Synchronization of Logics with Mixed Rules: Completeness Preservation

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Abstract. Several mechanisms for combining logics have appeared in the literature. Synchronization is one of the simplest: the language of the combined logic is the disjoint union of the given languages, but the class of models of the resulting logic is a subset of the cartesian product of the given classes of models (the interaction between the two logics is imposed by constraining the class of pairs of models). Herein, we give both a model-theoretic and a proof-theoretic account of synchronization as a categorial construction (using coproducts and cocartesian liftings). We also prove that soundness is preserved by possibly constrained synchronization and state sufficient conditions for preservation of model existence and strong completeness. We provide an application to the combination of dynamic logic and linear temporal logic.
Keywords: combination of logics, synchronization of logics, model existence, completeness, dynamic logic, temporal logic.

1 Introduction

There has been a recent growth of interest on the problem of combining logics and obtaining results about the preservation of logical properties like soundness, completeness, decidability, existence of finite models, *inter alia* (see for instance [14, 10, 4, 5, 2, 11]). Related work [13, 1, 3] motivated by applications in software engineering has addressed the problem of relating and combining institutions. The relevance to applications is very clear: the software development process is usually carried out in a heterogeneous environment of methods, languages, formalisms and platforms. For instance, we may use in the same project temporal logic and dynamic logic for specifying and verifying the envisaged software.

In a previous paper [12] we proposed two simple forms of combining logics: synchronization on formulae and synchronization on models. Herein, we continue the development of the latter mechanism by providing both proof and model theoretic categorial characterizations of synchronization. Furthermore, we prove that synchronizations preserve soundness and also model existence and completeness under suitable assumptions. This mechanism, though simple, is still useful as we explain via the example of combining dynamic logic and linear temporal logic. Indeed, synchronization seems to be the adequate form of combination in many cases when compared to other forms of combination, like temporalization [4] and fibring [6].

With synchronization, the language of the combined logic is the disjoint union of the given languages, but the class of models of the resulting logic is a subset of the cartesian product of the given classes of models: the interaction between the two logics is imposed by constraining the class of pairs of models.

In Section 2 we adopt the relevant notion of logic system as a tuple composed of language, models, satisfaction and inference rules, setting up the category we use in the rest of the paper.

In Section 3 we propose the basic mechanism of free synchronization of logic systems (with no interaction whatsoever). Synchronization without interaction is not interesting *per se*, but it is used in Section 4 in the definition of the useful mechanism of synchronization in the presence of interaction rules. We end Section 3 by showing that free synchronizations are coproducts.

In Section 4, before presenting the synchronization constrained by mixed inference rules, we first show how to extract a Hilbert system from a logic system by forgetting the semantic components. We show that the forgetful functor allows cocartesian liftings. These are used for the (categorial) definition of the constrained form of synchronization.

In Section 5 we concentrate on finding out sufficient conditions for proving that synchronizations preserve model existence and completeness since the preservation of soundness comes out already in Sections 3 and 4.

We illustrate the proposed combination mechanism by showing how to establish a useful combination of dynamic logic and linear temporal logic in Section 6 (remaining within the propositional context for the sake of economy of presentation).

2 Preliminaries

Since we are interested in both proof and model theoretic characterizations of synchronization, we need to work with a notion of logic system including both aspects. On one hand, we need models and satisfaction. On the other hand, we need inference rules. We also need an improper model that satisfies everything in order to be able to preserve completeness as we shall see. This implies no loss of generality because we can add to any logic such an improper model (if it does not exist already) without any impact on its semantic entailment. Moreover, we can assume that only one improper model exists (if there are several they can be collapsed). Therefore, we arrive at the following notion:

Definition 1. *A logic system is a tuple $\langle L, M, \Vdash, R\rangle$ where:*

- *L is a set;*
- *M is a pointed class with selected element z;*
- *$\Vdash \subseteq M \times L$ is a relation such that $\{\varphi \in L : m \Vdash \varphi\} = L$ iff $m = z$;*
- *$R \subseteq \wp_{\mathrm{fin}} L \times L$.*

Clearly, the elements of L are the *formulae* and the elements of M are the *models*, being z the *improper model*. Relation $\Vdash$ is the *satisfaction relation*. Each element $r = \langle Prem(r), Conc(r)\rangle$, of R is an inference *rule*: $Prem(r)$ is the (finite) set of premises and $Conc(r)$ is the conclusion. If $Prem(r) = \emptyset$ then r is said to be an *axiom*.

The entailment and derivation closures are defined as usual. We briefly review the basic definitions mainly to freeze the notation.

Definition 2. *Given a logic system, the entailment closure* $.^{\models} : \wp L \to \wp L$ *is defined as follows:* $\Psi^{\models} = \{\varphi \in L : m \Vdash \varphi \text{ whenever } m \Vdash \psi \text{ for every } \psi \in \Psi\}$.

Definition 3. *Given a logic system, we say that there is a derivation of* φ *from* Ψ *iff there is a sequence* $\psi_1, ..., \psi_m$ *of formulae such that:*

- ψ_m *is* φ;
- *for each* $i = 1, ..., m$, *the formula* ψ_i *is:*
  - *either an element of* Ψ;
  - *or* $Conc(r)$ *provided that* $r \in R$ *and every element of* $Prem(r)$ *occurs in the sequence* $\psi_1, ..., \psi_{i-1}$.

Definition 4. *Given a logic system, the derivation closure* $.^{\vdash} : \wp L \to \wp L$ *is defined as follows:* $\Psi^{\vdash} = \{\varphi \in L : \text{ there is a derivation of } \varphi \text{ from } \Psi\}$.

Definition 5. *A logic system is:*

- *sound iff* $\Psi^{\vdash} \subseteq \Psi^{\models}$ *for every* Ψ;
- *strongly complete iff* $\Psi^{\models} \subseteq \Psi^{\vdash}$ *for every* Ψ.

Definition 6. *A rule* r *in a logic system is said to be sound iff* $Conc(r) \in Prem(r)^{\models}$.

Proposition 7. A logic system is sound iff the rules are sound.

Proof:
Assume that the rules are sound and that $\varphi \in \Psi^{\vdash}$. We show that $\varphi \in \Psi^{\models}$ by induction on the length n of the derivation of φ from Ψ.
Base: $\varphi \in \Psi$. Then, of course $\varphi \in \Psi^{\models}$.
Step: φ is $Conc(r)$ for some $r \in R$. Then, $Prem(r) \subseteq \Psi^{\vdash}$ and so, using the induction hypothesis, $Prem(r) \subseteq \Psi^{\models}$. Moreover, $Conc(r) \in \Psi^{\models}$ using the soundness of r. QED

In the sequel, we also need the notions of consistency and model existence.

Definition 8. *A set* Ψ *of formulae is said to be consistent (with respect to derivation) iff* $\Psi^{\vdash} \neq L$.

Definition 9. *A logic system is said to have the model existence property iff for every consistent set* Ψ *of formulae there is a proper model* m *such that* $m \Vdash \psi$ *for every* $\psi \in \Psi$.

Proposition 10. (Strong) completeness implies the model existence property.

Proof:
Assume that $\varphi \notin \Psi^{\vdash}$. Using completeness, there exists $m \in M$ such that $m \Vdash \Psi$ and $m \nVdash \varphi$. Thus, m is proper and the result follows. QED

We shall also need some results concerning logic systems with negation. We just require that negation has the usual semantics ($m \Vdash (\neg \varphi)$ iff $m \nVdash \varphi$, for every proper model m) and fulfils a usual metatheorem (if $\varphi \in (\Psi \cup \{(\neg \varphi)\})^{\vdash}$ then $\varphi \in \Psi^{\vdash}$).

Proposition 11. Let $\langle L, M, \Vdash, R\rangle$ be a logic system with negation, $\varphi \in L$ and $\Psi \subseteq L$. If $\varphi \notin \Psi^{\vdash}$ then $\Psi \cup \{(\neg \varphi)\}$ is consistent.

Proof:
Easily $\varphi \notin (\Psi \cup \{(\neg \varphi)\})^{\vdash}$. QED

Proposition 12. For logic systems with negation, completeness is equivalent to model existence.

Proof:
Assume that model existence holds and that $\varphi \notin \Psi^{\vdash}$. It follows that $\Psi \cup \{(\neg \varphi)\}$ is consistent and there exists a proper $m \in M$ such that $m \Vdash \Psi \cup \{(\neg \varphi)\}$. Therefore $m \nVdash \varphi$ and $\varphi \notin \Psi^{\vDash}$. QED

Proposition 13. Let $\langle L, M, \Vdash, R\rangle$ be a sound logic system with negation and $\Psi \subseteq L$. Ψ is consistent iff there is no $\varphi \in L$ such that $\varphi, (\neg \varphi) \in \Psi^{\vdash}$.

Proof:
If Ψ is inconsistent it is trivial that any formula and its negation are derivable from Ψ. On the other hand, if $\varphi, (\neg \varphi) \in \Psi^{\vdash}$ then, by soundness, Ψ cannot have proper models. Therefore, Ψ is inconsistent. QED

Proposition 14. Let $\langle L, M, \Vdash, R\rangle$ be a logic system with negation and $\{\Psi_n\}_{n \in \mathbb{N}}$ a chain of consistent sets. Then, $\Psi = \bigcup_{n \in \mathbb{N}} \Psi_n$ is consistent.

Proof:
Suppose Ψ is inconsistent. Then, for some $\varphi \in L$, $\varphi, (\neg \varphi) \in \Psi^{\vdash}$. Since derivations are finite sequences, $\Psi^{\vdash} = \bigcup_{n \in \mathbb{N}} \Psi_n^{\vdash}$. Thus, there exist $n, m \in \mathbb{N}$ such that $\varphi \in \Psi_n^{\vdash}$ and $(\neg \varphi) \in \Psi_m^{\vdash}$. Therefore, $\varphi, (\neg \varphi) \in \Psi^{\vdash}_{max(m,n)}$ and $\Psi_{max(m,n)}$ is inconsistent. QED

We conclude this section with the definition of the category of sound logic systems that we shall use in the rest of the paper.

Definition 15. *A logic system morphism* $h : \langle L, M, \Vdash, R\rangle \to \langle L', M', \Vdash', R'\rangle$ *is a pair* $\langle \overrightarrow{h}, \overleftarrow{h}\rangle$ *where:*

- $\overrightarrow{h} : L \to L'$;

– $\overleftarrow{h} : M' \to M$;

such that:

1. $\overrightarrow{h}(Conc(r)) \in \overrightarrow{h}(Prem(r))^{\vdash'}$ *for each* $r \in R$;
2. $\overleftarrow{h}(z') = z$;
3. $m' \Vdash' \overrightarrow{h}(\varphi)$ *iff* $\overleftarrow{h}(m') \Vdash \varphi$.

Condition 1. imposes the preservation of inference rules. Condition 2. states that the improper model is preserved. Condition 3. is known as the satisfaction condition [7]. Note that these morphisms preserve entailment and derivation:

Proposition 16. Let $h : \langle L, M, \Vdash, R\rangle \to \langle L', M', \Vdash', R'\rangle$ be a logic system morphism. Then:

– $\overrightarrow{h}(\Psi^{\vDash}) \subseteq \overrightarrow{h}(\Psi)^{\vDash'}$;
– $\overrightarrow{h}(\Psi^{\vdash}) \subseteq \overrightarrow{h}(\Psi)^{\vdash'}$.

Proof:
1. Assume that $\varphi \in \Psi^{\vDash}$ and that $m' \Vdash' \overrightarrow{h}(\psi)$ for every $\psi \in \Psi$. Then, $\overleftarrow{h}(m') \Vdash \psi$ for every $\psi \in \Psi$, hence $\overleftarrow{h}(m') \Vdash \varphi$, so $m' \Vdash' \overrightarrow{h}(\varphi)$ and therefore $\overrightarrow{h}(\varphi) \in \overrightarrow{h}(\Psi)^{\vDash'}$.
2. Assume that $\varphi \in \Psi^{\vdash}$. We prove the result by induction on the length n of the derivation of φ from Ψ.
Base: $n = 1$. Then, $\varphi \in \Psi$, hence $\overrightarrow{h}(\varphi) \in \overrightarrow{h}(\Psi)$ and so $\overrightarrow{h}(\varphi) \in \overrightarrow{h}(\Psi)^{\vdash'}$.
Step: φ is $Conc(r)$ for some $r \in R$. Then, $Prem(r) \subseteq \Psi^{\vdash}$, therefore, by the induction hypothesis $\overrightarrow{h}(Prem(r)) \subseteq \overrightarrow{h}(\Psi)^{\vdash'}$ and so $\overrightarrow{h}(Conc(r)) \in \overrightarrow{h}(\Psi)^{\vdash'}$. QED

Proposition 17. Sound logic systems and their morphisms constitute the category *Lsy*.

Note that we might work with all logic systems (instead of concentrating on the sound ones). However, some results below only hold for sound systems (namely, the cocartesian lifting in Section 4). And anyway we are not interested in unsound logic systems.

3 Free synchronization

Before presenting the main construction of the paper (synchronization of logic systems with additional mixed rules) it is worthwhile to look first at free synchronization (without any interaction between the given logic systems). The basic idea is as follows. The language is the disjoint union of the given languages and the class of models is the cartesian product of the given classes of models. Satisfaction is componentwise. The rules are just put together.

Definition 18. *The free synchronization of the logic systems* $\langle L', M', \Vdash', R'\rangle$ *and* $\langle L'', M'', \Vdash'', R''\rangle$ *is the logic system*

$$\langle L', M', \Vdash', R'\rangle \oplus \langle L'', M'', \Vdash'', R''\rangle = \langle L, M, \Vdash, R\rangle$$

where:

- $L = L' \oplus L''$ *(disjoint union) with injections* i', i''*;*
- $M = M' \otimes M''$ *(cartesian product) with projections* p', p'' *and the improper model* $\langle z', z''\rangle$*;*
- $m \Vdash i'(\varphi')$ *iff* $p'(m) \Vdash' \varphi'$ *and* $m \Vdash i''(\varphi'')$ *iff* $p''(m) \Vdash'' \varphi''$*;*
- $R = i'(R') \cup i''(R'')$*.*

The notation we chose for the free synchronization is well justified by the following categorial characterization:

Proposition 19. Free synchronizations are coproducts in *Lsy*.

Proof:
1. Soundness. We just prove that each rule in R is sound. Let $r' \in R'$ and assume that $\langle m', m''\rangle \Vdash Prem(i'(r'))$. Then, $m' \Vdash' Prem(r')$ and, using the soundness of r', we have $m' \Vdash' Conc(r')$ and so $\langle m', m''\rangle \Vdash Conc(i'(r'))$.
2. Injections: $\langle i', p'\rangle : \langle L', M', \Vdash', R'\rangle \to \langle L', M', \Vdash', R'\rangle \oplus \langle L'', M'', \Vdash'', R''\rangle$ and $\langle i'', p''\rangle : \langle L'', M'', \Vdash'', R''\rangle \to \langle L', M', \Vdash', R'\rangle \oplus \langle L'', M'', \Vdash'', R''\rangle$.
3. Universal property: let $f' : \langle L', M', \Vdash', R'\rangle \to \langle L''', M''', \Vdash''', R'''\rangle$ and $f'' : \langle L'', M'', \Vdash'', R''\rangle \to \langle L''', M''', \Vdash''', R'''\rangle$ be any morphisms in *Lsy*. Then, the unique morphism $k : \langle L', M', \Vdash', R'\rangle \oplus \langle L'', M'', \Vdash'', R''\rangle \to \langle L''', M''', \Vdash''', R'''\rangle$ such that $k \circ \langle i', p'\rangle = f'$ and $k \circ \langle i'', p''\rangle = f''$ is $\langle \overrightarrow{k}, \overleftarrow{k}\rangle$ where $\overrightarrow{k} : L' \oplus L'' \to L'''$ such that $\overrightarrow{k}(i'(\varphi')) = \overrightarrow{f'}(\varphi')$ and $\overrightarrow{k}(i''(\varphi'')) = \overrightarrow{f''}(\varphi'')$ and $\overleftarrow{k} : M''' \to M' \otimes M''$ such that $\overleftarrow{k}(m''') = \langle \overleftarrow{f'}(m'''), \overleftarrow{f''}(m''')\rangle$. QED

4 Synchronization constrained by mixed rules

The basic idea is to enrich a free synchronization with additional mixed rules imposing interaction between the given logic systems. We go directly to the categorial definition (cocartesian lifting) since it is more elegant. But, to this end, we need first to introduce the category of Hilbert systems and a forgetful functor from logic systems to Hilbert systems.

Definition 20. *A Hilbert system is a pair* $\langle L, R\rangle$ *where:*

- L *is a set;*
- $R \subseteq \wp_{\mathrm{fin}} L \times L$*.*

Clearly, as before, we can extract a *derivation closure* from a Hilbert system.

Definition 21. *A Hilbert system morphism* $h : \langle L, R\rangle \to \langle L', R'\rangle$ *is a map* $h : L \to L'$ *such that* $h(Conc(r)) \in h(Prem(r))^{\vdash'}$ *for each* $r \in R$.

Proposition 22. Hilbert systems and their morphisms constitute the category *Hsy*.

Proposition 23. The maps

- $N(\langle L, M, \Vdash, R\rangle) = \langle L, R\rangle$
- $N(\langle \overrightarrow{h}, \overleftarrow{h}\rangle) = \overrightarrow{h}$

constitute the functor $N : Lsy \to Hsy$.

Proposition 24. For each $\langle L, M, \Vdash, R\rangle$ in *Lsy* and each morphism $h : \langle L, R\rangle \to \langle L', R'\rangle$ in *Hsy* such that h is bijective, the morphism

$$\langle h\rangle : \langle L, M, \Vdash, R\rangle \to \langle L', M', \Vdash', R'\rangle$$

where

- M' is the subclass of all $m \in M$ such that, for each $r' \in R'$,

$$m \Vdash h^{-1}(Conc(r')) \text{ if } m \Vdash h^{-1}(Prem(r'))$$

- $m' \Vdash' \varphi'$ iff $m' \Vdash h^{-1}(\varphi')$
- $\langle h\rangle = \langle h, \overleftarrow{h}\rangle$ where $\overleftarrow{h}$ is an inclusion

is cocartesian by N for h on $\langle L, M, \Vdash, R\rangle$.

Proof:
1. $\langle L', M', \Vdash', R'\rangle$ is a sound logic system: comes directly from the definition of M'.
2. $\langle h\rangle$ is a morphism in *Lsy*. The satisfaction condition holds taking into account the definition of $\Vdash'$.
3. Universal property. Let $f : \langle L, M, \Vdash, R\rangle \to \langle L'', M'', \Vdash'', R''\rangle$ be any morphism in *Lsy* and $g : \langle L', R'\rangle \to \langle L'', R''\rangle$ be any morphism in *Hsy* such that $g \circ h = \overrightarrow{f}$. Then,
(a) $\langle g, \overleftarrow{f}\rangle$ is a morphism in *Lsy*. We have to show that $\overleftarrow{f}(m'') \in M'$. Assume that $\overleftarrow{f}(m'') \Vdash h^{-1}(Prem(r'))$. So, using the satisfaction condition for f, we get $m'' \Vdash'' \overrightarrow{f}(h^{-1}(Prem(r')))$. Therefore $m'' \Vdash'' g(Prem(r'))$ and, using the soundness of $g(r')$, we get $m'' \Vdash'' g(Conc(r'))$. Hence, $m'' \Vdash'' \overrightarrow{f}(h^{-1}(Conc(r')))$ and, using the satisfaction condition for f, we get $\overleftarrow{f}(m'') \Vdash h^{-1}(Conc(r'))$.
(b) $\langle g, \overleftarrow{f}\rangle$ is the unique morphism such that $\langle g, \overleftarrow{f}\rangle \circ \langle h\rangle = f$: direct consequence.
QED

We denote by $h(\langle L, M, \Vdash, R\rangle)$ the codomain of the cocartesian morphism above.

We are finally ready to introduce synchronization constrained by mixed rules. We start by making precise what we mean by a mixed inference rule.

Definition 25. *Within the context of a free synchronization of two logic systems* $s' = \langle L', M', \Vdash', R' \rangle$ *and* $s'' = \langle L'', M'', \Vdash'', R'' \rangle$*, a mixed* $s' \to s''$*-rule is a pair* $x = \langle Prem(x), Conc(x) \rangle$ *where* $Prem(x) \in \wp_{\text{fin}} i'(L')$ *and* $Conc(x) \in i''(L'')$*.*

Definition 26. *Let* $\langle L', M', \Vdash', R' \rangle$ *and* $\langle L'', M'', \Vdash'', R'' \rangle$ *be sound logic systems,* X *a set of mixed* $s' \to s''$*-rules and* Y *a set of mixed* $s'' \to s'$*-rules both within their free synchronization. Then, the synchronization of these logic systems constrained by* X *and* Y *is the sound logic system*

$$\langle L', M', \Vdash', R' \rangle \overset{XY}{\oplus} \langle L'', M'', \Vdash'', R'' \rangle = id(\langle L', M', \Vdash', R' \rangle \oplus \langle L'', M'', \Vdash'', R'' \rangle)$$

corresponding to the cocartesian lifting for the identity map id *on* $L' \oplus L''$ *taken as the morphism in Hsy*

$$id : \langle L' \oplus L'', i'(R') \cup i''(R'') \rangle \to \langle L' \oplus L'', i'(R') \cup i''(R'') \cup X \cup Y \rangle.$$

As we shall see, it is useful to distinguish the special case of *one-way* constrained synchronization when for instance $Y = \emptyset$. In this case all the mixed rules are of type $s' \to s''$.

5 Preservation results

We have already seen that soundness is preserved by synchronization (both free and constrained). Therefore we can state:

Theorem 27. Soundness is preserved by both free and constrained synchronization.

It remains to investigate the preservation of completeness. To this end, we start by looking at the preservation of the model existence property by free synchronization, after two lemmas.

Lemma 28. Let $\langle L, M, \Vdash, R \rangle = \langle L', M', \Vdash', R' \rangle \oplus \langle L'', M'', \Vdash'', R'' \rangle$. Then, we have $i'(\Psi'^{\vdash'}) = i'(\Psi')^{\vdash}$ and $i''(\Psi'^{\vdash''}) = i''(\Psi'')^{\vdash}$.

Proof:
Assume that $\varphi \in i'(\Psi')^{\vdash}$. We prove by induction on the length n of the derivation of φ from $i'(\Psi')$ that $\varphi \in i'(\Psi'^{\vdash'})$.
Base: $n = 1$. Then, $\varphi \in i'(\Psi')$, hence φ is $i'(\varphi')$ for some $\varphi' \in \Psi'$ and so $i'(\varphi') \in i'(\Psi'^{\vdash'})$.
Step: φ is $i'(Conc(r'))$ for some $r' \in R'$. Then, $i'(Prem(r')) \subseteq i'(\Psi')^{\vdash}$ and, using the induction hypothesis, $i'(Prem(r')) \subseteq i'(\Psi'^{\vdash'})$. Therefore, $Prem(r') \subseteq \Psi'^{\vdash'}$, hence $Conc(r') \in \Psi'^{\vdash'}$ and so $i'(Conc(r')) \in i'(\Psi'^{\vdash'})$. QED

Therefore, free synchronization is a conservative extension of the two components. As an immediate corollary we get:

Lemma 29. Let $\langle L, M, \Vdash, R \rangle = \langle L', M', \Vdash', R' \rangle \oplus \langle L'', M'', \Vdash'', R'' \rangle$. Then, we have $i'(\Psi'^{\vdash'}) \cup i''(\Psi''^{\vdash''}) = (i'(\Psi') \cup i''(\Psi''))^{\vdash}$.

We now prove that both model existence and completeness are preserved by free synchronization.

Proposition 30. Given two logic systems s', s'' with the model existence property, their free synchronization has the model existence property.

Proof:
Let $s' \oplus s'' = \langle L', M', \Vdash', R' \rangle \oplus \langle L'', M'', \Vdash'', R'' \rangle = \langle L, M, \Vdash, R \rangle$ and assume that $\Psi = i'(\Psi') \cup i''(\Psi'') \subseteq L$ is consistent. If $i'(\varphi') \notin \Psi^{\vdash}$ then, $\varphi' \notin \Psi'^{\vdash'}$ and, using model existence in s', there exists $m' \in M'$ such that $m' \Vdash' \Psi'$. Thus, $\langle m', z'' \rangle \Vdash \Psi$. Analogously if $i''(\varphi'') \notin \Psi^{\vdash}$. QED

A similar argument is used for completeness.

Proposition 31. Given two (strongly) complete logic systems s', s'', their free synchronization is (strongly) complete.

Proof:
Let $s' \oplus s'' = \langle L', M', \Vdash', R' \rangle \oplus \langle L'', M'', \Vdash'', R'' \rangle = \langle L, M, \Vdash, R \rangle$ and assume that $\Psi = i'(\Psi') \cup i''(\Psi'') \subseteq L$. If $i'(\varphi') \notin \Psi^{\vdash}$ then $\varphi' \notin \Psi'^{\vdash'}$ and, using completeness of s', there exists $m' \in M'$ such that $m' \Vdash' \Psi'$ and $m' \nVdash' \varphi'$. Thus, $\langle m', z'' \rangle \in M$ shows that $i'(\varphi') \notin \Psi^{\vDash}$. Analogously if $i''(\varphi'') \notin \Psi^{\vdash}$. QED

The constrained case is harder. We start by looking at the simpler case of one-way constrained synchronization.

Proposition 32. Given two logic systems s' and s'' with the model existence property and a set X of $s' \to s''$-rules, their synchronization constrained by X and $\emptyset$ has the model existence property.

Proof:
Let $s' \overset{X\,\emptyset}{\oplus} s'' = \langle L', M', \Vdash', R' \rangle \overset{X\,\emptyset}{\oplus} \langle L'', M'', \Vdash'', R'' \rangle = \langle L, M''', \Vdash''', R''' \rangle$ and assume that $\Psi = i'(\Psi') \cup i''(\Psi'') \subseteq L$ is consistent. We consider two cases.
1. $i'(L') \nsubseteq \Psi^{\vdash'''}$. Then, Ψ' is consistent in s' and, using the model existence property, there exists a proper $m' \in M'$ such that $m' \Vdash' \Psi'$. Thus, $\langle m', z'' \rangle \in M'''$ since z'' satisfies every conclusion of rule in X and $\langle m', z'' \rangle \Vdash''' \Psi$.
2. $i'(L') \subseteq \Psi^{\vdash'''}$. Clearly, $i''(L'') \nsubseteq \Psi^{\vdash'''}$. Since only $s' \to s''$-rules are added, $\Phi'' = \Psi'' \cup \{\varphi'' \in L'' : i''(\varphi'') \text{ is } Conc(x) \text{ for some } x \in X\}$ is consistent in s''. By model existence, there exists a proper $m'' \in M''$ such that $m'' \Vdash'' \Phi''$. Thus, $m'' \Vdash'' \Psi''$ and $\langle z', m'' \rangle \in M'''$ since m'' satisfies every conclusion of rule in X. Therefore, $\langle z', m'' \rangle \Vdash''' \Psi$. QED

As far as completeness is concerned we shall need to work with logics with negation (recall that, in this case, model existence is equivalent to completeness). We also require that the constrained logic satisfies, at least, the metatheorem required for negation, although it may not be a logic with negation. Furthermore, we require that the set of mixed rules introduced is countable.

Note that the same applies to two-way synchronization and so we shall only do the proof in the general case.

Proposition 33. Given two logic systems s' and s'' with negation and the model existence property, a set X of $s' \to s''$-rules and a set Y of $s'' \to s'$-rules such that $X \cup Y$ is countable and the metatheorem of negation is fulfilled by $s = s' \overset{X\,Y}{\oplus} s''$, s has the model existence property.

Proof:
Let $s = s' \overset{X\,Y}{\oplus} s'' = \langle L', M', \Vdash', R' \rangle \overset{X\,Y}{\oplus} \langle L'', M'', \Vdash'', R'' \rangle = \langle L, M''', \Vdash''', R''' \rangle$ and assume that $\Psi = i'(\Psi') \cup i''(\Psi'') \subseteq L$ is consistent. We consider two cases.
1. If $i'(L') \subseteq \Psi^{\vdash'''}$ and $i''(L'') \not\subseteq \Psi^{\vdash'''}$ then the set $\Phi'' = \Psi'' \cup \{\varphi'' \in L'' : i''(\varphi'')$ is $Conc(x)$ for some $x \in X\}$ is consistent in s''. By model existence, there exists a proper $m'' \in M''$ such that $m'' \Vdash'' \Phi''$. Thus, $m'' \Vdash'' \Psi''$ and $\langle z', m'' \rangle \in M'''$ since every rule is satisfied. Therefore, $\langle z', m'' \rangle \Vdash''' \Psi$. Analogously if $i''(L'') \subseteq \Psi^{\vdash'''}$ and $i'(L') \not\subseteq \Psi^{\vdash'''}$.
2. If $i'(L') \not\subseteq \Psi^{\vdash'''}$ and $i''(L'') \not\subseteq \Psi{\vdash'''}$ then both Ψ' in s' and Ψ'' in s'' are consistent. Given an enumeration $r_1, r_2, \ldots$ of $X \cup Y$ we proceed as follows. Define $\Phi'_0 = \Psi'$ and $\Phi''_0 = \Psi''$ and then, inductively:

- if $r_{n+1} \in X$
  - $\Phi'_{n+1} = \Phi'_n$, $\Phi''_{n+1} = \Phi''_n \cup \{Conc(r_{n+1})\}$ if $Prem(r_{n+1}) \subseteq (\Phi'_n \cup \Phi''_n)^{\vdash'''}$;
  - $\Phi'_{n+1} = \Phi'_n \cup \{(\neg\, \psi')\}$, $\Phi''_{n+1} = \Phi''_n$ for a chosen $i'(\psi') \in Prem(r_{n+1})$ such that $i'(\psi') \notin (\Phi'_n \cup \Phi''_n)^{\vdash'''}$;
- *mutatis mutandis* if $r_{n+1} \in Y$.

Clearly, both $\Phi' = \bigcup_{n \in I\!N} \Phi'_n$ and $\Phi'' = \bigcup_{n \in I\!N} \Phi''_n$ are chains of consistent sets and hence consistent. By model existence on both s' and s'', there exist proper models $m' \in M'$ and $m'' \in M''$ such that $m' \Vdash' \Phi'$ and $m'' \Vdash'' \Phi''$. By construction, $\langle m', m'' \rangle \in M'''$ and $\langle m', m'' \rangle \Vdash''' \Psi$. QED

A similar argument is used to prove completeness preservation.

Proposition 34. Given two (strongly) complete logic systems s' and s'' with negation, a set X of $s' \to s''$-rules and a set Y of $s'' \to s'$-rules such that $X \cup Y$ is countable and the metatheorem of negation is fulfilled by $s = s' \overset{X\,Y}{\oplus} s''$, s is (strongly) complete.

Proof:
Let $s = s' \overset{X\,Y}{\oplus} s'' = \langle L', M', \Vdash', R' \rangle \overset{X\,Y}{\oplus} \langle L'', M'', \Vdash'', R'' \rangle = \langle L, M''', \Vdash''', R''' \rangle$. Let $\Psi = i'(\Psi') \cup i''(\Psi'') \subseteq L$ and assume, without loss of generality, that $i'(\varphi') \notin \Psi^{\vdash'''}$. Again we have to consider two cases.
1. $i''(L'') \subseteq \Psi^{\vdash'''}$. Then, $\Phi' = \Psi' \cup \{\psi' : i'(\psi')$ is $Conc(y)$ for some $y \in Y\}$ is such that $\varphi' \notin \Phi'^{\vdash'}$. By completeness in s', there exists $m' \in M'$ such that $m' \Vdash' \Phi'$ and $m' \nVdash \varphi'$. Thus, $\langle m', z'' \rangle \in M'''$ since every rule is satisfied and $\langle m', z'' \rangle$ shows that $i'(\varphi') \notin \Psi^{\vDash'''}$.
2. $i''(L'') \not\subseteq \Psi^{\vdash'''}$. Then, both $\Psi' \cup \{(\neg\, \varphi')\}$ in s' and Ψ'' in s'' are consistent. Given an enumeration $r_1, r_2, \ldots$ of $X \cup Y$ we proceed as in the previous proof. Let $\Phi'_0 = \Psi' \cup \{(\neg\, \varphi')\}$ and $\Phi''_0 = \Psi''$ and then, inductively:

- if $r_{n+1} \in X$
  - $\Phi'_{n+1} = \Phi'_n$, $\Phi''_{n+1} = \Phi''_n \cup \{Conc(r_{n+1})\}$ if $Prem(r_{n+1}) \subseteq (\Phi'_n \cup \Phi''_n)^{\vdash'''}$;
  - $\Phi'_{n+1} = \Phi'_n \cup \{(\neg \psi')\}$, $\Phi''_{n+1} = \Phi''_n$ for a chosen $i'(\psi') \in Prem(r_{n+1})$ such that $i'(\psi') \notin (\Phi'_n \cup \Phi''_n)^{\vdash'''}$;
- *mutatis mutandis* if $r_{n+1} \in Y$.

Clearly, both $\Phi' = \bigcup_{n \in I\!N} \Phi'_n$ and $\Phi'' = \bigcup_{n \in I\!N} \Phi''_n$ are chains of consistent sets and hence consistent. By model existence on both s' and s'', there exist proper models $m' \in M'$ and $m'' \in M''$ such that $m' \Vdash' \Phi'$ and $m'' \Vdash'' \Phi''$. By construction, $\langle m', m'' \rangle \in M'''$, and $\langle m', m'' \rangle \Vdash''' \Psi$. QED

Theorem 35. Model existence is preserved by both free and one-way constrained synchronization, and also by two-way constrained synchronization of logics with negation with a countable set of rules when the metatheorem of negation holds.

Theorem 36. (Strong) completeness is preserved by free synchronization, and also by constrained synchronization of logics with negation with a countable set of rules when the metatheorem of negation holds.

6 Application

We now illustrate the proposed mechanism with the constrained synchronization of the weakest (propositional) dynamic logic (see for instance [9]) and the (propositional) linear temporal logic (see for instance [8]). For the sake of simplicity we remain within the propositional realm, although the problem is of greater practical significance within the first order setting.

We assume that we are given a set of propositional symbols Π and a finite set of action symbols $\Delta = \{\delta_1, ..., \delta_n\}$ such that $\Pi \cap \Delta = \emptyset$. The problem is to combine the following logic systems:

- $D = \langle L', M', \Vdash', R' \rangle$ the dynamic logic system over the signature $\langle \Pi, \Delta \rangle$;
- $T = \langle L'', M'', \Vdash'', R'' \rangle$ the linear temporal logic system over the signature $\Pi \cup \Delta$.

With respect to D, recall that:

- $L' = \Pi \mid (\neg L') \mid (L' \Rightarrow L') \mid ([\Delta]L')$;
- each proper model is a triple $\langle S', d', v' \rangle$ with S' a non empty set (of states), $d' : \Delta \to \wp(S' \times S')$ and $v' : S' \to \wp\Pi$;
- satisfaction is straightforward; just note that:
  - $\langle S', d', v' \rangle, s' \Vdash' ([\delta]\varphi)$ iff $\langle S', d', v' \rangle, k' \Vdash' \varphi$ for every $k' \in S'$ such that $\langle s', k' \rangle \in d'(\delta)$;
- the rules are straightforward; for instance normality is as follows:
  - $\langle \emptyset, (([\delta](\varphi \Rightarrow \psi)) \Rightarrow (([\delta]\varphi) \Rightarrow ([\delta]\psi))) \rangle$.

With respect to T, recall that:

– $L'' = \Pi \cup \Delta \mid (\neg\, L'') \mid (L'' \Rightarrow L'') \mid (\mathsf{X}\, L'') \mid (\mathsf{G}\, L'')$;
– each proper model is a map $m'' : \mathbb{N} \to \wp(\Pi \cup \Delta)$;
– satisfaction is straightforward; just note that:
  - $m'', n'' \Vdash'' (\mathsf{X}\, \varphi)$ iff $m'', n'' + 1 \Vdash'' \varphi$;
  - $m'', n'' \Vdash'' (\mathsf{G}\, \varphi)$ iff $m'', k'' \Vdash'' \varphi$ for every $k'' \geq n''$;
– the rules are straightforward; for instance induction is as follows:
  - $\langle \emptyset, (\varphi \Rightarrow ((\mathsf{G}(\varphi \Rightarrow (\mathsf{X}\, \varphi))) \Rightarrow (\mathsf{G}\, \varphi))) \rangle$.

The idea is to use T as a temporal action logic where $m'', n'' \Vdash'' \delta$ means that action δ happens at instant n''.

If we need to write specifications using formulae from both logics, we should work on the synchronization of D and T constrained by $D \to T$-rules such as (where α is a propositional formula over the signature Π):

– $\langle \{i'(\alpha)\}, i''(\alpha) \rangle$;
– $\langle \{i'(([\delta]\alpha))\}, i''((\delta \Rightarrow (\mathsf{X}\, \alpha))) \rangle$;
– $\langle \emptyset, i''((\delta_1 \vee \ldots \vee \delta_n)) \rangle$.

The first rule imposes that propositional assertions in D carry over to T. The second rule brings into T some properties of actions specified within D. The third rule has the nature of a closure rule and it is important when proving temporal invariants using properties imposed in D.

As an example, within the synchronization of D and T constrained as above, we can derive $i''((\pi_1 \Rightarrow (\mathsf{G}(\pi_1 \vee \pi_2))))$ from the set $\{i'(([\delta_k]\pi_1)) : k \text{ odd}\} \cup \{i'(([\delta_k]\pi_2)) : k \text{ even}\}$ of assumptions.

In the opposite direction, we might want to impose the following $T \to D$-rules (where α is again a propositional formula over the signature Π):

– $\langle \{i''(\alpha)\}, i'(\alpha) \rangle$;
– $\langle \{i''((\mathsf{X}\, \alpha))\}, i'((([\delta_1]\alpha) \wedge \ldots \wedge ([\delta_n]\alpha))) \rangle$.

Note that once we have rules in both directions, a proof in the combination of D and T may be carried out by reasoning alternatively in both logics.

7 Concluding remarks

Motivated by software engineering applications, we have been investigating alternative mechanisms for combining logics. Herein, we present in detail model-theoretic and proof-theoretic accounts of one of the simplest mechanisms: constrained synchronization. In this form of combination, the language of the combined logic is the disjoint union of the given languages, but the class of models of the resulting logic is a subset of the cartesian product of the given classes of models: the interaction between the two logics is imposed by constraining the class of pairs of models via mixed inference rules. We show the usefulness of synchronization in a plain but meaningful example (putting together propositional dynamic logic and propositional linear temporal logic). Of course, setting up the desired set of mixed rules depends on the particular desired interplay between the pair of logics being considered and on the particular intended application of the synchronized logic and is therefore out of the scope of this paper.

Along the way we give categorial characterizations of both free synchronization (coproduct) and constrained synchronization (cocartesian lifting). We also establish that both preserve soundness and give sufficient conditions for preservation of model existence and strong completeness.

The results are general since nothing is assumed about the given logic systems to be combined, with the exception of model existence and completeness preservation of general constrained synchronization. We found sufficient conditions but whether they are necessary or not is an open problem.

In a related effort we are investigating categorial characterizations of another widely known form of combination: fibring [6, 11]. It seems that no single form of combination will be sufficient to deal with the problems that practitioners face in software development. Therefore, we believe that a deep understanding of all forms of combination already identified in the literature is much needed. Later on, it will also be useful to establish results relating different forms of combination.

Another interesting line of research that we are following is aimed at the computability issues of combination of logics. Besides the obvious question concerning the preservation of (semi)decidability other computability problems appear, namely concerning morphisms relating logics.

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