Fibring of Logics as a Categorial Construction

Amílcar Sernadas, Cristina Sernadas, Carlos Caleiro
Departamento de Matemática, IST
Av. Rovisco Pais, 1096 Lisboa, Portugal
email:{acs,css,ccal}@math.ist.utl.pt

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Abstract

Much attention has been given recently to the mechanism of fibring of logics, allowing free mixing of the connectives and using proof rules from both logics. Fibring seems to be a rather useful and general form of combination of logics that deserves detailed study. It is now well understood at the proof-theoretic level. However, the semantics of fibring is still insufficiently understood. Herein we provide a categorial definition of both proof-theoretic and model-theoretic fibring for logics without terms. To this end, we introduce the categories of Hilbert calculi, interpretation systems and logic system presentations. By choosing appropriate notions of morphism it is possible to obtain pure fibring as a coproduct. Fibring with shared symbols is then easily obtained by cocartesian lifting from the category of signatures. Soundness is shown to be preserved by these constructions. We illustrate the constructions within propositional modal logic.

Keywords: logic morphism, combination of logics, fibring, fibred semantics, preservation of soundness, modal logic.

1 Introduction

The problem of combining logics has attracted attention from both a theoretical point of view and a practical point of view. For a balanced presentation of the main theoretical motivations and issues see [5]. The mechanism of fibring two logics by allowing the free mixing of the symbols from both logics and the reasoning by applying inference rules from both logics is well understood [8]. Fibred semantics is another matter: as described in [9], it has an operational flavour and it is not clear at all the structure of the resulting models.

We present a novel semantics of fibring with explicit models that follows the intuitions on fibred semantics. To this end, the notion of interpretation system that we propose seems to have the right level of abstraction. However, in this paper we concentrate only on propositional-based logics, i.e., logics without terms and variable binding operators like quantification. The proposed approach seems to be workable in the more general case, but its complexity and the usefulness of propositional-based logics well justify presenting only the results on the simpler case.

We also present fibring at both proof-theoretic and model-theoretic levels and in two forms (unconstrained and constrained by sharing symbols) as categorial constructions, using coproducts and cocartesian liftings. Note that we allow the sharing of any symbols of the same arity, including logical connectives and modalities.

The use of the categorial machinery may discourage some readers but it is well justified. Indeed, we were very much helped by the categorial imperatives when fine tuning the abstraction for characterizing the semantics of fibring. Furthermore, the preservation results are clear corollaries of general properties of the morphisms. The usefulness of the categorial techniques in the area of the combination of logics has already been recognized in [12, 17, 18]. The notion of institution [10] is another important source for those interested in the use of categories in logic.

The category theoretic concepts we use can be found in the first few chapters of any textbook on category theory. We suggest [1], and specially [3] as far as (co)cartesian liftings are concerned. When establishing the category of interpretation systems we face some foundational issues that we do not discuss herein. The interested reader is referred to MacLane's standpoint on the matter [15]. For other approaches see [6].

In section 2, after establishing the appropriate category of Hilbert calculi, we show that unconstrained fibring appears as a coproduct. And we show that constrained fibring (by sharing symbols) appears as a cocartesian lifting from the category of signatures.

In section 3, we repeat the process for interpretation systems. Fibring at the semantic level is much more difficult to characterize but again we obtain it as a coproduct (in the unconstrained case) and as a cocartesian lifting (in the constrained case).

In section 4, we put together inference rules and models in the notion of logic system presentation. It is then straightforward to capitalize on the results of the two previous sections in order to characterize fibring with proofs and semantics hand in hand. In this section we consider several examples of logic system presentations and illustrate fibring within modal logic. We also address the problem of fibring intuitionistic and classical logics.

In section 5, after some preliminary results showing how derivation and entailment are transferred from the given logics into the resulting logic, we show that soundness is preserved.

We conclude the paper with some remarks on how to extend the results to logics with terms and variable binding operators, the open problem of proving the preservation by fibring of model existence and completeness, and the development of a generalized notion of interpretation system capable of representing general frames [20]. We also briefly discuss possible applications of the proposed techniques in software engineering and artificial intelligence.

2 Fibring of Hilbert calculi

In this section we present a proof-theoretic account of fibring as a categorial construction. We start by establishing the notion of signature that we need. Then, we introduce Hilbert calculi distinguishing between proof rules and derivation rules. This notion is rather general in the sense that it covers many kinds of calculi as long as they do not include terms and variable binding operators like quantification and abstraction.

After establishing the appropriate category of Hilbert calculi, we show that unconstrained fibring appears as a coproduct. And we show that constrained fibring (by sharing symbols) appears as a cocartesian lifting from the category of signatures.

Along the way, we compare our definitions with the intuitive notion of fibring as described for instance in [8, 9].

2.1 Signatures

At the syntactic level, the basic idea of fibring relies on the assumption that formulae in each logic are inductively built up from a certain set of atoms using constructors. In our notion of signature, we take atoms as nullary constructors. Therefore, we make no formal distinction between say propositional symbols and logical connectives.

Definition 2.1 A *signature* is a family $C = \{C_k\}_{k \in \mathbb{N}}$ where C_k is a set for every $k \in \mathbb{N}$.

The elements of C_k are called *constructors* of arity k. In the sequel, we assume fixed a set Ξ (of schema *variables*).

Definition 2.2 The set of *schema formulae* $L(C, \Xi)$ generated by a signature C and Ξ is inductively defined as follows:

- $c \in L(C, \Xi)$ provided that $c \in C_0$;
- $\xi \in L(C, \Xi)$ provided that $\xi \in \Xi$;
- $c(\gamma_1, \ldots, \gamma_k) \in L(C, \Xi)$ provided that $c \in C_k$ and $\gamma_1, \ldots, \gamma_k \in L(C, \Xi)$.

The elements of $L(C, \emptyset)$ are simply called *formulae*. We will use δ, γ for schema formulae and Δ, Γ for sets of schema formulae in $L(C, \Xi)$. Analogously, we will use φ, ψ for formulae and Φ, Ψ for sets of formulae.

Definition 2.3 A *substitution* on $L(C, \Xi)$ is a map $\sigma : \Xi \to L(C, \Xi)$. The *instance* of a schema formula γ by a substitution σ, denoted by $\gamma\sigma$, is the (schema) formula obtained from γ by simultaneously replacing each occurrence of ξ in γ by $\sigma(\xi)$ for every $\xi \in \Xi$.

We extend the notion of instance to sets of schema formulae: $\Gamma\sigma$ denotes the set $\{\gamma\sigma : \gamma \in \Gamma\}$.

Definition 2.4 A *signature morphism*

$$h : C \to C'$$

is a family $h = \{h_k\}_{k \in I\!N}$ where each h_k is a map from C_k to C'_k.

Definition 2.5 A signature morphism $h : C \to C'$ induces the following map $h : L(C, \Xi) \to L(C', \Xi)$:

- $h(c) = h_0(c)$;
- $h(\xi) = \xi$;
- $h(c(\gamma_1, \ldots, \gamma_k)) = h_k(c)(h(\gamma_1), \ldots, h(\gamma_k))$.

Prop/Definition 2.6 Signatures and their morphisms constitute the category *Sig*.

Clearly, *Sig* is the well known $Set/I\!N$ slice category of $I\!N$-indexed sets and index preserving maps. Therefore:

Proposition 2.7 The category *Sig* is small cocomplete.

In particular coproducts and pushouts exist in *Sig*. Both kinds of colimits will be used in the sequel. Coproducts allow us to put together two signatures without any sharing of constructors. Pushouts will be used for putting constructors together. We review here the coproduct, the special case of the pushout construction that we need, and the coequalizer (of course, all up to isomorphism).

- The coproduct of two signatures C' and C'' is the signature $C' \oplus C''$ endowed with injections $i' : C' \to C' \oplus C''$ and $i'' : C'' \to C' \oplus C''$ such that, for each $k \in I\!N$:
  - $(C' \oplus C'')_k$ is the disjoint union of C'_k and C''_k;
  - i'_k and i''_k are the injections of C'_k and C''_k into $(C' \oplus C'')_k$, respectively.
- The pushout of two injective signature morphisms with the same source $f' : C \to C'$ and $f'' : C \to C''$ is the signature $C' \overset{f'Cf''}{\oplus} C''$ endowed with the morphisms $g' : C' \to C' \overset{f'Cf''}{\oplus} C''$ and $g'' : C'' \to C' \overset{f'Cf''}{\oplus} C''$ such that, for each $k \in I\!N$:
  - $(C' \overset{f'Cf''}{\oplus} C'')_k$ is $C_k \cup i'_k(C'_k \setminus f'_k(C_k)) \cup i''_k(C''_k \setminus f''_k(C_k))$;
  - $g'_k(c') = \begin{cases} i'_k(c') & \text{if } c' \notin f'_k(C_k) \\ f'^{-1}_k(c') & \text{otherwise} \end{cases}$ and similarly for g''_k.
- The coequalizer of two signature morphisms with same source and target $f, g : C \to C'$ is the signature $C'/\equiv^{f,g}$ endowed with the morphism $q : C' \to C'/\equiv^{f,g}$ such that, for each $k \in I\!N$:

- $(C'/{\equiv^{f,g}})_k$ is the quotient set $C'_k/\equiv^{f,g}_k$, with $\equiv^{f,g}_k$ the smallest equivalence relation on C'_k containing $\{\langle f_k(c), g_k(c)\rangle : c \in C_k\}$;
- $q_k(c')$ is the $\equiv^{f,g}_k$ equivalence class of $c' \in C'_k$.

Note that pushouts exist even if we do not assume that f' and f'' are injective, but for the purposes of sharing we only use injective maps. Moreover, general pushouts are harder to present. Note also that every coequalizer is a family of surjective maps.

We shall use, later on, the fact that the pushout of two morphisms $f' : C \to C'$ and $f'' : C \to C''$ can be obtained by first making the coproduct $C' \oplus C''$ endowed with the injections $i' : C' \to C' \oplus C''$ and $i'' : C'' \to C' \oplus C''$ and then obtaining the coequalizer of $i' \circ f'$ and $i'' \circ f''$.

2.2 Hilbert calculi

We now introduce the notion of Hilbert calculus as an abstraction capturing the proof-theoretic aspects of a logic at the level of detail that we need: language constructors plus inference rules. It is convenient to distinguish among proof rules used for proving theorems and derivation rules for deriving consequences of a given set of hypotheses. This distinction is reflected at the model-theoretic level as we shall see in section 3.

Definition 2.8 A *Hilbert calculus* is a triple $\langle C, P, D\rangle$ where:

- C is a signature;
- $P \subseteq \wp_{\text{fin}}(L(C,\Xi)) \times L(C,\Xi)$;
- $D \subseteq (\wp_{\text{fin}}(L(C,\Xi)) \setminus \emptyset) \times L(C,\Xi)$

such that

- $D \subseteq P$.

Each element $r = \langle Prem(r), Conc(r)\rangle$, of P is a *rule schema*: $Prem(r)$ is the (finite) set of premises and $Conc(r)$ is the conclusion. If $Prem(r) = \emptyset$ then r is said to be an *axiom schema*; otherwise r is said to be a *proof rule schema*. Each element of D is also said to be a *derivation rule schema*. Clearly, P and D are, respectively, the proof-theoretic counterparts of validity and entailment.

Example 2.9 *Propositional modal logic.* A propositional (uni)modal Hilbert calculus over Π is a triple $\langle C, P, D\rangle$ such that:

- $C_0 = \Pi$, $C_1 = \{\neg, \Box\}$, $C_2 = \{\Rightarrow\}$;
- $P \supseteq \{\langle\emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle\emptyset, ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))\rangle,$
  $\langle\emptyset, (((\neg\,\xi_1) \Rightarrow (\neg\,\xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle\emptyset, ((\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2)))\rangle,$
  $\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle,$
  $\langle\{\xi_1\}, (\Box\xi_1)\rangle\}$;

- $D = \{\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\}$.

Note that we just require the axiom of normality (K) and necessitation [11].

In the sequel, unless otherwise stated, we assume fixed a Hilbert calculus $\langle C, P, D\rangle$.

Definition 2.10 We say that δ is *provable* from Γ iff there is a sequence $\gamma_1 \ldots \gamma_m \in L(C, \Xi)^+$ such that:

- γ_m is δ;
- for each $i = 1, \ldots, m$:
  - either $\gamma_i \in \Gamma$;
  - or there is a substitution σ_i such that γ_i is $Conc(r)\sigma_i$ for some $r \in P$ and every element of $Prem(r)\sigma_i$ occurs in the sequence $\gamma_1, \ldots, \gamma_{i-1}$.

When $\Gamma = \emptyset$ we just say that δ is provable. Note that we do not allow substitutions on hypotheses (the elements of Γ). Indeed, such substitutions do not make sense. For instance, from $\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}$ we want to be able to prove ξ_2, but not every formula as it would be possible by substitution on ξ_1. This remark also applies to derivability as defined below.

Definition 2.11 We say that δ is *derivable* from Γ iff there is a sequence $\gamma_1 \ldots \gamma_m \in L(C, \Xi)^+$ such that:

- γ_m is δ;
- for each $i = 1, \ldots, m$:
  - either $\gamma_i \in \Gamma$;
  - or γ_i is provable;
  - or there is a substitution σ_i such that γ_i is $Conc(r)\sigma_i$ for some $r \in D$ and every element of $Prem(r)\sigma_i$ occurs in the sequence $\gamma_1, \ldots, \gamma_{i-1}$.

When $\Gamma = \emptyset$ we just say that δ is derivable.

Prop/Definition 2.12 $\langle C, P, D\rangle$ induces:

- the *schema derivation operator* $.^{\vdash_\Xi} : 2^{L(C,\Xi)} \to 2^{L(C,\Xi)}$ such that
$$\Gamma^{\vdash_\Xi} = \{\delta \in L(C, \Xi) : \delta \text{ is derivable from } \Gamma\};$$
- the *derivation operator* $.^{\vdash} : 2^{L(C,\emptyset)} \to 2^{L(C,\emptyset)}$ such that
$$\Psi^{\vdash} = \{\varphi \in L(C, \emptyset) : \varphi \text{ is derivable from } \Psi\};$$
- the *theorem schemata* $\emptyset^{\vdash_\Xi} \subseteq L(C, \Xi)$;
- the *theorems* $\emptyset^{\vdash} \subseteq L(C, \emptyset)$.

Moreover, schema derivation and derivation are closure operators.

Proof:
Extensiveness, idempotence and monotonicity follow easily. And note that $\Psi^{\vdash} = \Psi^{\vdash_\Xi} \cap L(C, \emptyset)$. QED

Proposition 2.13 The following assertions hold:

- δ is provable from Γ whenever $\delta \in \Gamma^{\vdash_\Xi}$;
- δ is provable iff $\delta \in \emptyset^{\vdash_\Xi}$;
- $\delta\sigma$ is provable from $\Gamma\sigma$ whenever δ is provable from Γ;
- $\delta\sigma \in \emptyset^{\vdash_\Xi}$ whenever $\delta \in \emptyset^{\vdash_\Xi}$;
- $\delta\sigma \in \Gamma\sigma^{\vdash_\Xi}$ whenever $\delta \in \Gamma^{\vdash_\Xi}$.

Proof:
Straightforward, since $D \subseteq P$, and using the fact that $(\gamma\epsilon)\sigma = \gamma(\epsilon\sigma)$ with $\epsilon\sigma = \lambda\xi.\epsilon(\xi)\sigma$. QED

We are now ready to put forward the envisaged notion of Hilbert calculus morphism: a signature morphism that preserves the inference rules.

Definition 2.14 A *Hilbert calculus morphism*

$$h : \langle C, P, D\rangle \to \langle C', P', D'\rangle$$

is a morphism $h : C \to C'$ in *Sig* such that:

- $h(Conc(r))$ is provable from $h(Prem(r))$ in $\langle C', P', D'\rangle$ for every $r \in P$;
- $h(Conc(r))$ is derivable from $h(Prem(r))$ in $\langle C', P', D'\rangle$ for every $r \in D$.

It is clear that provability is just needed for rules in $P \setminus D$. If $r \in D \subseteq P$, its provability follows from its derivability.

Proposition 2.15 Let $h : \langle C, P, D\rangle \to \langle C', P', D'\rangle$ be a Hilbert calculus morphism. Then:

- $h(\delta)$ is provable from $h(\Gamma)$ whenever δ is provable from Γ;
- $h(\delta) \in \emptyset^{\vdash'_\Xi}$ whenever $\delta \in \emptyset^{\vdash_\Xi}$;
- $h(\delta) \in h(\Gamma)^{\vdash'_\Xi}$ whenever $\delta \in \Gamma^{\vdash_\Xi}$.

Proof:
Straightforward, by making use of the fact that $h(\gamma\sigma) = h(\gamma)\sigma^h$ with $\sigma^h = \lambda\xi.h(\sigma(\xi))$. QED

The proposition above is needed to establish the category of Hilbert calculi and it will also be useful later on.

Prop/Definition 2.16 Hilbert calculi and their morphisms constitute the category *Hil*.

In the sequel we shall need a mechanism for constructing a new Hilbert calculus from a given one along a given signature morphism. To this end we need the following (forgetful) functor:

Prop/Definition 2.17 The maps:

- $N(\langle C, P, D\rangle) = C$;
- $N(h : \langle C, P, D\rangle \to \langle C', P', D'\rangle) = h$

constitute the functor $N : Hil \to Sig$.

Proposition 2.18 For each $\langle C, P, D\rangle$ in *Hil* and each morphism $h : C \to C'$ in *Sig*, the morphism

$$h : \langle C, P, D\rangle \to \langle C', h(P), h(D)\rangle$$

is cocartesian by N for h on $\langle C, P, D\rangle$.

Proof:
1. h is obviously a morphism in *Hil*.
2. Universal property.
Assume that $g : \langle C, P, D\rangle \to \langle C'', P'', D''\rangle$ is a morphism in *Hil* and $f : C' \to C''$ is a morphism in *Sig* such that $f \circ h = g$. It is trivial to see that f is a morphism in *Hil*: $f(h(r)) = g(r)$ for every $r \in P \cup D$, and hence provability or derivability in $\langle C'', P'', D''\rangle$ are guaranteed, respectively, since g is a morphism. Of course, f is the only morphism in *Hil* such that $N(f) = f$ and $f \circ h = g$. QED

We denote the codomain of the cocartesian morphism above by $h(\langle C, P, D\rangle)$.

2.3 Unconstrained fibring

Intuitively, in the unconstrained fibring of two Hilbert calculi we have the constructors and the inference rules from both calculi. To this end, the schema variables are essential to making the construction precise.

Definition 2.19 Let $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ be Hilbert calculi. Then, their *unconstrained fibring* is

$$\langle C', P', D'\rangle \oplus \langle C'', P'', D''\rangle = \langle C' \oplus C'', i'(P') \cup i''(P''), i'(D') \cup i''(D'')\rangle$$

where i', i'' are the injections of the coproduct $C' \oplus C''$.

For examples, we refer the reader to section 4.

Proposition 2.20 Unconstrained fibrings are coproducts in *Hil*.

Proof:
1. The injections i' and i'' are morphisms in *Hil*.
2. Universal property.
Let $h' : \langle C', P', D' \rangle \to \langle C''', P''', D''' \rangle$, $h'' : \langle C'', P'', D'' \rangle \to \langle C''', P''', D''' \rangle$ be any morphisms in *Hil*. Let $k : C' \oplus C'' \to C'''$ be the unique morphism in *Sig* such that $k \circ i' = f'$ and $k \circ i'' = f''$. Trivially, k is a morphism in *Hil* and it is the unique such that $k \circ i' = f'$ and $k \circ i'' = f''$. QED

The coproduct construction captures the intuitive idea that fibring should extend the given logics in a minimal and conservative way.

2.4 Constrained fibring

In order to constrain the fibring (imposing some interaction between the two given Hilbert systems) we may have two approaches that can be used together: sharing of constructors and addition of new rules.

The technique of cocartesian lifting provides the means for sharing constructors: it provides a canonical Hilbert system guided by the sharing defined at the signature level.

Definition 2.21 Let $\langle C', P', D' \rangle$ and $\langle C'', P'', D'' \rangle$ be Hilbert calculi and $f' : C \to C'$, $f'' : C \to C''$ be injective signature morphisms. Then, their *constrained fibring by sharing* is:

$$\langle C', P', D' \rangle \overset{f'Cf''}{\oplus} \langle C'', P'', D'' \rangle = q(\langle C', P', D' \rangle \oplus \langle C'', P'', D'' \rangle)$$

where $q : C' \oplus C'' \to C' \overset{f'Cf''}{\oplus} C''$ is the coequalizer of $i' \circ f' : C \to C' \oplus C''$ and $i'' \circ f'' : C \to C' \oplus C''$.

For examples, we refer the reader to section 4.

Note that we allow sharing of both atoms (propositional symbols) and logic operators (connectives and modalities). Sharing of logic operators is reflected not only in the syntax of the fibred logic but it can also, in consequence, provide a way to impose some degree of interaction between the two logics. For instance, if we are fibring two modal logics we may impose that the two boxes are identified, obtaining a box that inherits the properties of the two original ones.

Adding possibly mixed rules afterwards raises no technical challenge at the pure proof-theoretic level and we omit further discussion on the subject. See [18] for a study of such rules but in the context of a simpler mechanism for combining logics.

3 Fibring of interpretation systems

In this section we present a model-theoretic account of fibring as a categorial construction. We start by introducing interpretation systems: intuitively, an interpretation system provides for a given signature of constructors a class of models and the means for interpreting the constructors in each model over a domain of points. Then, the evaluation of formulae can be carried out inductively on their structure using the interpretation given for the constructors.

After establishing the appropriate category of interpretation systems, we again show that unconstrained fibring appears as a coproduct. And we again show that constrained fibring (by sharing symbols) appears as a cocartesian lifting from the category of signatures.

Along the way, we compare our definitions with the operational notion of fibred semantics described in [8, 9].

3.1 Interpretation systems

We now introduce the notion of interpretation system as an abstraction capturing the model-theoretic aspects of a logic at the required level of detail: language constructors plus models with associated domains and interpretation maps for the constructors. To this end, it is convenient to start by defining what we mean by a structure on a given signature:

Definition 3.1 Given a signature C, a *C-structure* is a pair $\langle U, \nu \rangle$ where:

- U is a nonempty set;
- $\nu = \{\nu_k\}_{k \in I\!N}$ with each $\nu_k : C_k \to [(2^U)^k \to 2^U]$.

We denote by $Str(C)$ the class of all C-structures.

Definition 3.2 A *pre-interpretation system* is a triple $\langle C, M, A \rangle$ where:

- C is a signature;
- M is a class;
- $A : M \to Str(C)$ is a map.

The elements of M are the models and A maps each model to an "algebra" over C. In the sequel, we use the following notation: $A(m)$ for $\langle U_m, \nu_m \rangle$.

Definition 3.3 A pre-interpretation system $\langle C, M, A \rangle$ is an *interpretation system* provided that, for every $m_1 \in M$ and every bijection $f : U_{m_1} \cong V$ there exists $m_2 \in M$ such that $U_{m_2} = V$ and for every $k \in I\!N$, $c \in C_k$ and $b \in (2^V)^k$:

$$\nu_{m_2 k}(c)(b)(f(u)) = \nu_{m_1 k}(c)(b \circ f)(u).$$

We use the notations $m_2 = f(m_1)$ and $\langle U_{m_2}, \nu_{m_2} \rangle = f(\langle U_{m_1}, \nu_{m_1} \rangle)$.

Models m_1 and m_2 as above are called *equivalent*.

It is a simple task to close a pre-interpretation system with respect to model equivalence. The resulting interpretation system yields exactly the same entailment operators. A detailed proof will be given later on, still in this section.

Prop/Definition 3.4 Let $\langle C, M, A\rangle$ be a pre-interpretation system. Then, we define its *enrichment* to be the interpretation system $\langle C, \overline{M}, \overline{A}\rangle$ where:

- $\overline{M}$ is the class of all pairs $\langle m, f\rangle$ where $m \in M$ and $f : U_m \cong V$ is a bijection;
- $\overline{A}(\langle m, f\rangle) = f(A(m))$.

Example 3.5 *Propositional modal logic.* A propositional (uni)modal interpretation system over Π is a triple $\langle C, M, A\rangle$ defined as follows:

- C as in Example 2.9;
- M is a "rich" class of Kripke structures of the form $\langle W, S, V\rangle$ where W is a nonempty set, S is a relation over W and $V : \Pi \to 2^W$ is a valuation map;
- $A(\langle W, S, V\rangle) = \langle W, \nu\rangle$ where:
  - $\nu_0(\pi) = V(\pi)$;
  - $\nu_1(\neg)(b) = \lambda w \,.\, 1 - b(w)$;
  - $\nu_1(\Box)(b) = \lambda w \,.\, \Pi_{w' : \langle w, w'\rangle \in S}\ b(w')$;
  - $\nu_2(\Rightarrow)(b, b') = \lambda w \,.\, b(w) \leq b'(w)$.

Note that many logics immediately yield interpretation systems (i.e., they are closed under equivalence of models). That is the case, for instance, of propositional modal logic (above). Linear propositional temporal logic, however, does not meet this requirement. Its models are usually seen as being maps from its set of propositional symbols to $2^{\mathbb{N}}$ which corresponds to $\mathbb{N}$ structures. But clearly, any other denumerable set isomorphic to $\mathbb{N}$ would do, as a carrier.

We can extract out of an interpretation system the alternative satisfaction relations (between models and formulae) and hence the corresponding entailment operators.

In the sequel, unless otherwise stated, we assume fixed a (pre-)interpretation system $\langle C, M, A\rangle$.

Definition 3.6 A schema variable *assignment* into a model $m \in M$ is a map $\alpha : \Xi \to 2^{U_m}$. Then,

- the *schema interpretation map* by m for α is the map

$$[\![.]\!]_{\Xi}^{m\alpha} : L(C, \Xi) \to 2^{U_m}$$

  defined as follows:

  - $[\![\xi]\!]_{\Xi}^{m\alpha} = \alpha(\xi)$;
  - $[\![c]\!]_{\Xi}^{m\alpha} = \nu_{m0}(c)$;
  - $[\![c(\gamma_1, \ldots, \gamma_k)]\!]_{\Xi}^{m\alpha} = \nu_{mk}(c)([\![\gamma_1]\!]_{\Xi}^{m\alpha}, \ldots, [\![\gamma_k]\!]_{\Xi}^{m\alpha})$;

- the *interpretation map* by m is the map

$$[\![.]\!]^m : L(C,\emptyset) \to 2^{U_m}$$

such that $[\![.]\!]^m(\varphi) = [\![.]\!]^{m\alpha}_{\Xi}(\varphi)$ for any assignment α into m;

- the *schema contextual satisfaction relation* is as follows:

$$m\alpha u \Vdash_{p\Xi} \gamma \;\text{ iff }\; [\![\gamma]\!]^{m\alpha}_{\Xi}(u) = 1;$$

- the *schema floating satisfaction relation* is as follows:

$$m\alpha \Vdash_{\Xi} \gamma \;\text{ iff }\; [\![\gamma]\!]^{m\alpha}_{\Xi}(u) = 1 \text{ for every } u \in U_m;$$

- the *contextual satisfaction relation* is as follows:

$$mu \Vdash \varphi \;\text{ iff }\; [\![\varphi]\!]^m(u) = 1;$$

- the *floating satisfaction relation* is as follows:

$$m \Vdash \varphi \;\text{ iff }\; [\![\varphi]\!]^m(u) = 1 \text{ for every } u \in U_m.$$

Clearly, the interpretation of formulae (without schema variables) does not depend on the particular assignment chosen.

Prop/Definition 3.7 $\langle C, M, A\rangle$ induces:

- the *schema (contextual) entailment operator* $.^{\vDash_\Xi} : 2^{L(C,\Xi)} \to 2^{L(C,\Xi)}$ such that

$$\Gamma^{\vDash_\Xi} = \{\delta \in L(C,\Xi) : m\alpha u \Vdash_{\Xi} \delta \text{ whenever } m\alpha u \Vdash_{\Xi} \gamma \text{ for each } \gamma \in \Gamma\};$$

- the *(contextual) entailment operator* $.^{\vDash} : 2^{L(C,\emptyset)} \to 2^{L(C,\emptyset)}$ such that

$$\Psi^{\vDash} = \{\varphi \in L(C,\emptyset) : mu \Vdash \varphi \text{ whenever } mu \Vdash \psi \text{ for each } \psi \in \Psi\};$$

- the *schema floating entailment operator* $.^{\overline{\vDash}_\Xi} : 2^{L(C,\Xi)} \to 2^{L(C,\Xi)}$ such that

$$\Gamma^{\overline{\vDash}_\Xi} = \{\delta \in L(C,\Xi) : m\alpha \Vdash_{\Xi} \delta \text{ whenever } m\alpha \Vdash_{\Xi} \gamma \text{ for each } \gamma \in \Gamma\};$$

- the *floating entailment operator* $.^{\overline{\vDash}} : 2^{L(C,\emptyset)} \to 2^{L(C,\emptyset)}$ such that

$$\Psi^{\overline{\vDash}} = \{\varphi \in L(C,\emptyset) : m \Vdash \varphi \text{ whenever } m \Vdash \psi \text{ for each } \psi \in \Psi\}.$$

Moreover, schema contextual entailment, schema floating entailment, contextual entailment and floating entailment are closure operators.

Proof:
Extensiveness, idempotence and monotonicity follow easily. Just note that $\Psi^{\vDash} = \Psi^{\vDash_\Xi} \cap L(C,\emptyset)$ and $\Psi^{\overline{\vDash}} = \Psi^{\overline{\vDash}_\Xi} \cap L(C,\emptyset)$. QED

It is easy to prove that entailments are preserved by enrichment.

Proposition 3.8 A pre-interpretation system and its enrichment lead to the same entailment operators.

Proof:
Straightforward, using the fact that $[\![\gamma]\!]_{\Xi}^{m_1\alpha_1}(u) = [\![\gamma]\!]_{\Xi}^{f(m_1)\alpha_2}(f(u))$ with $\alpha_1 = \lambda\xi.\alpha_2(\xi) \circ f$. QED

We now turn our attention to the problem of defining an appropriate notion of interpretation system morphism. Clearly, such a morphism must relate the constructors and the models in a contravariant way. We also expected to have to map the points in the domains but, as it happened, it turns out that such a map is not needed since we are working with rich systems and therefore the change of points is not essential. Finally, the morphism must preserve the interpretation of the constructors in the obvious way.

Definition 3.9 An *interpretation system morphism*

$$h : \langle C, M, A\rangle \to \langle C', M', A'\rangle$$

is a pair $h = \langle \overrightarrow{h}, \overleftarrow{h}\rangle$ where:

- $\overrightarrow{h}: C \to C'$ is a morphism in *Sig*;
- $\overleftarrow{h}= \{\overleftarrow{h}_{m'u'}\}_{m'\in M', u'\in U_{m'}}$ with $\overleftarrow{h}_{m'u'}\in M$

such that for every $m' \in M'$ and $u' \in U_{m'}$:

- $U_{\overleftarrow{h}_{m'u'}} = \{v' \in U_{m'} : \overleftarrow{h}_{m'v'}=\overleftarrow{h}_{m'u'}\}$;
- letting $inc^h_{m'u'} : U_{\overleftarrow{h}_{m'u'}} \subseteq U_{m'}$ be the corresponding inclusion, for every $k \in I\!N$, $c \in C_k$ and $b' \in (2^{U_{m'}})^k$:

$$\nu_{m'k}(\overrightarrow{h}_k\,(c))(b')(u') = \nu_{\overleftarrow{h}_{m'u'}k}(c)(b' \circ inc^h_{m'u'})(u').$$

Note that each model of the target system is associated with a collection of source models. The first condition imposes that they induce a partition of the set of points of the target model. The second condition imposes a correspondence between the denotations of the constructors guided by the partition. Therefore, each target model can be seen as the "union" of a collection of source models.

Proposition 3.10 Let $h : \langle C, M, A\rangle \to \langle C', M', A'\rangle$ be an interpretation system morphism, $m' \in M'$ and $u' \in U_{m'}$. Then:

- $U_{\overleftarrow{h}_{m'u'}} \subseteq U_{m'}$;
- $u' \in U_{\overleftarrow{h}_{m'u'}}$;
- $v' \in U_{\overleftarrow{h}_{m'u'}}$ implies $\overleftarrow{h}_{m'v'}=\overleftarrow{h}_{m'u'}$ and $u' \in U_{\overleftarrow{h}_{m'v'}}$.

Prop/Definition 3.11 Interpretation systems and their morphisms constitute the category *Int*.

Proof:
1. Identities.
The identity over $\langle C, M, A\rangle$ is $id_{\langle C,M,A\rangle} = \langle id_C, \overleftarrow{id}\rangle$ where $\overleftarrow{id}_{mu}= m$ for every $m \in M$ and $u \in U_m$.
2. Composition.
Assume that $f : \langle C, M, A\rangle \to \langle C', M', A'\rangle$ and $g : \langle C', M', A'\rangle \to \langle C'', M'', A''\rangle$ are morphisms in *Int*. Then, $h = g \circ f$ is defined as follows:

- $\overrightarrow{h}=\overrightarrow{g} \circ \overrightarrow{f}$;
- $\overleftarrow{h}_{m''u''}=\overleftarrow{f}_{\overleftarrow{g}_{m''u''}u''}$.

We prove that h is indeed an interpretation system morphism.

- $U_{\overleftarrow{h}_{m''u''}} = \{v'' \in U_{m''} : \overleftarrow{h}_{m''v''}=\overleftarrow{h}_{m''u''}\}$.
  - If $v'' \in U_{\overleftarrow{h}_{m''u''}}$ then, by definition of $\overleftarrow{h}_{m''u''}$, $v'' \in U_{\overleftarrow{f}_{\overleftarrow{g}_{m''u''}u''}} \subseteq U_{\overleftarrow{g}_{m''u''}} \subseteq U_{m''}$. Therefore, $v'' \in U_{m''}$ and $\overleftarrow{g}_{m''v''}=\overleftarrow{g}_{m''u''}$. But then, also, $\overleftarrow{f}_{\overleftarrow{g}_{m''v''}v''}=\overleftarrow{f}_{\overleftarrow{g}_{m''u''}v''}=\overleftarrow{f}_{\overleftarrow{g}_{m''u''}u''}$ and so $\overleftarrow{h}_{m''v''}=\overleftarrow{h}_{m''u''}$.
  - If $v'' \in U_{m''}$ and $\overleftarrow{h}_{m''v''}=\overleftarrow{h}_{m''u''}$ then $v'' \in U_{\overleftarrow{f}_{\overleftarrow{g}_{m''v''}v''}} = U_{\overleftarrow{h}_{m''v''}} = U_{\overleftarrow{h}_{m''u''}}$.
- $\nu_{m''k}(\overrightarrow{h}_k(c))(b'')(u'') = \nu_{\overleftarrow{h}_{m''u''}k}(c)(b'' \circ inc^h_{m''u''})(u'')$.
  It is trivial that $inc^h_{m''u''} = inc^g_{m''u''} \circ inc^f_{\overleftarrow{g}_{m''u''}u''}$. Thus,
  $\nu_{m''k}(\overrightarrow{h}_k(c))(b'')(u'') =$
  $\quad\nu_{m''k}(\overrightarrow{g}_k(\overrightarrow{f}_k(c)))(b'')(u'') =$
  $\quad\nu_{\overleftarrow{g}_{m''u''}k}(\overrightarrow{f}_k(c))(b'' \circ inc^g_{m''u''})(u'') =$
  $\quad\nu_{\overleftarrow{f}_{\overleftarrow{g}_{m''u''}u''}k}(c)(b'' \circ inc^g_{m''u''} \circ inc^f_{\overleftarrow{g}_{m''u''}u''})(u'') =$
  $\nu_{\overleftarrow{h}_{m''u''}k}(c)(b'' \circ inc^h_{m''u''})(u'')$. QED

As before, we shall need a mechanism for constructing a new interpretation system from a given one along a given signature morphism. The (forgetful) functor is:

Prop/Definition 3.12 The maps:

- $N(\langle C, M, A\rangle) = C$;

- $N(h : \langle C, M, A\rangle \to \langle C', M', A'\rangle) = \overrightarrow{h}$

constitute the functor $N : \mathit{Int} \to \mathit{Sig}$.

Proposition 3.13 For each $\langle C, M, A\rangle$ in *Int* and each morphism $h : C \to C'$ in *Sig* such that each map h_k is surjective, the morphism

$$\langle h\rangle : \langle C, M, A\rangle \to \langle C', M', A'\rangle$$

where:

- M' is the subclass of all elements m of M such that:
  - $\nu_{mk}(c_1) = \nu_{mk}(c_2)$ whenever $h_k(c_1) = h_k(c_2)$, for each $k \in \mathbb{N}$ and $c_1, c_2 \in C_k$;
- $A'(m) = \langle U_m, \nu'\rangle$ where $\nu'_k(h_k(c)) = \nu_{mk}(c)$, for each $k \in \mathbb{N}$ and $c \in C_k$;
- $\overrightarrow{\langle h\rangle} = h$;
- $\overleftarrow{\langle h\rangle}_{mu} = m$

is cocartesian by N for h on $\langle C, M, A\rangle$.

Proof:
1. M' is trivially closed under equivalence.
2. Clearly, $\langle h\rangle$ is a morphism in *Int*.
3. Universal property.
Assume that $g : \langle C, M, A\rangle \to \langle C'', M'', A''\rangle$ is a morphism in *Int* and $f : C' \to C''$ is a morphism in *Sig* such that $f \circ h = \overrightarrow{g}$. We first show that for any $m'' \in M''$ and $u'' \in U_{m''}$, $m = \overleftarrow{g}_{m''u''} \in M'$.
Assume that $c_1, c_2 \in C_k$ and $h_k(c_1) = h_k(c_2)$. Then, $f_k(h_k(c_1)) = f_k(h_k(c_2))$ and so $\overrightarrow{g}_k\,(c_1) = \overrightarrow{g}_k\,(c_2) = c''$. Let $u'' \in U_m$, $b \in (2^{U_m})^k$ and consider any $b'' \in (2^{U_{m''}})^k$ such that $b = b'' \circ inc^{g}_{m''u''}$. Then, $\nu_{mk}(c_1)(b)(u'') = \nu_{m''k}(c'')(b'')(u'')$ and also $\nu_{mk}(c_2)(b)(u) = \nu_{m''k}(c'')(b'')(u'')$, therefore $\nu_{mk}(c_1) = \nu_{mk}(c_2)$.
Now consider $\langle f\rangle = \langle f, \overleftarrow{f}\rangle$ with $\overleftarrow{f}_{m''u''} = \overleftarrow{g}_{m''u''}$ for every $m'' \in M''$ and $u'' \in U_{m''}$. It is straightforward to check that $\langle f\rangle : \langle C', M', A'\rangle \to \langle C'', M'', A''\rangle$ is a morphism in *Int*. Moreover it is obviously the unique morphism such that $\langle f\rangle \circ \langle h\rangle = g$ and $N(\langle f\rangle) = f$. QED

We denote the codomain of the cocartesian morphism defined above by $h(\langle C, M, A\rangle)$.

3.2 Unconstrained fibring

We need to use the following (well known) results on fixed points originally due to Tarski and Kleene [14].

Proposition 3.14 Let $\langle U, \leq\rangle$ be a complete lattice, $u \in U$ and $f : U \to U$ a monotonic map such that $u \leq f(u)$. Then, the set $\{v \in U : u \leq v = f(v)\}$ has a minimum. Moreover, if f is continuous then, letting

- $f^0(u) = u$;
- $f^{n+1}(u) = f(f^n(u))$,

the minimum element is $\bigvee_{n \in \mathbb{N}} f^n(u)$.

Clearly, the minimum element above is the least fixed point of f which is greater or equal than u. In the sequel, we denote it by $\mathrm{lfp}(f, u)$.

We are now ready to propose the envisaged definition of unconstrained fibring of two given interpretation systems. As before we obtain fibring as a coproduct. Reflected on the proposed notion of morphism of interpretation systems, and therefore captured by coproducts, is the concept of fibring function [9]: the fibring function for each fibred model is given, implicitly, by the injection morphisms.

Note that the logic resulting from the fibring still has the structure of an interpretation system contrarily to the "operational" description of fibred semantics given elsewhere.

Prop/Definition 3.15 Let $\langle C', M', A' \rangle$ and $\langle C'', M'', A'' \rangle$ be interpretation systems. Then, their *unconstrained fibring* $\langle C', M', A' \rangle \oplus \langle C'', M'', A'' \rangle$, is the interpretation system $\langle C' \oplus C'', M, A \rangle$ defined as follows using the injections $i' : C' \to C' \oplus C''$ and $i'' : C'' \to C' \oplus C''$ in *Sig*:

- M is the class of all pairs $m = \langle U, \tau \rangle$ such that:
  - U is a nonempty set;
  - $\tau = \{\tau_u\}_{u \in U}$ where each $\tau_u = \langle \tau'_u, \tau''_u \rangle$ such that $\tau'_u \in M'$, $\tau''_u \in M''$;
  - $U_{\tau'_u} = \{v \in U : \tau'_v = \tau'_u\}$, $U_{\tau''_u} = \{v \in U : \tau''_v = \tau''_u\}$ for each $u \in U$;
  - letting $O : 2^U \to 2^U$ be such that

    $$O(V) = \bigcup_{v \in V} (U_{\tau'_v} \cup U_{\tau''_v})$$

    then, $\mathrm{lfp}(O, \{u\}) = U$ for every $u \in U$.

- $A(\langle U, \tau \rangle) = \langle U, \nu \rangle$ where:
  - $\nu_k(i'_k(c'))(b)(u) = \nu_{\tau'_u k}(c')(b \circ inc'_u)(u)$
    being $inc'_u : U_{\tau'_u} \subseteq U$ the corresponding inclusion;
  - $\nu_k(i''_k(c''))(b)(u) = \nu_{\tau''_u k}(c'')(b \circ inc''_u)(u)$
    being $inc''_u : U_{\tau''_u} \subseteq U$ the corresponding inclusion.

Proof:
1. O is obviously monotonic. Moreover, $u \in U_{\tau'_u}$ and $\{u\} \subseteq O(\{u\})$ for every $u \in U$.
2. M is closed under equivalence.
Let $m_1 = \langle U, \tau \rangle \in M$ and $f : U \cong V$ be a bijection. We can exhibit a model $m_2 = \langle V, \theta \rangle$ such that $A(m_2) = f(A(m_1))$. It is enough to make $\theta'_{f(u)} = f'_u(\tau'_u)$

and $\theta''_{f(u)} = f''_u(\tau''_u)$ where $f'_u : U_{\tau'_u} \cong f(U_{\tau'_u})$ and $f''_u : U_{\tau''_u} \cong f(U_{\tau''_u})$ are the corresponding restrictions of f. QED

The fixed point requirement is needed for the uniqueness of the morphism in the universal property of the coproduct (see below). But it has no effect over entailment. Clearly, each "disconnected" model can be safely replaced by its "connected" parts.

In the sequel, whenever $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ are understood, we denote models of their unconstrained fibring by $m = \langle U_m, \tau_m\rangle$.

Proposition 3.16 Let $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ be interpretation systems. Then, the pairs

- $i' = \langle \overrightarrow{i'}, \overleftarrow{i'}\rangle : \langle C', M', A'\rangle \to \langle C', M', A'\rangle \oplus \langle C'', M'', A''\rangle$;
- $i'' = \langle \overrightarrow{i''}, \overleftarrow{i''}\rangle : \langle C'', M'', A''\rangle \to \langle C', M', A'\rangle \oplus \langle C'', M'', A''\rangle$

such that:

- $\overrightarrow{i'}, \overrightarrow{i''}$ are the injections of $C' \oplus C''$ in *Sig*;
- $\overleftarrow{i'}_{mu} = \tau'_{mu}$ and $\overleftarrow{i''}_{mu} = \tau''_{mu}$

are morphisms in *Int*.

Proof:
The verification is straightforward just noting that for each m, $inc'_u = inc^{i'}_{mu}$ and $inc''_u = inc^{i''}_{mu}$. QED

Note the separation of the m''' model into its "connected" substructures in the proof of the following result.

Proposition 3.17 Let $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ be interpretation systems. Then, $\langle C', M', A'\rangle \oplus \langle C'', M'', A''\rangle$ is a coproduct in *Int*.

Proof:
1. The injections are i' and i'' as defined above.
2. Universal property.
Let $f' : \langle C', M', A'\rangle \to \langle C''', M''', A'''\rangle$ and $f'' : \langle C'', M'', A''\rangle \to \langle C''', M''', A'''\rangle$ be morphisms in *Int*. We start by proving that there is a morphism g from the unconstrained fibring to $\langle C''', M''', A'''\rangle$ such that $g \circ i' = f'$ and $g \circ i'' = f''$. Consider the tuple $g = \langle \overrightarrow{g}, \overleftarrow{g}\rangle$ where:

- $\overrightarrow{g}$ is the unique morphism in *Sig* such that $\overrightarrow{g} \circ \overrightarrow{i'} = \overrightarrow{f'}$ and $\overrightarrow{g} \circ \overrightarrow{i''} = \overrightarrow{f''}$;
- letting $E_{m'''} : 2^{U_{m'''}} \to 2^{U_{m'''}}$ be such that

$$E_{m'''}(V) = \bigcup_{v''' \in V} (U_{\overleftarrow{f'}_{m'''v'''}} \cup U_{\overleftarrow{f''}_{m'''v'''}})$$

then, $\overleftarrow{g}_{m'''u'''} = \langle U, \tau\rangle$ where:

– $U = \mathrm{lfp}(E_{m'''}, \{u'''\})$;

– $\tau_{v'''} = \langle \overleftarrow{f'}_{m'''v'''}, \overleftarrow{f''}_{m'''v'''} \rangle$.

i. $\overleftarrow{g}_{m'''u'''} \in M$.

- It is straightforward that $E_{m'''}$ is monotonic. Moreover, $u''' \in U_{\overleftarrow{f'}_{m'''u'''}}$ and so, $\{u'''\} \subseteq E_{m'''}(\{u'''\})$ and $U \subseteq U_{m'''}$ is well defined.

- τ is obviously well defined.

- $U_{\tau'_{v'''}} = \{w''' \in U : \tau'_{w'''} = \tau'_{v'''}\}$.
  By definition $\tau'_{x'''} = \overleftarrow{f'}_{m'''x'''}$ thus, $\{w''' \in U : \tau'_{w'''} = \tau'_{v'''}\} \subseteq U_{\tau'_{v'''}}$. But, for $v''' \in U$, if $w''' \in U_{\overleftarrow{f'}_{m'''v'''}}$ then $w''' \in E_{m'''}(U) = U$.
  Analogously for $U_{\tau''_{v'''}} = \{w''' \in U : \tau''_{w'''} = \tau''_{v'''}\}$.

- $\mathrm{lfp}(O, \{v'''\}) = U$.
  $E_{m'''}$ is continuous. In fact,
  $E_{m'''}(\bigcup_{k\in K} V_k) =$
  $\bigcup_{v''' \in (\bigcup_{k\in K} V_k)} (U_{\overleftarrow{f'}_{m'''v'''}} \cup U_{\overleftarrow{f''}_{m'''v'''}}) =$
  $\bigcup_{k\in K} \bigcup_{v''' \in V_k} (U_{\overleftarrow{f'}_{m'''v'''}} \cup U_{\overleftarrow{f''}_{m'''v'''}}) =$
  $\bigcup_{k\in K} E_{m'''}(V_k)$.
  Moreover, for any $V \subseteq U \subseteq U_{m'''}$, $O(V) = E_{m'''}(V)$. Indeed, $O(V) = \bigcup_{v''' \in V} (U_{\tau'_{v'''}} \cup U_{\tau''_{v'''}}) = \bigcup_{v''' \in V} (U_{\overleftarrow{f'}_{m'''v'''}} \cup U_{\overleftarrow{f''}_{m'''v'''}}) = E_{m'''}(V)$. Since $v''' \in U$, i.e., $v''' \in E^n_{m'''}(\{u'''\})$ for some $n \in \mathbb{N}$, it is enough to prove, by induction on n, that also $u''' \in E^n_{m'''}(\{v'''\})$.

  Base:
  $v''' \in E^0_{m'''}(\{u'''\}) = \{u'''\}$ iff $v''' = u'''$.
  Step:
  $v''' \in E^{n+1}_{m'''}(\{u'''\}) = E_{m'''}(E^n_{m'''}(\{u'''\}))$ iff $v''' \in E_{m'''}(\{w'''\})$ for some $w''' \in E^n_{m'''}(\{u'''\})$. Clearly, $v''' \in E_{m'''}(\{w'''\}) = (U_{\overleftarrow{f'}_{m'''w'''}} \cup U_{\overleftarrow{f''}_{m'''w'''}})$ implies that also $w''' \in E_{m'''}(\{v'''\}) = (U_{\overleftarrow{f'}_{m'''v'''}} \cup U_{\overleftarrow{f''}_{m'''v'''}})$. So, by induction hypothesis, $u''' \in E^n_{m'''}(\{w'''\}) \subseteq E^n_{m'''}(E_{m'''}(\{v'''\})) = E^{n+1}_{m'''}(\{v'''\})$.

ii. g is a morphism in *Int*.

- $U_{\overleftarrow{g}_{m'''u'''}} = \{v''' \in U_{m'''} : \overleftarrow{g}_{m'''v'''} = \overleftarrow{g}_{m'''u'''}\}$.
  If $\overleftarrow{g}_{m'''v'''} = \overleftarrow{g}_{m'''u'''}$, it is trivial that $v''' \in \mathrm{lfp}(E_{m'''}, \{v'''\}) = U_{\overleftarrow{g}_{m'''v'''}} = U_{\overleftarrow{g}_{m'''u'''}}$. On the other hand, if $v''' \in U_{\overleftarrow{g}_{m'''u'''}}$ then obviously $v''' \in E^n_{m'''}(\{u'''\})$ for some $n \in \mathbb{N}$. As before, also, $u''' \in E^n_{m'''}(\{v'''\})$ and thus, $\mathrm{lfp}(E_{m'''}, \{v'''\}) = \mathrm{lfp}(E_{m'''}, \{u'''\})$. Clearly, then, $\overleftarrow{g}_{m'''v'''} = \overleftarrow{g}_{m'''u'''}$.

- $\nu_{m'''k}(\overrightarrow{g}_k(c))(b''')(u''') = \nu_{\overleftarrow{g}_{m'''u'''}k}(c)(b''' \circ inc^g_{m'''u'''})(u''')$.
  Let $c = \overrightarrow{i'}_k(c')$ with $c' \in C'_k$. Easily,

$\nu_{m'''k}(\overrightarrow{g}_k\,(c))(b''')(u''') =$
$\quad \nu_{m'''k}(\overrightarrow{g}_k\,(\overrightarrow{i'}_k\,(c')))(b''')(u''') =$
$\quad \nu_{m'''k}(\overrightarrow{f'}_k\,(c'))(b''')(u''') =$
$\quad \nu_{\overleftarrow{f'}_{m'''u'''}k}(c')(b''' \circ inc^{f'}_{m'''u'''})(u''') =$
$\quad \nu_{\tau'_{u'''}k}(c')(b''' \circ inc^{g}_{m'''u'''} \circ inc'_{u'''})(u''') =$
$\quad \nu_{\overleftarrow{g}_{m'''u'''}k}(\overrightarrow{i'}_k\,(c'))(b''' \circ inc^{g}_{m'''u'''})(u''') =$
$\nu_{\overleftarrow{g}_{m'''u'''}k}(c)(b''' \circ inc^{g}_{m'''u'''})(u''')$.

Analogously for $c = \overrightarrow{i''}_k\,(c'')$ with $c'' \in C''_k$.

iii. $g \circ i' = f'$ and $g \circ i'' = f''$.
By definition, $\overrightarrow{g}$ is the unique such that $\overrightarrow{g} \circ \overrightarrow{i'} = \overrightarrow{f'}$ and $\overrightarrow{g} \circ \overrightarrow{i''} = \overrightarrow{f''}$. Moreover, it is immediate that $\overleftarrow{i'}_{\overleftarrow{g}_{m'''u'''}u'''} = \tau'_{u'''} = \overleftarrow{f'}_{m'''u'''}$ and $\overleftarrow{i''}_{\overleftarrow{g}_{m'''u'''}u'''} = \tau''_{u'''} = \overleftarrow{f''}_{m'''u'''}$.

iv. Uniqueness.
We prove that $\overleftarrow{g}_{m'''u'''}$ could not be other than $\mathrm{lfp}(E_{m'''}, \{u'''\})$. Necessarily, $\overleftarrow{g}_{m'''u'''} = \langle U, \tau\rangle$ with $u''' \in U \subseteq U_{m'''}$ and τ determined as above by the composition requirement. Also, $O(U) = E_{m'''}(U) \subseteq U$, which implies that $\mathrm{lfp}(E_{m'''}, \{u'''\}) \subseteq U$. But it is required that $U = \mathrm{lfp}(O, \{u'''\})$, so $U = \mathrm{lfp}(E_{m'''}, \{u'''\})$. QED

The coproduct construction faithfully captures the intuitive idea of fibred semantics as described for instance in [9].

We delay the illustration of fibring of interpretation systems to the next section.

3.3 Constrained fibring

Again, interesting applications will be constrained: in general, we are interested in sharing constructors. As before, the technique of cocartesian lifting provides the means for constructing the envisaged system guided by the sharing at the signature level.

Definition 3.18 Let $\langle C', M', A'\rangle$ and $\langle C'', M'', A''\rangle$ be interpretation systems and $f' : C \to C', f'' : C \to C''$ be injective signature morphisms. Then, their *constrained fibring by sharing* is as follows:

$$\langle C', M', A'\rangle \overset{f'Cf''}{\oplus} \langle C'', M'', A''\rangle = q(\langle C', M', A'\rangle \oplus \langle C'', M'', A''\rangle)$$

where $q : C' \oplus C'' \to C' \overset{f'Cf''}{\oplus} C''$ is the coequalizer in *Sig* of $i' \circ f' : C \to C' \oplus C''$ and $i'' \circ f'' : C \to C' \oplus C''$.

Recall that the coequalizer in *Sig* is a family of surjective maps as required by the lifting.

For examples, we refer the reader to section 4.

Since sharing of non nullary constructors is also possible, we have here a generalization to all constructors of the idea of dovetailing proposed by Gabbay: shared atoms have the same interpretation, i.e., they become indistinguishable.

Further constraining may be achieved by restricting the class of models in the fibred logic. Of course, then it can be rather difficult or impossible to find a proof-theoretic counterpart of such a restriction.

4 Fibring of logic system presentations

We now put together Hilbert calculi and interpretation systems to form logic system presentations. This is straightforward as expected. At the end of this section we present several examples and illustrate the fibring construction.

4.1 Logic system presentations

Definition 4.1 A *logic system presentation* is a tuple $\langle C, M, A, P, D\rangle$ where:

- $\langle C, M, A\rangle$ is an interpretation system;
- $\langle C, P, D\rangle$ is a Hilbert calculus.

Definition 4.2 A *logic system presentation morphism*

$$h : \langle C, M, A, P, D\rangle \to \langle C', M', A', P', D'\rangle$$

is an interpretation system morphism $\langle \overrightarrow{h}, \overleftarrow{h}\rangle : \langle C, M, A\rangle \to \langle C', M', A'\rangle$ such that $\overrightarrow{h}: \langle C, P, D\rangle \to \langle C', P', D'\rangle$ is an Hilbert calculus morphism.

Prop/Definition 4.3 Logic system presentations and their morphisms constitute the category *Lsp*.

The mechanism of cocartesian lifting is again used for constructing a new logic system presentation from a given one along a given signature morphism. The (forgetful) functor, now, is:

Prop/Definition 4.4 The maps:

- $N(\langle C, M, A, P, D\rangle) = C$;
- $N(h : \langle C, M, A, P, D\rangle \to \langle C', M', A', P', D'\rangle) = \overrightarrow{h}$

constitute the functor $N : Lsp \to Sig$.

Proposition 4.5 For each $\langle C, M, A, P, D\rangle$ in *Lsp* and each morphism $h : C \to C'$ in *Sig* such that each map h_k is surjective, the morphism

$$\langle h\rangle : \langle C, M, A, P, D\rangle \to \langle C', M', A', P', D'\rangle$$

where:

- $\langle C', M', A' \rangle = h(\langle C, M, A \rangle)$;
- $\langle C', P', D' \rangle = h(\langle C, P, D \rangle)$

is cocartesian by N for h on $\langle C, M, A, P, D \rangle$.

We denote the codomain of the cocartesian morphism defined above by $h(\langle C, M, A, P, D \rangle)$.

4.2 Fibring

Fibring of logic system presentations results from the combined effect of fibring their Hilbert calculi and interpretation systems.

Prop/Definition 4.6 Let p' and p'' be, respectively, logic system presentations $\langle C', M', A', P', D' \rangle$ and $\langle C'', M'', A'', P'', D'' \rangle$. Then, their *unconstrained fibring* $p = p' \oplus p''$, is the logic system presentation $\langle C, M, A, P, D \rangle$ such that:

- $C = C' \oplus C''$;
- $\langle C, M, A \rangle$ is the unconstrained fibring of $\langle C', M', A' \rangle$ and $\langle C'', M'', A'' \rangle$;
- $\langle C, P, D \rangle$ is the unconstrained fibring of $\langle C', P', D' \rangle$ and $\langle C'', P'', D'' \rangle$.

Moreover, p is a coproduct in *Lsp* with injections $i' : p' \to p$ and $i'' : p'' \to p$ such that:

- i' and i'' are the injections of $\langle C', M', A' \rangle$ and $\langle C'', M'', A'' \rangle$, respectively, into their coproduct $\langle C, M, A \rangle$ in *Int*;
- $\vec{i'}$ and $\vec{i''}$ are the injections of $\langle C', P', D' \rangle$ and $\langle C'', P'', D'' \rangle$, respectively, into their coproduct $\langle C, P, D \rangle$ in *Hil*.

Definition 4.7 Let $p' = \langle C', M', A', P', D' \rangle$ and $p'' = \langle C'', M'', A'', P'', D'' \rangle$ be logic system presentations and $f' : C \to C', f'' : C \to C''$ be injective signature morphisms. Then, their *constrained fibring by sharing* is as follows:

$$p' \overset{f'Cf''}{\oplus} p'' = q(p' \oplus p'')$$

where $q : C' \oplus C'' \to C' \overset{f'Cf''}{\oplus} C''$ is the coequalizer in *Sig* of $i' \circ f' : C \to C' \oplus C''$ and $i'' \circ f'' : C \to C' \oplus C''$.

4.3 Examples

Example 4.8 *Propositional modal logic.* The K propositional (uni)modal logic system presentation $p_\Pi^{KML} = \langle C, M, A, P, D \rangle$ over Π is defined as follows:

- $\langle C, P, D \rangle$ as in Example 2.9, without further proof rules;
- $\langle C, M, A \rangle$ as in Example 3.5, where M is the class of all Kripke structures.

Example 4.9 *Propositional logic.* Let Π be a set of propositional symbols. The propositional logic system presentation $p_\Pi^{PL} = \langle C, M, A, P, D\rangle$ over Π is defined as follows:

- $C_0 = \Pi$, $C_1 = \{\neg\}$, $C_2 = \{\Rightarrow\}$;
- M is the class of all pairs $\langle U, V\rangle$ such that:
  - U is a singleton;
  - $V : \Pi \to \{0, 1\}$;
- $A(\langle U, V\rangle) = \langle U, \nu\rangle$ where:
  - $\nu_0(\pi)(u) = V(\pi)$;
  - $\nu_1(\neg)(b)(u) = 1 - b(u)$;
  - $\nu_2(\Rightarrow)(b, b')(u) = b(u) \leq b'(u)$

  where u is the unique element of U;
- $P = \{\langle \emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle \emptyset, ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))\rangle,$
  $\langle \emptyset, (((\neg\, \xi_1) \Rightarrow (\neg\, \xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\}$;
- $D = \{\langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\}$.

Example 4.10 *Propositional linear temporal logic.* Let Π be a set (of propositional symbols). The propositional linear temporal logic system presentation $p_\Pi^{TL} = \langle C, M, A, P, D\rangle$ based on Π is defined as follows:

- $C_0 = \Pi$, $C_1 = \{\neg, \mathsf{X}, \mathsf{G}\}$, $C_2 = \{\Rightarrow\}$;
- M is the class of all pairs $\langle U, V\rangle$ such that:
  - U is isomorphic to $I\!N$;
  - $V : \Pi \to 2^U$;
- $A(\langle U, V\rangle) = \langle U, \nu\rangle$ where:
  - $\nu_0(\pi) = V(\pi)$;
  - $\nu_1(\neg)(b) = \lambda u \,.\, 1 - b(u)$;
  - $\nu_1(\mathsf{X})(b) = \lambda u \,.\, b(u + 1)$;
  - $\nu_1(\mathsf{G})(b) = \lambda u \,.\, \Pi_{u \leq u'} b(u')$;
  - $\nu_2(\Rightarrow)(b, b') = \lambda u \,.\, b(u) \leq b'(u)$;

- $P = \{\langle\emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle\emptyset, ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))\rangle,$
  $\langle\emptyset, (((\neg\,\xi_1) \Rightarrow (\neg\,\xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle\emptyset, ((\mathsf{X}(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\mathsf{X}\,\xi_1) \Rightarrow (\mathsf{X}\,\xi_2)))\rangle,$
  $\langle\emptyset, ((\mathsf{G}(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\mathsf{G}\,\xi_1) \Rightarrow (\mathsf{G}\,\xi_2)))\rangle,$
  $\langle\emptyset, ((\mathsf{X}(\neg\,\xi_1)) \Rightarrow (\neg(\mathsf{X}\,\xi_1)))\rangle,$
  $\langle\emptyset, ((\neg(\mathsf{X}\,\xi_1)) \Rightarrow (\mathsf{X}(\neg\,\xi_1)))\rangle,$
  $\langle\emptyset, ((\mathsf{G}\,\xi_1) \Rightarrow \xi_1)\rangle,$
  $\langle\emptyset, ((\mathsf{G}\,\xi_1) \Rightarrow (\mathsf{X}(\mathsf{G}\,\xi_1)))\rangle,$
  $\langle\emptyset, ((\mathsf{G}(\xi_1 \Rightarrow (\mathsf{X}\,\xi_1))) \Rightarrow (\xi_1 \Rightarrow (\mathsf{G}\,\xi_1)))\rangle,$
  $\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle,$
  $\langle\{\xi_1\}, (\mathsf{X}\,\xi_1)\rangle,$
  $\langle\{\xi_1\}, (\mathsf{G}\,\xi_1)\rangle\};$

- $D = \{\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\}$.

The axiomatization presented is taken from [11].

We now illustrate the proposed constructions by fibring two modal logics. While rich enough to bring in to play all the details of the constructions, it is still a sufficiently simple case that can be easily understood.

Example 4.11 *Fibring of modal logics.* Let Π be a set of propositional symbols. Consider the propositional modal $KB4$ logic system presentation

$$p_\Pi^{KB4} = \langle C, M', A', P', D\rangle$$

as in Examples 2.9 and 3.5 where:

- M' is the class of all symmetric and transitive Kripke structures;
- P' further includes:
  B: $(\xi_1 \Rightarrow (\Box(\diamond\xi_1)))$,
  4: $((\Box\xi_1) \Rightarrow (\Box(\Box\xi_1)))$;

and the propositional modal KD logic system presentation

$$p_\Pi^{KD} = \langle C, M'', A'', P'', D\rangle$$

also as in Examples 2.9 and 3.5 where:

- M'' is the class of all serial Kripke structures;
- P'' further includes:
  D: $((\Box\xi_1) \Rightarrow (\diamond\xi_1))$.

Their unconstrained fibring

$$p_\Pi^{KB4} \oplus p_\Pi^{KD} = \langle i'(C) \cup i''(C), M, A, i'(P') \cup i''(P''), i'(D) \cup i''(D)\rangle$$

is such that:

- M is the class of all fibred models $\langle U, \tau \rangle$ such that τ', respectively τ'', induce symmetric and transitive, respectively serial, Kripke structures over U.

- $A(\langle U, \tau \rangle) = (U, \nu)$ with:

  - $\nu_1(i'(\Box))(b) = \lambda u \,.\, \Pi_{u' : \langle u, u' \rangle \in S'_u} b(u')$ where $\tau'_u = \langle U_{\tau'_u}, S'_u, V'_u \rangle$;
  - $\nu_1(i''(\Box))(b) = \lambda u \,.\, \Pi_{u' : \langle u, u' \rangle \in S''_u} b(u')$ where $\tau''_u = \langle U_{\tau''_u}, S''_u, V''_u \rangle$.

  In fact, τ' induces $\langle W', S', V' \rangle$ where:

- $W' = U$;

- $S' = \bigcup_{u \in U} S'_u$;

- $V' = \lambda i'(\pi). \bigcup_{u \in U} V'_u(\pi)$;

and τ'' induces $\langle W'', S'', V'' \rangle$ where:

- $W'' = U$;

- $S'' = \bigcup_{u \in U} S''_u$;

- $V'' = \lambda i''(\pi). \bigcup_{u \in U} V''_u(\pi)$.

Things become interesting when we start sharing symbols. Let us look at what happens if all the propositional constructors (Π, $\neg$ and $\Rightarrow$) are shared. In this case, we get a bimodal logic, where the two (independent) modalities $i'(\Box) = \Box'$ and $i''(\Box) = \Box''$ have the properties inherited from above.

The Kripke structure associated to it is $\langle W, S', S'', V \rangle$ where:

- $W = W' = W'' = U$;

- $V = \lambda \pi . V'(i'(\pi)) = \lambda\ \pi . V''(i''(\pi))$.

We can prove, for instance $((\Box''(\Box'(\diamond''\xi))) \Rightarrow (\diamond''(\Box'(\Box'(\diamond''\xi)))))$.

1.	$((\Box'(\diamond''\xi)) \Rightarrow (\Box'(\Box'(\diamond''\xi))))$	$4'$
2.	$(\Box''((\Box'(\diamond''\xi)) \Rightarrow (\Box'(\Box'(\diamond''\xi)))))$	Nec$''$: 1
3.	$((\Box''((\Box'(\diamond''\xi)) \Rightarrow (\Box'(\Box'(\diamond''\xi)))))\Rightarrow$ $((\Box''(\Box'(\diamond''\xi))) \Rightarrow (\Box''(\Box'(\Box'(\diamond''\xi))))))$	K''
4.	$((\Box''(\Box'(\diamond''\xi))) \Rightarrow (\Box''(\Box'(\Box'(\diamond''\xi)))))$	MP: 2,3
5.	$((\Box''(\Box'(\Box'(\diamond''\xi)))) \Rightarrow (\diamond''(\Box'(\Box'(\diamond''\xi)))))$	D''
6.	$((\Box''(\Box'(\diamond''\xi))) \Rightarrow (\diamond''(\Box'(\Box'(\diamond''\xi)))))$	Syllogism: 4, 5

Furthermore, if we also share the boxes, we get a unimodal logic $S5$.

The induced Kripke structure is now $\langle W, S, V \rangle$ where:

- $S = S' = S''$.

In fact S is an equivalence relation on U. Reflexivity follows from transitivity and symmetry of S', and seriality of S''.

It is a well known result in modal logic that we can, in fact, prove T:

1.	$((\Box\xi) \Rightarrow (\Box(\Box\xi)))$	4
2.	$((\Box(\Box\xi)) \Rightarrow (\diamond(\Box\xi)))$	D
3.	$((\Box\xi) \Rightarrow (\diamond(\Box\xi)))$	Syllogism: 1, 2
4.	$((\diamond(\Box\xi)) \Rightarrow \xi)$	KB theorem
5.	$((\Box\xi) \Rightarrow \xi)$	Syllogism: 3, 4

We now consider the problem raised by trying to combine intuitionistic logic with classical logic. It is well known [9, 7] that by just putting together the rules from both calculi the intuitionistic component collapses into classical logic. The following example presents the propositional intuitionistic logic system presentation and shows what happens when we fibre it with the classical propositional logic system presentation.

Example 4.12 *The case of intuitionistic logic.* Let Π be a set of propositional symbols. The propositional intuitionistic logic system presentation $p_\Pi^{IL} = \langle C, M, A, P, D\rangle$ over Π is outlined below. The signature is as follows:

- $C_0 = \Pi$, $C_1 = \{\backsim\}$, $C_2 = \{\rightarrow, \wedge, \vee\}$.

The proof rules of the Hilbert calculus can be taken for instance from [4]. The only derivation rule is *modus ponens*. Note that P includes the axiom schema:

- $\langle \emptyset, (\xi_1 \rightarrow (\xi_2 \rightarrow \xi_1))\rangle$

It is worthwhile to ponder on the meaning of this axiom schema. It states that for every instance of ξ_1 and ξ_2 we obtain a theorem.

The envisaged interpretation system is similar to a modal logic one with the proviso that truth is *persistent* under accessibility. Therefore we would expect the following semantics:

- M is the class of all triples $\langle W, S, V\rangle$ where:
  - W is a non empty set;
  - $S \subseteq W^2$ is reflexive and transitive;
  - $V : \Pi \rightarrow 2^W$

  and such that:
  - $w' \in V(\pi)$ whenever $\langle w, w'\rangle \in S$ and $w \in V(\pi)$;
- $A(\langle W, S, V\rangle) = \langle W, \nu\rangle$ where:
  - $\nu_0(\pi) = V(\pi)$;

– $\nu_1(\backsim)(b) = \lambda w\,.\,\Pi_{w':\langle w,w'\rangle \in S}\ 1 - b(w')$;
– $\nu_2(\rightarrow)(b,b') = \lambda w\,.\,\Pi_{w':\langle w,w'\rangle \in S}\ b(w') \leq b'(w')$;
– and the others are as in classical logic.

However, this is not quite so! Indeed, the axiom schema mentioned above is not sound for every model in M. In order to make it sound, we have to restrict M to the class of models $\langle W, S, V\rangle$ where S is the identity relation over W. Therefore, we are reduced to working with "classical" models where the intuitionistic negation and implication of course become classical. Naturally, if we combine by fibring this logic presentation system with the classical logic presentation system we obtain classical logic.

In alternative, if we insist in faithfully representing intuitionistic semantics, we must constrain the set of admissible assignments (accepting only persistent assignments of ξ_1). But this would require a different notion of interpretation system where the structure of each model would be a triple $\langle U, \mathcal{V}, \nu\rangle$ with $\mathcal{V} \subseteq 2^U$ and $\nu_k(c) : \mathcal{V}^k \rightarrow \mathcal{V}$. Each assignment would be of the form $\alpha : \Xi \rightarrow \mathcal{V}$ as required. It is outside the scope of this paper to investigate such a generalized notion of interpretation system. In the case at hand of intuitionistic logic, $\mathcal{V}$ should be the set of all *persistent* maps $b \in 2^W$ such that if $w \in b$ then $w' \in b$ for every w' such that $\langle w, w'\rangle \in S$.

The example above shows that our notion of interpretation system is not general enough to represent intuitionistic logic: we only have classical models. The fact that after fibring we get classical logic is therefore due to this limitation and not to any inadequacy of the proposed notion of fibring.

In section 6 we outline future work aimed at overcoming this limitation along the lines of the proposed generalization.

5 Preservation properties

In this section, after establishing some preliminary results on how derivation and entailment are transferred, we prove that soundness is preserved by both forms of fibring (unconstrained and constrained).

We assume fixed a logic system presentation $p = \langle C, M, A, P, D\rangle$. We denote its *schema language* by $L_{p\Xi} = L(C, \Xi)$ and its *language* by $L_p = L(C, \emptyset)$.

Definition 5.1 Given p,

- *schema derivation* $.^{\vdash_{p\Xi}} : 2^{L_{p\Xi}} \rightarrow 2^{L_{p\Xi}}$,
- *derivation* $.^{\vdash_p} : 2^{L_p} \rightarrow 2^{L_p}$,
- *theorem schemata* $\emptyset^{\vdash_{p\Xi}}$ and
- *theorems* $\emptyset^{\vdash_p}$

are the operators and sets induced by the Hilbert calculus $\langle C, P, D\rangle$ and,

- *schema (contextual) entailment* $.^{\vDash_{p\Xi}} : 2^{L_{p\Xi}} \rightarrow 2^{L_{p\Xi}}$,

- *(contextual) entailment* $.^{\models_p} : 2^{L_p} \to 2^{L_p}$,
- *schema floating entailment* $.^{\overline{\models}_{p\Xi}} : 2^{L_{p\Xi}} \to 2^{L_{p\Xi}}$ and
- *floating entailment* $.^{\overline{\models}_p} : 2^{L_p} \to 2^{L_p}$

are the operators induced by the interpretation system $\langle C, M, A\rangle$.

5.1 Derivation

All the results of this subsection are concerned with the proof-theoretic part of a logic system presentation. They could, of course, have been stated and proved with respect to Hilbert calculi, instead.

As we have seen before, morphisms preserve derivations as well as schema derivations. Hence, for logic system presentations we have:

Proposition 5.2 Let $p = \langle C, M, A, P, D\rangle$ and $p' = \langle C', M', A', P', D'\rangle$ be logic system presentations. If $h : p \to p'$ is a logic system presentation morphism, then:

- $\vec{h}\,(\Gamma^{\vdash_{p\Xi}}) \subseteq \vec{h}\,(\Gamma)^{\vdash_{p'\Xi}}$;
- $\vec{h}\,(\Psi^{\vdash_p}) \subseteq \vec{h}\,(\Psi)^{\vdash_{p'}}$.

As corollaries, both unconstrained and constrained fibrings preserve derivations.

Corollary 5.3 Let $p' = \langle C', M', A', P', D'\rangle$ and $p'' = \langle C'', M'', A'', P'', D''\rangle$ be logic system presentations and $p = p' \oplus p''$ their unconstrained fibring with inclusions i' and i''. Then, both $\vec{i'}$ and $\vec{i''}$ preserve derivations.

Corollary 5.4 Let $p' = \langle C', M', A', P', D'\rangle$ and $p'' = \langle C'', M'', A'', P'', D''\rangle$ be logic system presentations, $p = p' \oplus p''$ their unconstrained fibring, $f' : C \to C'$ and $f'' : C \to C''$ injective signature morphisms, $p''' = p' \overset{f'Cf''}{\oplus} p''$ their corresponding constrained fibring and q the coequalizer of $\vec{i'} \circ f'$ and $\vec{i''} \circ f''$. Then, q preserves derivations.

5.2 Entailment

All the definitions and results of this subsection are concerned with the model-theoretic part of a logic system presentation. They could, of course, have been stated and proved with respect to interpretation systems, instead.

We look first at how morphisms treat satisfaction:

Proposition 5.5 Let $p = \langle C, M, A, P, D\rangle$ and $p' = \langle C', M', A', P', D'\rangle$ be logic system presentations and $h : p \to p'$ a logic system presentation morphism. Then, for every $\gamma \in L_{p\Xi}$, $\varphi \in L_p$, $m' \in M'$, $u' \in U_{m'}$ and α into m':

- $m'\alpha u' \Vdash_{p'\Xi} \vec{h}\,(\gamma)$ iff $\overleftarrow{h}_{m'u'}\ \alpha^h_{m'u'} u' \Vdash_{p\Xi} \gamma$;

- $m'u' \Vdash_{p'} \overrightarrow{h}(\varphi)$ iff $\overleftarrow{h}_{m'u'}\ u' \Vdash_{p\Xi} \varphi$;
- $m'\alpha \Vdash_{p'\Xi} \overrightarrow{h}(\gamma)$ iff $\overleftarrow{h}_{m'u'}\ \alpha^h_{m'u'} \Vdash_{p\Xi} \gamma$ for every $u' \in U_{m'}$;
- $m' \Vdash_{p'} \overrightarrow{h}(\varphi)$ iff $\overleftarrow{h}_{m'u'} \Vdash_{p\Xi} \varphi$ for every $u' \in U_{m'}$,

where $\alpha^h_{m'u'} = \lambda\xi.\alpha(\xi) \circ inc^h_{m'u'}$.

Proof:
Straightforward, using the fact that $[\![\overrightarrow{h}(\gamma)]\!]^{m'\alpha}_{p'\Xi}(u') = [\![\gamma]\!]^{\overleftarrow{h}_{m'u'}\alpha^h_{m'u'}}_{p\Xi}(u')$. QED

Now we can establish that morphisms preserve both contextual and floating entailment:

Proposition 5.6 Let $h : p \to p'$ be a logic system presentation morphism, then:

- $\overrightarrow{h}(\Gamma^{\theta_{p\Xi}}) \subseteq \overrightarrow{h}(\Gamma)^{\theta_{p'\Xi}}$;
- $\overrightarrow{h}(\Psi^{\theta_p}) \subseteq \overrightarrow{h}(\Psi)^{\theta'_{p'}}$,

where θ is either $\vDash$ (contextual entailment) or $\overline{\vDash}$ (floating entailment).

As corollaries, both unconstrained and constrained fibrings preserve entailment (contextual or floating, with or without schema variables).

Corollary 5.7 Let $p' = \langle C', M', A', P', D' \rangle$ and $p'' = \langle C'', M'', A'', P'', D'' \rangle$ be logic system presentations and $p = p' \oplus p''$ their unconstrained fibring with inclusions i' and i''. Then, both $\overrightarrow{i'}$ and $\overrightarrow{i''}$ preserve entailment.

Corollary 5.8 Let $p' = \langle C', M', A', P', D' \rangle$ and $p'' = \langle C'', M'', A'', P'', D'' \rangle$ be logic system presentations, $p = p' \oplus p''$ their unconstrained fibring with inclusions i' and i'', $f' : C \to C'$ and $f'' : C \to C''$ injective signature morphisms, $p''' = p' \overset{f'Cf''}{\oplus} p''$ their corresponding constrained fibring and q the coequalizer of $\overrightarrow{i'} \circ f'$ and $\overrightarrow{i''} \circ f''$. Then, q preserves entailment.

5.3 Soundness

We are now ready to tackle the problem of verifying if soundness is preserved by fibring. Actually, we prove that if the inference rules are sound in the given logic system presentations, then the inference rules are still sound in the unconstrained and constrained fibred logics.

Definition 5.9 We say that p is:

- *weakly sound* iff $\emptyset^{\vdash_{p\Xi}} \subseteq \emptyset^{\vDash_{p\Xi}}$;
- *sound* iff $\Gamma^{\vdash_{p\Xi}} \subseteq \Gamma^{\vDash_{p\Xi}}$ for every $\Gamma \subseteq L_{p\Xi}$.

Clearly, weak soundness (respectively, soundness) without schema variables, i.e., $\emptyset^{\vdash_p} \subseteq \emptyset^{\models_p}$ (respectively, $\Psi^{\vdash_p} \subseteq \Psi^{\models_p}$ for every $\Psi \subseteq L_p$) follows from general weak soundness (respectively, soundness).

We could have defined analogous notions of soundness based on floating consequence (although it is not usual). However, it is a trivial task to prove that $\Gamma^{\models_{p\Xi}} \subseteq \Gamma^{\overline{\models}_{p\Xi}}$. Therefore, soundness (for contextual consequence) implies soundness for floating consequence. Nevertheless, note also that $\emptyset^{\models_p} = \emptyset^{\overline{\models}_p}$. Still, soundness for floating consequence is weaker and we could drop the special requirement of soundness for derivation rules defined below.

Definition 5.10 A proof rule r of p is *sound* iff $Conc(r) \in Prem(r)^{\overline{\models}_{p\Xi}}$.

Definition 5.11 A derivation rule r of p is *sound* iff $Conc(r) \in Prem(r)^{\models_{p\Xi}}$.

In order to proceed towards proving the preservation of soundness we need the following technical result:

Proposition 5.12 Let $\gamma \in L_{p\Xi}$, $m \in M$, $u \in U_m$, σ a substitution on $L_{p\Xi}$ and α an assignment into m. Then,

- $m\alpha u \Vdash_{p\Xi} \gamma\sigma$ iff $m\beta^\sigma_\alpha u \Vdash_{p\Xi} \gamma$;
- $m\alpha \Vdash_{p\Xi} \gamma\sigma$ iff $m\beta^\sigma_\alpha \Vdash_{p\Xi} \gamma$,

where $\beta^\sigma_\alpha = \lambda\xi.[\![\sigma(\xi)]\!]^{m\alpha}_{p\Xi}$.

Proof:
Straightforward, using the fact that $[\![\gamma\sigma]\!]^{m\alpha}_{p\Xi}(u) = [\![\gamma]\!]^{m\beta^\sigma_\alpha}_{p\Xi}(u)$. QED

Proposition 5.13 If p has sound proof rules then p is weakly sound.

Proof:
Since $\emptyset^{\vdash_{p\Xi}} = \{\delta \in L_{p\Xi} : \delta \text{ is provable}\}$ and $\emptyset^{\models_{p\Xi}} = \emptyset^{\overline{\models}_{p\Xi}}$, we prove, equivalently, that $\{\delta \in L_{p\Xi} : \delta \text{ is provable}\} \subseteq \emptyset^{\overline{\models}_{p\Xi}}$. For the purpose, assume that δ is provable, $m \in M$ and α is an assignment into m. We prove, by induction on the length of the proof of δ, that $m\alpha \Vdash_{p\Xi} \delta$.
Base:
δ is $Conc(r)\sigma$ where $Prem(r) = \emptyset$ and $r \in P$. Using the soundness of r, we get that $Conc(r) \in \emptyset^{\overline{\models}_{p\Xi}}$, and thus $m\beta^\sigma_\alpha \Vdash_{p\Xi} Conc(r)$. It then follows that $m\alpha \Vdash_{p\Xi} Conc(r)\sigma$.
Step:
δ is $Conc(r)\sigma$ for some $r \in P$. Then, for each $\gamma \in Prem(r)$, $\gamma\sigma$ is provable and the induction hypothesis leads to $m\alpha \Vdash_{p\Xi} \gamma\sigma$. Clearly, then, $m\beta^\sigma_\alpha \Vdash_{p\Xi} \gamma$ for each $\gamma \in Prem(r)$ and using the soundness of r, $m\beta^\sigma_\alpha \Vdash_{p\Xi} Conc(r)$. Immediately, $m\alpha \Vdash_{p\Xi} Conc(r)\sigma$. QED

As a trivial consequence of the above, $\emptyset^{\vdash_p} \subseteq \emptyset^{\models_p}$.

Proposition 5.14 If p has sound rules (proof and derivation) then p is sound.

Proof:
Assume that $\delta \in \Gamma^{\vdash_{p\Xi}}$, $m \in M$, $u \in U_m$ and α is an assignment into m such that $m\alpha u \Vdash_{p\Xi} \gamma$ for every $\gamma \in \Gamma$. We prove, by induction on the length of the derivation of δ from Γ, that $m\alpha u \Vdash_{p\Xi} \delta$.
Base:
i. $\delta \in \Gamma$ and trivially $m\alpha u \Vdash_{p\Xi} \delta$.
ii. δ is provable. Since $\{\delta \in L_{p\Xi} : \delta \text{ is provable}\} = \emptyset^{\vdash_{p\Xi}} \subseteq \emptyset^{\vDash_{p\Xi}}$, $m\alpha u \Vdash_{p\Xi} \delta$.
Step:
δ is $Conc(r)\sigma$ for some $r \in D$. Then, $Prem(r)\sigma \subseteq \Gamma^{\vdash_{p\Xi}}$ and so, by the induction hypothesis , $m\alpha u \Vdash_{p\Xi} \gamma\sigma$ for each $\gamma \in Prem(r)$. Thus, $m\beta^{\sigma}_{\alpha} u \Vdash_{p\Xi} \gamma$ for each $\gamma \in Prem(r)$ and, using the soundness of r, $m\beta^{\sigma}_{\alpha} u \Vdash_{p\Xi} Conc(r)$. Therefore, $m\alpha u \Vdash_{p\Xi} Conc(r)\sigma$. QED

Note that, in fact, the soundness of the rules of p is strictly stronger than the soundness of p itself. In fact, soundness of p implies the soundness of all its derivation rules. However, as far as proof rules are concerned, the only guarantee is that every application of a rule in a proof from $\emptyset$ is "sound".

In general, for soundness preservation purposes, it is necessary that proof rules are indeed sound. This is not a severe restriction since in most (all?) known cases, the soundness of a proof system results from the soundness of its rules. In any case, any unsound rule in a sound logic system presentation can be safely replaced by all its useful instances (which are sound!).

Moreover, preservation of soundness through fibring must be related with the corresponding inclusion or cocartesian morphisms. And, by definition, morphisms preserve rules rather than just provability and derivability, which is essential to the notion of fibring as explained in the motivation.

Proposition 5.15 Let $p' = \langle C', M', A', P', D' \rangle$ and $p'' = \langle C'', M'', A'', P'', D'' \rangle$ be logic system presentations and $p = p' \oplus p''$ their unconstrained fibring with inclusions i' and i''. Then, if p' and p'' both have sound rules (proof and derivation) then p has sound rules (proof and derivation).

Proof:
1. Proof rules.
By definition, the proof rules of p are $P = \overrightarrow{i'}(P') \cup \overrightarrow{i''}(P'')$. Let $r \in P$.
i. r is $\overrightarrow{i'}(r')$ for some $r' \in P'$.
Knowing that r' is sound in p', $Conc(r') \in \overline{Prem(r')^{\vDash_{p'\Xi}}}$, it is an immediate consequence that $\overrightarrow{i'}(Conc(r')) \in \overrightarrow{i'}(\overline{Prem(r')^{\vDash_{p'\Xi}}}) \subseteq \overrightarrow{i'}(Prem(r'))^{\vDash_{p\Xi}}$. Thus, $Conc(r) \in \overline{Prem(r)^{\vDash_{p\Xi}}}$ and r is sound in p.
ii. r is $\overrightarrow{i''}(r'')$ for some $r'' \in P''$.
Analogous, using i''.
2. Derivation rules.
By definition, the derivation rules of p are $D = \overrightarrow{i'}(D') \cup \overrightarrow{i''}(D'')$. Let $r \in D$.
i. r is $\overrightarrow{i'}(r')$ for some $r' \in D'$.
Knowing that r' is sound in p', $Conc(r') \in Prem(r')^{\vDash_{p'\Xi}}$, it is an immediate consequence that $\overrightarrow{i'}(Conc(r')) \in \overrightarrow{i'}(Prem(r')^{\vDash_{p'\Xi}}) \subseteq \overrightarrow{i'}(Prem(r'))^{\vDash_{p\Xi}}$. Thus,

$Conc(r) \in Prem(r)^{\models_p \Xi}$ and r is sound in p.
ii. r is $\overrightarrow{i''}(r'')$ for some $r'' \in D''$.
Analogous, using i''. QED

Proposition 5.16 Let $p' = \langle C', M', A', P', D' \rangle$ and $p'' = \langle C'', M'', A'', P'', D'' \rangle$ be logic system presentations, $p = p' \oplus p''$ their unconstrained fibring with inclusions i' and i'', $f' : C \to C'$ and $f'' : C \to C''$ injective signature morphisms and $p''' = p' \overset{f'Cf''}{\oplus} p''$ their corresponding constrained fibring. Then, if p' and p'' both have sound rules (proof and derivation) then p''' has sound rules (proof and derivation).

Proof:
1. Proof rules.
By definition, the proof rules of p''' are $P''' = q(P)$, where P are the proof rules of p. Let $r''' = q(r) \in P'''$. From the previous result, $r \in P$ is sound, $Conc(r) \in Prem(r)^{\models_p \Xi}$, and thus $q(Conc(r)) \in q(Prem(r)^{\models_p \Xi}) \subseteq q(Prem(r))^{\models_{p'''} \Xi}$. So, $Conc(r''') \in Prem(r''')^{\models_{p'''} \Xi}$ and r''' is sound.
2. Derivation rules.
By definition, the derivation rules of p''' are $D''' = q(D)$, where D are the derivation rules of p. Let $r''' = q(r) \in D'''$. From the previous result, $r \in D$ is sound, $Conc(r) \in Prem(r)^{\models_p \Xi}$, and thus $q(Conc(r)) \in q(Prem(r)^{\models_p \Xi}) \subseteq q(Prem(r))^{\models_{p'''} \Xi}$. So, $Conc(r''') \in Prem(r''')^{\models_{p'''} \Xi}$ and r''' is sound. QED

6 Concluding remarks

We made a categorial characterization of fibring at both proof-theoretic and model-theoretic levels, providing a novel fibred semantics with explicit models.

At the proof-theoretic level, after establishing the appropriate category of Hilbert calculi, we showed that unconstrained fibring appears as a coproduct. And we showed that constrained fibring (by sharing both propositional symbols and logical operators) appears as a cocartesian lifting from the category of signatures.

At the model-theoretic level, we were able to repeat the process for interpretation systems, providing a semantics of fibring with explicit models. Fibring at the semantic level turned out to be much more difficult to characterize but again we obtained it as a coproduct (in the unconstrained case) and as a cocartesian lifting (in the constrained case).

By putting together inference rules and models in the notion of logic system presentation, we were able to characterize fibring with proofs and semantics hand in hand. At this point, we illustrated the proposed constructions and gave a simple but sufficiently meaningful example of fibring within modal logic. We also discussed in detail the problem of fibring intuitionistic and classical logics showing a limitation of the proposed notion of interpretation system.

Finally, after some preliminary results showing how derivation and entailment are transferred from the given logics into the resulting logic, we showed that soundness is preserved by fibring.

We should stress again the main limitations of the paper. We concentrated herein only on propositional-based logics, i.e., logics without terms and variable binding operators like quantification. The proposed approach seems to be workable in the more general case, but we feel that the reduced complexity and the usefulness of propositional-based logics well justify presenting only the results on the simpler case. For preliminary results on the topic of fibring of more complex logics see [19]. Therein, rules are considered to be composed of requirements, premises and conclusion. Requirements are used to express constraints such as "term free for variable in formula". They may also be used for "weakening" the axiom $(\xi_1 \rightarrow (\xi_2 \rightarrow \xi_1))$ of intuitionistic logic by imposing that ξ_1 may only be replaced by a persistent formula (as proposed by [7] when fibring intuitionistic and classical logic).

The other limitation concerns the fact that we cannot faithfully represent the semantics of intuitionistic logic. But the proposed approach seems to be workable in the more general case of interpretation systems capable of representing general frames as sketched at the end of section 4. Such general frames will encompass intuitionistic logics and are also interesting from the point of view of completeness preservation by fibring.

Indeed, another important line of work is concerned with additional preservation results, for instance on model existence and completeness. It seems that the techniques used in [13] can be adapted.

The connection of our approach to the approach presented in [12] also seems to be a worthwhile line of future research, now that we have some results for logics with variables, terms and binding operators [19].

We would like to end with a few words on applications. We expect that the work on fibring will have great impact in some application areas such as software engineering and artificial intelligence. Indeed, in both areas it is necessary in many situations to work with several formalisms (read logics) for specifying and reasoning about systems, since some aspects of those systems are better described in one formalism and other aspects in a quite different formalism [2, 16]. For instance, in software engineering, dynamic logic and temporal logic are frequently used in the same project. Fibring can help a lot in such cases, since it provides a rich environment where both kinds of reasoning can be made. Furthermore, fibring seems to be a more general combination than those that have been considered in practice so far.

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