Fibring: Completeness Preservation*

Alberto Zanardo[1] Amilcar Sernadas[2] Cristina Sernadas[2]

[1] Dipartimento di Matematica Pura ed Applicata, University of Padova, Italy

[2] CMA, Departamento de Matemática, IST, Portugal

February 17, 2000

Abstract

A completeness theorem is established for logics with congruence endowed with general semantics (in the style of general frames). As a corollary, completeness is shown to be preserved by fibring logics with congruence provided that congruence is retained in the resulting logic. The class of logics with equivalence is shown to be closed under fibring and to be included in the class of logics with congruence. Thus, completeness is shown to be preserved by fibring logics with equivalence and general semantics. An example is provided showing that completeness is not always preserved by fibring logics endowed with standard (non general) semantics. A categorial characterization of fibring is provided using coproducts and cocartesian liftings.

1 Introduction

Much attention has been recently given to the problems of combining logics and obtaining transference results. Besides leading to very interesting applications whenever it is necessary to work with different logics at the same time, combination of logics is of interest on purely theoretical grounds [BdR97].

*This paper was mainly developed during a three month visit of the first author to the Centro de Matemática Aplicada of the Instituto Superior Técnico of the Universidade Técnica de Lisboa. The visit was entirely supported by FCT.

Among the different techniques for combining logics, fibring [Gab96a, Gab96b, Gab99] deserves close study. When fibring two given logics we produce a logic where we allow the free mixing of the connectives from both logics and we use the proof rules from both logics. In [SSC99] an explicit semantics is provided for fibring. Therein, soundness is shown to be preserved by fibring, but the preservation of completeness is left as an open problem.

Herein, we concentrate on the problem of preservation of completeness. The final result of the paper is a positive answer to this question with reasonable requirements on the two given logics ("full" semantics and availability of "equivalence"). Since, "equivalence" implies "congruence" and the former is preserved by fibring, the preservation of strong completeness is a consequence of the completeness theorem for general semantics: every logic system with "full" semantics and with "congruence" is strongly complete.

In Section 2 we briefly review Hilbert calculi and their fibrings (free and constrained by sharing symbols). In Section 3 we introduce general interpretation systems and their fibrings. The notion of general interpretation systems generalizes the notion of interpretation system as adopted in [SSC99]. This generalization follows the style of general frames for modal logic (see for example [HC96]), herein in a more general context.

In Section 4, we bring together Hilbert calculi and general interpretation systems in order to establish the appropriate notion of general logic system where we can study completeness. We present two examples from modal logic, using the standard (non general) semantics. The first example shows a case of preservation of weak completeness by fibring as a direct corollary of the results in [KW91]. The second example, based upon [Wol96], shows a case where weak completeness is not preserved.

In Section 5, we obtain a (strong) completeness theorem for logic systems with full general semantics and with congruence, using an adapted Henkin construction. As a corollary, completeness is shown to be preserved by fibring logics with congruence provided that congruence is retained in the resulting logic. Unfortunately, congruence is not always preserved by fibring as shown by the counterexample provided at the end of Section 5.

In Section 6, we introduce the notion of logic with equivalence and show that the class of logics with equivalence is closed under fibring and is a subclass of the class of logics with congruence. Thus, completeness is shown to be preserved by fibring logics with equivalence and general semantics. The class of logics with equivalence is a very wide one and includes classical, intuitionistic, minimal and all modal propostional logics, as well as any other extension of basic logic ([SBF]). Along the way, we also establish other preservation results like the preservation of the metatheorem of deduction by fibring.

At both proof-theoretic and model-theoretic levels, we provide a categorial characterization of fibring. The categorial constructions corresponding to fibrings of (general) interpretation systems was much simplified herein compared to [SSC99] thanks to the possibility of working, without any loss of generality, with interpretation systems closed for disjoint unions (of the world spaces). This simplification is clear in the elegant notion of morphism between general interpretation systems. The use of categorial constructions (coproducts and cocartesian liftings) for characterizing mechanisms for combining logics was already advocated in [SSC97a, SSC97b] for a simpler form of combination (synchronization). The reader less inclined to category theory may skip the subsections on categorial constructions without any loss of continuity. Anyway, only elementary notions are borrowed from category theory. They are presented for instance in [BW90].

2 Hilbert calculi

Since we work only with propositional-based logics, the following notion of signature is enough: a *signature* is a family $C = \{C_k\}_{k\in\mathbb{N}}$ where each C_k is a set. The elements of C_k are called *constructors* of arity k.

Given a signature C and a set Ξ (of *schema variables*), we can construct formulae as follows: the set $L(C,\Xi)$ of *schema formulae* is the smallest set which contains $C_0 \cup \Xi$ and is closed under constructors in C, that is, if $c \in C_k$ and $\gamma_1,\ldots,\gamma_k \in L(C,\Xi)$, then $c(\gamma_1,\ldots,\gamma_k) \in L(C,\Xi)$. The elements of $L(C,\emptyset)$ are called *formulae*. The elements of Ξ are the (atomic) schema formulae for which substitution is allowed: a *substitution* is a function $\sigma : \Xi \to L(C,\Xi)$. Given any schema formula γ, the *instance* of γ by the substitution σ is denoted by $\gamma\sigma$ and is the result of simultaneously replacing each $\xi \in \Xi$ in γ by $\sigma(\xi)$. For sets Γ of schema formulae, we will write $\Gamma\sigma$ to denote $\{\gamma\sigma : \gamma \in \Gamma\}$.

In the sequel, the letter C always denotes a given signature and the set Ξ is assumed to be fixed once for all; we assume also that Ξ is disjoint from each C_k. Given two signatures C', C'', we denote by $C' \cap C''$ their *intersection*: for each $k \in \mathbb{N}$, $(C' \cap C'')_k = C'_k \cap C''_k$. Mutatis mutandis, we denote by $C' \cup C''$ their *union*: for each $k \in \mathbb{N}$, $(C' \cup C'')_k = C'_k \cup C''_k$.

Definition 2.1 *A* Hilbert calculus *is a triple $\langle C, P, D \rangle$ in which (1) P is a subset of $\wp_{\text{fin}} L(C, \Xi) \times L(C, \Xi)$, (2) D is a subset of $(\wp_{\text{fin}} L(C, \Xi) \setminus \emptyset) \times L(C, \Xi)$, and (3) $D \subseteq P$.*

Given any $r = \langle \Gamma, \gamma \rangle$ in P, the (finite) set Γ is the set of premises of r and γ is the conclusion; we will often write $r = \langle \mathit{Prem}(r), \mathit{Conc}(r) \rangle$. If $\mathit{Prem}(r) = \emptyset$, then r is said to be an *axiom schema*; otherwise, it is said to be a *proof rule schema*. Each r in D is said to be a *derivation rule schema*. We delay until the example below an explanation of the advantages of distinguishing between proof and derivation rules.

Definition 2.2 *We say that $\delta \in L(C, \Xi)$ is* provable *from $\Gamma \subseteq L(C, \Xi)$ in the Hilbert calculus $H = \langle C, P, D \rangle$ (in symbols, $\Gamma \vdash^p_H \delta$) iff there is a sequence $\gamma_1, \ldots, \gamma_m \in L(C, \Xi)^+$ such that $\gamma_m = \delta$ and, for $i = 1$ to m, either*

(1) $\gamma_i \in \Gamma$, *or*

(2) *there exist a rule $r \in P$ and a substitution σ such that $\mathit{Conc}(r)\sigma = \gamma_i$ and $\mathit{Prem}(r)\sigma \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.*

When $\Gamma = \emptyset$, we say that δ is provable.

Definition 2.3 *We say that $\delta \in L(C, \Xi)$ is* derivable *from $\Gamma \subseteq L(C, \Xi)$ in the Hilbert calculus $H = \langle C, P, D \rangle$ (in symbols, $\Gamma \vdash^d_H \delta$) iff there is a sequence $\gamma_1, \ldots, \gamma_m \in L(C, \Xi)^+$ such that $\gamma_m = \delta$ and, for $i = 1$ to m, either*

(1) $\gamma_i \in \Gamma$, *or*

(2) γ_i *is provable, or*

(3) *there exist a rule $r \in D$ and a substitution σ such that $\mathit{Conc}(r)\sigma = \gamma_i$ and $\mathit{Prem}(r)\sigma \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.*

When no confusion can arise, we write $\vdash^p$ for $\vdash^p_H$ as well as $\vdash^d$ for $\vdash^d_H$. As usual, at the left side of $\vdash^p$ or of $\vdash^d$, the set $\Gamma \cup \{\delta_1, \ldots, \delta_n\}$ is often written as $\Gamma, \delta_1, \ldots, \delta_n$.

A Hilbert calculus $\langle C, P, D\rangle$ induces in a natural way a *provability operator* $(\cdot)^{\vdash^p}$ and a *derivability operator* $(\cdot)^{\vdash^d}$ which are maps from $\wp L(C, \Xi)$ into $\wp L(C, \Xi)$:

$$\Gamma^{\vdash^p} = \{\delta \in L(C, \Xi) : \Gamma \vdash^p \delta\} \tag{2.1}$$

$$\Gamma^{\vdash^d} = \{\delta \in L(C, \Xi) : \Gamma \vdash^d \delta\} \tag{2.2}$$

Clearly, every derivable formula from a set Γ is also provable from Γ. The next example illustrates the need for distinguishing between "proof" and "derivation".

Example 2.4 *In a* modal Hilbert calculi *we have that $P \supseteq P_0 \cup P_\Box$ and $D = D_0$, where, for any complete set $\{\mathsf{ax}_1, \ldots, \mathsf{ax}_n\}$ of schema axioms for propositional logic in the language $L(C \setminus C_0, \Xi)$,*

$$\begin{array}{rcl} P_0 & = & \{\ \langle \emptyset, \mathsf{ax}_1\rangle, \ldots, \langle \emptyset, \mathsf{ax}_n\rangle, \langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\ \}; \\ P_\Box & = & \{\ \langle \emptyset, ((\Box_i(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box_i\xi_1) \Rightarrow (\Box_i\xi_2)))\rangle, \\ & & \langle \{\xi_1\}, (\Box_i\xi_1)\rangle\ \}; \\ D_0 & = & \{\ \langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\ \}. \end{array}$$

Proof rules are to be sound with respect to floating (global) entailment whereas derivation rules are to be sound with respect to contextual (local) entailment (see Section 3).

Both $(\cdot)^{\vdash^p}$ and $(\cdot)^{\vdash^d}$ are closure operators. We say that a set of formulae Γ is p-deductively closed (resp. d-deductively closed) iff $\Gamma = \Gamma^{\vdash^p}$ (resp. $\Gamma = \Gamma^{\vdash^d}$).

Definition 2.5 *The* fibring *of the Hilbert calculi $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ is the Hilbert calculus*

$$\langle C', P', D'\rangle \cup \langle C'', P'', D''\rangle = \langle C' \cup C'', P' \cup P'', D' \cup D''\rangle$$

Whenever $C' \cap C''$ is the empty signature, the fibring is said to be unconstrained. *Otherwise, it is said to be* constrained by sharing the symbols in *$C' \cap C''$.*

Categorial characterization

We now briefly sketch a categorial characterization of both unconstrained and constrained fibring. To this end, we start by introducing the relevant categories (of signatures and of Hilbert calculi).

A *signature morphism* $h : C \to C'$ is a family $\{h_k\}_{k\in\mathbb{N}}$ in which each h_k is a map from C_k into C'_k. That is, signature morphisms translate constructors preserving their arities. Signatures and their morphisms constitute the category *Sig*. This category is finitely cocomplete (in the sense that it has all finite colimits). In particular, the category has coproducts and pushouts. It is worthwhile to describe in some detail these constructions as they are used in the sequel.

The *coproduct* of two signatures C' and C'' is the signature $C' \oplus C''$ endowed with injections $i' : C' \to C' \oplus C''$ and $i'' : C'' \to C' \oplus C''$ such that, for each $k \in \mathbb{N}$,

- $(C' \oplus C'')_k$ is the disjoint union of C'_k and C''_k, and
- i'_k and i''_k are the injections of C'_k and of C''_k into $(C' \oplus C'')_k$, respectively.

In the special case where the two signatures C', C'' are disjoint, we can consider i', i'' to be inclusions and completely disregard them. Therefore, in that special case, we can identify the coproduct with the union of signatures (up to isomorphism).

Assuming that $f' : C \to C'$ and $f'' : C \to C''$ are injective signature morphisms, the *pushout* of f' and f'' is the signature $C' \overset{f'Cf''}{\oplus} C''$ endowed with the morphisms $g' : C' \to C' \overset{f'Cf''}{\oplus} C''$ and $g'' : C'' \to C' \overset{f'Cf''}{\oplus} C''$ such that, for each $k \in \mathbb{N}$,

- $(C' \overset{f'Cf''}{\oplus} C'')_k$ is $C_k \cup i'_k(C'_k \setminus f'_k(C_k)) \cup i''_k(C''_k \setminus f''_k(C_k))$ and
- $g'_k(c') = \begin{cases} i'_k(c') & \text{if } c' \notin f'_k(C_k) \\ f_k^{-1}(c') & otherwise \end{cases}$ and similarly for g''_k.

Note that pushouts exist for any pair f', f'' of diverging morphisms, but we are interested in pushouts only for sharing constructors. To this end, it is sufficient to consider pushouts of injective morphisms as above.

In the special case where the intersection of the two signatures C', C'' is C, we can consider f', f'' to be inclusions and, again, completely disregard them. Therefore, in that special case, we can identify the pushout with the union of signatures (up to isomorphism).

It is well known in elementary category theory that pushouts can be calculated using coproducts and coequalizers. In the case at hand we have: the pushout $C' \overset{f'Cf''}{\oplus} C''$ is the codomain of the coequalizer q of the two signature morphisms $i' \circ f'$ and $i'' \circ f''$.

We now proceed to develop the category of Hilbert calculi.

Definition 2.6 A *Hilbert calculus morphism*

$$h : \langle C, P, D\rangle \to \langle C', P', D'\rangle$$

is a morphism $h : C \to C'$ in *Sig* such that:

- $h(Conc(r))$ is provable from $h(Prem(r))$ for every $r \in P$;
- $h(Conc(r))$ is derivable from $h(Prem(r))$ for every $r \in D$.

It is straightforward to verify (by induction on the length of the proof/derivation) that every Hilbert calculus morphism preserves proofs and derivations:

$$\begin{cases} \text{if } \Gamma \vdash^p \delta \text{ then } h(\Gamma) \vdash'^p h(\delta) \\ \text{if } \Gamma \vdash^d \delta \text{ then } h(\Gamma) \vdash'^d h(\delta) \end{cases} \tag{2.3}$$

Hilbert calculi and their morphisms constitute the category *Hil*. Furthermore, the maps:

- $N(\langle C, P, D\rangle) = C$;
- $N(h : \langle C, P, D\rangle \to \langle C', P', D'\rangle) = h$

constitute the forgetful functor $N :$ *Hil* $\to$ *Sig*. This functor is quite useful for relating Hilbert calculi with their underlying signatures. More interestingly, given a Hilbert calculus over C and signature morphism from C to C', we can build in a canonical way the corresponding Hilbert calculus over C', as follows (using a cocartesian lifting):

Proposition 2.7 For each $\langle C, P, D\rangle$ in *Hil* and each morphism $h : C \to C'$ in *Sig*, the morphism $h : \langle C, P, D\rangle \to \langle C', h(P), h(D)\rangle$ is cocartesian by N for h on $\langle C, P, D\rangle$.

Proof: It is trivial to verify that h is a Hilbert calculus morphism. Furthermore, it is straightforward to verify the universal property of the cocartesian lifting: given a Hilbert calculus morphism $f : \langle C, P, D\rangle \to \langle C'', P'', D''\rangle$ and a signature morphism $g : C' \to C''$ such that $g \circ h = f$, there is a unique Hilbert calculus morphism $j : \langle C', P', D'\rangle \to \langle C'', P'', D''\rangle$ such that $j \circ h = f$. Just take j to be g and check that it is indeed a Hilbert calculus morphism.■

We denote the codomain of this cocartesian morphism by $h(\langle C, P, D\rangle)$. We are ready at last to provide the envisaged categorial characterization: unconstrained fibring is a coproduct and constrained fibring is obtained by cocartesian lifting. As expected, this caracterization is quite simple for Hilbert calculi, but, more significantly, it will be replicated at the semantic level following precisely the same approach.

Proposition 2.8 Let $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ be Hilbert calculi. Then, their *unconstrained fibring* $\langle C', P', D'\rangle \oplus \langle C'', P'', D''\rangle$ is the coproduct

$$\langle C' \oplus C'', i'(P') \cup i''(P''), i'(D') \cup i''(D'')\rangle$$

endowed with the injections i', i''.

Proof: It is trivial to verify that i', i'' are Hilbert calculus morphisms and that the universal property of the coproduct holds. ■

Proposition 2.9 Let $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ be Hilbert calculi and $f' : C \to C', f'' : C \to C''$ be injective signature morphisms. Then, their *constrained fibring by sharing* $\langle C', P', D'\rangle \overset{f'Cf''}{\oplus} \langle C'', P'', D''\rangle$ is

$$q(\langle C', P', D'\rangle \oplus \langle C'', P'', D''\rangle)$$

where q is the coequalizer of $i' \circ f'$ and $i'' \circ f''$.

Proof: It is trivial to verify that q is a Hilbert calculus morphism and that the universal property of the cocartesian lifting holds. ■

Clearly, when C is the intersection of C', C'' and we consider f', f'' to be the inclusions, this construction leads to $\langle C', P', D'\rangle \cup \langle C'', P'', D''\rangle$, as defined in purely set-theoretic terms before.

3 General interpretation systems

Towards presenting the notion of fibring at the model-theoretic level, we introduce first the interpretation structures we need in order to be able to provide a semantics for a wide class of propositional-based logics.

Definition 3.1 *A* C-structure *is a triple* $\langle U, \mathcal{B}, \nu \rangle$ *in which* U *is a non-empty set,* $\mathcal{B}$ *is a non-empty subset of* $\wp U$, *and* $\nu = \{\nu_k\}_{k \in \mathbb{N}}$ *is a family of functions such that*

$$\nu_k : C_k \to [\mathcal{B}^k \to \mathcal{B}]$$

The class of all C*-structures will be denoted by* $Str(C)$*.*

The set U is called the set of *points*. The set $\mathcal{B}$ is called the set of *admissible valuations*. And ν provides the *interpretation* of the constructors in the signature C. The use of a subset of $\wp U$ as the set of admissible valuations has a long tradition which starts with the Henkin general semantics for second order logic and for the theory of types ([Hen50]) and passes through the *general frames* in modal and temporal logics [Ben83, Ben85, HC96, CZ97]. If, in a C-structure $S = \langle U, \mathcal{B}, \nu \rangle$, the set $\mathcal{B}$ is $\wp U$, we say that S is *standard*. Such standard C-structures were adopted in [SSC99]. We need these more general structures in order to obtain a completeness result later on.

Definition 3.2 *A* pre-interpretation system *is a triple* $\langle C, M, A \rangle$ *in which* M *is a class and* A *is a map from* M *into* $Str(C)$*.*

The elements of M in a pre-interpretation system are called *models*, and, for $m \in M$, $A(m)$ will be also written as $\langle U_m, \mathcal{B}_m, \nu_m \rangle$.

Example 3.3 In the Kripke semantics for propositional intuitionistic logic, a model m is tuple $\langle W, R, \mathcal{B}, V \rangle$ in which W is a set, R is a *reflexive* and *transitive* relation on W, $\mathcal{B}$ is the set of all R-closed subsets of W[1], and V is a function from Π into $\mathcal{B}$. Clearly, $U_m = W$ and $\mathcal{B}_m = \mathcal{B}$. Furthermore, ν_m

[1] A set X is R-closed whenever, for all $x, y \in W$, $x \in X$ and xRy imply $y \in X$.

is defined by

$$\begin{array}{lcl}\nu_{m2}(\wedge)(b_1, b_2) & = & b_1 \cap b_2 \\ \nu_{m2}(\vee)(b_1, b_2) & = & b_1 \cup b_2 \\ \nu_{m1}(\neg)(b_1) & = & \cup\, b : b \in \mathcal{B} \text{ and } b \subseteq W \setminus b_1 \\ \nu_{m2}(\Rightarrow)(b_1, b_2) & = & \cup\, b : b \in \mathcal{B} \text{ and } b \subseteq (W \setminus b_1) \cup (b_2)\end{array}$$

where b_1 and b_2 range over $\mathcal{B}$. Since arbitrary unions and finite intersections of R-closed sets are R-closed sets, we have that $\nu_{mk}(c)(b)$ belongs to $\mathcal{B}$ for all $b \in \mathcal{B}^k$. If R is an equivalence relation, then $\mathcal{B}$ turns out to be a Boolean subalgebra of $\wp\mathcal{B}$ and $\langle W, R, \mathcal{B}, V\rangle$ is a model of classical logic.

Two C-structures $\langle U, \mathcal{B}, \nu\rangle$ and $\langle U', \mathcal{B}', \nu'\rangle$ are said to be *isomorphic* (in symbols $\langle U, \mathcal{B}, \nu\rangle \cong \langle U', \mathcal{B}', \nu'\rangle$) iff there is a bijection f from U onto U' such that, for every $k \in \mathbb{N}$, $c \in C_k$, $b' \in \mathcal{B}'^k$, and $u \in U$,[2]

$$\nu'_k(c)(b') = \nu_k(c)(f^{-1}(b')) \tag{3.1}$$

and

$$\mathcal{B} = \{f^{-1}(b') : b' \in \mathcal{B}'\} \tag{3.2}$$

Prop/Definition 3.4 *A pre-interpretation system $\langle C, M, A\rangle$ is an* interpretation system *iff it is closed under isomorphic images and disjoint unions*[3]*; that is,*

- *if $\langle U, \mathcal{B}, \nu\rangle \cong \langle U', \mathcal{B}', \nu'\rangle$ and $\langle U, \mathcal{B}, \nu\rangle = A(m)$ for some $m \in M$, then there exists a $m' \in M$ such that $A(m') = \langle U', \mathcal{B}', \nu'\rangle$, and*

- *if $U_n \cap U_{n'} = \emptyset$ for all $n \neq n'$ in a subset N of M, then there exists an $m \in M$ such that*

$$U_m = \cup_{n \in N} U_n \tag{3.3}$$

$$\mathcal{B}_m = \{b \in \wp U_m : b \cap U_n \in \mathcal{B}_n \textit{ for all } n \in N\} \tag{3.4}$$

[2]If $X = \langle X_1, \ldots, X_k\rangle$ is a k-tuple of sets and the domain of the function g contains $\cup X_i$, then we abbreviate $\langle g(X_1), \ldots, g(X_k)\rangle$ by $g(X)$. Similarly, $X \cap Y^k$ will abbreviate $\langle X_1 \cap Y, \ldots, X_k \cap Y\rangle$.

[3]In [SSC99], interpretations systems were required to be closed only under isomorphic images and included only standard C-structures.

and, for every $k \in \mathbb{N}$ *and* $b \in \mathcal{B}_m^k$,

$$\nu_{mk}(c)(b) = \bigcup_{n \in N} \nu_{nk}(c)(b \cap U_n^k) \tag{3.5}$$

Proof. Equality (3.4) guarantees that $\nu_{mk}(c)$ is defined for every $b \in \mathcal{B}_m^k$ and, since the sets U_n are pairwise disjoint, $\nu_{mk}(c)(b)$ is an element of $\mathcal{B}_m$. ∎

For m and N as in this definition, we will say that m is the *union of the elements* of N and that $\langle U_m, \mathcal{B}_m, \nu_m \rangle$ is the *union of the C-structures* in N.

If $\langle C, M, A \rangle$ is an interpretation system, then M is obviously a proper class. Given any pre-interpretation system $\langle C, M, A \rangle$, we can always obtain the smallest interpretation system $\langle C, M, A \rangle^c$ containing it, by making it closed under isomorphic images and disjoint unions in the obvious way (using (3.3-5) as definitions of U_m, $\mathcal{B}_m$ and ν_{mk}). We shall see below that this *closure* is not different in any essential way from the original pre-interpretation system with respect to the semantics of the logic at hand.

Let $S = \langle U, \mathcal{B}, \nu \rangle$ be any C-structure. A *variable assignment* over S is a map $\alpha : \Xi \to \mathcal{B}$. The *interpretation map* (of $L(C, \Xi)$ in S with the assignment α) is a function

$$[\![\cdot]\!]_\alpha^S : L(C, \Xi) \to \mathcal{B}$$

defined by

$$\begin{aligned} &[\![c]\!]_\alpha^S = \nu_{m0}(c), \\ &[\![\xi]\!]_\alpha^S = \alpha(\xi), \\ &[\![c(\delta_1, \ldots, \delta_k)]\!]_\alpha^S = \nu_{mk}(c)([\![\delta_1]\!]_\alpha^S, \ldots, [\![\delta_k]\!]_\alpha^S). \end{aligned} \tag{3.6}$$

The *contextual satisfaction relation* $\Vdash$ is defined by

$$S\alpha u \Vdash \delta \quad \text{iff} \quad u \in [\![\delta]\!]_\alpha^S \tag{3.7}$$

where δ ranges over $L(C, \Xi)$ and u ranges over U. On the basis of this definition, we can define the *floating satisfaction relation* as follows: $S\alpha \Vdash \delta$ means that $S\alpha u \Vdash \delta$ holds for every $u \in U$, that is, $[\![\delta]\!]_\alpha^S = U$. As usual, we will write $(\cdot) \Vdash \Gamma$ as an abbreviation of: for every $\gamma \in \Gamma$, $(\cdot) \Vdash \gamma$.

The *entailment operators* $(\cdot)^{\models^p_S}$ and $(\cdot)^{\models^d_S}$ (relative to the C-structure S) are functions from $\wp L(C,\Xi)$ into $\wp L(C,\Xi)$.

$$\Gamma^{\models^p_S} \;=\; \{\delta \,:\, \forall\alpha\,(S\alpha \Vdash \Gamma \Rightarrow S\alpha \Vdash \delta)\} \tag{3.8}$$

$$\Gamma^{\models^d_S} \;=\; \{\delta \,:\, \forall\alpha\forall u\,(S\alpha u \Vdash \Gamma \Rightarrow S\alpha u \Vdash \delta)\} \tag{3.9}$$

where the superscripts p and d are meant to remind that the operators $(\cdot)^{\models^p_S}$ and $(\cdot)^{\models^d_S}$ are the semantical counterparts of 'proof' and 'derivation'. Clearly, $\Gamma^{\models^d_S} \subseteq \Gamma^{\models^p_S}$.

If $\langle C, M, A\rangle$ is a pre-interpretation system then we will write $[\![\cdot]\!]^m_\alpha$ instead of $[\![\cdot]\!]^{A(m)}_\alpha$ and similarly for $m\alpha u \Vdash (\cdot)$, $m\alpha \Vdash (\cdot)$, $(\cdot)^{\models^p_m}$, and $(\cdot)^{\models^d_m}$; moreover, variable assignments over $A(m)$ will be referred to as variable assignments over m. The entailment operators for $\langle C, M, A\rangle$ are defined by

$$\Gamma^{\models^p} \;=\; \{\delta \,:\, \forall m{\in}M,\, \forall\alpha\,(m\alpha \Vdash \Gamma \Rightarrow m\alpha \Vdash \delta)\} \tag{3.10}$$

$$\Gamma^{\models^d} \;=\; \{\delta \,:\, \forall m{\in}M,\, \forall\alpha,\, \forall u{\in}U_m\,(m\alpha u \Vdash \Gamma \Rightarrow m\alpha u \Vdash \delta)\} \tag{3.11}$$

Proposition 3.6 below will be frequently used in the paper and shows that the entailment operators of a pre-interpretation system are the same as those of its closure. The proofs of that proposition and of the following auxiliary lemma are straightforward.

Lemma 3.5 *Assume that:* (1) $\langle C, M, A\rangle$ *is an interpretation system,* (2) N *and* m *are as in Definition 3.4,* (3) α *is a variable assignment over* m*, and* (4) *for every* $n \in N$*,* α_n *is the restriction of* α *to* n*, that is, for every* $\xi \in \Xi$*,* $\alpha_n(\xi) = \alpha(\xi) \cap U_n$*.*

Then, for every $\varphi \in L(C,\Xi)$ *and every* $n \in N$*,*

$$[\![\varphi]\!]^n_{\alpha_n} \;=\; [\![\varphi]\!]^m_\alpha \cap U_n \tag{3.12}$$

and

$$[\![\varphi]\!]^m_\alpha \;=\; \bigcup_{n\in N} [\![\varphi]\!]^n_{\alpha_n} \tag{3.13}$$

Proposition 3.6 *Assume that* (1) $\langle C, M', A'\rangle = \langle C, M, A\rangle^c$ *is the closure of a pre-interpretation system* $\langle C, M, A\rangle$*,* (2) $(\cdot)^{\models^p}$*,* $(\cdot)^{\models^d}$*,* $(\cdot)^{\models'^p}$ *and* $(\cdot)^{\models'^d}$ *are the entailment operators induced by* $\langle C, M, A\rangle$ *and* $\langle C, M', A'\rangle$*, respectively. Then,* $(\cdot)^{\models^p} = (\cdot)^{\models'^p}$ *and* $(\cdot)^{\models^d} = (\cdot)^{\models'^d}$*.*

Definition 3.7 *Given any set $\mathcal{B}$ of admissible assignments in a C-structure $\langle U, \mathcal{B}, \nu\rangle$, we say that $\mathcal{B}'$ is a ν-*subalgebra *of $\mathcal{B}$ iff $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{B}'$ is closed under the operations $\nu_k(c)$ for all $k \in \mathbb{N}$ and $c \in C_k$. An interpretation system $\langle C, M, A\rangle$ is said to be* closed under subalgebras *iff, for every $m \in M$ and every ν_m-subalgebra $\mathcal{B}'$ of $\mathcal{B}_m$, there is a model $m' \in M$ such that $U_{m'} = U_m$, $\mathcal{B}_{m'} = \mathcal{B}'$, and, for all $k \in \mathbb{N}$ and $c \in C_k$, $\nu_{m'}(c) = \nu_m(c)|_{\mathcal{B}'}$*

The *closure under subalgebras* of any interpretation system is defined in the obvious way. The following proposition is analogous to Proposition 3.6 and shows that also closing an interpretation system under subalgebras has no effect on the entailment operators.

Proposition 3.8 *Assume that* (1) *$\langle C, M', A'\rangle$ is the closure under subalgebras of the interpretation system $\langle C, M, A\rangle$,* (2) *$(\cdot)^{\vDash^p}$, $(\cdot)^{\vDash^d}$, $(\cdot)^{\vDash'^p}$ and $(\cdot)^{\vDash'^d}$ are the entailment operators induced by $\langle C, M, A\rangle$ and $\langle C, M', A'\rangle$, respectively. Then, $(\cdot)^{\vDash^p} = (\cdot)^{\vDash'^p}$ and $(\cdot)^{\vDash^d} = (\cdot)^{\vDash'^d}$.*

Convention. From now on, unless otherwise stated, all interpretation systems are assumed to be closed under subalgebras.

Prop/Definition 3.9 *The fibring of the interpretation systems $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ is the interpretation system*

$$\langle C', M', A'\rangle \cup \langle C'', M'', A''\rangle = \langle C' \cup C'', M, A\rangle$$

where:

- *M is the subclass of $M' \times M''$ composed of the pairs $\langle m', m''\rangle$ such that:*
  - *$U_{m'} = U_{m''}$; $\mathcal{B}_{m'} = \mathcal{B}_{m''}$;*
  - *$\nu_{m'k}(c)(b) = \nu_{m''k}(c)(b)$*
    for every $c \in (C' \cap C'')_k$ and $b \in \mathcal{B}^k_{m'}$ $(= \mathcal{B}^k_{m''})$;
- *$A(\langle m', m''\rangle) = \langle U, \mathcal{B}, \nu\rangle$ where:*
  - *$U = U_{m'}$ $(= U_{m''})$; $\mathcal{B} = \mathcal{B}_{m'}$ $(= \mathcal{B}_{m''})$;*
  - *$\nu_k(c') = \nu_{m'k}(c')$ for each $c' \in C'_k$;*
  - *$\nu_k(c'') = \nu_{m''k}(c'')$ for each $c'' \in C''_k$.*

Whenever $C' \cap C''$ is the empty signature, the fibring is said to be unconstrained. *Otherwise, it is said to be* constrained by sharing the symbols in $C' \cap C''$.

Proof. We have to prove that $\langle C' \cup C'', M, A\rangle$ is closed under unions and subalgebras.

Consider any subset N of M such that $U_{n_1} \cap U_{n_2} = \emptyset$ for all $n_1 \neq n_2$ in N and let $S_N = \langle U_N, \mathcal{B}_N, \nu_N\rangle$ be the union of the $(C' \cup C'')$-structures in N.

Let N' and N'' be the subsets of M' and M'' consisting respectively of first and of second components in elements of N. Since $U_{\langle m',m''\rangle} = U_{m'} = U_{m''}$ for all pairs $\langle m', m''\rangle$ in M, we have that $U_{n'_1} \cap U_{n'_2} = \emptyset$ for all $n'_1 \neq n'_2$ in N' and $U_{n''_1} \cap U_{n''_2} = \emptyset$ for all $n''_1 \neq n''_2$ in N''. Thus, we can consider the unions m' and m'' of the models in N' and in N''; these unions are elements of M' and of M'', respectively, and $U_{m'} = U_{m''}$.

For every $\langle n', n''\rangle \in N$, we have $\mathcal{B}_{n'} = \mathcal{B}_{n''}$. Then, by (3.4), $\mathcal{B}_{m'} = \mathcal{B}_{m''}$ and hence the pair $m = \langle m', m''\rangle$ belongs to M.

It is straightforward to check that $U_m = U_N$ and, using (3.4), that $\mathcal{B}_m = \mathcal{B}_N$. By (3.5) and the definition of $A(\langle m', m''\rangle)$ above, we have also $\nu_m = \nu_N$. This concludes the proof that $\langle C' \cup C'', M, A\rangle$ is closed under unions.

In order to show that $\langle C' \cup C'', M, A\rangle$ is closed under subalgebras, we have only to observe that any $\nu_{\langle m',m''\rangle}$-subalgebra of $\mathcal{B}_{\langle m',m''\rangle}$ is also a $\nu_{m'}$-subalgebra of $\mathcal{B}_{m'}$ and a $\nu_{m''}$-subalgebra of $\mathcal{B}_{m''}$, and that M' and M'' are closed under subalgebras. ∎

Example 3.10 Assume that $\langle C', M', A'\rangle$ is an interpretation system in which M' is the class of all models for intuitionistic propositional logic and that M'' in $\langle C'', M'', A''\rangle$ is the class of all models for classical propositional logic (see Example 3.3). Consider the fibring $\langle C, M, A\rangle$ of $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$, where we assume that only the elements of $C'_0 = C''_0$ are shared. Write $\Rightarrow'$, $\Rightarrow''$, $\neg'$, $\neg''$, ... for the other constructors in C' and C''.

For every $m = \langle m', m''\rangle$ in M, the sets $\mathcal{B}_{m'}$, $\mathcal{B}_{m''}$, and $\mathcal{B}_m$ coincide. In particular, they are Boolean subalgebras of $\wp U_m$ $(= \wp U_{m'} = \wp U_{m''})$ because such is $\mathcal{B}_{m''}$, and hence $\mathcal{B}_{m'}$ is closed under complementation. This implies that the functions $\nu_{m'}(\Rightarrow')$ and $\nu_{m'}(\neg')$, which are defined as in Example 3.3, coincide with the classical $\nu_{m''}(\Rightarrow'')$ and $\nu_{m''}(\neg'')$. Thus, by Definition 3.9,

$\nu_m(\Rightarrow') = \nu_m(\Rightarrow'')$, $\nu_m(\neg') = \nu_m(\neg'')$, and so on, and m turns out to be, as a matter of fact, isomorphic to m''.

Categorial characterization

We start by defining the appropriate notion of morphism between interpretation systems. Then we proceed to define, within the category of interpretation systems, unconstrained fibring as a coproduct and constrained fibring by cocartesian lifting following, now at the semantic level, the categorial approach already adopted for Hilbert calculi.

Definition 3.11 *An* interpretation system morphism

$$h : \langle C, M, A\rangle \to \langle C', M', A'\rangle$$

is a pair $h = \langle \overline{h}, \underline{h}\rangle$ where:

- *$\overline{h} : C \to C'$ is a morphism in Sig;*
- *$\underline{h} : M' \to M$;*

such that for every $m' \in M'$:

- *$U_{\underline{h}(m')} = U_{m'}$; $\mathcal{B}_{m'} = \mathcal{B}_{\underline{h}(m')}$;*
- *for every $k \in \mathbb{N}$ and $c \in C_k$, $\nu_{m'k}(\overline{h}_k(c)) = \nu_{\underline{h}(m')k}(c)$.*

Interpretation systems together with their morphisms constitute the category *Int*. Furthermore, the maps:

- $N(\langle C, M, A\rangle) = C$;
- $N(h : \langle C, M, A\rangle \to \langle C', M', A'\rangle) = \overline{h}$

constitute the functor $N : \mathit{Int} \to \mathit{Sig}$. This functor is rather useful for relating interpretation systems with their underlying signatures. More interestingly, given an interpretation system over C and signature morphism from C to C', we can build in a canonical way the corresponding interpretation system over C', as follows (using a cocartesian lifting):

Proposition 3.12 For each $\langle C, M, A\rangle$ in *Int* and each surjective morphism $h : C \to C'$ in *Sig*, the morphism

$$\hat{h} : \langle C, M, A\rangle \to \langle C', M', A'\rangle$$

where:

- M' is the subclass of M such that: $\nu_{mk}(c_1) = \nu_{mk}(c_2)$ whenever $h(c_1) = h(c_2)$;
- $A'(m) = \langle U_m, \mathcal{B}_m, \nu'\rangle$ where, for every $k \in \mathbb{N}$ and $c \in C_k$, $\nu'_k(h_k(c)) = \nu_{mk}(c)$;
- $\hat{h} = \langle h, inc\rangle$ where inc is the inclusion of M' into M

is cocartesian by N for h on $\langle C, M, A\rangle$.

Proof: Straightforward, following the method sketched in the proof of the corresponding result for Hilbert calculi (proposition 2.7). ∎

We denote the codomain of the cocartesian morphism by $h(\langle C, M, A\rangle)$. We are ready now to provide the envisaged categorial characterization: unconstrained fibring is a coproduct and constrained fibring is obtained by cocartesian lifting.

Proposition 3.13 Let $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ be interpretation systems. Then, their *unconstrained fibring* $\langle C', M', A'\rangle \oplus \langle C'', M'', A''\rangle$ is the coproduct

$$\langle C' \oplus C'', M, A\rangle$$

where:

- M is the subclass of $M' \times M''$ composed of the pairs $\langle m', m''\rangle$ such that:
  - $U_{m'} = U_{m''}$; $\mathcal{B}_{m'} = \mathcal{B}_{m''}$;
- $A(\langle m', m''\rangle) = \langle U, \mathcal{B}, \nu\rangle$ where:
  - $U = U_{m'} (= U_{m''})$; $\mathcal{B} = \mathcal{B}_{m'} (= \mathcal{B}_{m''})$;

– $\nu_k(i'(c')) = \nu_{m'k}(c')$ for each $c' \in C'_k$;

– $\nu_k(i''(c'')) = \nu_{m''k}(c'')$ for each $c'' \in C''_k$

endowed with the injections:

- $\langle i', p'\rangle$ where i' is the injection $C' \to C' \oplus C''$ and p' is the projection $M \to M'$;

- $\langle i'', p''\rangle$ where i'' is the injection $C'' \to C' \oplus C''$ and p'' is the projection $M \to M''$.

Proof: It is straightforward to verify that $\langle i', p'\rangle, \langle i'', p''\rangle$ are interpretation system morphisms and that the universal property of the coproduct holds. ■

Proposition 3.14 Let $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ be interpretation systems and $f' : C \to C', f'' : C \to C''$ be injective signature morphisms. Then, their *constrained fibring by sharing* $\langle C', M', A'\rangle \overset{f'Cf''}{\oplus} \langle C'', M'', A''\rangle$ is

$$q(\langle C', M', A'\rangle \oplus \langle C'', M'', A''\rangle)$$

where q is the coequalizer of $i' \circ f'$ and $i'' \circ f''$.

Proof: Note that coequalizers are surjective in the category of signatures (since coequalizers are epimorphisms in any category). It is trivial to verify that q is an interpretation system morphism and that the universal property of the cocartesian lifting holds. ■

Clearly, when C is the intersection of C', C'' and we consider f', f'' to be the inclusions, this construction leads to $\langle C', M', A'\rangle \cup \langle C'', M'', A''\rangle$, as defined in purely set-theoretic terms before.

4 General logic systems

Definition 4.1 *A* logic system presentation *is a tuple $\langle C, M, A, P, D\rangle$ in which $\langle C, P, D\rangle$ is a Hilbert calculus and $\langle C, M, A\rangle$ is an interpretation system.*

We will often abbreviate 'logic system presentation' by l.s.p.. Given any l.s.p. $\mathcal{L} = \langle C, M, A, P, D\rangle$, we will denote by $(\cdot)^{\vdash^p_{\mathcal{L}}}$, $(\cdot)^{\vdash^d_{\mathcal{L}}}$ the proof-theoretic consequence operators relative to $\langle C, P, D\rangle$ and by $(\cdot)^{\models^p_{\mathcal{L}}}$, $(\cdot)^{\models^d_{\mathcal{L}}}$ the model-theoretic entailment operators relative to $\langle C, M, A\rangle$.

A logic system presentation $\mathcal{L}$ is said to be

- *p-sound* iff, for all $\Gamma \subseteq L(C, \Xi)$, $\Gamma^{\vdash^p_{\mathcal{L}}} \subseteq \Gamma^{\models^p_{\mathcal{L}}}$.
- *p-complete* iff, for all $\Gamma \subseteq L(C, \Xi)$, $\Gamma^{\vdash^p_{\mathcal{L}}} \supseteq \Gamma^{\models^p_{\mathcal{L}}}$.
- *d-sound* iff, for all $\Gamma \subseteq L(C, \Xi)$, $\Gamma^{\vdash^d_{\mathcal{L}}} \subseteq \Gamma^{\models^d_{\mathcal{L}}}$.
- *d-complete* iff, for all $\Gamma \subseteq L(C, \Xi)$, $\Gamma^{\vdash^d_{\mathcal{L}}} \supseteq \Gamma^{\models^d_{\mathcal{L}}}$.

For $\Gamma = \emptyset$, the inclusions above are referred to as *weak* (p- or d-) soundness and completeness. It is worth noticing that weak p-soundness and weak d-soundness coincide, as well as weak p-completeness and weak d-completeness.

Given the l.s.p.'s $\mathcal{L}' = \langle C', M', A', P', D'\rangle$ and $\mathcal{L}'' = \langle C'', M'', A'', P'', D''\rangle$, their fibring is defined in the obvious way: $\mathcal{L}' \cup \mathcal{L}''$ is the logic system presentation $\langle C' \cup C'', M, A, P' \cup P'', D' \cup D''\rangle$ in which M and A fulfill Definition 3.9. $\mathcal{L}' \cup \mathcal{L}''$ is said to be *unconstrained* or *constrained by sharing* the elements of $C' \cap C''$ according to whether $C' \cap C''$ is empty or not.

Many of the results presented in the sequel regard particular l.s.p.'s, that will be called *full* (see Definition 4.4 below). In these structures, M is the largest class allowed by the rules in P and D.

Definition 4.2 *A C-structure $S = \langle U, \mathcal{B}, \nu\rangle$ is a* structure for *the Hilbert calculus $\langle C, P, D\rangle$ iff, for all Γ, δ*

$$\langle \Gamma, \delta\rangle \in P \quad \Rightarrow \quad \forall \alpha\, [S\alpha \Vdash \Gamma \textit{ implies } S\alpha \Vdash \delta]$$

$$\langle \Gamma, \delta\rangle \in D \quad \Rightarrow \quad \forall \alpha, u\, [S\alpha u \Vdash \Gamma \textit{ implies } S\alpha u \Vdash \delta]$$

where α ranges over the set of variable assignments over S and u ranges over U. A model m in the interpretation system $\langle C, M, A\rangle$ is a model for *the Hilbert calculus $\langle C, P, D\rangle$ iff $A(m)$ is a structure for $\langle C, P, D\rangle$.*

The following proposition is proved in [SSC99] for standard semantics. The adaptation of the proof to general semantics is obvious.

Proposition 4.3 *If every model in $\mathcal{L} = \langle C, M, A, P, D\rangle$ is a model for $\langle C, P, D\rangle$, then $\mathcal{L}$ is p-sound and d-sound.*

Of course, it makes no sense to say that m is a model for a given Hilbert calculus if an interpretation system to which m belongs, and hence a function A, are not provided. In particular cases, however, the set-theoretical structure of m provides a *preferred* related structure; this holds, for instance, for the models of intuitionistic logic considered in Example 3.3 and for the modal models consider below. In these cases, by 'a model of' a given Hilbert calculus, we will always mean a model such that its preferred structure is a structure for that Hilbert calculus.

Definition 4.4 *A logic system presentation $\langle C, M, A, P, D\rangle$ is* full [*resp.* standard full] *iff the image of the function A is the class of all structures [resp. standard structures] for $\langle C, P, D\rangle$.*

Given any Hilbert calculus $\langle C, P, D\rangle$, there are many full [standard full] l.s.p.'s for $\langle C, P, D\rangle$. If $\langle C, M, A, P, D\rangle$ and $\langle C, M', A', P, D\rangle$ are two l.s.p.'s of this kind, however, there is a natural one-to-one correspondence $m \to m'$ between M and M' such that $A(m)$ and $A'(m')$ coincide. Then, it makes sense to refer to *the* full [standard full] l.s.p. of $\langle C, P, D\rangle$.

Propositions 3.6 and 3.8 imply that full l.s.p.'s are closed under unions and under subalgebras. A *standard* full l.s.p., instead, in general is not closed under subalgebras, unless, for every model m, $\wp U_m$ has no proper ν_m-subalgebra.

Lemma 4.5 *Let $\langle C', M', A', P', D'\rangle$ and $\langle C'', M'', A'', P'','' D\rangle$ be l.s.p.'s and assume that every model in M' $[M'']$ is a model for $\langle C', P', D'\rangle$ $[\langle C'', P'', D''\rangle]$. Then, every model in $\langle C', M', A', P', D'\rangle \cup \langle C'', M'', A'', P'','' D''\rangle$ is a model for $\langle C' \cup C'', P' \cup P'', D' \cup D''\rangle$.*

Proof. Let m be any model in $\langle C', M', A', P', D'\rangle \cup \langle C'', M'', A'', P'','' D''\rangle$. Then, $m = \langle m', m''\rangle$ for suitable models $m' \in M'$ and $m'' \in M''$, and $A(m)$

is defined according to Definition 3.9. The sets U_m, $U_{m'}$, and $U_{m''}$ coincide, as well as the sets $\mathcal{B}_m, \mathcal{B}_{m'}$ and $\mathcal{B}_{m''}$. Thus, every variable assignment over m can be viewed as a variable assignment over m' and as a variable assignment over m'', and hence the implications in Definition 4.2 hold for $P = P' \cup P''$ as well as for $D = D' \cup D''$. ■

Theorem 4.6 *Let $\mathcal{L}' = \langle C', M', A', P', D' \rangle$ and $\mathcal{L}'' = \langle C'', M'', A'', P'', D'' \rangle$ be full [resp. standard full] l.s.p.'s. Then, the fibring $\langle C, M, A, P, D \rangle = \mathcal{L}' \cup \mathcal{L}''$ is full [resp. standard full].*

Proof. Let $S = \langle U, \mathcal{B}, \nu \rangle$ be any structure for $\langle C, P, D \rangle$ and consider the $C'-$ and $C''-$structures S' and S'', defined by: $U' = U'' = U$, $\mathcal{B}' = \mathcal{B}'' = \mathcal{B}$, and $\nu' = \nu|_{C'}$, and $\nu'' = \nu|_{C''}$. Since $C = C' \cup C''$ and $D = D' \cup D''$, S' and S'' are respectively structures for $\langle C', P', D' \rangle$ and for $\langle C'', P'', D'' \rangle$, and hence there are $m' \in M'$ and $m'' \in M''$ such that $A'(m') = S'$ and $A''(m'') = S''$ because $\mathcal{L}'$ and $\mathcal{L}''$ are full.

The models m' and m'' fulfill the conditions of Definition 3.9 and hence we can consider the element $\langle m', m'' \rangle$ of M. Definition 3.9 implies also that $A(\langle m', m'' \rangle) = S$ and this proves that every structure for $\langle C, P, D \rangle$ is $A(m)$ for an $m \in M$. The converse inclusion is given by Lemma 4.5.

As for standard full l.s.p.'s, we have only to observe that, for $U_{m'} = U_{m''} = U_m$, $\wp(U_m) = \wp(U_{m'}) = \wp(U_{m''})$. ■

As a consequence of the proof of this theorem and of Proposition 4.3, we have that fibrings preserve soundness.

We turn our attention now to the problem of preservation of completeness by fibring. The examples considered below are based on results proved in [KW91] and in [Wol96] and concern modal and temporal logics endowed with standard semantics; thus, for every model m considered in the rest of this section, $\mathcal{B}_m = \wp(U_m)$ is always assumed and we may omit $\mathcal{B}_m$ in the presentation of these models.

4.1 Two examples from modal logic

Definition 4.7 *We say that C is a* modal signature *based on the set* Π *iff:* 1) $C_0 = \Pi$, 2) $C_1 = \{\neg\} \cup \{\Diamond_i\}_{i \in I}$, *where I is a set,* 3) $C_2 = \{\wedge\}$, *and* 4)

$C_k = \emptyset$ *for each* $k > 2$.

The set $\{\Diamond_i\}_{i\in I}$ in a modal signature C will be referred to as the set of *modalities* in C. If, as a limit case, this set is empty, then we have that C is the signature of propositional logic based on Π. The fibring $C' \cup C''$ of two modal signatures is a modal signature.

A (standard) *Kripke model* for the modal signature C is a tuple $\langle W, \{S_i\}_{i\in I}, V\rangle$ in which W is a nonempty set, each S_i is a binary relation on W (the *accessibility relation* for the operator $\Diamond_i$), and $V : \Pi \to \wp W$. The *Kripke* C*-structure* based on the model $\langle W, \{S_i\}_{i\in I}, V\rangle$ is the triple $\langle W, \wp W, \nu\rangle$ in which:

E0	$\nu_0(\pi)$	$=$	$V(\pi)$, *for each* $\pi \in \Pi$,
E1	$\nu_1(\neg)(b)$	$=$	$W - b$,
E2i	$\nu_1(\Diamond_i)(b)$	$=$	$\{\, w \in W : \exists w' \,(wS_iw' \text{ and } w' \in b)\,\}$,
E3	$\nu_2(\wedge)(b, b')$	$=$	$b \cap b'$.

If, like in Example 2.4, $\Box_i$ is used as primitive modal operator instead of $\Diamond_i$, then the rule corresponding to E2i is $\nu_1(\Box_i)(b) = \{\, w \in W : \forall w' \,(wS_iw' \to w' \in b)\,\}$ and this set is $\nu_1(\neg)\nu_1(\Diamond_i)\nu_1(\neg)(b)$. In the sequel, we will shift freely from the $\Diamond$ notation to the $\Box$ notation and vice-versa.

The operators $\nu_1(\Diamond_i)$ and the relations S_i are interdefinable by means of the following equivalence which is a consequence of (E2i).

$$wS_iw' \quad \text{iff} \quad w \in \nu_1(\Diamond_i)(\{w'\}) \tag{4.1}$$

Notice that this equivalence is meaningful because, we are here considering standard interpretation systems and hence each singleton $\{w'\}$ is always an admissible valuation.

Proposition 4.8 *Let* $\langle C, M, A\rangle$ *a pre-interpretation system in which every element of* M *is a Kripke model. Then, every* C*-structure in the closure of* $\langle C, M, A\rangle$ *is isomorphic to a Kripke structure.*

Proof. We have to show that, given any set N of Kripke models such that $U_n \cap U_{n'}$ for all $n \neq n'$ in it, the C-structure $\langle U_m, \mathcal{B}_m, \nu_m\rangle$ defined by means of (3.3-5) is isomorphic to a Kripke structure. Since each $n \in N$ is a standard model, by (3.4), we have also $\mathcal{B}_m = \wp(U_m)$. Equality (3.5) and straightforward boolean operations show that E0,1,3 hold for ν_m.

Assume now that each $n \in N$ has the form $\langle W_n, \{S_{ni}\}_{i\in I}, V_n\rangle$. For every $i \in I$, define the binary relation S_{mi} on U_m by $S_{mi} = \cup_{n\in N} S_{ni}$. By (3.5) and E2i, we have

$$\nu_{m1}(\Diamond_i)(b) = \bigcup_{n\in N} \{\, w \in W_n : \exists w' \,(w S_{ni} w' \text{ and } w' \in b \cap W_n)\,\} \tag{4.2}$$

Since the sets W_n are pairwise disjoint, this equality implies

$$\nu_{m1}(\Diamond_i)(b) = \{\, w \in U_m : \exists w' \,(w S_{mi} w' \text{ and } w' \in b)\,\} \tag{4.3}$$

and hence ν_{m1} fulfills E2i. ■

Definition 4.9 *We say that $\langle C, M, A\rangle$ is a* modal interpretation system *whenever, for every $m \in M$, there is a Kripke model $m' = \langle W', \{S'_i\}_{i\in I}, V'\rangle$, such that $A(m)$ is the Kripke C-structure of m'.*

According to this definition, given any model m in a modal interpretation system, we can always assume that it is a Kripke model and that $A(m)$ is the corresponding Kripke structure. By Proposition 4.8, any union of Kripke structures is (isomorphic to) a Kripke structure; moreover, the accessibility relations in the union are the set-theoretical unions of the corresponding accessibility relations. This proves the following proposition.

Proposition 4.10 *If $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ are modal interpretation systems, then $\langle C'\cup C'', M, A\rangle = \langle C', M', A'\rangle \cup \langle C'', M'', A''\rangle$ is a modal interpretation system and, for every $m = \langle m', m''\rangle$ in M and every modality $\Diamond_i$ in $C'\cup C''$, $S_{mi} = S_{m'i}$ or $S_{mi} = S_{m''i}$, according to whether $\Diamond_i$ is a modality in C' or in C''. In particular, if $\Diamond_i \in (C' \cap C'')_1$, then $S_{mi} = S_{m'i} = S_{m''i}$.*

Corollary 4.11 *Assume $\langle C'\cup C'', M, A\rangle = \langle C', M', A'\rangle \cup \langle C'', M'', A''\rangle$, where $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ are modal interpretation systems. Let I, I', and I'' be respectively the index sets for modalities in $C'\cup C''$, C', and C'', so that $I = I' \cup I''$. Then, the elements of M are the tuples $\langle W, \{S_i\}_{i\in I}, V\rangle$ such that $\langle W, \{S_i\}_{i\in I'}, V\rangle$ is an element of M' and $\langle W, \{S_i\}_{i\in I''}, V\rangle$ is an element of M''.*

Examples. 1) If $I' = I''$, then $M = M' \cap M''$. 2) If $I' \cap I'' = \emptyset$ (so that $C' \cap C''$ is the propositional logic signature) and M' and M'' are respectively the classes of *all* Kripke models for C' and C'', then M is the class of *all* Kripke models for $C' \cup C''$.

Definition 4.12 *Given any modal signature C, we say that $\langle C, P, D\rangle$ is a* modal calculus *whenever*

$$P \supseteq \bigcup_{i \in I} P_{\Box_i} \cup P_0 \quad \textit{and} \quad D = D_0 \tag{4.4}$$

where the sets P_0, $P_{\Box_i}$, and D_0 are defined as in Example 2.4.

Furthermore, a modal logic system presentation *is a tuple $\langle C, M, A, P, D\rangle$ in which $\langle C, M, A\rangle$ is a modal interpretation system and $\langle C, P, D\rangle$ is a modal calculus.*

By Definition 2.5, the (possibly constrained) fibring of two modal calculi is still a modal calculus; it is (isomorphic to) the smallest modal calculus which contains the other two. Thus, fibrings of modal l.s.p.'s are modal l.s.p.'s.

In order to be able to apply the results proved in [KW91] and [Wol96], in the next two subsections we assume that, for all modal logic system presentations $\langle C, M, A, P, D\rangle$ considered therein,

$$\textit{no element of } C_0 \text{ occurs in some rule in } P \text{ or in } D \tag{4.5}$$

It must be also observed that the possibility of transferring the completeness result of [KW91] to fibring, uses in an essential way the assumption that interpretation systems are closed under disjoint unions.

In order to avoid trivializing the problems, we also assume consistency; that is, $\emptyset^{\vdash^p} \neq L(C, \Xi)$ (which is equivalent to $\emptyset^{\vdash^d} \neq L(C, \Xi)$), for all provability and derivability operators considered in the next subsections.

An example of completeness preservation[4].

The following theorem is a consequence of Theorem 1 in [KW91].

[4] In [KW91], the results considered in this subsection are proved for modal logics with only one modality; in Section 9, however, the authors show how to extend the theorems to arbitrary modal logics.

Theorem 4.13 *Assume that* (1) $\langle C', P', D'\rangle$ *and* $\langle C'', P'', D''\rangle$ *are modal Hilbert calculi,* (2) *the signatures* C' *and* C'' *have disjoint sets of modalities, and* (3) $(\cdot)^{\vdash'^p}$, $(\cdot)^{\vdash''^p}$, *and* $(\cdot)^{\vdash^p}$ *are respectively the schema provability operators of* $\langle C', P', D'\rangle$, $\langle C'', P'', D''\rangle$, *and* $\langle C', P', D'\rangle \cup \langle C'', P'', D''\rangle$. *Then,* $\emptyset^{\vdash'^p} = \emptyset^{\vdash^p} \cap L(C', \Xi)$ *and* $\emptyset^{\vdash''^p} = \emptyset^{\vdash^p} \cap L(C'', \Xi)$.

Corollary 4.14 *Assume that:* (1) $\mathcal{L}' = \langle C', M', A', P', D'\rangle$ *and* $\mathcal{L}'' = \langle C'', M'', A'', P'', D''\rangle$ *are modal l.s.p.'s,* (2) C' *and* C'' *have disjoint sets of modalities, and* (3) $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}'' (= \langle C' \cup C'', M, A, P, D\rangle)$ *is weakly p-complete. Then,* $\mathcal{L}'$ *and* $\mathcal{L}''$ *are weakly p-complete.*

Proof. Let $(\cdot)^{\vdash'^p}$, $(\cdot)^{\vdash''^p}$, and $(\cdot)^{\vdash^p}$ be the schema provability operators of $\mathcal{L}'$, $\mathcal{L}''$, and $\mathcal{L}$, respectively. Assume that the formula δ in $L(C', \Xi)$ does not belong to $\emptyset^{\vdash'^p_{\mathcal{L}}}$; by Theorem 4.13, $\delta \notin \emptyset^{\vdash^p_{\mathcal{L}}}$, which implies $\delta \notin \emptyset^{\vDash^p_{\mathcal{L}}}$ because $\mathcal{L}$ is weakly complete. Then, there exists a model $m \in M$, an element u of U_m, and schema variable assignment α such that $u \notin [\![\delta]\!]^m_\alpha$. We can assume $m = \langle U_m, \{S'_i\}_{i\in I} \cup \{S''_j\}_{j\in J}, V\rangle$, where $\{S'_i\}_{i\in I}$ and $\{S''_j\}_{j\in J}$ are the (disjoint) families of accessibility relations for modalities in C' and in C'', respectively.

Let m' be $\langle U_m, \{S'_i\}_{i\in I}, V\rangle$. Since no modality $\diamond''_j$ in C'' occurs in δ, $[\![\delta]\!]^m_\alpha$ does not depend on the relations S''_j and it is equal to $[\![\delta]\!]^{m'}_\alpha$. To conclude the proof, we have only to observe that, by Corollary 4.11, the model m' is an element of M'. ■

The converse of this corollary is a consequence of the main theorem (Theorem 7) in [KW91]; in the language of fibring, this theorem can be read as:

Theorem 4.15 *Let the modal l.s.p.* $\mathcal{L}'$ *and* $\mathcal{L}''$ *be as in Corollary 4.14 and assume in addition that they are full and weakly p-complete. Then, the full modal l.s.p. for* $\langle C' \cup C'', P' \cup P'', D' \cup D''\rangle$ *is weakly p-complete.*

Proposition 4.16 *Let the modal l.s.p.* $\mathcal{L}'$ *and* $\mathcal{L}''$ *be as in Corollary 4.14 and assume in addition that they are full, then the constrained fibring* $\mathcal{L}' \cup \mathcal{L}'' = \langle C' \cup C'', M, A, P' \cup P'', D' \cup D''\rangle$ *is full.*

Proof. Let $m = \langle U_m, \{S'_i\}_{i\in I} \cup \{S''_j\}_{j\in J}, V\rangle$ be any Kripke model for the calculus $\langle C' \cup C'', P' \cup P'', D' \cup D''\rangle$ and consider the models $m' = \langle U_m, \{S'_i\}_{i\in I}, V\rangle$

and $m'' = \langle U_m, \{S''_j\}_{j\in J}, V\rangle$, which are models for, respectively, $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$. Then, $m' \in M'$ and $m'' \in M''$ because $\mathcal{L}'$ and $\mathcal{L}''$ are full. By Corollary 4.11, $m \in M$. ∎

Theorem 4.17 *Fibrings of full modal l.s.p.'s with disjoint sets of modalities preserve weak completeness.*

An example of non-preservation of completeness in temporal logic.

Definition 4.18 *Given a modal signature C with $\{\Diamond_i\}_{i\in I}$ as set of modalities, the* temporal signature *based on C is the modal signature C^t in which $C^t_1 \setminus \{\neg\}$ is $\{\Diamond_i\}_{i\in I} \cup \{\Diamond^-_i\}_{i\in I}$, where each $\Diamond^-_i$ is a new modality.*

The elements $\Diamond_i$ and $\Diamond^-_i$ of C_1, in a temporal signature, are usually referred to as the i-th *future* and *past* operators. Of course, every modal signature in which the set of modalities is infinite or has a finite even number of elements, can be viewed as a temporal signature; things become more interesting when we define the notion of *temporal logic system presentation.*

Definition 4.19 *Given a temporal signature C^t, we will say that $\mathcal{L} = \langle C^t, M, A, P, D\rangle$ is a* temporal logic system presentation *whenever:* (1) *$\mathcal{L}$ is a modal logic system presentation,* (2) *every $m = \langle U, \{S_i\}_{i\in I} \cup \{S^-_i\}_{i\in I}, V\rangle$ in M is a* temporal model, *that is, for every $i \in I$, $S^-_i = (S_i)^{-1}$, and* (3) *$\langle C^t, P, D\rangle$ is a* temporal calculus, *that is, P contains $\langle \emptyset, \xi \Rightarrow \Box_i \Diamond^-_i \xi\rangle$ and $\langle \emptyset, \xi \Rightarrow \Box^-_i \Diamond_i \xi\rangle$ for every $i \in I$.*

It can be easily verified that any Kripke model for the temporal calculus $\langle C^t, P, D\rangle$ is a temporal model.

Definition 4.20 *The* basic temporal logic system presentation K_{C^t} *for the temporal signature C^t is the tuple $\langle C^t, M_B, A_B, P_B, D_B\rangle$ in which:* (1) *M_B is the class of all temporal models for C^t,* (2) *$D_B = D_0$ (see 4.4), and* (3)

$$P_B \;=\; P_0 \cup \bigcup_{i\in I} \Big(P_{\Box_i} \cup P_{\Box^-_i} \cup \{\langle \emptyset, \xi \Rightarrow \Box_i \Diamond^-_i \xi\rangle, \langle \emptyset, \xi \Rightarrow \Box^-_i \Diamond_i \xi\rangle\}\Big)$$

Thus, in the basic temporal logic system presentation K_{C^t}, we have that $\langle C^t, P_B, D_B\rangle$ and $\langle C^t, M_B, A_B\rangle$ are respectively the smallest temporal calculus and the largest temporal interpretation system for C^t. Classical temporal logic results give that K_{C^t} is sound and complete.

Following [Wol96], the *minimal temporal extension* of a modal logic can be obtained by 'duplicating' the modal operators and by adding the basic axioms for temporal logic of Def. 4.20. Formally:

Definition 4.21 *Given any modal logic system presentation $\mathcal{L} = \langle C, M, A, P, D\rangle$ with modalities $\Box_i$ $(i \in I)$, the* minimal temporal extension *of $\mathcal{L}$ is the temporal logic system presentation $\mathcal{L}^t = \langle C^t, M^t, A^t, P^t, D^t\rangle$ in which:* (1) *M^t is the class of all $m^t = \langle W, \{S_i\}_{i\in I} \cup \{S_i^-\}_{i\in I}, V\rangle$ such that $m = \langle W, \{S_i\}_{i\in I}, V\rangle$ belongs to M,* (2) *$D^t = D_0$, and* (3)

$$P^t = P \cup \bigcup_{i\in I} \left(P_{\Box_i^-} \cup \{\langle \emptyset, \xi \Rightarrow \Box_i \Diamond_i^- \xi\rangle, \langle \emptyset, \xi \Rightarrow \Box_i^- \Diamond_i \xi\rangle\}\right)$$

In [Wol96], a modal logic Λ is considered which is complete for validity with respect to a given class of Kripke frames, and it is shown that the minimal temporal extension (therein denoted by $\Lambda^+.t$) of Λ is not complete for the same class of frames viewed as temporal frames. Since the basic temporal logic system presentation of any temporal signature is complete, the following proposition allows to view Wolter's result as an example of non-preservation of completeness.

Proposition 4.22 *For every modal logic system presentation $\mathcal{L} = \langle C, M, A, P, D\rangle$, the minimal temporal extension $\mathcal{L}^t$ of $\mathcal{L}$ is $\mathcal{L} \cup K_{C^t}$.*

Proof. By Definitions 4.20 and 4.21, P^t is $P \cup P_B$ and D^t is $D \cup D_B$ and hence, by Definition 2.5, $\langle C^t, P^t, D^t\rangle = \langle C, P, D\rangle \cup \langle C^t, P_B, D_B\rangle$. The equality $\langle C^t, M^t, A^t\rangle = \langle C, M, A\rangle \cup \langle C^t, M_B, A_B\rangle$ is a consequence of the same definitions, and of Corollary 4.11. ∎

5 L.s.p.'s with congruence. Completeness.

Definition 5.1 (a) *A Hilbert calculus $\langle C, P, D\rangle$ is said to be a* Hilbert calculus with congruence *iff the following* Metatheorem of Congruence *holds: For* *every p-deductively closed set Γ, every $k > 0$, every $c \in C_k$, and formulae $\delta_1, \ldots, \delta_k, \delta'_1, \ldots, \delta'_k$ in $L(C, \Xi)$,*

$$\frac{\Gamma, \delta_i \vdash^d \delta'_i \ \text{ and } \ \Gamma, \delta'_i \vdash^d \delta_i \ \text{ for } i = 1, \ldots, k}{\Gamma, c(\delta_1, \ldots, \delta_k) \vdash^d c(\delta'_1, \ldots, \delta'_k)} \tag{MTC}$$

(b) *A l.s.p. $\langle C, M, A, P, D\rangle$ is said to be a* l.s.p. with congruence *iff $\langle C, P, D\rangle$ is a Hilbert calculus with congruence.*

Observation. Many well-known logics with a propositional basis enjoy the MTC. It is worth noticing that, if Γ were not required to be p-deductively closed, then MTC would fail in a wide class of logics, including modal and temporal logics. In fact, for $k = 1$ and $\Gamma = \{\delta_1, \delta'_1\}$, the antecedent of MTC holds trivially in every Hilbert calculus, while, in modal logic, we do not have $\Gamma, \Box\delta_1 \vdash^d \Box\delta'_1$.

In the sequel, we use the metalinguistic expression

$$E_\Gamma(\delta, \delta') \quad \equiv_{\text{def}} \quad \Gamma, \delta \vdash^d \delta' \ \text{ and } \ \Gamma, \delta' \vdash^d \delta \tag{5.1}$$

Of course, from the premises of the MTC we can also infer $\Gamma, c(\delta_1, \ldots, \delta_k) \vdash^d c(\delta'_1, \ldots, \delta'_k)$ and, therefore, the MTC can be stated as follows:

$$\frac{E_\Gamma(\delta_i, \delta'_i) \ \ \textit{for } i = 1, \ldots, k}{E_\Gamma(c(\delta_1, \ldots, \delta_k), c(\delta'_1, \ldots, \delta'_k))}$$

Definition 5.2 *Within a given Hilbert calculus $\langle C, P, D\rangle$:*
(a) *For any $\Gamma \subseteq L(C, \Xi)$ and $\delta \in L(C, \Xi)$, we say that* (1) Γ *is a* p-non-δ *set iff $\Gamma \not\vdash^p \delta$,* (2) Γ *is* p-consistent *iff it is a* p-non-δ *set for some formula δ,* (3) Γ *is a* maximal p-non-δ *set iff it is a p-non-δ and $\Gamma' \vdash^p \delta$ for every proper extension Γ' of Γ,* (4) Γ *is* maximal p-consistent *iff it is a maximal* p-non-δ *set for some formula δ.*
(b) d-non-δ *sets,* d-consistent *sets, and sets maximal for these properties are defined similarly, by replacing $\vdash^p$ by $\vdash^d$.*

We will abbreviate 'maximal p-consistent set' 'maximal d-consistent set' by m.p-c.s. and m.d-c.s.. Every m.p-c.s. is p-deductively closed. A classical Lindenbaum construction shows that every *p-non-δ* set can be extended to a maximal p-*non*-δ set, and hence that any p-consistent set can be extended to a m.p-c.s.. These properties hold similarly for m.d-c.s.'s.

Main Henkin Construction.

In this construction, $\langle C, P, D\rangle$ is any fixed Hilbert calculus with congruence, the operators $(\cdot)^{\vdash^p}$ and $(\cdot)^{\vdash^d}$ are the corresponding provability and derivability operators, and Γ_0 be any fixed p-consistent set. Moreover, we assume that Γ_0 is p-deductively closed. The Henkin construction of the C-structure $S = \langle U, \mathcal{B}, \nu\rangle$ for $\langle C, P, D\rangle$ is described below.

Let U be the set of all m.d-c.s.'s which contain Γ_0. Clearly, U is non-empty because Γ_0 is p-consistent and hence it is also d-consistent. For every formula $\gamma \in L(C, \Xi)$, set

$$|\gamma| = \{\, u \in U : \gamma \in u \,\}$$

Lemma 5.3 *For every formula γ,*

$$\gamma \in \Gamma_0 \quad \textit{iff} \quad |\gamma| = U$$

Proof. The implication from the left to the right is trivial because each element of U contains Γ_0. Assume $\gamma \notin \Gamma_0$. This implies $\Gamma_0 \not\vdash^p \gamma$ and $\Gamma_0 \not\vdash^d \gamma$ because $(\cdot)^{\vdash^d} \subseteq (\cdot)^{\vdash^p}$. Then, Γ_0 is a d-*non*-γ set and any maximal d-*non*-γ extension of it belongs to $U \setminus |\gamma|$. ∎

Lemma 5.4 *For all formulae γ, γ',*

$$E_{\Gamma_0}(\gamma, \gamma') \quad \textit{iff} \quad |\gamma| = |\gamma'| \tag{5.2}$$

Proof. If $E_{\Gamma_0}(\gamma, \gamma')$ does not hold, then, by (5.1), either $\Gamma_0, \gamma' \not\vdash^d \gamma$ or $\Gamma_0, \gamma \not\vdash^d \gamma'$. In the first case, $\Gamma_0 \cup \{\gamma'\}$ is a d-*non*-γ set and hence there is a m.d-c.s. $u \in U$ which contains γ' but does not contain γ; in the second case, we can reach the opposite conclusion. In both cases, we have $|\gamma| \neq |\gamma'|$.

If, conversely, $E_{\Gamma_0}(\gamma, \gamma')$ holds, (5.1) yields that γ' belongs to every d-deductively closed set which contains $\Gamma_0 \cup \{\gamma\}$ and vice-versa; thus, every $u \in U$ which contains γ contains also γ' and vice-versa. ∎

Set

$$\mathcal{B} = \{\, b \subseteq U \text{ such that } b = |\gamma| \text{ for some formula } \gamma \,\}$$

Thus, arbitrary elements of $\mathcal{B}$ will be written as $|\gamma|$. For $\gamma \in C_0$, set

$$\nu_0(\gamma) = |\gamma|$$

For $k > 0$ and $c \in C_k$, let $\nu_k(c)$ be the element of $\mathcal{B}^k \to \mathcal{B}$ defined by

$$\nu_k(c)(|\gamma_1|, \ldots, |\gamma_k|) \;=\; |c(\gamma_1, \ldots, \gamma_k)|$$

The functions $\nu_k(c)$ are well-defined. If, in fact, $|\gamma_i| = |\gamma_i'|$ holds for $i = 1$ to k, then, Lemma 5.4 implies that $E_{\Gamma_0}(\gamma_i, \gamma_i')$ holds for $i = 1, \ldots, k$, which implies in turn $E_{\Gamma_0}(c(\gamma_1, \ldots, \gamma_k), c(\gamma_1', \ldots, \gamma_k'))$ by MTC. Using Lemma 5.4 again, we have $|c(\gamma_1, \ldots, \gamma_k)| = |c(\gamma_1', \ldots, \gamma_k')|$.

This concludes the definition of the C-structure S.

Lemma 5.5 *Assume that* (1) *the set* $\{\xi_1, \ldots, \xi_n\}$ *contains the elements of* Ξ *occurring in the formula* γ, (2) α *is a variable assignment over* S *such that, for* $i = 1$ *to* n, $\alpha(\xi_i) = |\delta_i|$, *and* (3) $\gamma(\xi_i/\delta_i)$ *is obtained from* γ *by replacing each* ξ_i *occurring in it by* δ_i. *Thus,*

$$[\![\gamma]\!]^S_\alpha \;=\; |\gamma(\xi_i/\delta_i)|$$

Proof. If $\gamma \in C_0$, then $\gamma(\xi_i/\delta_i) = \gamma$ and the thesis is the definition of ν_0. If γ is ξ_i, then $[\![\gamma]\!]^S_\alpha = \alpha(\xi_i) = |\delta_i| = |\gamma(\xi_i/\delta_i)|$.

Let γ be $c(\gamma_1, \ldots, \gamma_k)$ and assume inductively that the thesis holds for $\gamma_1, \ldots, \gamma_k$. Then, the following equalities hold

$$\begin{aligned}&[\![\gamma]\!]^S_\alpha = [\![c(\gamma_1, \ldots, \gamma_k)]\!]^S_\alpha = \nu_k(c)([\![\gamma_1]\!]^S_\alpha, \ldots, [\![\gamma_k]\!]^S_\alpha) = \\ &\nu_k(c)(|\gamma_1(\xi_i/\delta_i)|, \ldots, |\gamma_k(\xi_i/\delta_i)|) = |c((\gamma_1(\xi_i/\delta_i), \ldots, \gamma_k(\xi_i/\delta_i))|\end{aligned}$$

but $c((\gamma_1(\xi_i/\delta_i), \ldots, \gamma_k(\xi_i/\delta_i))$ is $\gamma(\xi_i/\delta_i)$. ■

In particular, if α_0 is the variable assignment such that $\alpha_0(\xi) = |\xi|$ for every $\xi \in \Xi$, then

$$[\![\gamma]\!]^S_{\alpha_0} \;=\; |\gamma| \tag{5.3}$$

Theorem 5.6 *The C-structure S is a structure for $\langle C, P, D\rangle$.*

Proof. Assume $r = \langle\{\gamma_1, \dots, \gamma_k\}, \gamma\rangle \in P$, and $[\![\gamma_j]\!]^S_\alpha = U$ for $j = 1$ to k. Let $\{\xi_1, \dots, \xi_n\}$ be the (possibly empty) set of elements of Ξ occurring in r and assume $\alpha(\xi_i) = |\delta_i|$. By Lemma 5.5, $|\gamma_j(\xi_i/\delta_i)| = U$ for each j, and, by Lemma 5.3, $\gamma_j(\xi_i/\delta_i) \in \Gamma_0$. This set is p-deductively closed and hence, by clause (2) in Definition 2.2, $\gamma(\xi_i/\delta_i)$ belongs to it and $|\gamma(\xi_i/\delta_i)| = U$. Lemma 5.5 yields $[\![\gamma]\!]^S_\alpha = U$.

Assume now $r = \langle\{\gamma_1, \dots, \gamma_k\}, \gamma\rangle \in D$, and $u \in [\![\gamma_j]\!]^S_\alpha$ for $j = 1$ to k. By Lemma 5.5, $u \in |\gamma_j(\xi_i/\delta_i)|$ which is equivalent to $\gamma_j(\xi_i/\delta_i) inu$. This set is d-deductively closed and hence clause (3) in Definition 2.3 implies $\gamma(\xi_i/\delta_i) \in u$. By Lemma 5.5, we have $u \in [\![\gamma]\!]^S_\alpha$. ∎

End of Main Henkin Construction.

Theorem 5.7 *Every full logic system presentation with congruence $\mathcal{L} = \langle C, M, A, P, D\rangle$ is p-complete and d-complete.*

Proof. Assume $\delta \notin (\Gamma)^{\vdash^p_{\mathcal{L}}}$ and let Γ_0 be $(\Gamma)^{\vdash^p_{\mathcal{L}}}$, so that Γ_0 fulfills the conditions of the Main Henkin Construction. Define a C-structure $S = \langle U, \mathcal{B}, \nu\rangle$ starting from Γ_0 like in that construction; by Theorem 5.6, S is a structure for $\langle C, P, D\rangle$ and hence, since $\mathcal{L}$ is full, there is a model $m \in M$ such that $A(m) = S$. Since $\delta \notin (\Gamma)^{\vdash^p_{\mathcal{L}}}$, we have $\delta \notin (\Gamma_0)^{\vdash^p_{\mathcal{L}}}$ and $\delta \notin (\Gamma_0)^{\vdash^d_{\mathcal{L}}}$; thus, we can consider a maximal *d-non-δ* extension of Γ_0 and this set is an element u of U. Let α_0 the variable assignment considered in (5.3), so that $[\![\gamma]\!]^m_{\alpha_0} = U$ for every $\gamma \in \Gamma_0$, but $[\![\delta]\!]^m_{\alpha_0} \subseteq U \setminus \{u\}$ and hence $\delta \notin (\Gamma)^{\vDash^p_{\mathcal{L}}}$.

Assume $\delta \notin (\Gamma)^{\vdash^d_{\mathcal{L}}}$, so that, in particular $\delta \notin (\emptyset)^{\vdash^p_{\mathcal{L}}}$. Let Γ_0 be $(\emptyset)^{\vdash^p_{\mathcal{L}}}$ and define the structure S starting from this Γ_0 like above. By the assumption, the set $(\Gamma)^{\vdash^d_{\mathcal{L}}}$, which contains Γ_0, can be extended to a maximal *d-non-δ* set $u \in U$. For α_0 as above, we have $u \in [\![\gamma]\!]^S_{\alpha_0}$ for every $\gamma \in \Gamma$, but $u \notin [\![\delta]\!]^S_{\alpha_0}$. Then, we can proceed in the same way as in the first part of the proof to conclude $\delta \notin (\Gamma)^{\vDash^d_{\mathcal{L}}}$. ∎

By Theorem 4.6, the fibring of full l.s.p.'s is full and, hence, if the MTC is also preserved, the previous theorem implies that completeness is preserved as well. Therefore, it is important to investigate whether congruence is preserved by fibring. Actually, it is not always preserved as the following counterexample shows.

Example 5.8 Consider the Hilbert calculi $H' = \langle C', P', D'\rangle$ and $H'' = \langle C'', P'', D''\rangle$ defined as follows:

$$\begin{array}{l} C'_0 = \{\pi_0, \pi_1, \pi_2\};\ \ C'_1 = \{c\};\ \ C'_k = \emptyset \text{ for } k \geq 2; \\ P' = \{\langle\{\xi\}, c(\xi)\rangle\} \text{ with } \xi \in \Xi;\quad D' = \emptyset; \\ C''_0 = \{\pi_0, \pi_1, \pi_2\};\ \ C''_k = \emptyset \text{ for } k \geq 1; \\ P'' = D'' = \{\langle\{\pi_0, \pi_1\}, \pi_2\rangle, \langle\{\pi_0, \pi_2\}, \pi_1\rangle\} \end{array}$$

Clearly, MTC holds in both. In $\langle C'', P'', D''\rangle$, it holds vacuously since we have no constructors of arity greater than zero. In $\langle C', P', D'\rangle$, assume that Γ is p-deductively closed, $\Gamma, \delta_1 \vdash^d \delta_2$, and $\Gamma, \delta_2 \vdash^d \delta_1$. We have to establish $\Gamma, c(\delta_1) \vdash^d c(\delta_2)$. Since $D' = \emptyset$ and using $\Gamma, \delta_1 \vdash^d \delta_2$, we have to consider only two cases: (1) $\delta_2 \in \Gamma$: then, since Γ is p-deductively closed, $c(\delta_2) \in \Gamma$ and we are done by the extensiveness of derivation; (2) δ_2 is δ_1: then, we have the conclusion directly by the extensiveness of derivation.

But the MTC does not hold in the constrained fibring H of H' and H'' by sharing π_0, π_1, π_2. In fact, consider $\Gamma = \{\pi_0\}^{\vdash^p} = \{c^n(\pi_0) : n \geq 0\}$. Then, we have $E_\Gamma(\pi_1, \pi_2)$, but we do not have $E_\Gamma(c(\pi_1), c(\pi_2))$. ∎

Fortunately, there is a wide class of logics for which MTC is preserved by fibring as shown in the next section.

6 L.s.p.'s with equivalence. Preservation results.

Definition 6.1 (a) *A Hilbert calculus $\langle C, P, D\rangle$ is said to be a* Hilbert calculus with implication $\Rightarrow$ *iff C_2 contains $\Rightarrow$ and the* Metatheorems of Modus Ponens (MTMP) *and the* Metatheorem of Deduction (MTD) *hold: For every p-deductively closed $\Gamma \subseteq L(C, \Xi)$ and $\delta_1, \delta_2 \in L(C, \Xi)$,*

$$\frac{\Gamma \vdash^d (\delta_1 \Rightarrow \delta_2)}{\Gamma, \delta_1 \vdash^d \delta_2} \qquad \text{(MTMP)}$$

$$\frac{\Gamma, \delta_1 \vdash^d \delta_2}{\Gamma \vdash^d (\delta_1 \Rightarrow \delta_2)} \qquad \text{(MTD)}$$

(b) *A l.s.p.* $\langle C, M, A, P, D\rangle$ *is said to be a* l.s.p. with implication *iff* $\langle C, P, D\rangle$ *is a Hilbert calculus with implication.*

Proposition 6.2 *The fibring of Hilbert calculi with implication is a Hilbert calculus with implication, provided that the implication is shared.*

Proof. Let us consider the Hilbert calculi $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ both with implication $\Rightarrow$ and their fibring $\langle C, P, D\rangle$.

(1) MTMP holds in the fibring.

Note that MTMP holds in a Hilbert calculus iff $\{(\xi_1 \Rightarrow \xi_2)\}^{\vdash^p}, \xi_1 \vdash^d \xi_2$. In fact, from MTMP we get this derivation by choosing $\Gamma = \{(\xi_1 \Rightarrow \xi_2)\}^{\vdash^p}$, $\delta_1 = \xi_1$ and $\delta_2 = \xi_2$. Conversely, if $(\delta_1 \Rightarrow \delta_2)$ is derived from Γ, then, by monotonicity, it is also derived from $\Gamma \cup \{\delta_1\}$. Since Γ is p-deductively closed, the set $\{(\delta_1 \Rightarrow \delta_2)\}^{\vdash^p}$ is contained in Γ, and hence δ_2 can be derived from $\Gamma \cup \{\delta_1\}$ using the instantiation $\{(\delta_1 \Rightarrow \delta_2)\}^{\vdash^p}, \delta_1, \vdash^d \delta_2$ of $\{(\xi_1 \Rightarrow \xi_2)\}^{\vdash^p}, \xi_1, \vdash^d \xi_2$.

Therefore, it is trivial that the presence of MTMP in at least one of the two given Hilbert calculi guarantees that the MTMP holds in the fibring. Indeed, by (2.3), since there is a morphism from each of the two given Hilbert calculi to the fibring, $\{(\xi_1 \Rightarrow \xi_2)\}^{\vdash^p}, \xi_1, \vdash^d \xi_2$ will hold also in the fibring.

(2) MTD holds in the fibring.

Observe that MTD holds in a Hilbert calculus with MTMP iff (i) $\vdash^d (\xi \Rightarrow \xi)$; (ii) $\{\xi_1\}^{\vdash^p} \vdash^d (\xi_2 \Rightarrow \xi_1)$; and (iii) for each derivation rule $r = \langle\{\gamma_1, \ldots, \gamma_k\}, \gamma\rangle$, we have $\{(\xi \Rightarrow \gamma_1), \ldots, (\xi \Rightarrow \gamma_k)\}^{\vdash^p} \vdash^d (\xi \Rightarrow \gamma)$, where $\xi \in \Xi$ does not occur in r. From MTD we obtain (i) by chosing $\Gamma = \emptyset^{\vdash^p}$ and $\delta_1 = \delta_2 = \xi$, taking into account that $\emptyset^{\vdash^p} \vdash^d (\xi \Rightarrow \xi)$ is equivalent to $\vdash^d (\xi \Rightarrow \xi)$. From MTD we obtain (ii) with $\Gamma = \{\xi_1\}^{\vdash^p}$, $\delta_1 = \xi_2$ and $\delta_2 = \xi_1$. In order to obtain (iii), choose $\Gamma = \{(\xi \Rightarrow \gamma_1), \ldots, (\xi \Rightarrow \gamma_k)\}^{\vdash^p}$, $\delta_1 = \xi$ and $\delta_2 = \gamma$. Using the MTMP, from $\Gamma \cup \{\xi\}$ we derive $\gamma_1, \ldots, \gamma_k$. So, using the rule r, from $\Gamma \cup \{\xi\}$ we derive γ, and, by the MTD, we get (iii). Conversely, assume (i), (ii) and (iii) and $\Gamma, \delta_1 \vdash^d \delta_2$. We establish $\Gamma \vdash^d (\delta_1 \Rightarrow \delta_2)$ by complete induction on the length of the given derivation sequence for $\Gamma, \delta_1 \vdash^d \delta_2$. If

δ_2 is in Γ or is equal to δ_1, we use a suitable instantiation of (ii) or (i), respectively. If δ_2 results by application of a rule r from previous $\gamma_1, \dots, \gamma_k$, then, by the induction hypothesis, from Γ we derive $(\delta_1 \Rightarrow \gamma_1), \dots, (\delta_1 \Rightarrow \gamma_k)$. Since derivability implies provability and Γ is p-deductively closed, we have $\{(\delta_1 \Rightarrow \gamma_1), \dots, (\delta_1 \Rightarrow \gamma_k)\}^{\vdash^p} \subseteq \Gamma$. Thus, by (iii) and monotonicity, from Γ we derive $(\delta_1 \Rightarrow \delta_2)$.
Therefore, it is trivial that if MTD holds in both given Hilbert calculi then it will hold in their fibring. Indeed, if (i) and (ii) hold in at least one of them, they will also hold in the fibring, again using (2.3). However, (iii) is required to hold in both of them since the fibring has the derivation rules of both. For each rule $r = \langle \{\gamma_1, \dots, \gamma_k\}, \gamma \rangle$ from each of the given calculi, we shall have $\{(\xi \Rightarrow \gamma_1), \dots, (\xi \Rightarrow \gamma_k)\}^{\vdash^p} \vdash^d (\xi \Rightarrow \gamma)$ in the fibring invoking once again (2.3). ∎

Note that, in this proposition, the assumption of sharing the implication is needed only for showing the preservation of the MTD, more specifically, in order to guarantee that (iii) above encompasses all derivation rules in the fibring.

Definition 6.3 (a) *A Hilbert calculus $\langle C, P, D \rangle$ with implication $\Rightarrow$ is said to be a* Hilbert calculus with equivalence $\Leftrightarrow$ *iff* C_2 *contains* $\Leftrightarrow$ *and the* Metatheorem of Biconditionality 1 (MTB1)*,* Metatheorem of Biconditionality 2 (MTB2)*, and the* Metatheorem of Substitution of Equivalents (MTSE) *hold: For every p-deductively closed $\Gamma \subseteq L(C, \Xi)$ and $\delta_1, \delta_2, \epsilon \in L(C, \Xi)$,*

$$\frac{\Gamma \vdash^d (\delta_1 \Rightarrow \delta_2) \qquad \Gamma \vdash^d (\delta_2 \Rightarrow \delta_1)}{\Gamma \vdash^d (\delta_1 \Leftrightarrow \delta_2)} \tag{MTB1}$$

$$\frac{\Gamma \vdash^d (\delta_1 \Leftrightarrow \delta_2)}{\Gamma \vdash^d (\delta_1 \Rightarrow \delta_2) \qquad \Gamma \vdash^d (\delta_2 \Rightarrow \delta_1)} \tag{MTB2}$$

$$\frac{\Gamma \vdash^d (\delta_1 \Leftrightarrow \delta_2)}{\Gamma \vdash^d (\epsilon \Leftrightarrow \epsilon')} \tag{MTSE}$$

where ϵ' is obtained from ϵ by replacing one or more occurrences of δ_1 by δ_2.
(b) *A l.s.p. $\langle C, M, A, P, D \rangle$ is said to be a* l.s.p. with equivalence *iff $\langle C, P, D \rangle$ is a Hilbert calculus with equivalence.*

Lemma 6.4 *In a Hilbert calculus with implication, the MTSE holds iff, for every p-deductively closed $\Gamma \subseteq L(C, \Xi)$, $\delta_1, \ldots, \delta_k, \delta'_1, \ldots, \delta'_k, \in L(C, \Xi)$, $k > 0$, and $c \in C_k$*

$$\frac{\Gamma \vdash^d (\delta_i \Leftrightarrow \delta'_i) \ \text{ for } i = 1, \ldots, k}{\Gamma \vdash^d (c(\delta_1, \ldots, \delta_k) \Leftrightarrow c(\delta'_1, \ldots, \delta'_k))} \qquad (\text{MTSE}')$$

Proof. MTSE′ results by k applications of a special case of the MTSE. MTSE results from MTSE′ by a straightforward induction on the length of ϵ. (Note that in a calculus with equivalence we have $\vdash^d (\xi \Leftrightarrow \xi)$, since we have $\vdash^d (\xi \Rightarrow \xi)$ in every calculus with implication.) ∎

Proposition 6.5 *The fibring of Hilbert calculi with equivalence is a Hilbert calculus with equivalence, provided that both the implication and the equivalence are shared.*

Proof. Let us consider two Hilbert calculi $\langle C', P', D' \rangle$ and $\langle C'', P'', D'' \rangle$, both with equivalence, and their fibring $\langle C, P, D \rangle$. According to Proposition 6.2, this fibring is a Hilbert calculus with implication. It remains to verify that MTB1, MTB2 and MTSE′ are preserved (if so, thanks to the previous Lemma, so will MTSE).

(1) MTB1 holds in the fibring.

Observe that MTB1 holds in a Hilbert calculus iff $\{(\xi_1 \Rightarrow \xi_2), (\xi_2 \Rightarrow \xi_1)\}^{\vdash^p} \vdash^d (\xi_1 \Leftrightarrow \xi_2)$. This derivation follows from MTB1 for $\Gamma = \{(\xi_1 \Rightarrow \xi_2), (\xi_2 \Rightarrow \xi_1)\}^{\vdash^p}$, $\delta_1 = \xi_1$, and $\delta_2 = \xi_2$. Conversely, since derivability implies provability and Γ is p-deductively closed, the assumptions of MTB1 imply $\{(\delta_1 \Rightarrow \delta_2), (\delta_2 \Rightarrow \delta_1)\}^{\vdash^p} \subseteq \Gamma$. Hence, the instantiation $\{(\delta_1 \Rightarrow \delta_2), (\delta_2 \Rightarrow \delta_1)\}^{\vdash^p} \vdash^d (\delta_1 \Leftrightarrow \delta_2)$ of $\{(\xi_1 \Rightarrow \xi_2), (\xi_2 \Rightarrow \xi_1)\}^{\vdash^p} \vdash^d (\xi_1 \Leftrightarrow \xi_2)$ gives the conclusion of MTB1. Therefore, by (2.3), it is trivial that the presence of MTB1 in at least one of the two given calculi guarantees that the MTB1 holds in their fibring.

(2) MTB2 holds in the fibring.

Note that MTB2 holds in a Hilbert calculus iff $\{(\xi_1 \Leftrightarrow \xi_2)\}^{\vdash^p} \vdash^d (\xi_1 \Rightarrow \xi_2)$ and $\{(\xi_1 \Leftrightarrow \xi_2)\}^{\vdash^p} \vdash^d (\xi_2 \Rightarrow \xi_1)$. These derivations follows from MTB2 for $\Gamma = \{(\xi_1 \Leftrightarrow \xi_2)\}^{\vdash^p}$, $\delta_1 = \xi_1$, and $\delta_2 = \xi_2$. Conversely, as in (1), the assumption of MTB2 implies $\{(\delta_1 \Leftrightarrow \delta_2)\}^{\vdash^p} \subseteq \Gamma$, from which the conclusions of MTB2 follow by the obvious instantiations of $\{(\xi_1 \Leftrightarrow \xi_2)\}^{\vdash^p} \vdash^d (\xi_1 \Rightarrow \xi_2)$

and of $\{(\xi_1 \Leftrightarrow \xi_2)\}^{\vdash^p} \vdash^d (\xi_2 \Rightarrow \xi_1)$. Therefore, we obtain the preservation using again (2.3).

(3) MTSE$'$ holds in the fibring.

The preservation is established on the basis of (2.3), taking into account that MTSE$'$ holds in a Hilbert calculus iff $\{(\xi_i \Leftrightarrow \xi'_i) : i = 1, \ldots, k\}^{\vdash^p} \vdash^d (c(\xi_1, \ldots, \xi_k) \Leftrightarrow c(\xi'_1, \ldots, \xi'_k))$ for every constructor c of arity greater than zero. Each of these derivations follows from MTSE$'$ by choosing $\Gamma = \{(\xi_i \Leftrightarrow \xi'_i) : i = 1, \ldots, k\}^{\vdash^p}$, $\delta_i = \xi_i$ and $\delta'_i = \xi'_i$. In fact, each $(\xi_i \Leftrightarrow \xi'_i)$ is derived from Γ and hence, using the MTSE$'$, we establish that $(c(\xi_1, \ldots, \xi_k) \Leftrightarrow c(\xi'_1, \ldots, \xi'_k))$ is derived from Γ. Conversely, assume that each $(\delta_i \Leftrightarrow \delta'_i)$ is derived from the p-deductively closed set Γ. This implies that $\{(\delta_i \Leftrightarrow \delta'_i) : i = 1, \ldots, k\}^{\vdash^p} \subseteq \Gamma$. Using the appropriate instantiation of $\{(\xi_i \Leftrightarrow \xi'_i) : i = 1, \ldots, k\}^{\vdash^p} \vdash^d (c(\xi_1, \ldots, \xi_k) \Leftrightarrow c(\xi'_1, \ldots, \xi'_k))$ we establish that $(c(\delta_1, \ldots, \delta_k) \Leftrightarrow c(\delta'_1, \ldots, \delta'_k))$ is derived from Γ. ■

Note that the assumption of sharing the equivalence is needed only for showing the preservation of the MTSE$'$, more specifically, in order to guarantee that $\{(\xi_i \Leftrightarrow \xi'_i) : i = 1, \ldots, k\}^{\vdash^p} \vdash^d (c(\xi_1, \ldots, \xi_k) \Leftrightarrow c(\xi'_1, \ldots, \xi'_k))$ holds for all non 0-ary constructors in the fibring.

Observation. The previous preservation results could have been proved also for the stronger version of the metatheorems considered above, in which Γ ranges over arbitrary subsets of $L(C, \Xi)$, not only over p-deductively closed sets. The proofs of the new preservation properties can be obtained from those of Proposition 6.2 and 6.5 by simply replacing every set of the form $\Sigma^{\vdash^p}$ by Σ.

Theorem 6.6 *The MTC holds in a Hilbert calculus with equivalence.*

Proof. Assume that Γ is p-deductively closed and that $\Gamma, \delta_i \vdash^d \delta'_i$ and $\Gamma, \delta'_i \vdash^d \delta_i$ for $i = 1, \ldots, k$. Then: 1) $\Gamma \vdash^d (\delta_i \Rightarrow \delta'_i)$ and $\Gamma \vdash^d (\delta'_i \Rightarrow \delta_i)$ by MTD, 2) $\Gamma \vdash^d (\delta_i \Leftrightarrow \delta'_i)$ by MTB1, 3) $\Gamma \vdash^d (c(\delta_1, \ldots, \delta_k) \Leftrightarrow c(\delta'_1, \ldots, \delta'_k))$ by MTSE$'$, 4) $\Gamma \vdash^d (c(\delta_1, \ldots, \delta_k) \Rightarrow c(\delta'_1, \ldots, \delta'_k))$ by MTB2, and 5) $\Gamma, c(\delta_1, \ldots, \delta_k) \vdash^d c(\delta'_1, \ldots, \delta'_k)$, which is the thesis, by MTMP. ■

This theorem provides a very wide class of Hilbert calculi in which MTC holds. Classical, intuitionistic, and modal logics are logics with equivalence and hence they belong to this class. With respect to this, it is important to observe that these logics can be viewed as extensions of the logic called *basic logic* in [SBF], which turns out to be a logic with equivalence as well (where $\Leftrightarrow$ is defined on the basis of $\Rightarrow$ and of $\&$). This implies that other extensions of basic logic (like, e.g., linear logic and quantum logic) are also logics with equivalence and hence they enjoy MTC.

Proposition 6.5 shows that the class of Hilbert calculi with equivalence is closed under fibrings if implication and equivalence are shared, and hence this kind of fibrings preserves also MTC. Incidentally, Example 5.8 shows that the calculi H' and H'' considered therein, as well as their fibring H, cannot be endowed with equivalence without changing the provability and derivability in a substantial way.

A trivial consequence of Theorems 4.6 and 5.7 is that p-completeness and d-completeness of full l.s.p.'s with MTC is preserved by fibring whenever MTC is preserved. Thus, Proposition 6.5 and Theorem 6.6 imply the following theorem of preservation of completeness, which holds for all logics considered above.

Theorem 6.7 *The fibring by sharing implication and equivalence of two full l.s.p.'s with equivalence is p-complete and d-complete.*

7 Concluding remarks

We obtained a (strong) completeness theorem for logic systems with full general semantics and with congruence, using an adapted Henkin construction. As a corollary, completeness was shown to be preserved by fibring logics with congruence provided that congruence is retained in the resulting logic. Although congruence is not always preserved by fibring, we were able to establish a sufficient condition for its preservation. This condition holds in every extension of basic logic, including classical, intuitionistic, minimal and all modal propositional logics.

General semantics was motivated by realizing that (weak) completeness is not always preserved when fibring logics with standard (non general) semantics. In the end, working with general semantics allowed us to obtain the envisaged preservation result for strong completeness. It remains an open problem if the positive result in Subsection 4.1 concerning the preservation of weak completeness can be improved.

General semantics is also interesting because it allows the faithful representation of the semantics of intuitionistic logic (contrarily to standard semantics as discussed in [SSC99]). However, the fibring of intuitionistic logic with classical logic still leads to the collapsing of all connectives into classical logic. This is a well known problem with current accounts of fibring [Gab99]. At the proof-theoretic level it is possible to avoid the collapse by constraining the use of the axiom $(\xi_1 \to (\xi_2 \to \xi_1))$ as proposed in [dCH96]. But at the model-theoretic level it is an open problem how to avoid the collapse.

It should be noted that it is necessary to be able to impose requirements in inference rules in order to be able to constrain their use. Therefore, the solution proposed in [dCH96] implies a more complex notion of Hilbert calculus, along the same lines in [SSCM99] where inference rules with requirements were introduced for another reason (for dealing with the problems of logics with variables, terms and binding operators).

Future work is planned at extending the transference result established in this paper to such more complex logics with variables, terms and binding operators.

Acknowledgments

The authors wish to express their deep gratitude to an anonymous referee of a previous version of the paper who made valuable suggestions regarding the MTC making our results applicable to a much wider class of logics. The authors are also grateful to Carlos Caleiro, José Carmo and Giovanni Sambin for many helpful discussions on relevant topics. This work was partially supported by *Fundação para a Ciência e a Tecnologia*, the PRAXIS XXI Projects PCEX/P /MAT/46/96 ACL, PRAXIS/P/MAT/10002/1998 ProbLog and 2/2.1/TIT/ 1658/95 LogComp, as well as by the ESPRIT IV Working Groups 22704 ASPIRE and 23531 FIREworks.

References

[BdR97] P. Blackburn and M. de Rijke. Why combine logics? *Studia Logica*, 59(1):5–27, 1997.

[Ben83] J. van Benthem. *The Logic of Time*. D. Reidel P. C., Dordrecht, 1983.

[Ben85] J. van Benthem. *Modal Logic and Classical Logic*. Bibliopolis, Napoli, 1985.

[BW90] M. Barr and C. Wells. *Category Theory for Computing Science*. Prentice Hall, 1990.

[CZ97] A. Chagrov and M. Zakhryaschev. *Modal Logic*. Clarendon Press, Oxford, 1997.

[dCH96] L. F. del Cerro and A. Herzig. Combining classical and intuitionistic logic. In F. Baader and K. Schulz, editors, *Frontiers of Combining Systems*, pages 93–102. Kluwer Academic Publishers, 1996.

[Gab96a] D. Gabbay. Fibred semantics and the weaving of logics: part 1. *Journal of Symbolic Logic*, 61(4):1057–1120, 1996.

[Gab96b] D. Gabbay. An overview of fibred semantics and the combination of logics. In F. Baader and K. Schulz, editors, *Frontiers of Combining Systems*, pages 1–55. Kluwer Academic Publishers, 1996.

[Gab99] D. Gabbay. *Fibring logics*. Oxford University Press, 1999.

[HC96] G. E. Hughes and M. J. Cresswell. *A New Introduction to Modal Logic*. Routledge, London, 1996.

[Hen50] L. Henkin. Completeness in the theory of types. *Journal of Symbolic Logic*, 15:81–91, 1950.

[KW91] M. Kracht and F. Wolter. Properties of independently axiomatizable bi-modal logics. *Journal of Symbolic Logic*, 56(4):1469–1485, 1991.

[SBF] G. Sambin, G. Battilotti, and C. Faggian. Basic logic: Reflection, symmetry, visibility. *Journal of Symbolic Logic*. To appear.

[SSC97a] A. Sernadas, C. Sernadas, and C. Caleiro. Synchronization of logics. *Studia Logica*, 59(2):217–247, 1997.

[SSC97b] A. Sernadas, C. Sernadas, and C. Caleiro. Synchronization of logics with mixed rules: Completeness preservation. In *Algebraic Methodology and Software Technology - AMAST'97*, pages 465–478. Springer-Verlag, LNCS 1349, 1997.

[SSC99] A. Sernadas, C. Sernadas, and C. Caleiro. Fibring of logics as a categorial construction. *Journal of Logic and Computation*, 9(2):149–179, 1999.

[SSCM99] A. Sernadas, C. Sernadas, C. Caleiro, and T. Mossakowski. Categorial fibring of logics with terms and binding operators. In M. de Rijke and D. Gabbay, editors, *Frontiers of Combining Systems - FroCoS'98*. Research Studies Press, 1999. In print.

[Wol96] F. Wolter. A counterexample in tense logic. *Notre Dame Journal of Formal Logic*, 37(2):167–173, 1996.