definition:

PROPOSITION 94. *Given $\mathcal{S}'$ and $\mathcal{S}''$ as above, $\mathcal{B} \in Str(\mathcal{S}' + \mathcal{S}'')$ if and only if:*

- $\mathcal{B}|_{C'} \in Str(\mathcal{S}')$;
- $\mathcal{B}|_{C''} \in Str(\mathcal{S}'')$.

For illustration, consider the following example of constrained fibring.

EXAMPLE 95. (*Deontic paraconsistent logic*) Let us see if we can recover $\mathcal{C}_1^D$ as a fibring, as in Example 85, but now at the semantic level. As before, we want to share the propositional symbols, conjunction, disjunction and implication, and so we rename the negation connectives of D and $\mathcal{C}_1$ to $\neg_\mathrm{D}$ and $\neg_{\mathcal{C}_1}$, respectively. The fibred interpretation system presentation ends up having two negations: a paraconsistent negation and a classical negation inherited from D. Clearly, the derived (classical) strong negation $\lambda y_1 \,.\, ((\neg_{\mathcal{C}_1} y_1) \wedge y_1^\circ)$ inherited from $\mathcal{C}_1$ collapses into $\neg_\mathrm{D}$. In order to recover $\mathcal{C}_1^D$, we have to add one additional meta-axiom on valuations to the previously obtained fibred interpretation system presentation:

- $( \ \to v(y_1^\circ) \leq v((\Box\, y_1)^\circ))$.

Using the terminology introduced in [Carnielli and Coniglio, 1999], this procedure can be seen as a *splitting* of $\mathcal{C}_1^D$ in the components D and $\mathcal{C}_1$. This idea is also in the spirit of the broad meaning of fibring, as described in [Gabbay, 1999], Chapter 1. △

*Categorial perspective*

We first define the category of interpretation system presentations. Just note that any signature morphism $h : C \to C'$ freely extends to a map from the meta-language over $C$ to the meta-language over $C'$ that we also denote by $h$.

DEFINITION 96. An *interpretation system presentation morphism* $h : \langle C, S \rangle \to \langle C', S' \rangle$ is a signature morphism $h : C \to C'$ such that $h(S) \subseteq S'^{\bullet}$.

Due to this definition, it is straightforward to verify that interpretation system presentation morphisms preserve both local and global entailment.

Interpretation system presentations and their morphisms constitute a category **Isp**. As in previous situations, we can now characterize the fibring of interpretation system presentations as colimits in **Isp**.

PROPOSITION 97. *Let $\mathcal{S}'$ and $\mathcal{S}''$ be interpretation system presentations. Their unconstrained fibring $\mathcal{S}' + \mathcal{S}''$ is a coproduct in the category **Isp**.*

PROPOSITION 98. *Let $f' : \mathcal{S}^0 \to \mathcal{S}'$ and $f'' : \mathcal{S}^0 \to \mathcal{S}''$ be interpretation system presentation morphisms, where $\mathcal{S}^0 = \langle C' \cap C'', \emptyset \rangle$. The constrained fibring of $\mathcal{S}'$ and $\mathcal{S}''$ sharing $\mathcal{S}^0$ is a pushout in the category* ***Isp***.

## 4.3 Preservation results

Besides recovering fibring in this wider context we can also prove that this extended notion of fibring preserves soundness and completeness under reasonable conditions. The completeness transfer result generalizes the ones established before and is obtained using a new adequacy preservation technique exploiting the properties of the meta-logic, in this case CEQ. We should stress that the present approach is not just an adaptation of previous work but it involves the conceptual breakthrough of dropping the widely accepted principle of truth-functionality. We start by introducing logic system presentations with both a deductive component and a (possibly) non-truth-functional semantic component.

DEFINITION 99. A *logic system presentation* is a tuple $\mathcal{L} = \langle C, S, R_\ell, Rg \rangle$ where the pair $\langle C, S \rangle$ constitutes an interpretation system presentation and the triple $\langle C, R_\ell, R_{\mathrm{g}} \rangle$ constitutes a deductive system.

EXAMPLE 100. The logic systems for $\mathcal{C}_1$ and D will be denoted by $\mathcal{L}^{\mathcal{C}_1}$ and $\mathcal{L}^{\mathrm{D}}$, respectively, and their fibring while sharing the propositional symbols, conjunction, disjunction and implication, as in 95, will be denoted by $\mathcal{L}^{\mathcal{C}_1+\mathrm{D}}$. △

Soundness and completeness are also defined as expected. For simplicity, given $\mathcal{L} = \langle C, S, R_\ell, Rg \rangle$, we use $\vDash^d_{\mathcal{L}}$ and $\vdash^d_{\mathcal{L}}$ to denote $\vDash^d_{\mathcal{S}}$ and $\vdash^d_{\mathcal{D}}$, respectively, where $\mathcal{S} = \langle C, S \rangle$ and $\mathcal{D} = \langle C, R_\ell, R_{\mathrm{g}} \rangle$, for $d$ equal to g or $\ell$.

EXAMPLE 101. The logic system presentations $\mathcal{L}^{\mathcal{C}_1}$ and $\mathcal{L}^{\mathrm{D}}$ are sound and complete. △

Can we say the same about their fibring? The answer to this question can be checked at the very end of this section, after we establish the meaningful preservation results.

In order to deal with local reasoning at the meta-level, in the sequel, we shall take advantage of the following two schema variable substitutions:

- $\sigma^{+1}$ such that $\sigma^{+1}(\xi_i) = \xi_{i+1}$ for every $i \geq 1$;
- $\sigma^{-1}$ such that $\sigma^{-1}(\xi_1) = \xi_1$ and $\sigma^{-1}(\xi_i) = \xi_{i-1}$ for every $i \geq 2$.

Note that if $\gamma$ is a schema formula then $\gamma\sigma^{+1}$ is a variant of $\gamma$ where $\xi_1$ does not occur. Furthermore, easily, $\gamma\sigma^{+1}\sigma^{-1} = \gamma$.

*Preservation of soundness*

First we analyze what can be obtained proof-theoretically within CEQ. Given an interpretation system presentation $\mathcal{S} = \langle C, S \rangle$, we adopt the following abbreviations, where $\Gamma \cup \{\delta\} \subseteq sL(C)$:

- $\Gamma \vdash^{\mathrm{g}}_{\mathcal{S}} \delta$ for $S^{\bullet} \cup \{( \to v(\gamma) = \top) : \gamma \in \Gamma\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\delta) = \top$;
- $\Gamma \vdash^{\ell}_{\mathcal{S}} \delta$ for $S^{\bullet} \cup \{( \to v(\xi_1) \leq v(\gamma\sigma^{+1})) : \gamma \in \Gamma\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\xi_1) \leq v(\delta\sigma^{+1})$.

PROPOSITION 102. *Given an interpretation system presentation* $\mathcal{S} = \langle C, S \rangle$ *and* $\Gamma \cup \{\delta\} \subseteq sL(C)$, *we have:*

- $\Gamma \vDash^{\mathrm{g}} \mathcal{S}\delta$ *iff* $\Gamma \vdash^{\mathrm{g}}_{\mathcal{S}} \delta$;
- $\Gamma \vDash^{\ell}_{\mathcal{S}} \delta$ *iff* $\Gamma \vdash^{\ell}_{\mathcal{S}} \delta$.

This is an immediate consequence of the completeness of CEQ. In the local case it is essential to note that, since schema variables cannot occur in $S^{\bullet}$, we can freely change the denotation of schema variables given by an interpretation structure $\mathcal{B} \in Str(\mathcal{S})$ (namely according to $\sigma^{+1}$ or $\sigma^{-1}$) and still obtain an algebra in $Str(\mathcal{S})$. The fact that $\xi_1$ cannot occur in schema formulas instantiated by $\sigma^{+1}$ does the rest.

Soundness preservation follows easily from this result.

THEOREM 103. *Soundness is preserved by fibring.*

*Preservation of completeness*

To achieve completeness preservation results, we again take advantage of the completeness of the meta-logic CEQ, as proved for instance in [Goguen and Meseguer, 1985; Meseguer, 1998], by encoding the relevant part of the deductive system of CEQ in the deductive system of the object logic. For the envisaged encoding we need to assume that the logic system at hand is sufficiently expressive.

DEFINITION 104. A logic system presentation $\mathcal{L} = \langle C, S, R_{\ell}, R_{\mathrm{g}} \rangle$ is said to be *rich* if:

1. there exist derivable constants $\mathbf{t}, \mathbf{f}$ and binary connectives $\wedge, \vee, \Rightarrow$ in $C$;
2. $S^{\bullet} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\mathbf{t}) = \top$;
3. $S^{\bullet} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\mathbf{f}) = \bot$;
4. $S^{\bullet} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(y_1 \wedge y_2) = v(y_1) \sqcap v(y_2)$;

5. $S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(y_1 \vee y_2) = v(y_1) \sqcup v(y_2)$;

6. $S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(y_1 \Rightarrow y_2) = v(y_1) \sqsupset v(y_2)$;

7. $\langle \{\xi_1, \xi_1 \Rightarrow \xi_2\}, \xi_2 \rangle \in R_\ell$.

EXAMPLE 105. Both logic systems $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{D}}$, as well as many other common logics, are rich. In D, $\mathbf{f}$ can be defined as $(\neg\, \mathbf{t})$. △

Within a rich logic system it is possible to translate from the meta-logic level to the object logic level. A ground term of sort $\tau$ over $\Sigma(C, \Xi)$ is mapped to a schema formula in $sL(C)$ according to the following rules:

$v(\gamma)^*$ is $\gamma$;

$\top^*$ is $\mathbf{t}$;

$\bot^*$ is $\mathbf{f}$;

$(-t)^*$ is $(t^* \Rightarrow \mathbf{f})$;

$(t_1 \sqcap t_2)^*$ is $(t_1^* \wedge t_2^*)$;

$(t_1 \sqcup t_2)^*$ is $(t_1^* \vee t_2^*)$;

$(t_1 \sqsupset t_2)^*$ is $(t_1^* \Rightarrow t_2^*)$.

Moreover, a ground $\tau$-equation eq of the form $(t_1 = t_2)$ is translated to $t_1^* \equiv t_2^*$. Finally, if $E$ is a set of ground $\tau$-equations, then $E^*$ denotes the set $\{\mathrm{eq}^* : \mathrm{eq} \in E\}$.

LEMMA 106. *Let $\mathcal{L}$ be a rich logic system presentation and $t$ a ground $\tau$-term over $\Sigma(C, \Xi)$. Then:*

$$S^\bullet \vdash^{CEQ}_{\Sigma(C,\Xi)} v(t^*) = t.$$

In a rich logic system it is also easy to show that if $t_1$ and $t_2$ are ground $\tau$-terms and $\mathcal{B}$ is an interpretation structure, then $[\![t_1]\!]_{\mathcal{B}} \leq_{\mathcal{B}} [\![t_2]\!]_{\mathcal{B}}$ if and only if $v_{\mathcal{B}}([\![t_1^* \Rightarrow t_2^*]\!]_{\mathcal{B}}) = \top_{\mathcal{B}}$, and $[\![t_1]\!]_{\mathcal{B}} = [\![t_2]\!]_{\mathcal{B}}$ if and only if $v_{\mathcal{B}}([\![t_1^* \Leftrightarrow t_2^*]\!]_{\mathcal{B}}) = \top_{\mathcal{B}}$.

Under certain conditions, described below, one can encode the relevant part of the meta-reasoning into the object calculus.

DEFINITION 107. A rich logic system presentation $\mathcal{L}$ is said to be *equationally appropriate* if for every conditional equation $(\mathrm{eq}_1\ \&\ \ldots\ \&\ \mathrm{eq}_n \rightarrow \mathrm{eq})$ in $S^\bullet$ and every ground substitution $\theta$:

$$\{(\mathrm{eq}_1\theta)^*, \ldots, (\mathrm{eq}_n\theta)^*\} \vdash^{\mathrm{g}}_{\mathcal{L}} (\mathrm{eq}\,\theta)^*.$$

Now, the following result follows:

PROPOSITION 108. *Let $\mathcal{L}$ be a rich logic system presentation. Then, $\mathcal{L}$ is complete if and only if it is equationally appropriate.*

The proof of this fact can be found in [Caleiro *et al.*, 2003a]. The equivalence between completeness and equational appropriateness for rich systems will be used below for showing that completeness is preserved by fibring rich systems, but this equivalence may also be useful for establishing the completeness of logics endowed with a semantics presented by conditional equations. Indeed, it is a much easier task to verify equational appropriateness than to establish completeness directly.

Finally, we consider the problem of preservation of completeness by fibring, taking advantage of the technical machinery presented before on the encoding of the meta-logic in the object calculus. The result capitalizes on the following two lemmas.

LEMMA 109. *Richness is preserved by fibring provided that conjunction, disjunction, implication,* verum *and* falsum *are shared.*

LEMMA 110. *Equational appropriateness is preserved by fibring provided that conjunction, disjunction, implication,* verum *and* falsum *are shared.*

THEOREM 111. *Given two rich, sound and complete logic system presentations, their fibring while sharing conjunction, disjunction, implication,* verum *and* falsum *is also sound and complete.*

EXAMPLE 112. By fibring while sharing conjunction, disjunction, implication, *verum*, and consequently *falsum*, the logic system presentations $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{D}}$ we obtain a new modal paraconsistent logic system presentation $\mathcal{L}_{\mathcal{C}_1+\mathrm{D}}$ that is sound and complete. Observe that if we add to $\mathcal{L}_{\mathcal{C}_1+\mathrm{D}}$:

- $( \ \rightarrow v(y_1^\circ) \leq v((\Box\, y_1)^\circ))$ as a valuation axiom; and
- $(\xi_1^\circ \Rightarrow (\Box\, \xi_1)^\circ)$ as a local axiom schema;

we still obtain a sound and complete logic system presentation that is equivalent to the system $\mathcal{C}_1^D$ of [da Costa and Carnielli, 1988] both at the proof-theoretic and the semantic levels. △

This example illustrates the range of applicability of our soundness and completeness preservation results, shown to hold even in the wider context of non-truth functional logics.

## 5 FIBRING FIRST-ORDER BASED LOGICS

Extending the definition of fibring to first-order based logics raises new technical problems at both the semantic and the deductive levels. In this section, based on [Sernadas *et al.*, 2002a], we avoid the categorial machinery to simplify the exposition, which even so is a bit complex. We remark however that categorial intuitions have always been behind all the constructions, and that all the definitions and results can be recast perhaps with a little effort in full-fledged categorial terms.

At the semantic level, the problem is to find a suitable notion of semantic structure encompassing a wide class of logics having as special cases logics as distinct as modal propositional logic and classical quantifier logic. To this end, quantifiers are dealt with as special modalities for which assignments play the role of worlds. As will be seen later on, they are distinguished by the concepts of vertically and horizontally persistent first-order based deductive systems. From the point of view of fibring, it is very natural to look at quantifiers as modalities. At the deductive level, the new problem faced by fibring, with respect to propositional fibring described in Section 2, is the need to deal with side constraints in inference rules.

## *5.1 Deductive systems*

We start by describing the language of first-order based logics. That is, what we accept as being a first-order based signature and how the language is generated by a signature.

We assume given once and for all three denumerable sets: $X$ (the set of (quantification) *variables*), $\Theta$ (the set of *term schema variables*) and $\Xi$ (the set of *formula schema variables*). We also assume as fixed the *equality symbol* $=$ and the *inequality symbol* $\neq$. The schema variables (or meta-variables) will be used for writing for example schematic inference rules, following the approach in the previous chapters.

DEFINITION 113. A *first-order based signature* $\Sigma$ is a tuple $\langle I, F, P, C, Q, O\rangle$ where $I$ is a set (of *individual symbols*), and $F = \{F_k\}_{k\in\mathbb{N}}$, $P = \{P_k\}_{k\in\mathbb{N}}$, $C = \{C_k\}_{k\in\mathbb{N}}$, $Q = \{Q_k\}_{k\in\mathbb{N}^+}$ and $O = \{O_k\}_{k\in\mathbb{N}^+}$ are families of sets (of *function symbols*, *predicate symbols*, *connectives*, *quantifiers* and *modalities*, respectively).

In order to avoid ambiguities, it is assumed that the sets $P_0$, $C_0$ and $\Xi$ are pairwise disjoint, as well as the sets $I$, $F_0$, $X$ and $\Theta$. For the same reason, it is assumed that, for each $k$ in $\mathbb{N}^+$, the sets $C_k$ and $O_k$ are disjoint.

Let $S$ denote the set $\{\tau, \phi\}$, where $\tau$ and $\phi$ are the (meta) sorts of terms and formulas, respectively. Given a first-order based signature $\Sigma$, define the family $G = \{G_{\vec{s}s}\}_{\vec{s}\in S^*, s\in S}$ of sets of *generators* based on the signature and on the meta-variables. Consider the $S$-sorted free algebra induced by $G$. Then we denote by $sT(\Sigma, X, \Theta)$ the carrier of sort $\tau$ and refer to its elements as $\Sigma$-*terms* (or, simply, *terms*), and by $sL(\Sigma, X, \Theta, \Xi)$ the carrier of sort $\phi$ and refer to its elements as $\Sigma$-*formulas* (or, simply, *formulas*). Furthermore, we denote by $T(\Sigma, X)$ and $L(\Sigma, X)$, respectively, the sets of terms and formulas written without schema variables.

A *substitution* $\rho$ over the fob signature $\Sigma$ maps each term schema variable $\theta$ to a term $\theta\rho$ in $T(\Sigma, X)$ and each formula schema variable $\xi$ to a formula $\xi\rho$ in $L(\Sigma, X)$. A *schema substitution* $\sigma$ over the first-order based signature $\Sigma$ maps each term schema variable $\theta$ to a schema term $\theta\sigma$ in $sT(\Sigma, X, \Theta)$ and each formula schema variable $\xi$ to a schema formula $\xi\sigma$ in $sL(\Sigma, X, \Theta, \Xi)$.

Similarly to the previous sections we present the deductive component as a Hilbert-style system. However, the problem is now much more complex because rules in first-order based logics frequently have side constraints like "provided that a term is free for a variable in a formula". Such constraints correspond to the following abstractions (adapted from [Sernadas *et al.*, 2000]):

DEFINITION 114. A *proviso over a first-order based signature* $\Sigma$ is a map from the set of substitutions to $\{0, 1\}$. A *proviso* $\pi$ is a family $\{\pi_\Sigma\}_{\Sigma \in \text{fobSig}}$, where fobSig is the class of all first-order based signatures and each $\pi_\Sigma$ is a proviso over the first-order based signature $\Sigma$, such that $\pi_{\Sigma'}(\rho) = \pi_\Sigma(\rho)$ for every substitution $\rho$ over $\Sigma$ whenever $\Sigma' \supseteq \Sigma$.

Intuitively, we have $\pi_\Sigma(\rho) = 1$ iff the substitution $\rho$ over $\Sigma$ is allowed. The *unit proviso* $\mathbf{1}$ maps at each signature $\Sigma$ every substitution over $\Sigma$ to 1. And the *zero proviso* $\mathbf{0}$ maps at each signature $\Sigma$ every substitution over $\Sigma$ to 0. Given a proviso $\pi$ we say that $\pi_\Sigma$ is the $\Sigma$-*instance* of $\pi$. When no confusion arises we write $\pi(\rho)$ for $\pi_\Sigma(\rho)$.

DEFINITION 115. A *rule* over the first-order based signature $\Sigma$ is a triple $\langle \Psi, \eta, \pi \rangle$ where $\Psi$ is a finite set of formulas (the set of *premises*), $\eta$ is a formula (the *conclusion*) and $\pi \in \mathit{Prov}$ (the *constraint*).

One can reasonably find strange that, in the previous definition, the last component of a rule over the signature $\Sigma$ is not a proviso over $\Sigma$, but a whole family $\pi$. This fact has technical reasons; namely, we want to be able to consider a rule over $\Sigma$ also as a rule over $\Sigma'$, where $\Sigma'$ is a richer signature. In this case, we need to know how the proviso works on substitutions over $\Sigma'$. It is worth observing that we loose no generality by endowing a rule with just one proviso.

DEFINITION 116. A *first-order based deductive system* $\mathcal{D}$ is a tuple $\langle \Sigma, R_\ell, R_{\text{Qg}}, R_{\text{Og}}, R_{\text{g}} \rangle$ where $\Sigma$ is a first-order based signature, $R_\ell$ is a set of rules over $\Sigma$ (the *local rules*), $R_{\text{Qg}} \supseteq R_\ell$ is a set of rules over $\Sigma$ (the *quantifier global rules*), $R_{\text{Og}} \supseteq R_\ell$ is a set of rules over $\Sigma$ (the *modal global rules*) and $R_{\text{g}} \supseteq R_{\text{Qg}} \cup R_{\text{Og}}$ is a set of rules over $\Sigma$ (the *global rules*).

The distinction between global and local rules (as already discussed in Section 2) is understood in terms of two semantic entailments introduced in Definition 126. The distinction between quantifier and modal global rules will be used only at the proof-theoretic level. We delay its justification until we address later on in this subsection the problem of defining precisely what we mean by a vertically and a horizontally persistent logic. But to this end we need first to introduce the notion of Q-proof and O-proof. In the following, when there is no ambiguity, we will assume that the components of a first-order based deductive system $\mathcal{D}$ are $\Sigma$, $R_\ell$, $R_{\mathrm{Qg}}$, $R_{\mathrm{Og}}$ and $R_{\mathrm{g}}$.

Logics are often endowed with uniform deductive calculi in the sense that their rules do not depend on the signature at hand. More precisely:

DEFINITION 117. A first-order based deductive system $\langle \Sigma, R_\ell, R_{\mathrm{Qg}}, R_{\mathrm{Og}}, R_{\mathrm{g}}\rangle$ is said to be *uniform* if for every first-order based signature $\Sigma$ and proof rule $\langle \Psi, \eta, \pi\rangle$, $\pi_\Sigma(\rho) = \pi_\Sigma(\rho')$, where, for each $\theta \in \Theta$ and $\xi \in \Xi$, $\rho'(\theta)$ and $\rho'(\xi)$ are respectively obtained from $\rho(\theta)$ and $\rho(\xi)$ by replacing some occurrences of $i$ by $x$, provided that $x$ is fresh in $\Psi\rho \cup \{\eta\rho\}$.

The uniform condition above is expected in a signature-independent framework since individual symbols belong to the signature. However, it may happen that a logic has some individual symbols that are present in all signatures. Even in this case, the uniform condition imposes that provisos should be blind to them.

Before defining precisely the four notions of inference within the context of a first-order based deductive system, we need to say what we mean by applying a schema substitution to an instance of a proviso. Given a proviso $\pi$ and a schema substitution $\sigma$ over a first-order based signature $\Sigma$, we denote by $\pi_\Sigma\sigma$ the map such that $(\pi_\Sigma\sigma)(\rho) = \pi_\Sigma(\sigma\rho)$. Obviously, $\mathbf{1}_\Sigma\sigma = \mathbf{1}_\Sigma$ and $\mathbf{0}_\Sigma\sigma = \mathbf{0}_\Sigma$. Furthermore, for every substitution $\rho$ over a first-order based signature $\Sigma$ we have that either $\pi_\Sigma\rho = \mathbf{1}_\Sigma$ or $\pi_\Sigma\rho = \mathbf{0}_\Sigma$, depending on whether $\pi_\Sigma(\rho) = 1$ or $\pi_\Sigma(\rho) = 0$, respectively.

DEFINITION 118. Within the context of a first-order based deductive system $\mathcal{D}$:

(i) A *global deduction of* $\varphi$ *from* $\Gamma$ *constrained by* a proviso $\pi$ over $\Sigma$, written $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D}} \varphi : \pi$, is a sequence $\langle \varphi_1, \pi_1\rangle, \ldots, \langle \varphi_n, \pi_n\rangle$ of pairs such that: $\pi \neq \mathbf{0}_\Sigma$, $\varphi$ is $\varphi_n$ and $\pi$ is $\pi_n$, and for each $i = 1, \ldots, n$ either $\varphi_i \in \Gamma$ and $\pi_i = \mathbf{1}_\Sigma$ or there is a rule $\langle \Psi, \eta, \pi'\rangle \in R_{\mathrm{g}}$ and a schema substitution $\sigma$ over $\Sigma$ such that $\varphi_i$ is $\eta\sigma$, $\Psi\sigma = \{\varphi_{j_1}, \ldots, \varphi_{j_k}\} \subseteq \{\varphi_1, \ldots, \varphi_{i-1}\}$ and $\pi_i = \pi_{j_1} * \cdots * \pi_{j_k} * \pi'_\Sigma\sigma$.

(ii) A *quantifier global deduction of* $\varphi$ *from* $\Gamma$ *constrained by* a proviso $\pi$ over $\Sigma$, written $\Gamma \vdash^{\mathrm{Qg}}_{\mathcal{D}} \varphi : \pi$, is a sequence $\langle \varphi_1, \pi_1\rangle, \ldots, \langle \varphi_n, \pi_n\rangle$ of pairs such that: $\pi \neq \mathbf{0}_\Sigma$, $\varphi$ is $\varphi_n$ and $\pi$ is $\pi_n$, and for each $i = 1, \ldots, n$ either $\varphi_i \in \Gamma$ and $\pi_i = \mathbf{1}_\Sigma$ or $\varphi_i$ is globally derived from the empty set constrained by $\pi'$ and $\pi_i = \pi'$ or there is a rule $\langle \Psi, \eta, \pi'\rangle \in \mathrm{R}_{\mathrm{Qg}}$ and a schema substitution $\sigma$ such that $\varphi_i$ is $\eta\sigma$, $\Psi\sigma = \{\varphi_{j_1}, \ldots, \varphi_{j_k}\} \subseteq \{\varphi_1, \ldots, \varphi_{i-1}\}$ and $\pi_i = \pi_{j_1} * \cdots * \pi_{j_k} * \pi'_\Sigma\sigma$. Analogously we define *modal global deduction* and *local deduction*.

We now proceed to identify interesting classes of first-order based deductive systems. When proving the completeness theorem as shown in [Sernadas *et al.*, 2002a] we need to assume that we are working with systems in these classes. This assumption is not too restrictive since first-order based logics tend to be endowed with such systems.

We start by defining vertically and horizontally persistent first-order based deductive systems. These notions show the need for the distinction between quantifier and modal global rules. First we introduce some useful provisos over $\Sigma$. Given a set $\Psi$ of schema formulas over $\Sigma$, $\mathrm{cfo}_\Sigma(\Psi) = \lambda\rho.\ \wedge_{\psi\in\Psi} \mathrm{cfo}_\Sigma(\xi)[\xi/\psi](\rho)$ and $\mathrm{rig}_\Sigma(\Psi) = \lambda\rho.\ \wedge_{\psi\in\Psi} \mathrm{rig}_\Sigma(\xi)[\xi/\psi](\rho)$, where $[\xi/\psi]$ denotes the schema substitution $\Sigma$ that replaces $\xi$ by $\psi$.

DEFINITION 119. A first-order based deductive system $\mathcal{D}$ is *vertically persistent*, VP, if $\Gamma^{\vdash^{\mathrm{g}}}, \Psi \vdash^{\mathrm{Qg}}_{\mathcal{D}} \varphi : \pi{*}\mathrm{cfo}_\Sigma(\Psi)$ whenever $\Gamma^{\vdash^{\mathrm{g}}}, \Psi \vdash^{\ell}_{\mathcal{D}} \varphi : \pi{*}\mathrm{cfo}_\Sigma(\Psi)$, and if $\Gamma^{\vdash^{\mathrm{g}}}, \Psi \vdash^{\mathrm{Og}}_{\mathcal{D}} \varphi : \pi * \mathrm{rig}_\Sigma(\Psi)$ whenever $\Gamma^{\vdash^{\mathrm{g}}}, \Psi \vdash^{\ell}_{\mathcal{D}} \varphi : \pi * \mathrm{rig}_\Sigma(\Psi)$. We say that $\mathcal{D}$ is *persistent* if it is both vertically and horizontally persistent.

Intuitively, in a persistent deductive system, whatever we can Qg-prove from a set of closed first-order formulas, we can also locally prove from the same set; and whatever we can Og-prove from a set of rigid formulas we can locally prove from the same set. That is, quantifier proof rules do not bring anything new from a set of closed first-order formulas and modal proof rules do not bring anything new from a set of rigid formulas.

DEFINITION 120. A first-order based deductive system $\mathcal{D}$ is said to be *congruent* if (i) for every Qg-deductively closed set $\Gamma'$ and Og-deductively closed set $\Gamma''$ holds $\Gamma', \Gamma'', c(\varphi_1, \ldots, \varphi_k) \vdash^{\ell}_{\mathcal{D}} c(\varphi'_1, \ldots, \varphi'_k) : \pi$ whenever $\Gamma', \Gamma'', \varphi_i \vdash^{\ell}_{\mathcal{D}} \varphi'_i : \pi$ and $\Gamma', \Gamma'', \varphi'_i \vdash^{\ell}_{\mathcal{D}} \varphi_i : \pi$ for $i = 1, \ldots, k$; (ii) for every Qg-deductively closed set $\Gamma$ holds $\Gamma, qx\,(\varphi_1, \ldots, \varphi_k) \vdash^{\ell}_{\mathcal{D}} qx\,(\varphi'_1, \ldots, \varphi'_k) : \pi$ if $\Gamma, \varphi_i \vdash^{\ell}_{\mathcal{D}} \varphi'_i : \pi$ and $\Gamma, \varphi'_i \vdash^{\ell}_{\mathcal{D}} \varphi_i : \pi$ for $i = 1, \ldots, k$; and (iii) for every Og-deductively closed set $\Gamma$ holds $\Gamma, o(\varphi_1, \ldots, \varphi_k) \vdash^{\ell}_{\mathcal{D}} o\,(\varphi'_1, \ldots, \varphi'_k) : \pi$ whenever $\Gamma, \varphi_i \vdash^{\ell}_{\mathcal{D}} \varphi'_i : \pi$ and $\Gamma, \varphi'_i \vdash^{\ell}_{\mathcal{D}} \varphi_i : \pi$ for $i = 1, \ldots, k$.

It is easy to understand why the set is required to be Qg-deductively closed or Og-deductively closed. Observe that, in first-order logic, for $\Gamma = \{\varphi, \psi\}$ we have $\Gamma, \varphi \vdash^{\ell} \psi$ and $\Gamma, \psi \vdash^{\ell} \varphi$, but in general we do not have $\Gamma, \forall x\, \varphi \vdash^{\ell} \forall x\, \psi$. And, in modal logic, for $\Gamma = \{\varphi, \psi\}$ we have $\Gamma, \varphi \vdash^{\ell} \psi$ and $\Gamma, \psi \vdash^{\ell} \varphi$, but in general we do not have $\Gamma, \Box\, \varphi \vdash^{\ell} \Box\, \psi$.

DEFINITION 121. A first-order based deductive system $\mathcal{D}$ is said to be *for equality* if for every set $\Gamma$ of formulas, formula $\varphi$ and terms $t, t_1, t'_1, \ldots, t_k, t'_k$: 1. $\vdash^{\ell}_{\mathcal{D}} t = t$; 2. $t_1 = t_2 \vdash^{\ell}_{\mathcal{D}} t_2 = t_1$; 3. $t_1 = t_2, t_2 = t_3 \vdash^{\ell}_{\mathcal{D}} t_1 = t_3$; 4.(i) $\Gamma \vdash^{\ell}_{\mathcal{D}} f(t_1, \ldots, t_k) = f(t'_1, \ldots, t'_k) : \pi$ whenever $\Gamma \vdash^{\ell}_{\mathcal{D}} t_i = t'_i : \pi$ for $i = 1, \ldots, k$; 4.(ii) $\Gamma, p(t_1, \ldots, t_k) \vdash^{\ell}_{\mathcal{D}} p(t'_1, \ldots, t'_k) : \pi$ whenever $\Gamma \vdash^{\ell}_{\mathcal{D}} t_i = t'_i : \pi$ for $i = 1, \ldots, k$; 5. $\Gamma \vdash^{\ell}_{\mathcal{D}} \varphi : \pi$ whenever $\Gamma, t = i \vdash^{\ell}_{\mathcal{D}} \varphi : \pi$, where $i$ does not occur in the rules of $\mathcal{D}$ and $\pi(\rho) = 0$ whenever $i$ occurs in $\Gamma\rho$ or in $\varphi\rho$.

Clauses 1-4 impose that equality is a congruence relation. Clause 5 expresses a well known derived rule in ordinary first-order logic with equality that is reasonable to assume of any first-order based logic for equality.

DEFINITION 122. A first-order based deductive system $\mathcal{D}$ for equality is said to be *for inequality* if for every formula $\varphi$, terms $t_1$ and $t_2$, and set $\Gamma$ of formulas, 1. $\Gamma \vdash^{\ell}_{\mathcal{D}} \varphi : \pi$ whenever $\Gamma \vdash^{\ell}_{\mathcal{D}} t_1 = t_2 : \pi$ and $\Gamma \vdash^{\ell}_{\mathcal{D}} t_1 \neq t_2 : \pi$; and 2. $\Gamma \vdash^{\ell}_{\mathcal{D}} \varphi : \pi$ whenever $\Gamma, t_1 = t_2 \vdash^{\ell}_{\mathcal{D}} \varphi : \pi$ and $\Gamma, t_1 \neq t_2 \vdash^{\ell}_{\mathcal{D}} \varphi : \pi$.

Clauses 1 and 2 relate inequality with equality as expected when nothing is assumed about the available connectives.

## *5.2 Interpretation systems*

In this case, instead of working with abstract algebras of truth values we only work with algebras of sets. This seems to be the right abstraction for dealing with the quantifiers and modalities. We also look at quantifiers as modal operators where assignments play the role of worlds.

Thus the semantics of quantification is established by looking at different points sharing the same world (by varying the assignment). Vice-versa, the semantics of modalities is obtained by looking at different points sharing the same assignment (by varying the world).

The value of a variable should depend only on the choice of the assignment. Thus we must have a fixed universe of individuals across the different worlds. But, we may still vary the scope of quantification from one world to another, since we do not assume that the set of assignments at a given world is composed of all functions from variables to individuals. Connectives can be expected to be independent of both assignments and worlds. However, we choose to be more general here for technical reasons related to the proof of the completeness theorem (all the details in [Sernadas *et al.*, 2002a]). Finally, function and predicate symbols are by default flexible (they may depend on the world at end). Of course, as usual they are constant (they do not depend on the assignment at hand). It is also convenient to have individual symbols that are both constant (independent of the assignment) and rigid (independent of the world).

DEFINITION 123. A *first-order based structure* over $\Sigma$ is a tuple $\langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [\cdot]\rangle$ with the following components:

- $U$ is a nonempty set (of *points*);
- $\Delta$ is a nonempty set (of *assignments*) and $W$ is a nonempty set (of *worlds*);
- $\alpha : U \to \Delta$ and $\omega : U \to W$;
- $D$ is a nonempty set (of *individuals*);
- $\mathcal{E} \subseteq D^U$ is a set (of *individual concepts*) and $\mathcal{B} \subseteq \wp(U)$ is a set (of *truth values*), such that $U \in \mathcal{B}$;
- the *interpretation map* $[\cdot]$ is a function defined by means of the following clauses 1) to 9), where

$$U_\delta = \{u \in U : \alpha(u) = \delta\}, \quad \mathcal{B}_\delta = \{b \cap U_\delta : b \in \mathcal{B}\},$$
$$U_w = \{u \in U : \omega(u) = w\}, \quad \mathcal{B}_w = \{b \cap U_w : b \in \mathcal{B}\},$$
$$U_{w\delta} = U_w \cap U_\delta, \quad \mathcal{B}_{w\delta} = \{b \cap U_{w\delta} : b \in \mathcal{B}\},$$

1) $[x] = \{[x]_\delta\}_{\delta \in \Delta}$ where $[x]_\delta \in D$ for $x \in X$;

2) $[i] = \{[i]_\delta\}_{\delta \in \Delta}$ where $[i]_\delta \in D$ for $i \in I$, and $[i]_{\alpha(u)} = [i]_{\alpha(u')}$ whenever $u, u' \in U_w$ for some $w \in W$;

3) $[f] = \{[f]_w\}_{w \in W}$ where $[f]_w : D^k \to D$ for $f \in F_k$;

4) $[=] : D^2 \to 2$ is the diagonal relation;

5) $[\neq] : D^2 \to 2$ is the complement of the diagonal relation;

6) $[p] = \{[p]_w\}_{w \in W}$ where $[p]_w : D^k \to 2$ for $p \in P_k$;

7) $[c] = \{[c]_{w\delta}\}_{w \in W,\, \delta \in \Delta}$ where $[c]_{w\delta} : (\mathcal{B}_{w\delta})^k \to \mathcal{B}_{w\delta}$ for $c \in C_k$;

8) $[qx] = \{[qx]_w\}_{w \in W}$ where $[qx]_w : (\mathcal{B}_w)^k \to \mathcal{B}_w$ for $q \in Q_k$ and $x \in X$;

9) $[o] = \{[o]_\delta\}_{\delta \in \Delta}$ where $[o]_\delta : (\mathcal{B}_\delta)^k \to \mathcal{B}_\delta$ for $o \in O_k$.

Finally, the sets $\mathcal{E}$ and $\mathcal{B}$ considered above are assumed to be such that the following derived functions are well defined:

i) $\widehat{x} : \to \mathcal{E}$ by $\widehat{x}(u) = [x]_{\alpha(u)}$; $\quad \widehat{i} : \to \mathcal{E}$ by $\widehat{i}(u) = [i]_{\alpha(u)}$;

ii) $\widehat{f} : \mathcal{E}^k \to \mathcal{E}$ by $\widehat{f}(e_1, \dots, e_k)(u) = [f]_{\omega(u)}(e_1(u), \dots, e_k(u))$;

iii) $\widehat{=} : \mathcal{E}^2 \to \mathcal{B}$ by $\widehat{=}(e_1, e_2)(u) = [=](e_1(u), e_2(u))$;

iv) $\widehat{\neq} : \mathcal{E}^2 \to \mathcal{B}$ by $\widehat{\neq}(e_1, e_2)(u) = [\neq](e_1(u), e_2(u))$;

v) $\widehat{p} : \mathcal{E}^k \to \mathcal{B}$ by $\widehat{p}(e_1, \dots, e_k)(u) = [p]_{\omega(u)}(e_1(u), \dots, e_k(u))$;

vi) $\widehat{c} : \mathcal{B}^k \to \mathcal{B}$ by
$\widehat{c}(b_1, \dots, b_k)(u) = [c]_{\omega(u)\alpha(u)}(b_1 \cap U_{\omega(u)\alpha(u)}, \dots, b_k \cap U_{\omega(u)\alpha(u)})(u)$;

vii) $\widehat{qx} : \mathcal{B}^k \to \mathcal{B}$ by
$\widehat{qx}(b_1, \dots, b_k)(u) = [qx]_{\omega(u)}(b_1 \cap U_{\omega(u)}, \dots, b_k \cap U_{\omega(u)})(u)$;

viii) $\widehat{o} : \mathcal{B}^k \to \mathcal{B}$ by
$\widehat{o}(b_1, \dots, b_k)(u) = [o]_{\alpha(u)}(b_1 \cap U_{\alpha(u)}, \dots, b_k \cap U_{\alpha(u)})(u)$.

Terms are interpreted in $\mathcal{E}$: the denotation of a term may vary with the assignment and the world at hand. Formulas are interpreted in $\mathcal{B}$: the denotation of a formula may also vary with the point at hand. The standard choices for the sets $\mathcal{E}$ and $\mathcal{B}$ are $D^U$ and $\wp U$, respectively.

The interpretation $[x]$ depends only on the assignment at hand. The interpretation $[i]$ also depends only on the assignment, but, furthermore, it must be constant within a given world. Naturally, $[f]$ and $[p]$ depend only on the world at hand. Equality and inequality are given their standard interpretations. On the other hand, one might expect $[c]$ to be invariant since that is the case in the most usual first-order based logic (modal first-order logic). However, we make it dependent on the pair world-assignment for technical reasons. Concerning the interpretation of quantifiers, we made $[qx]$ dependent only on the world at hand having in mind the possibility of different ranges of quantification on different worlds. Finally, the interpretation $[o]$ of a modality $o$ depends only on the assignment at hand.

It is worthwhile to extend these comments to the algebraic operations $\hat{\cdot}$ induced by the interpretation of the symbols. The definition of the functions $\widehat{f}$ and $\widehat{p}$ imply that the truth of formulas depends on the world at hand already at the atomic level (and not only as a consequence of the semantics for the modal operators).

Given a first-order based structure, it is straightforward to define two global and local satisfaction relations.

DEFINITION 124. Given a first-order based structure $s = \langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [\,\cdot\,]\rangle$ over a signature $\Sigma$: 1. $[\![\cdot]\!]_\tau^s : T(\Sigma, X) \to \mathcal{E}$ is inductively defined by $[\![t]\!]_\tau^s = \widehat{t}$, for $t \in X \cup I$ and $[\![f(t_1, \ldots, t_k)]\!]_\tau^s = \widehat{f}([\![t_1]\!]_\tau^s, \ldots, [\![t_k]\!]_\tau^s)$, for $f \in F_k$, $k \geq 0$, 2. $[\![\cdot]\!]_\phi^s : L(\Sigma, X) \to \mathcal{B}$ is inductively defined in the same way as $[\![\cdot]\!]_\tau^s$, using the $\widehat{p}$'s, $\widehat{c}$'s, $\widehat{qx}$'s and $\widehat{o}$'s as well as taking into account $[\![\cdot]\!]_\tau^s$, 3. $s \Vdash^{\mathrm{g}} \gamma$ iff $[\![\gamma]\!]_\phi^s = U$, and 4. for every $u \in U$, $su \Vdash^\ell \gamma$ iff $u \in [\![\gamma]\!]_\phi^s$.

We might look directly at $\Sigma$-structures as models of the first-order based language over $\Sigma$. But we prefer to allow the possibility of working with other kinds of models, as long as a mechanism for extracting a structure from a model is available. The methodological advantage is obvious: we may then use the original models of an already known logic and just show how to get a structure from each of those models.

DEFINITION 125. A *first-order based interpretation system* is a triple $\langle \Sigma, M, A\rangle$ where $M$ is a class (of *models*) and $A$ maps each $m$ in $M$ to a structure over $\Sigma$.

Within the context of an interpretation system, we freely replace $A(m)$ by $m$, writing for instance $[\![\cdot]\!]_\tau^m$ instead of $[\![\cdot]\!]_\tau^{A(m)}$ and $mu \Vdash^{\mathrm{d}} \gamma$ for $A(m)u \Vdash^{\mathrm{d}} \gamma$, for d equal to g or $\ell$.

DEFINITION 126. Given an interpretation system $\mathcal{I}$, we define $\Gamma \vDash_{\mathcal{I}}^{\mathrm{g}} \varphi$ if for every $m \in M$ if $m \Vdash^{\mathrm{g}} \gamma$ for every $\gamma \in \Gamma$ then $m \Vdash^{\mathrm{g}} \varphi$, and we define

$\Gamma \vDash^{\ell}_{\mathcal{I}} \varphi$ if for every $m \in M$ and $u \in U$ at $A(m)$ if $mu \Vdash^{\ell} \gamma$ for every $\gamma \in \Gamma$ then $mu \Vdash^{\ell} \varphi$.

### *5.3 Preservation results*

Herein we sketch how to achieve the preservation of soundness and strong global completeness by fibring. The complete exposition can be consulted in [Sernadas *et al.*, 2002a].

A *first-order based logic system* is a tuple $\langle \Sigma, M, A, R_{\ell}, R_{\mathrm{Qg}}, R_{\mathrm{Og}}, R_{\mathrm{g}} \rangle$ where $\langle \Sigma, R_{\ell}, R_{\mathrm{Qg}}, R_{\mathrm{Og}}, R_{\mathrm{g}} \rangle$ is a first-order based deductive system and $\langle \Sigma, M, A \rangle$ is a first-order based interpretation system. The notions of global completeness and soundness are defined as before but only for non-schematic formulas. In this way, we avoid to deal with provisos at the semantic level.

We now define when a structure is appropriate for a first-order based deductive system over the same signature. Based on this notion, we establish when a deductive system is sound for an interpretation system.

DEFINITION 127. Let $s$ be a first-order based structure over $\Sigma$ and $\mathcal{D}$ be a first-order based deductive system such that, for every substitution $\rho$ and for every $\langle \Psi, \eta, \pi \rangle \in R_{\mathrm{g}}$, $s \Vdash^{\mathrm{g}}_{\Sigma} \eta\rho$ whenever $s \Vdash^{\mathrm{g}}_{\Sigma} \psi\rho$ for every $\psi \in \Psi$ and $\pi(\rho) = 1$, and for every $\langle \Psi, \eta, \pi \rangle \in R_{\ell}$ and $u \in U$, $su \Vdash^{\ell}_{\Sigma} \eta\rho$ whenever $su \Vdash^{\ell}_{\Sigma} \psi\rho$ for every $\psi \in \Psi$ and $\pi(\rho) = 1$. Then, $s$ is said to be *appropriate for*, or simply *for*, the deductive system $\mathcal{D}$. A structure $s$ is said to be *appropriate for*, or simply *for*, the logic system $\mathcal{L}$ when it is *appropriate for* the deductive system of $\mathcal{L}$. If $A(m)$ is appropriate for the first-order based deductive system $\mathcal{D}$ for every model $m$ in the first-order based interpretation system $\mathcal{I}$ then the deductive system $\mathcal{D}$ is said to be *sound* for $\mathcal{I}$.

Before defining the fibring of two logic systems, we adapt the concept of reduct of a structure under an inclusion of signatures introduced in the previous sections to the first-order based case.

DEFINITION 128. Given first-order based signatures $\Sigma \subseteq \Sigma'$ and a first-order based structure $s'$ over $\Sigma'$, the *reduct* of $s'$ to $\Sigma$ is the first-order based structure $s'|_{\Sigma}$ over $\Sigma$ equal to $\langle U', \Delta', W', \alpha', \omega', D', \mathcal{E}', \mathcal{B}', [\cdot]'|_{\Sigma} \rangle$.

We are now in condition to define the fibring of first-order based logic systems.

DEFINITION 129. The *fibring of first-order based logic systems* $\mathcal{L}'$ and $\mathcal{L}''$ denoted by

$$\mathcal{L}' + \mathcal{L}''$$

is the logic system $\langle \Sigma, M, A, R_{\ell}, R_{\mathrm{Qg}}, R_{\mathrm{Og}}, R_{\mathrm{g}} \rangle$ where $\Sigma$ is $\Sigma' \cup \Sigma''$, $M$ is the class of all first-order based structures $s$ over $\Sigma$ such that $s|_{\Sigma'} \in A'(M')$ and $s|_{\Sigma''} \in A''(M'')$, $s$ is appropriate for $\langle \Sigma, R_{\ell}, R_{\mathrm{Qg}}, R_{\mathrm{Og}}, R_{\mathrm{g}} \rangle$, $A(s) = s$ for each $s$ in $M$, and $R_{\ell}$ is ${R_{\ell}}' \cup {R_{\ell}}''$, $R_{\mathrm{Qg}}$ is $R'_{\mathrm{Qg}} \cup R''_{\mathrm{Qg}}$, $R_{\mathrm{Og}}$ is $R'_{\mathrm{Og}} \cup R''_{\mathrm{Og}}$, and $R_{\mathrm{g}}$ is ${R_{\mathrm{g}}}' \cup {R_{\mathrm{g}}}''$.

The logic system for first-order modal logic K described in [Sernadas *et al.*, 2002a] is an example of a logic system that can be obtained via fibring. It is the result of fibring pure first-order logic and modal logic enriched with variables, individual symbols, equality and inequality.

*Preservation of soundness*

Note that first-order based fibring should be defined in the context of logic systems, due to the fact that each structure in the logic system resulting from the fibring should be appropriate for the deductive system. This is a necessary requirement for the preservation of appropriateness and so to the preservation of soundness, as showed in [Sernadas *et al.*, 2002a]. It may happen, in fact, that $s|_{\Sigma'}$ is appropriate for a rule $r'$ in $\mathcal{D}$, but $s$ is not appropriate for $r'$ in the rules of $\mathcal{D}$ or of $\mathcal{D}'$: in the richer language there can be new instances of $r'$. An example is the first-order logic axiom $\xi \Rightarrow \forall x \xi$ ($x$ is not free in $\xi$), which can be falsified if the language contains modalities.

*Preservation of completeness*

Strong global completeness is preserved by fibring under some natural assumptions that are fulfilled in a wide class of logics encompassing the most common first-order based logics. The idea is to show that the logic system resulting from the fibring of two other logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$ is full, congruent, persistent, uniform, and for equality and inequality, whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy those properties.

Fibring two full logic systems is still a full logic system, and the fibring of two uniform logic systems is still a uniform logic system. But in general, however, the same does not happen with congruence, as shown in Section 2.

Recall that the fibring of logic systems with implication is a logic system with implication, provided that implication is shared, and the fibring of logic systems with equivalence is a logic system with equivalence, provided that both implication and equivalence are shared. Moreover it is proved that a logic system with equivalence is congruent. The preservation of persistence is obtained by the same process as well as the properties of a logic system be with equality and inequality.

## 6 LOOKING TOWARDS THE FUTURE

In this guided tour, we presented the topic of combining logics and the issues raised by fibring in a simple yet stimulating context. We gave emphasis to the underlying categorial structures, and established several basic properties of this very rich combination mechanism, stressing the importance of soundness and completeness preservation results.

As already mentioned, fibring can and has been defined and investigated in much more complex situations. Fibring was investigated in the context of labelled natural deduction type deductive systems for propositional based logics [Rasga *et al.*, 2002] and for first-order based logics [Rasga, 2003], in higher-order based logics [Coniglio *et al.*, 2003], and in connection with the theory of institutions with applications to software engineering, cf. [Caleiro *et al.*, 2001; Caleiro *et al.*, 2003b]. In all such areas one can find challenging problems where new and perhaps unexpected techniques can be applied.

Current research is directed at widening the universe where fibring can be defined and at establishing other preservation results, like sufficient conditions for the preservation of interpolation properties [Carnielli and Sernadas, 2004], and a new way of solving the collapsing problem in the global context using a variant called cryptofibring [Caleiro and Ramos, 2004]. In [Caleiro, 2000] the issue of combining logics was also addressed at a more abstract level as an operation on structural consequence systems with structured syntax, from which the relative hardness of the problem of completeness preservation by fibring, when compared to soundness preservation, could be justified. In this setting, cryptofibring identifies collapsing situations as particular cases of non-conservativeness, adding to the abstract characterization put forward in [Sernadas *et al.*, 2002b] in the context of modulated fibring, thus endorsing the initial desideratum of characterizing the fibring of two logics as the smallest logic over the combined language that conservatively extends both of them (cf. [Gabbay, 1999]).

New topics still to be addressed would include fibring of logics endowed with proof-theoretical components of different nature, as for example fibring logics presented through tableau system and sequent calculus. This would require facing the problem of defining convenient meta-theoretical environments able to encompass all of them. It is also envisaged, in the proof-theoretical front, to study the preservation of cut elimination properties by fibring logics endowed with sequent calculi. On the other hand, from the model-theoretical viewpoint, the preservation of other meta-theoretical properties like weak completeness, decidability, or the finite model property are also worth being investigated. The reverse operation of fibring, in the direction of splitting a logic in terms of less complex components, can also be incorporated in the research domain of combining logics (cf. [Carnielli and Coniglio, 1999]).

Fibring can also be useful in areas like security and authentication logics [Caleiro *et al.*, in print], or spatial logics; for example, capitalizing on its preservation results, fibring promises a new look at probabilistic logics [Mateus *et al.*, 2001] and also at quantum logics as in [Mateus and Sernadas, 2004; Mateus and Sernadas, in print].

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## APPENDIX: CONNECTION WITH POINT BASED SEMANTICS

There is a deep conceptual gap between the point based semantics of fibring and the semantics we proposed. We now briefly review this process. Rather than just putting forth the definitions, we chose to give a panoramic view of the difficulties involved and the way they can be solved, building on the intuitions behind D. Gabbay's original idea, and going from the first very concrete definition, based on Kripke-style interpretation structures, to the current much more abstract and general approach, based on ordered algebras.

The relationship between the notion of fibring previously defined and the original idea in [Gabbay, 1996a] certainly deserves a careful analysis. Of course, such a study will have the purpose of providing a historical account to the development of fibring as a methodology for combining logics. But beyond that, we hope that by the end of this explanation the reader will have acquired a much better overall understanding of the problem of combining logics and the technical difficulties involved.

To get into the level of abstraction considered in [Gabbay, 1996a], we need to restrict ourselves to interpretation systems based on Kripke-like structures (K-structures, for short).

DEFINITION 130. A *K-interpretation structure* over $C$ is a pair $\mathcal{W} = \langle W, \nu \rangle$ where $W$ is a non-empty set and $\langle \wp(W), \nu \rangle$ is a $C$-algebra. We denote by $KStr(C)$ the class of all K-interpretation structures over $C$.

The set $W$ of *worlds* induces the space of truth values $\wp(W)$, ordered by inclusion, whose top element is precisely $W$. In this way, we can identify a K-interpretation structure $\mathcal{W}$ as a special way of presenting the interpretation structure $\overline{\mathcal{W}} = \langle \wp(W), \subseteq, \nu, W \rangle$ in the sense of Section 2. To disambiguate, we sometimes denote $\nu$ by $\nu_{\mathcal{W}}$.

DEFINITION 131. A *K-interpretation system* is a pair $\mathcal{KI} = \langle C, \mathcal{K}\rangle$ where $C$ is a signature and $\mathcal{K} \subseteq KStr(C)$.

Of course, a K-interpretation system $\mathcal{KI} = \langle C, \mathcal{K}\rangle$ can also be seen as a special way of presenting the interpretation system $\overline{\mathcal{KI}} = \langle C, \overline{\mathcal{K}}\rangle$. Under this assumption, it is interesting to note that we can recover the usual Kripke-like notions of local and global reasoning for each $\mathcal{KI}$, if we adopt the corresponding general definitions for $\overline{\mathcal{KI}}$ (cf. definitions 23 and 22). Given $\mathcal{W} \in \mathcal{K}$, define the *local satisfaction* relation at $w \in W$ by $\mathcal{W}, w \Vdash_{\mathcal{KI}} \varphi$ if $w \in [\![\varphi]\!]_{\mathcal{W}}$. Analogously, define the *global satisfaction* by $\mathcal{W} \Vdash_{\mathcal{KI}} \varphi$ if $\mathcal{W}, w \Vdash_{\mathcal{KI}} \varphi$ for every $w \in W$. As usual we simply write $\Vdash$ instead of $\Vdash_{\mathcal{KI}}$ when $\mathcal{KI}$ is clear from the context. Then:

- *globally*: $\Psi \vDash^{\mathrm{g}}_{\overline{\mathcal{KI}}} \varphi$ if and only if for every $\mathcal{W} \in \mathcal{K}$, if $\mathcal{W} \Vdash \psi$ for each $\psi \in \Psi$ then $\mathcal{W} \Vdash \varphi$;
- *locally*: $\Psi \vDash^{\ell}_{\overline{\mathcal{KI}}} \varphi$ if and only if for every $\mathcal{W} \in \mathcal{K}$ and $w \in W$, if $\mathcal{W}, w \Vdash \psi$ for each $\psi \in \Psi$ then $\mathcal{W}, w \Vdash \varphi$.

In the sequel, we also use $\vDash^{\mathrm{g}}_{\mathcal{KI}}$ and $\vDash^{\ell}_{\mathcal{KI}}$ to denote $\vDash^{\mathrm{g}}_{\overline{\mathcal{KI}}}$ and $\vDash^{\ell}_{\overline{\mathcal{KI}}}$, respectively.

Gabbay's original idea for the semantics of fibring [Gabbay, 1996a; Gabbay, 1996b; Gabbay, 1999] was based on the notion of *fibring function*, and assumed that both logics had a Kripke-like semantics. In this case, the fibring function $F$ would provide, at any moment, a way to map models and worlds from one logic to the other, and back again. Suppose that $\varphi'$ is a formula and $c'$ a unary constructor of the first logic system, given by $\mathcal{KI}'$, and $c''$ a unary constructor of the second logic, given by $\mathcal{KI}''$. To evaluate $c'(c''(\varphi'))$ in the combined logic we should proceed as follows.

1. Take a model $\mathcal{W}'$ of the first logic.
2. Typically, the satisfaction of $c'(c''(\varphi'))$ at $\mathcal{W}'$ will depend on some condition involving the *unknown* satisfaction of $c''(\varphi')$ at $\mathcal{W}'$.
3. For each world $w' \in W'$, instead of $\mathcal{W}', w' \Vdash' c''(\varphi')$, apply the fibring function $F$ to obtain $F(\mathcal{W}', w') = \langle \mathcal{W}'', w''\rangle$ where $\mathcal{W}''$ is a model of the second logic and $w'' \in W''$, and use $\mathcal{W}'', w'' \Vdash'' c''(\varphi')$.
4. Again, the satisfaction of $c''(\varphi')$ at $\mathcal{W}''$ will depend on some condition involving the *unknown* satisfaction of $\varphi'$ at $\mathcal{W}''$.
5. For each world $u'' \in W''$, instead of $\mathcal{W}'', u'' \Vdash'' \varphi'$, apply the fibring function $F$ to obtain $F(\mathcal{W}'', u'') = \langle \mathcal{U}', u'\rangle$ where $\mathcal{U}'$ is a model of the first logic and $u' \in U'$, and use $\mathcal{U}', u' \Vdash' \varphi'$.

This idea is intuitively appealing. However, it is not obvious how to accommodate this operational view based on the fibring function into a meaningful definition of fibred model. Things get even harder if, as we advocate, we further require fibring to be a universal construction between K-interpretation systems. In that case, how to characterize the resulting system $\mathcal{KI}' + \mathcal{KI}''$? And what is the relevant notion of K-interpretation system morphism?

The first solution to all these questions, proposed in [Sernadas *et al.*, 1999], was based on considering fibred models that could be partitioned, simultaneously, into clouds of disjoint models from each of the logics.

The relevant notion of morphism, described below, captures precisely these requirements and is essentially equivalent to the one presented in [Sernadas *et al.*, 1999]. Its apparent simplicity when compared in the technical detail with the notion introduced in [Sernadas *et al.*, 1999] is the result of the maturation of the essential abstract ideas underlying fibring over the time elapsed thus far.

DEFINITION 132. Assume that $\mathcal{KI}' = \langle C', \mathcal{K}' \rangle$ and $\mathcal{KI} = \langle C, \mathcal{K} \rangle$ are two K-interpretation systems. A *K-interpretation system morphism* $\langle h, g \rangle : \mathcal{KI}' \to \mathcal{KI}$ consists of a signature morphism $h : C' \to C$ and a map $g$ that associates to each $\mathcal{W} \in \mathcal{K}$ a set $g(\mathcal{W}) \subseteq \mathcal{K}'$, such that the following conditions are satisfied, assuming that $g(\mathcal{W}) = \{\mathcal{W}'_i : i \in I\}$:

- $W = \bigcup_{i \in I} W'_i$;
- $W'_{i_1} \cap W'_{i_2} = \emptyset$ if $i_1 \neq i_2$;
- for each $i \in I$, $c' \in C'_k$ and $X_1, \ldots, X_k \in \wp(W)$,

$$\nu_{\mathcal{W}}(h(c'))(X_1, \ldots, X_k) \cap W'_i = \nu_{\mathcal{W}'_i}(c')(X_1 \cap W'_i, \ldots, X_k \cap W'_i).$$

Note that such a notion a morphism indeed forces each model of $\mathcal{KI}$ to be seen as a union of disjoint models of $\mathcal{KI}'$ where the interpretation of each constructor from $C'$ is also preserved.

K-interpretation systems and their morphisms constitute the category **KInt**, with identity and composition as defined in [Sernadas *et al.*, 1999].

As explained before, each K-interpretation system $\mathcal{KI}$ can be seen as a way of presenting the interpretation system $\overline{\mathcal{KI}}$. Yet, it is not so clear at this stage how to understand a K-interpretation system morphism $\langle h, g \rangle : \mathcal{KI}' \to \mathcal{KI}$ as a presentation of some morphism from $\overline{\mathcal{KI}'}$ to $\overline{\mathcal{KI}}$.

The fibring of K-interpretation systems can now be characterized.

DEFINITION 133. The *fibring of K-interpretation systems* $\mathcal{KI}' = \langle C', \mathcal{K}' \rangle$ *and* $\mathcal{KI}'' = \langle C'', \mathcal{K}'' \rangle$ is the interpretation system $\mathcal{KI}' + \mathcal{KI}'' = \langle C' \cup C'', \mathcal{K} \rangle$ where $\mathcal{K}$ is the class of all K-interpretation structures $\mathcal{W}$ over $C' \cup C''$ that can be built from sets of structures $\{\mathcal{W}'_i : i \in I\} \subseteq \mathcal{K}'$ and $\{\mathcal{W}''_j : j \in J\} \subseteq \mathcal{K}''$ satisfying:

- $W = \bigcup_{i \in I} W'_i = \bigcup_{j \in J} W''_j$;
- $W'_{i_1} \cap W'_{i_2} = \emptyset$ if $i_1 \neq i_2$, and $W''_{j_1} \cap W''_{j_2} = \emptyset$ if $j_1 \neq j_2$;
- no proper subsets $I_0 \subset I$ and $J_0 \subset J$ fulfill $\bigcup_{i \in I_0} W'_i = \bigcup_{j \in J_0} W''_j$;
- if $w \in W'_i \cap W''_j$, $c \in C'_k \cup C''_k$ and $X_1, \ldots, X_k \in \wp(W)$ then

$$w \in \nu_{\mathcal{W}'_i}(c)(X_1 \cap W'_i, \ldots, X_k \cap W'_i)$$

if and only if

$$w \in \nu_{\mathcal{W}''_j}(c)(X_1 \cap W''_j, \ldots, X_k \cap W''_j),$$

by defining $\mathcal{W} = \langle W, \nu \rangle$ as follows:

- for each $i \in I$, $c' \in C'_k$ and $X_1, \ldots, X_k \in \wp(W)$,

$$\nu_{\mathcal{W}}(c')(X_1, \ldots, X_k) \cap W'_i = \nu_{\mathcal{W}'_i}(c')(X_1 \cap W'_i, \ldots, X_k \cap W'_i);$$

- for each $j \in J$, $c'' \in C''_k$ and $X_1, \ldots, X_k \in \wp(W)$,

$$\nu_{\mathcal{W}}(c'')(X_1, \ldots, X_k) \cap W''_j = \nu_{\mathcal{W}''_j}(c'')(X_1 \cap W''_j, \ldots, X_k \cap W''_j).$$

As before, the fibring of interpretation systems is defined under the assumption that the common subsignature is shared. In fact, for every shared constructor $c \in C' \cap C''$, the definition above implies that the two clouds of models $\{\mathcal{W}'_i : i \in I\}$ and $\{\mathcal{W}''_j : j \in J\}$ agree on their interpretation. Note that, according to the previously mentioned operational description, we can recognize the fibring function $F$ associated to the fibred model $\mathcal{W}$ as mapping each pair $\langle \mathcal{W}'_i, w \rangle$ such that $w \in W'_i$ to the pair $\langle \mathcal{W}''_j, w \rangle$ where $j$ is the unique element of $J$ such that $w \in W''_j$, and vice-versa.

To make fibring robust with respect to the particular "names" of the worlds in each structure, it is useful to work under the assumption that the K-interpretation systems being combined are *closed under isomorphisms*. Rigorously, given a signature $C$, a K-structure $\mathcal{W} \in KStr(C)$ and a bijection $b : W \to U$, we define the isomorphic copy $b(\mathcal{W})$ of $\mathcal{W}$ to be the K-structure $\mathcal{U}$ over $C$ such that $\mathcal{U} = \langle U, \nu_{\mathcal{U}} \rangle$ with $\nu_{\mathcal{U}}(c)(X_1, \ldots, X_k) = b(\nu_{\mathcal{W}}(c)(b^{-1}(X_1), \ldots, b^{-1}(X_k)))$ for each $c \in C_k$ and $X_1, \ldots, X_k \in \wp(U)$. We say that $\mathcal{KI} = \langle C, \mathcal{K} \rangle$ is *closed for isomorphisms* if for every $\mathcal{W} \in \mathcal{K}$ and bijection $b$ from $W$ it is also the case that $b(\mathcal{W}) \in \mathcal{K}$. It is a trivial fact that closing a given K-interpretation system under isomorphisms has no effect on its entailment operators. Moreover, it is clear that if $\mathcal{KI}'$ and $\mathcal{KI}''$ are closed under isomorphisms then so is $\mathcal{KI}' + \mathcal{KI}''$.

Once again, given a signature morphism $h : C' \to C$ and a K-interpretation structure $\mathcal{W}$ over $C$, we denote by $\mathcal{W}|_h$ the *h-reduct* of $\mathcal{W}$, that is, the K-interpretation structure over $C'$ given by $\langle W, \nu_{\mathcal{W}} \circ h\rangle$. This construction induces a map $\cdot|_h : \mathit{KStr}(C) \to \mathit{KStr}(C')$ that goes along well with our previous definitions. Indeed, one can easily show that $\overline{\mathcal{W}|_h} = \overline{\mathcal{W}}|_h$. As before, in the particular case when $C' \subseteq C$ and we consider the inclusion morphism $i : C' \to C$, we shall also use $\mathcal{W}|_{C'}$ instead of $\mathcal{W}|_i$.

Now, let $\mathcal{KI}' = \langle C', \mathcal{K}'\rangle$ and $\mathcal{KI}'' = \langle C'', \mathcal{K}''\rangle$ be two K-interpretation systems, and consider the trivial injections $\langle i', f'\rangle : \mathcal{KI}^0 \to \mathcal{KI}'$ and $\langle i'', f''\rangle : \mathcal{KI}^0 \to \mathcal{KI}''$ of the canonical K-interpretation system $\mathcal{KI}^0 = \langle C, \mathit{KStr}(C)\rangle$ over their common subsignature, where $f'(\mathcal{W}') = \{\mathcal{W}'|_C\}$ and $f''(\mathcal{W}'') = \{\mathcal{W}''|_C\}$. The fibred K-interpretation system $\mathcal{KI}' + \mathcal{KI}''$ together with the inclusions $\langle j', g'\rangle : \mathcal{KI}' \to \mathcal{KI}' + \mathcal{KI}''$ and $\langle j'', g''\rangle : \mathcal{KI}'' \to \mathcal{KI}' + \mathcal{KI}''$ is a pushout in **KInt** of $\langle i', f'\rangle$ and $\langle i'', f''\rangle$, where for each $\mathcal{W}$ built as above we have $g'(\mathcal{W}) = \{\mathcal{W}'_i : i \in I\}$ and $g''(\mathcal{W}) = \{\mathcal{W}''_j : j \in J\}$. As before, $\mathcal{KI}' + \mathcal{KI}''$ is a coproduct of $\mathcal{KI}'$ and $\mathcal{KI}''$ whenever their common subsignature $C$ is empty ($\langle \emptyset, \mathit{KStr}(\emptyset)\rangle$ is the initial object of **KInt**).

Note that the minimality of the partitions required by the condition that no proper subsets $I_0 \subset I$ and $J_0 \subset J$ fulfill $\bigcup_{i \in I_0} W'_i = \bigcup_{j \in J_0} W''_j$ is essential to guarantee the universal property of the construction. Indeed, a fibred model cannot be decomposable into two fibred models.

This first successful universal characterization of Kripke-like fibred semantics can actually be made much simpler if we just assume that the models of the logics being combined are already *closed for unions*. This simplification was first proposed in [Zanardo *et al.*, 2001], where it was also noted that closing a given K-interpretation system for unions simply does not change its entailment operators. Rigorously, given a signature $C$ and a set $\{\mathcal{W}_i : i \in I\} \subseteq \mathit{KStr}(C)$ of pairwise disjoint K-structures, that is, $W_{i_1} \cap W_{i_2} = \emptyset$ if $i_1 \neq i_2$, we define their union to be the K-structure $\mathcal{W} = \bigcup_{i \in I} \mathcal{W}_i$ over $C$ such that $\mathcal{W} = \langle W, \nu_{\mathcal{W}}\rangle$ with $W = \bigcup_{i \in I} W_i$, and $\nu_{\mathcal{W}}(c)(X_1, \dots, X_k) \cap W_i = \nu_{\mathcal{W}_i}(c)(X_1 \cap W_i, \dots, X_k \cap W_i)$ for each $c \in C_k$ and $X_1, \dots, X_k \in \wp(W)$. We say that $\mathcal{KI} = \langle C, \mathcal{K}\rangle$ is *closed for unions* if for every disjoint set $\{\mathcal{W}_i : i \in I\} \subseteq \mathcal{K}$ it is also the case that $\bigcup_{i \in I} \mathcal{W}_i \in \mathcal{K}$.

DEFINITION 134. Let $\mathcal{KI}' = \langle C', cK'\rangle$ and $\mathcal{KI} = \langle C, \mathcal{K}\rangle$ be two K-interpretation systems. A *simple K-interpretation system morphism* $\langle h, g\rangle : \mathcal{KI}' \to \mathcal{KI}$ consists of a signature morphism $h : C' \to C$ and a map $g : \mathcal{K} \to \mathcal{K}'$, such that the following conditions are satisfied for every $\mathcal{W} \in \mathcal{K}$, assuming that $g(\mathcal{W}) = \mathcal{W}'$:

- $W = W'$;
- for each $c' \in C'_k$ and $X_1, \dots, X_k \in \wp(W)$,

$$\nu_{\mathcal{W}}(h(c'))(X_1, \dots, X_k) = \nu_{\mathcal{W}'}(c')(X_1, \dots, X_k).$$

Note that such a notion a morphism indeed forces each model of $\mathcal{KI}$ to correspond to a model of $\mathcal{KI}'$ where the interpretation of each constructor from $C'$ is preserved.

At the light of simple morphisms and closure for unions it is now easier to understand the difficulty in understanding a K-interpretation system morphism $\langle h, g\rangle : \mathcal{KI}' \to \mathcal{KI}$ as a presentation of some morphism from $\overline{\mathcal{KI}'}$ to $\overline{\mathcal{KI}}$. What we would need would be an operation on interpretation systems that would mimic the closure for unions of K-interpretation systems. More precisely, what we need is to work with interpretation systems that are *closed for products*. Rigorously, given a signature $C$ and a set $\{\mathcal{B}_i : i \in I\} \subseteq Str(C)$ of interpretation structures, we define their product to be the structure $\mathcal{B} = \prod_{i\in I} \mathcal{B}_i$ over $C$ such that $\mathcal{B} = \langle B, \leq, \top, \nu\rangle$ with $B = \prod_{i\in I} B_i$, $\langle x_i\rangle_{i\in I} \leq \langle y_i\rangle_{i\in I}$ if each $x_i \leq_i y_i$, $\top = \langle \top_i\rangle_{i\in I}$, and $\nu(c)(\langle x_{1,i}\rangle_{i\in I}, \ldots, \langle x_{k,i}\rangle_{i\in I}) = \langle \nu_i(c)(x_{1,i}, \ldots, x_{k,i})\rangle_{i\in I}$ for each $c \in C_k$ and $\langle x_{1,i}\rangle_{i\in I}, \ldots, \langle x_{k,i}\rangle_{i\in I} \in \prod_{i\in I} B_i$. We say that an interpretation system $\mathcal{I} = \langle C, \mathcal{A}\rangle$ is *closed for products* if for every set $\{\mathcal{B}_i : i \in I\} \subseteq \mathcal{A}$ it is also the case that $\prod_{i\in I} \mathcal{B}_i \in \mathcal{A}$. Once again, note also that closing a given interpretation system for products does not change its entailment operators.

Notably, given a set $\{\mathcal{W}_i : i \in I\} \subseteq KStr(C)$, it is not difficult to conclude that $\overline{\bigcup_{i\in I} \mathcal{W}_i}$ is isomorphic to $\prod_{i\in I} \overline{\mathcal{W}_i}$. Thus, if $\mathcal{KI}$ is closed for unions then it immediately follows that $\overline{\mathcal{KI}}$ is closed for products. Therefore, we can now understand a simple K-interpretation system morphism $\langle h, g\rangle : \mathcal{KI}' \to \mathcal{KI}$ as a presentation of the morphism $\overline{\langle h, g\rangle} = h : \overline{\mathcal{KI}'} \to \overline{\mathcal{KI}}$. Note that the conditions on a simple morphism require precisely that $g(\mathcal{W}) = \mathcal{W}|_g$ for each model $\mathcal{W}$ of $\mathcal{KI}$. Now, it is a small step to ckeck that $\overline{\mathcal{KI}' + \mathcal{KI}''}$ and $\overline{\mathcal{KI}'} + \overline{\mathcal{KI}''}$ coincide, which justifies our general definition of fibring in the wider setting of interpretation systems.

REMARK 135. K-interpretation systems and their simple morphisms constitute a large subcategory **sKInt** of **KInt**. Furthermore, the functor $\overline{\cdot}$ : **sKInt** $\to$ **Int** transforms simple fibrings into fibrings. $\triangle$

The fibring of K-interpretation systems can now be given a much simpler characterization.

DEFINITION 136. The *simple fibring of K-interpretation systems* $\mathcal{KI}' = \langle C', \mathcal{K}'\rangle$ *and* $\mathcal{KI}'' = \langle C'', \mathcal{K}''\rangle$ is the interpretation system $\mathcal{KI}' \oplus \mathcal{KI}'' = \langle C' \cup C'', \mathcal{K}\rangle$ where $\mathcal{K}$ is the class of all K-interpretation structures $\mathcal{W}$ over $C' \cup C''$ that can be built from interpretations structures $\mathcal{W}' \in \mathcal{K}'$ and $\mathcal{W}'' \in \mathcal{K}''$ satisfying:

- $W = W' = W''$;
- if $c \in C'_k \cup C''_k$ and $X_1, \ldots, X_k \in \wp(W)$ then

$$\nu_{\mathcal{W}'}(c)(X_1, \ldots, X_k) = \nu_{\mathcal{W}''}(c)(X_1, \ldots, X_k),$$

by defining $\mathcal{W} = \langle W, \nu_{\mathcal{W}} \rangle$ as follows:

- for each $c' \in C'_k$ and $X_1, \ldots, X_k \in \wp(W)$,
$$\nu_{\mathcal{W}}(c')(X_1, \ldots, X_k) = \nu_{\mathcal{W}'}(c')(X_1, \ldots, X_k);$$
- for each $c'' \in C''_k$ and $X_1, \ldots, X_k \in \wp(W)$,
$$\nu_{\mathcal{W}}(c'')(X_1, \ldots, X_k) = \nu_{\mathcal{W}''}(c'')(X_1, \ldots, X_k).$$

Given two K-interpretation systems $\mathcal{KI}'$ and $\mathcal{KI}''$ both closed for unions, it can easily be proved that their simple fibring $\mathcal{KI}' \oplus \mathcal{KI}''$ coincides with their fibring $\mathcal{KI}' + \mathcal{KI}''$. In any case, as explained above, the corresponding entailment operators coincide.

Now, let $\mathcal{KI}' = \langle C', \mathcal{K}' \rangle$ and $\mathcal{KI}'' = \langle C'', \mathcal{K}'' \rangle$ be two K-interpretation systems, and consider the trivial injections $\langle i', \cdot|_C \rangle : \mathcal{KI}^0 \to \mathcal{KI}'$ and $\langle i'', \cdot|_C \rangle : \mathcal{KI}^0 \to \mathcal{KI}''$ of the canonical Krike-like interpretation system $\mathcal{KI}^0 = \langle C, KStr(C) \rangle$ over their common subsignature. The simply fibred K-interpretation system $\mathcal{KI}' \oplus \mathcal{KI}''$ together with the inclusions $\langle j', g' \rangle : \mathcal{KI}' \to \mathcal{KI}' \oplus \mathcal{KI}''$ and $\langle j'', g'' \rangle : \mathcal{KI}'' \to \mathcal{KI}' \oplus \mathcal{KI}''$ is a pushout in **sKInt** of $\langle i', \cdot|_C \rangle$ and $\langle i'', \cdot|_C \rangle$, where for each $\mathcal{W}$ built as above we have $g'(\mathcal{W}) = \mathcal{W}'$ and $g''(\mathcal{W}) = \mathcal{W}''$. As before, $\mathcal{KI}' \oplus \mathcal{KI}''$ is a coproduct of $\mathcal{KI}'$ and $\mathcal{KI}''$ whenever their common subsignature $C$ is empty ($\langle \emptyset, KStr(\emptyset) \rangle$ is the initial object of **sKInt**).

This Kripke-like semantic view is still a bit restrictive. In general, there is no reason to suppose that interesting logics should be endowed with K-interpretation structures. Moreover, general completeness results for modal logics are only possible if we consider general Kripke structures, or alternatively, modal algebras. Still, by now, it should be easy to bridge the intuitive gap to the broader algebraic setting initially proposed. A final word is due, however, concerning the fibring function. In the Kripke-like setting developed above, namely now in the simple case, the fibring function $F$ associated to each fibred structure $\mathcal{W}$ built from $\mathcal{W}'$ and $\mathcal{W}''$ is now mapping each pair $\langle \mathcal{W}', w \rangle$ to $\langle \mathcal{W}'', w \rangle$ and back. Indeed, if we take into account that the corresponding space of truth values is $\wp(W)$ in the three cases, the fibring function is also providing a way of identifying the truth-value in $\mathcal{W}''$ of a given formula $\varphi''$ of the second logic with a truth-value in $\mathcal{W}'$. This is precisely the game played, in the general setting, by the fibred interpretation structures.

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## BIBLIOGRAPHY

[Armando, 2002] A. Armando, editor. *Frontiers of combining systems*, volume 2309 of *Lecture Notes in Computer Science*, Berlin, 2002. Springer-Verlag. Lecture Notes in Artificial Intelligence.

[Baader and Schulz, 1996] F. Baader and K. U. Schulz, editors. *Frontiers of combining systems*, volume 3 of *Applied Logic Series*. Kluwer Academic Publishers, Dordrecht, 1996. Papers from the First International Workshop (FroCoS'96) held in Munich, March 26-29, 1996.

[Beckert and Gabbay, 1998] B. Beckert and D. Gabbay. Fibring semantic tableaux. In *Automated reasoning with analytic tableaux and related methods*, volume 1397 of *Lecture Notes in Computer Science*, pages 77–92. Springer Verlag, 1998.

[Birkhoff, 1967] G. Birkhoff. *Lattice Theory*. AMS Colloquium Publications, 1967.

[Blackburn and de Rijke, 1997] P. Blackburn and M. de Rijke. Why combine logics? *Studia Logica*, 59(1):5–27, 1997.

[Caleiro and Ramos, 2004] C. Caleiro and J. Ramos. Cryptofibring. In Carnielli et al. [2004], pages 87–92.

[Caleiro *et al.*, 1999] C. Caleiro, C. Sernadas, and A. Sernadas. Parameterisation of logics. In J. Fiadeiro, editor, *Recent trends in algebraic development techniques - Selected papers*, volume 1589 of *Lecture Notes in Computer Science*, pages 48–62. Springer-Verlag, 1999.

[Caleiro *et al.*, 2001] C. Caleiro, P. Mateus, J. Ramos, and A. Sernadas. Combining logics: Parchments revisited. In M. Cerioli and G. Reggio, editors, *Recent Trends in Algebraic Development Techniques - Selected Papers*, volume 2267 of *Lecture Notes in Computer Science*, pages 48–70. Springer-Verlag, 2001.

[Caleiro *et al.*, 2003a] C. Caleiro, W. A. Carnielli, M. E. Coniglio, A. Sernadas, and C. Sernadas. Fibring non-truth-functional logics: Completeness preservation. *Journal of Logic, Language and Information*, 12(2):183–211, 2003.

[Caleiro *et al.*, 2003b] C. Caleiro, P. Gouveia, and J. Ramos. Completeness results for fibred parchments: Beyond the propositional base. In M. Wirsing, D. Pattinson, and R. Hennicker, editors, *Recent Trends in Algebraic Development Techniques - Selected Papers*, volume 2755 of *Lecture Notes in Computer Science*, pages 185–200. Springer-Verlag, 2003.

[Caleiro *et al.*, in print] C. Caleiro, L. Viganò, and D. Basin. Metareasoning about security protocols using distributed temporal logic. *Electronic Notes in Theoretical Computer Science*, in print. Presented at IJCAR'04 ARSPA Workshop.

[Caleiro, 2000] C. Caleiro. *Combining Logics*. PhD thesis, IST, Universidade Técnica de Lisboa, 2000.

[Carnielli and Coniglio, 1999] W.A. Carnielli and M.E. Coniglio. A categorial approach to the combination of logics. *Manuscrito*, 22(2):69–94, 1999.

[Carnielli and Marcos, 1999] W. A. Carnielli and J. Marcos. Limits for paraconsistency calculi. *Notre Dame Journal of Formal Logic*, 40(3):375–390, 1999.

[Carnielli and Marcos, 2001] W. A. Carnielli and J. Marcos. Tableau systems for logics of formal inconsistency. In H. R. Arabnia, editor, *Proceedings of the International Conference on Artificial Intelligence (IC-AI'2001)*, pages 848–852. CSREA Press, Athens, GA, USA, 2001.

[Carnielli and Marcos, 2002] W. A. Carnielli and J. Marcos. A taxonomy of C-systems. In W. A. Carnielli, M. E. Coniglio, and I. M. L. D' Ottaviano, editors, *Proceedings of the 2nd World Congress on Paraconsistency 2000*, pages 1–94. Marcel Dekker, 2002.

[Carnielli and Sernadas, 2004] W. A. Carnielli and C. Sernadas. Preservation of interpolation features by fibring. Technical report, Center for Logic and Computation, DM, Instituto Superior Técnico, 2004.

[Carnielli *et al.*, 2004] W. A. Carnielli, F. M. Dionísio, and P. Mateus, editors. *Proceedings of CombLog'04 - Workshop on Combination of Logics: Theory and Applications*. IST Press, Lisbon, 2004.

[Carnielli, 1987] W. A. Carnielli. Systematization of finite many-valued logics through the method of tableaux. *Journal of Symbolic Logic*, 52(2):473–493, 1987.

[Coniglio and Carnielli, 2002] M. E. Coniglio and W. A. Carnielli. Transfers between logics and their applications. *Studia Logica*, 72(3):367–400, 2002.

[Coniglio *et al.*, 2003] M. E. Coniglio, A. Sernadas, and C. Sernadas. Fibring logics with topos semantics. *Journal of Logic and Computation*, 13(4):595–624, 2003.

[da Costa and Alves, 1977] N.C.A. da Costa and E. Alves. A semantical analysis of the calculi $\mathcal{C}_n$. *Notre Dame Journal of Formal Logic*, 18(4):621–630, 1977.

[da Costa and Carnielli, 1988] N.C.A. da Costa and W. Carnielli. Paraconsistent deontic logic. *Philosophia - The Philosophical Quarterly of Israel*, 16(3/4):293–305, 1988.

[da Costa, 1963] N.C.A. da Costa. *Sistemas Formais Inconsistentes*. Cathedra Thesis, UFPR. Published by Editora UFPR in 1993, Brazil, 1963.

[de Rijke and Blackburn, 1996] M. de Rijke and P. Blackburn, editors. *Special issue on combining logics*, volume 37(2) of *Notre Dame Journal of Formal Logic*. University of Notre Dame, 1996.

[del Cerro and Herzig, 1996] L. Fariñas del Cerro and A. Herzig. Combining classical and intuitionistic logic. In Baader and Schulz [1996], pages 93–102.

[Finger and Gabbay, 1992] M. Finger and D. Gabbay. Adding a temporal dimension to a logic system. *Journal of Logic, Language and Information*, 1(3):203–233, 1992.

[Finger and Gabbay, 1996] M. Finger and D. Gabbay. Combining temporal logic systems. *Notre Dame Journal of Formal Logic*, 37(2):204–232, 1996.

[Finger and Weiss, 2002] M. Finger and M. A. Weiss. The unrestricted combination of temporal logic systems. *Logic Journal of the IGPL*, 10(2):165–189, 2002.

[Gabbay and de M. Rijke, 2000] D. Gabbay and de M. Rijke, editors. *Frontiers of combining systems. 2*, volume 7 of *Studies in Logic and Computation*. Research Studies Press Ltd., Baldock, 2000. Papers from the 2nd International Workshop (FroCoS'98) held at the University of Amsterdam, Amsterdam, October 2-4, 1998.

[Gabbay and Pirri, 1997] D. Gabbay and F. Pirri, editors. *Special issue on combining logics*, volume 59(1,2) of *Studia Logica*. Kluwer Academic Publishers, 1997.

[Gabbay and Shehtman, 1998] D. Gabbay and V. Shehtman. Products of modal logics. I. *Logic Journal of the IGPL*, 6(1):73–146, 1998.

[Gabbay and Shehtman, 2000] D. Gabbay and V. Shehtman. Products of modal logics. II. Relativised quantifiers in classical logic. *Logic Journal of the IGPL*, 8(2):165–210, 2000.

[Gabbay and Shehtman, 2002] D. Gabbay and V. Shehtman. Products of modal logics. III. Products of modal and temporal logics. *Studia Logica*, 72(2):157–183, 2002.

[Gabbay *et al.*, 2003] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: Theory and applications*. Studies in Logic and the Foundations of Mathematics, 148. Elsevier, 2003.

[Gabbay, 1996a] D. Gabbay. Fibred semantics and the weaving of logics: part 1. *Journal of Symbolic Logic*, 61(4):1057–1120, 1996.

[Gabbay, 1996b] D. Gabbay. An overview of fibred semantics and the combination of logics. In Baader and Schulz [1996], pages 1–55.

[Gabbay, 1999] D. Gabbay. *Fibring logics*. Oxford University Press, 1999.

[Goguen and Meseguer, 1985] J. Goguen and J. Meseguer. Completeness of many-sorted equational logic. *Houston Journal of Mathematics*, 11(3):307–334, 1985.

[Gottwald, 2001] S. Gottwald. *A treatise on many-valued logics*. Research Studies Press, 2001.

[Governatori *et al.*, 2002] G. Governatori, V. Padmanabhan, and A. Sattar. On fibring semantics for BDI logics. In S. Flesca and G. Ianni, editors, *Logics in computer science - JELIA*, volume 2424 of *Lecture Notes in Artificial Intelligence*, pages 198–210. Springer Verlag, 2002.

[Hähnle, 2001] Reiner Hähnle. Advanced many-valued logics. In *Handbook of philosophical logic, Vol. 2*, pages 297–395. Kluwer Academic Publishers, Dordrecht, 2001.

[Hughes and Cresswell, 1996] G. Hughes and M. Cresswell. *A New Introduction to Modal Logic*. Routledge, London, 1996.

[Kirchner and Ringeissen, 2000] H. Kirchner and C. Ringeissen, editors. *Frontiers of combining systems*, volume 1794 of *Lecture Notes in Computer Science*, Berlin, 2000. Springer-Verlag. Lecture Notes in Artificial Intelligence.

[Kracht and Wolter, 1991] M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56(4):1469–1485, 1991.

[Kracht and Wolter, 1997] M. Kracht and F. Wolter. Simulation and transfer results in modal logic – a survey. *Studia Logica*, 59(2):149–177, 1997.

[Kracht, 1999] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[Lewin *et al.*, 1991] R. Lewin, I. Mikenberg, and M. Schwarze. $\mathcal{C}_1$ is not algebraizable. *Notre Dame Journal of Formal Logic*, 32(4):609–611, 1991.

[Loparić and Alves, 1980] A. Loparić and E. Alves. The semantics of the systems $\mathcal{C}_n$ of da Costa. In A. Arruda, N.C.A. da Costa, and A. Sette, editors, *Proceedings of the 3rd Brazilian Conference on Mathematical Logic*, pages 161–172. Sociedade Brasileira de Lógica, 1980.

[Mac Lane, 1998] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1998.

[Marx, 1999] M. Marx. Complexity of products of modal logics. *Journal of Logic and Computation*, 9(2):197–214, 1999.

[Mateus and Sernadas, 2004] P. Mateus and A. Sernadas. Exogenous quantum logic. In Carnielli et al. [2004], pages 141–150.

[Mateus and Sernadas, in print] P. Mateus and A. Sernadas. Reasoning about quantum systems. In J. Alferes and J. Leite, editors, *Logics in Artificial Intelligence, Ninth European Conference, JELIA'04*, Lecture Notes in Artificial Intelligence. Springer-Verlag, in print.

[Mateus *et al.*, 2001] P. Mateus, A. Pacheco, J. Pinto, A. Sernadas, and C. Sernadas. Probabilistic situation calculus. *Annals of Mathematics and Artificial Intelligence*, 32(1/4):393–431, 2001.

[Meseguer, 1998] J. Meseguer. Membership algebra as a logical framework for equational specification. In F. Parisi-Presicce, editor, *Recent Developments in Algebraic Development Techniques - Selected Papers*, volume 1376 of *Lecture Notes in Computer Science*, pages 18–61. Springer-Verlag, 1998.

[Mortensen, 1980] C. Mortensen. Every quotient algebra for $\mathcal{C}_1$ is trivial. *Notre Dame Journal of Formal Logic*, 21:694–700, 1980.

[Rasga *et al.*, 2002] J. Rasga, A. Sernadas, C. Sernadas, and L. Viganò. Fibring labelled deduction systems. *Journal of Logic and Computation*, 12(3):443–473, 2002.

[Rasga, 2003] J. Rasga. *Fibring Labelled First-order Based Logics*. PhD thesis, IST, Universidade Técnica de Lisboa, 2003.

[Sernadas and Sernadas, 2003] A. Sernadas and C. Sernadas. Combining logic systems: Why, how, what for? *CIM Bulletin*, 15:9–14, December 2003.

[Sernadas *et al.*, 1997] A. Sernadas, C. Sernadas, and C. Caleiro. Synchronization of logics. *Studia Logica*, 59(2):217–247, 1997.

[Sernadas *et al.*, 1999] A. Sernadas, C. Sernadas, and C. Caleiro. Fibring of logics as a categorial construction. *Journal of Logic and Computation*, 9(2):149–179, 1999.

[Sernadas *et al.*, 2000] A. Sernadas, C. Sernadas, C. Caleiro, and T. Mossakowski. Categorial fibring of logics with terms and binding operators. In Gabbay and de M. Rijke [2000], pages 295–316.

[Sernadas *et al.*, 2002a] A. Sernadas, C. Sernadas, and A. Zanardo. Fibring modal first-order logics: Completeness preservation. *Logic Journal of the IGPL*, 10(4):413–451, 2002.

[Sernadas *et al.*, 2002b] C. Sernadas, J. Rasga, and W. A. Carnielli. Modulated fibring and the collapsing problem. *Journal of Symbolic Logic*, 67(4):1541–1569, 2002.

[Thomason, 1984] R. H. Thomason. Combinations of tense and modality. In *Handbook of philosophical logic, Vol. II*, volume 165 of *Synthese Library*, pages 135–165. Reidel, Dordrecht, 1984.

[van Danlen, 1986] D. van Danlen. Intuitionistic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Vol. III*, pages 225–339. D. Reidel Publishing Company, 1986.

[Wolter and Zakharyaschev, 2000] F. Wolter and M. Zakharyaschev. Temporalizing description logics. In Gabbay and de M. Rijke [2000], pages 379–401.

[Wolter, 1998] F. Wolter. Fusions of modal logics revisited. In *Advances in modal logic, Vol. 1*, volume 87 of *CSLI Lecture Notes*, pages 361–379. CSLI Publ., 1998.

[Zanardo *et al.*, 2001] A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *Journal of Symbolic Logic*, 66(1):414–439, 2001.