A Theory-based Topological Notion of Institution*

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Abstract. By adopting theories as primitive components of a logic and recognizing that formulae are just presentation details we arrive at the concept of topological institution. In a topological institution, we have, for each signature, a frame of theories, a set of interpretation structures and a satisfaction relation. More precisely, we have, for each signature, a topological system. We show how to extract a topological institution from a given institution and establish an adjunction. Illustrations are given within the context of equational logic. We study the compositionality of theories. Formulae are recovered when we establish a general technique for presenting topological institutions. Topological institutions with finitely observable theories are shown to be useful in temporal monitoring applications where we would like to be able to characterize the properties of the system that can be monitored. Namely, an invariant property ($\mathbf{G}\varphi$) cannot be monitored because it cannot be positively established in finite time. On the contrary, a reactivity property ($\mathbf{F}\varphi$) can be positively established in finite time.

1 Introduction

Institutions [4, 5], π-institutions [3, 2] and entailment systems [9] are abstractions that subsume the informal notion of logic system with its varying alphabets. Institutions are semantically based, contrarily to π-institutions and entailment systems. In all cases the central notion of theory is not primitive. Formulae and their models or formulae and consequence relations are taken as the starting point.

However, the (categorial) framework of institutions has proved to be quite useful in software specification and in knowledge representation precisely because it provides the means for composing and structuring theories (and their presentations). In both domains of application we need to be able to describe and reason about "properties" of the system at hand. To this end, a bijection is assumed between theories and properties. Clearly, formulae are used just for presenting properties. Thus, *the central concept is the theory, not the formula.* And indeed all interesting institutional developments deal with theories.

Therefore, one may wonder why the theory is not the primitive concept in institutions. We could easily envisage an alternative structure built around theories (properties) instead of formulae (presentations of properties). Such a structure should

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include signatures (alphabets) plus theories and their semantics. The introduction of this concept is the main goal of the paper.

The basic problem is to find a suitable formula-independent notion of theory. Furthermore, we must also decide upon a suitable structure for the theories over a given signature. Just a set seems to be too little since we do have operations on theories. Maybe a complete lattice, recognizing that we have all joins (and meets) of theories over a given signature. But as we shall see, this would be too much and, on computability grounds, we adopt instead the (weaker) frame structure.

Indeed, a parallel development within the broad area of domain theory provides the starting point: the notion of topological system proposed in [14]. Vickers arrived at this concept by further exploring the consequences of Smyth's dogma: *open sets are semidecidable properties* [13]. Topological systems subsume both topological spaces and locales. A topological system includes a frame of "opens", a set of "points" and a "satisfaction" relation between points and opens. Topological system morphisms satisfy the constraint $x \Vdash_D \Omega\, h(a')$ iff $\mathrm{pt}\, h(x) \Vdash_{D'} a'$ that is strongly reminiscent of the satisfaction condition of institutions.

If we identify the "opens" with formulae we are dealing with geometric logic, as discussed in [14, 15]. But there is an obvious alternative that we explore in this paper: if we restrict our attention to "semidecidable properties" then theories are the opens! Therefore, by working with topological systems we find a structure where we can deal with theories that are not necessarily built from formulae. That is, we are able to deal in this way with *theories as first class citizens*.

Finitary topological institutions promise to be useful in monitoring and querying. We examine herein how to set up a weak version of a temporal logic suitable for monitoring. In database monitoring applications [8] we would like to be able to characterize the properties of the system that can be monitored. Namely, an invariant property ($\mathbf{G}\varphi$) cannot be monitored because it cannot be positively established in finite time. On the contrary, a reactivity property ($\mathbf{F}\varphi$) can be positively established in finite time. Herein, we set up the envisaged logic as a finitary topological institution.

After introducing the concept of topological institution in Section 2 we characterize the relationship between theories and models and we proceed to recover the results on colimits of theories. In Section 3 we start by showing how to extract a topological institution out of an institution and establish an adjunction between the categories of topological institutions and institutions. In Section 4 we consider how to present topological institutions and show how deductive systems can be used to this end. We also examine possible applications of topological institutions with finitely observable theories, namely to temporal monitoring. The proofs omitted herein and further material can be found in [12].

2 Topological institutions

2.1 Topological systems

The interested reader should consult [6, 14] for a full development of the theory of frames, locales and topological spaces. Herein we follow the approach in [14], namely with respect to the introduction of the notion of topological system and starting from there. The main component of a topological system is its frame of "opens". Hence it is worthwhile to recall that a *frame* is a poset $F = \langle |F|, \leq_F \rangle$ such that each subset $S \subseteq |F|$ has a join $\bigvee S$, and binary meets distribute over joins: $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$. In the sequel we use the two-element frame $\mathbf{2}$ with elements $\mathbf{t}$ and $\mathbf{f}$ such that $\mathbf{f} \leq_{\mathbf{2}} \mathbf{t}$. A *frame morphism* $h : F \rightarrow F'$ is a map from $|F|$ into $|F'|$ preserving all joins and finite meets (thus, a monotone map). Frames and their morphisms constitute a category that we denote by *Fr*. We are now ready to bring in the notion proposed by Vickers:

Definition 1. A *topological system* D is a triple $\langle \mathrm{pt}\, D, \Omega\, D, \Vdash_D \rangle$ where:

- $\mathrm{pt}\, D$ is a set;
- $\Omega\, D$ is a frame;
- $\Vdash_D \subseteq \mathrm{pt}\, D \times \Omega\, D$ such that[3]:
  - for any $S \subseteq \Omega\, D$, $x \Vdash_D \bigvee S$ iff $x \Vdash_D a$ for some $a \in S$;
  - for any finite $S \subseteq \Omega\, D$, $x \Vdash_D \bigwedge S$ iff $x \Vdash_D a$ for every $a \in S$.

The elements of $\mathrm{pt}\, D$ are called *points*. The elements of $\Omega\, D$ are called *opens*. The relation $\Vdash_D$ is named *satisfaction relation*. Note that the last property on $\Vdash$ leads to monotonicity: if $x \Vdash_D a$ and $a \leq_{\Omega\, D} b$ then $x \Vdash_D b$.

Definition 2. A *topological system morphism* $h : D \rightarrow D'$ is a pair $\langle \mathrm{pt}\, h, \Omega\, h \rangle$ where:

- $\mathrm{pt}\, h : \mathrm{pt}\, D \rightarrow \mathrm{pt}\, D'$ is a map;
- $\Omega\, h : \Omega\, D' \rightarrow \Omega\, D$ is a frame morphism such that $x \Vdash_D \Omega\, h(a')$ iff $\mathrm{pt}\, h(x) \Vdash_{D'} a'$ for every $x \in \mathrm{pt}\, D$ and $a' \in \Omega\, D'$.

By looking at points as "models" and at opens as "formulae", the condition will remind the reader familiar with the notion of institution [4, 5] of the satisfaction condition. Topological systems and their morphisms constitute a category that we denote by TSy. Topological systems with frame F establish a full subcategory TSy_F of TSy.

We now proceed to identify topological spaces among topological systems, following the idea in [14]. A topological space is a special topological system where the opens are subsets of points. And there is a canonical construction for building a topological space out of any given topological system. As we shall see, this construction helps in the characterization of the relationship between properties and models.

A *topological space* X is a topological system $\langle \mathrm{pt}\, X \Omega\, X, \Vdash_X \rangle$ such that there is an inclusion frame morphism $\Omega\, X \hookrightarrow \wp \mathrm{pt}\, X$[4], and $\Vdash_X$ is set membership ($x \Vdash_X a$ iff $x \in a$). Topological spaces constitute a full subcategory of TSy that we denote by TSp[5]. Note that for a topological space morphism $h : X \rightarrow X'$ we have that $\Omega\, h = (\mathrm{pt}\, h)^{-1}$. We denote by $IncSp$ the inclusion functor $TSp \hookrightarrow TSy$.

[3] We write $x \Vdash_D a$ instead of $\langle x, a \rangle \in \Vdash_D$ saying that x *satisfies* a.

[4] That is, unions of opens are open and finite intersections of opens are open. Clearly, an infinite meet of opens (still open) may not coincide with the corresponding intersection.

[5] The category TSp is isomorphic to the traditional category Top of topological spaces and continuous maps not presented as special cases of topological systems and their morphisms.

Definition 3. A topological system is said to be *spatial* iff it is isomorphic to a topological space.

Definition 4. Let D be a topological system and $a \in \Omega\, D$. The *extension* of a is the set $ext(a) = \{x \in \mathrm{pt}\, D : x \Vdash_D a\}$.

Clearly, if we consider points to be interpretation structures and opens to be properties then the extension of a is the set of models of a.

Definition 5. Given a topological system D we define its *spatialization* $\mathrm{Spat}D$ as the topological space such that:

- $\mathrm{pt}\ \mathrm{Spat}D = \mathrm{pt}\ D$;
- $\Omega\,\mathrm{Spat}D = \langle\{ext(a) : a \in \Omega\, D\}, \subseteq\rangle$.

It is easy to verify that we do obtain a frame with the inclusion ordering on the extensions. Moreover, we have soundness: if $a \leq_{\Omega\, D} a'$ then $ext(a) \subseteq ext(a')$.

Definition 6. Given a topological system D, its *TSp-coreflection* is the topological system morphism $e : \mathrm{Spat}D \to D$ defined as follows:

- $\mathrm{pt}\, e(x) = x$;
- $\Omega\, e(a) = ext(a)$.

This terminology is justified by the following result [12]:

Theorem 7. *TSp is a coreflective subcategory of TSy.*

It follows that a topological system D is spatial iff its *TSp-coreflection* is iso iff for every $a, a' \in \Omega\, D$ if $ext(a) \subseteq ext(a')$ then $a \leq_{\Omega\, D} a'$. That is, we have completeness iff the topological system is spatial. Using the universal property of the coreflections, it is straightforward to verify that the spatialization construction extends to a functor $Spat : TSy \to TSp$. Furthermore, $Spat$ is right adjoint to $IncSp$.

2.2 Basic concept and theories

As envisaged, we want for each signature a topological system with the theories, the interpretation structures and the satisfaction relation for that signature. Therefore:

Definition 8. A *topological institution* T is a pair $\langle \mathrm{sg}T, \mathrm{sy}T\rangle$ where

- $\mathrm{sg}T$ is a category (of signatures);
- $\mathrm{sy}T : \mathrm{sg}T \to TSy^{op}$ is a functor.

The opens in $\Omega\,\mathrm{sy}T(\Sigma)$ are called *theories over the signature* Σ, the points in $\mathrm{pt}\ \mathrm{sy}T(\Sigma)$ are called *interpretation structures* over Σ and $\Vdash_{\mathrm{sy}T(\Sigma)}$ the *satisfaction relation* over Σ. In the sequel T will always denote a topological institution.

Definition 9. A topological institution T is *spatial* iff, for each signature Σ, the topological system $\mathrm{sy}T(\Sigma)$ is spatial.

Definition 10. A *topological institution morphism* $\tau : T \rightarrow T'$ is a pair $\langle \mathrm{sg}\tau, \mathrm{sy}\tau \rangle$ where

- $\mathrm{sg}\tau : \mathrm{sg}T \rightarrow \mathrm{sg}T'$ is functor;
- $\mathrm{sy}\tau : \mathrm{sy}T \rightarrow \mathrm{sg}\tau ; \mathrm{sy}T'$ is a natural transformation.

Topological institutions and their morphisms constitute a category: TIn. It is interesting to explore the spatialization construction for topological institutions, but we refrain herein to establish the resulting coreflection.

Definition 11. The *model topological institution* $\mathrm{Mod}T$ induced by T is as follows:

- $\mathrm{sgMod}T = \mathrm{sg}T$;
- $\mathrm{syMod}T$ is such that:
  - $\mathrm{syMod}T(\Sigma) = \mathrm{Spatsy}T(\Sigma)$;
  - $\mathrm{syMod}T(\sigma) : \mathrm{syMod}T(\Sigma') \rightarrow \mathrm{syMod}T(\Sigma)$ is the unique morphism in TSy such that $e'; \mathrm{sy}T(\sigma) = \mathrm{syMod}T(\sigma); e$ where e and e' are the TSp-coreflection morphisms for $\mathrm{sy}T_\Sigma$ and $\mathrm{sy}T_{\Sigma'}$ respectively.

Note that the morphism $\mathrm{syMod}T(\sigma)$ is unique because of the universal property of the coreflection e. Each theory th of $\mathrm{sy}T(\Sigma)$ is replaced in $\mathrm{syMod}T(\Sigma)$ by the set of interpretation structures $ext(th)$ that satisfy th. Therefore, if T is spatial then we have for each Σ: $th \leq_{\mathrm{sy}T(\Sigma)} th'$ iff $ext(th) \subseteq ext(th')$.

We now look at the "flat" category of theories in T. As expected, we recover the result that (finite) colimits of signatures can be lifted to (finite) colimits of theories.

Definition 12. A *theory* is a pair $th = \langle \Sigma, a \rangle$ where $\Sigma \in |\mathrm{sg}T|$ and $a \in \Omega\,\mathrm{sy}T(\Sigma)$.

Definition 13. Let $th = \langle \Sigma, a \rangle$ and $th' = \langle \Sigma', a' \rangle$ be theories. A *morphism* from th into th' is a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ such that $a' \leq_{\Omega\,\mathrm{syT}(\Sigma')} \Omega\,\mathrm{sy}T(\sigma)(a)$.

Theories and theory morphisms constitute a category: Th_T. Furthermore, assuming that $Sg : Th_T \rightarrow \mathrm{sg}T$ is the functor mapping theories and theory morphisms into their signatures and signature morphisms, $\langle Th_T, Sg \rangle$ is a concrete category over $\mathrm{sg}T$. It is straightforward to establish [12]:

Theorem 14. *If* $\mathrm{sg}T$ *has finite colimits so does* Th_T.

We consider for instance how coproducts are lifted. Let $th = \langle \Sigma, a \rangle$ and $th' = \langle \Sigma', a' \rangle$ be theories and $\langle \Sigma + \Sigma', inj_\Sigma, inj_{\Sigma'} \rangle$ be "the" coproduct in $\mathrm{sg}T$ of Σ and Σ'. Then, the triple $\langle th + th', inj_\Sigma, inj_{\Sigma'} \rangle$ where $th + th' = \langle \Sigma + \Sigma', \Omega\,\mathrm{syT}(inj_\Sigma)(a) \wedge \Omega\,\mathrm{syT}(inj_{\Sigma'})(a') \rangle$ is "the" coproduct in Th_T of th and th'.

3 Institutions versus topological institutions

Before we can show how to extract a topological institution from any given institution, we need to see how to complete a meet-semilattice, while respecting (some of) the existing joins, in order to obtain a frame, using Johnstone's technique of coverages [6].

3.1 Ideal completion

We proceed to review Johnstone's technique, characterizing the envisaged universal property (of the canonical construction) as a reflection. To this end, we introduce the category of sites and its subcategory of frame sites.

Definition 15. A *site* Y is a pair $\langle \mathrm{ms}Y, \mathrm{cv}Y\rangle$ where:

- $\mathrm{ms}Y$ is a meet-semilattice, that is, a poset $\langle |\mathrm{ms}Y|, \leq_{\mathrm{ms}Y}\rangle$ such that every finite subset $S \subseteq |\mathrm{ms}Y|$ has a meet;
- $\mathrm{cv}Y$ is *coverage* on $\mathrm{ms}Y$, that is, a map $\mathrm{cv}Y : |\mathrm{ms}Y| \to \wp\wp|\mathrm{ms}Y|$ such that:
  - $\mathrm{cv}Y(a) \subseteq \wp{\downarrow}(a)$[6];
  - for every $a' \leq a$, if $S \in \mathrm{cv}Y(a)$ then $\{s \wedge a' : s \in S\} \in \mathrm{cv}Y(a')$.

Definition 16. A *site morphism* $h : Y \to Y'$ is a meet-semilattice morphism from $\mathrm{ms}Y$ into $\mathrm{ms}Y'$[7] such that if $S \in \mathrm{cv}Y(a)$ then $h(S) \in \mathrm{cv}Y'(h(a))$.

Sites and their morphisms constitute a category that we denote by *Sit*. Some sites correspond to frames as we proceed to explain.

Definition 17. A *frame site* is a site $Y = \langle \mathrm{ms}Y, \mathrm{cv}Y\rangle$ such that $\mathrm{ms}Y$ is a frame, and $S \in \mathrm{cv}Y(a)$ iff $a = \bigvee S$.

We denote by *FrSit* the full subcategory of *Sit* whose objects are the frame sites. We denote by *IncFr* the inclusion functor $FrSit \hookrightarrow Sit$. Clearly, the category *FrSit* is isomorphic to *Fr*. We now show how to extract a canonical frame site from any given site by ideal completion.

Definition 18. A Y*-ideal* is a lower closed[8] subset I of $|\mathrm{ms}Y|$ such that, for every $a \in |\mathrm{ms}Y|$ and $S \in \mathrm{cv}Y(a)$, if $S \subseteq I$ then $a \in I$.

Definition 19. Given a site Y, the pair $\mathrm{Idl}Y = \langle\{I : I \text{ is a } Y\text{-ideal}\}, \subseteq\rangle$ is a frame called the *ideal completion* of Y.

We denote by $\mathrm{SIdl}Y$ the corresponding frame site.

Definition 20. Given a site Y, its *FrSit-reflection* is the site morphism $q : Y \to \mathrm{SIdl}Y$ induced by the map from $|\mathrm{ms}Y|$ into $|\mathrm{msSIdl}Y|$ defined as follows:

- $q(a) = \bigcap\{I \in |\mathrm{msSIdl}Y| : a \in I\}$.

It is necessary to verify that the indicated map does induce a site morphism. That is to say, it should preserve finite meets and the coverage[9]. The reflection name is justified by the following result [12]:

Theorem 21. *FrSit is a reflective subcategory of Sit.*

Using the universal property of reflections, it is straightforward to verify that the ideal completion construction extends to a functor $Idl : Sit \to FrSit$. Furthermore, *Idl* is left adjoint to *IncFr*.

[6] ${\downarrow}(a)$ denotes as usual the set $\{a' : a' \leq a\}$

[7] A map from $|\mathrm{ms}Y|$ into $|\mathrm{ms}Y'|$ preserving finite meets.

[8] If $a' \leq a$ and $a \in I$ then $a' \in I$.

[9] It should transform covers to joins.

3.2 Institutions induce topological institutions

Given a signature Σ in some institution, the idea is to establish a topological system where the opens are the theories over Σ, the points are the interpretation structures over Σ and the satisfaction relation is as given for Σ. However, this naive workplan has to be refined, mainly because satisfaction does not always respect joins of theories as required in topological systems. In the sequel I will always denote an elementary institution $\langle Sig_I, Sen_I, Int_I, \Vdash_I\rangle$[10].

If $\Phi \subseteq Sen_I(\Sigma)$ and $\psi \in Sen_I(\Sigma)$ then we say that Φ *semantically entails* ψ, written $\Phi \vDash_{I_\Sigma} \psi$, iff, for every $x \in Int_I(\Sigma)$, if $x \Vdash_{I_\Sigma} \varphi$ for every $\varphi \in \Phi$ then $x \Vdash_{I_\Sigma} \psi$. We denote by $\Phi^{\vDash_{I_\Sigma}}$ the set $\{\psi \in Sen_I(\Sigma) : \Phi \vDash_{I_\Sigma} \psi\}$ of formulae semantically entailed by Φ. We denote the class of all theories by $|Th_I|$ and, for each signature Σ, the class of all theories over Σ by $|Th_{I_\Sigma}|$. We say that $x \in Int(\Sigma)$ satisfies a theory $th = \langle \Sigma, \Phi\rangle$, indicated by $x \Vdash_{I_\Sigma} th$ or $x \Vdash_{I_\Sigma} \Phi$, iff $x \Vdash_{I_\Sigma} \varphi$ for all $\varphi \in \Phi$. Clearly:

Proposition 22. *Consider the binary relation $\leq_{I_\Sigma}$ in $|Th_{I_\Sigma}|$ such that $\langle \Sigma, \Phi_1\rangle \leq_{I_\Sigma} \langle \Sigma, \Phi_2\rangle$ iff $\Phi_1 \vDash_{I_\Sigma} \Phi_2$. The pair $Th^{op}_{I_\Sigma} = \langle |Th_{I_\Sigma}|, \leq_{I_\Sigma}\rangle$ is a complete lattice.*

For each $S \subseteq |Th_{I_\Sigma}|$, $\bigvee S = \bigcap S$ and $\bigwedge S = (\bigcup S)^{\vDash_{I_\Sigma}}$. The bottom theory is $\langle \Sigma, Sen_I(\Sigma)\rangle$ and the top theory is $\langle \Sigma, \emptyset^{\vDash_{I_\Sigma}}\rangle$. We use the *op* notation since $th_1 \leq_{I_\Sigma} th_2$ iff the set of formulae of th_2 is included in the set of formulae of th_1.

Note that, in general $\langle Int_I(\Sigma), Th^{op}_{I_\Sigma}, \Vdash_{I_\Sigma}\rangle$ is not a topological system. Indeed, we do not know if we do have a frame (distributivity?) and whether every join is respected by the satisfaction relation: $x \Vdash_{I_\Sigma} \bigvee S$ iff $x \Vdash_{I_\Sigma} s$ for some $s \in S$. Therefore, we proceed to complete (using the coverage technique) the meet-semilattice of theories while preserving only the joins that are respected by the satisfaction relation. Afterwards, we extend the satisfaction relation to the new theories.

Definition 23. A set $S \subseteq |Th^{op}_{I_\Sigma}|$ is said to be *join-acceptable* by $\Vdash_{I_\Sigma}$ iff the following condition holds: $x \Vdash_{I_\Sigma} \bigvee S$ iff $x \Vdash_{I_\Sigma} th$ for some $th \in S$.

Example 1. Let EqI be the equational elementary institution. A non-empty $S \subseteq |Th^{op}_{EqI_\Sigma}|$ is join-acceptable iff S has a maximum theory.

The following result [12] shows how to set up a site with the theories of the institution at hand:

Proposition 24. *Let $\mathrm{cv}_{I_\Sigma} : Th^{op}_{I_\Sigma} \to \wp\wp Th^{op}_{I_\Sigma}$ be as follows: for each theory th, $S \in \mathrm{cv}_{I_\Sigma}(th)$ iff $\bigvee S = th$ and S is join-acceptable. Then, the pair $Y_{I_\Sigma} = \langle Th^{op}_{I_\Sigma}, \mathrm{cv}_{I_\Sigma}\rangle$ is a site and the $FrSit$-reflection $q_{I_\Sigma} : Y_{I_\Sigma} \to SIdlY_{I_\Sigma}$ is injective.*

Example 2. In the simple case of EqI, for each theory $\langle \Sigma, E\rangle$ we have:

- $\mathrm{cv}_{EqI_\Sigma}(\langle \Sigma, E\rangle) = \{S \subseteq |Th^{op}_{EqI_\Sigma}| : max(S) = \langle \Sigma, E\rangle\}$:
- $q_{EqI_\Sigma}(\langle \Sigma, E\rangle) = \{\langle \Sigma, E'\rangle \in |Th^{op}_{EqI_\Sigma}| : E' \vDash_{EqI_\Sigma} E\}$

For more complex examples see [12] namely the case of propositional logic.

[10] By "elementary" (or pre-institution as it is called in [10]) we mean that the functor Int_I goes into Set^{op}.

Definition 25. The *frame of theories* induced by a given elementary institution I is $\overline{Th}^{op}_{I_\Sigma} = IdlY_{I_\Sigma}$.

Example 3. In $\overline{Th}^{op}_{EqI_\Sigma}$ we find a new theory for each set $S \subseteq |Th^{op}_{EqI_\Sigma}|$ that is not join-acceptable. For instance, consider the set $\{\langle\Sigma, E_1\rangle, \langle\Sigma, E_2\rangle\}$ of equational theories such that $E_1 \not\vDash_{EqI_\Sigma} E_2$ and $E_2 \not\vDash_{EqI_\Sigma} E_1$. Then, we get the new theory $\bigvee\{q_{EqI_\Sigma}(\langle\Sigma, E_1\rangle), q_{EqI_\Sigma}(\langle\Sigma, E_2\rangle)\}$ in $\overline{Th}^{op}_{EqI_\Sigma}$ that appears below the theory $q_{EqI_\Sigma}(\langle\Sigma, E_1 \cap E_2\rangle)$.

In short, in order to set up a topological system for each signature of an institution, we start by identifying the join-acceptable sets of theories. These sets are then reflected in the coverage used for the ideal completion of the meet-semilattice of theories. It remains to be seen that it is possible to extend the satisfaction relation to the new theories while ensuring that all joins are respected by satisfaction.

But before we proceed with this extension note that the ideal completion adds only the necessary joins besides the acceptable ones: "no junk". Also, all existing theories are preserved as different from each other: "no confusion".

Lemma 26. *For each $\Sigma \in |Sig_I|$ and $x \in Int_I(\Sigma)$ let $sat^x_{I_\Sigma} : Th^{op}_{I_\Sigma} \to \mathbf{2}$ be the meet-semilattice morphism defined as follows: $sat^x_{I_\Sigma} = \lambda th.$ if $x \Vdash_{I_\Sigma} th$ then $\mathbf{t}$ else $\mathbf{f}$. Let $\mathrm{S}\mathbf{2}$ be the frame site corresponding to the frame $\mathbf{2}$. Then, $sat^x_{I_\Sigma} : Y_{I_\Sigma} \to \mathrm{S}\mathbf{2}$ is a site morphism.*

Proposition 27. *Let $\overline{sat}^x_{I_\Sigma} : \overline{Th}^{op}_{I_\Sigma} \to \mathrm{S}\mathbf{2}$ be the unique frame (site) morphism such that $q_{I_\Sigma}; \overline{sat}^x_{I_\Sigma} = sat^x_{I_\Sigma}$. For each $ith \in |\overline{Th}^{op}_{I_\Sigma}|$ let $x \Vdash_{I_\Sigma} ith$ iff $\overline{sat}^x_{I_\Sigma}(ith) = \mathbf{t}$. Then, the triple $D_{I_\Sigma} = \langle Int_I(\Sigma), \overline{Th}^{op}_{I_\Sigma}, \Vdash_{I_\Sigma}\rangle$ is a spatial topological system.*

Proof. Assume without loss of generality that $ith = \bigvee q_{I_\Sigma}(S)$. Then $x \Vdash_{I_\Sigma} ith$ iff $\overline{sat}^x_{I_\Sigma}(ith) = \mathbf{t}$ iff $\bigvee_{s\in S} \overline{sat}^x_{I_\Sigma}(q_{I_\Sigma}(s)) = \mathbf{t}$ iff $\overline{sat}^x_{I_\Sigma}(q_{I_\Sigma}(s)) = \mathbf{t}$ for some $s \in S$ iff $x \Vdash_{I_\Sigma} q_{I_\Sigma}(s)$ for some $s \in S$. A similar reasoning applies to finite meets.

Let $ith \vDash_{I_\Sigma} ith'$ iff for every $x \in Int_I(\Sigma)$ if $x \Vdash_{I_\Sigma} ith$ then $x \Vdash_{I_\Sigma} ith'$ (or in other words $ext(ith) \subseteq ext(ith')$). Assume without loss of generality that $ith = \bigvee\{q_{I_\Sigma}(s) : s \in S\}$ and $ith' = \bigvee\{q_{I_\Sigma}(s') : s' \in S'\}$. For every $x \in Int_I(\Sigma)$ if $\overline{sat}^x_{I_\Sigma}(q_{I_\Sigma}(s)) = \mathbf{t}$ then $\overline{sat}^x_{I_\Sigma}(q_{I_\Sigma}(s')) = \mathbf{t}$ for some $s' \in S'$. Moreover, for every $x \in Int_I(\Sigma)$ if $sat^x_{I_\Sigma}(s) = \mathbf{t}$ then $sat^x_{I_\Sigma}(s') = \mathbf{t}$ for some $s' \in S'$. Hence $s \vDash_{I_\Sigma} \bigvee S'$ for every $s \in S$ and so $\bigvee S \leq_{I_\Sigma} \bigvee S'$. For every $r' \subset q_{I_\Sigma}(\bigvee S')$ we have $r' \subseteq ith'$ and so, in particular, $q_{I_\Sigma}(s) \subseteq ith'$ for every $s \in S$. Moreover, $ith \subseteq ith'$. □

The spatiality of the resulting topological system (for each signature) is only to be expected since theories are semantically defined in institutions. In order to obtain the induced topological institution, it remains to check that each signature morphism produces a topological system morphism (the proofs that can be found in [12]).

Lemma 28. *For each signature morphism $\sigma : \Sigma \to \Sigma'$, let $\Omega\, h_{I_\sigma} : |Y_{I_\Sigma}| \to |Y_{I_{\Sigma'}}|$ be such that $\Omega\, h_{I_\sigma}(\langle\Sigma, \Phi\rangle) = \langle\Sigma', \{Sen_I(\sigma)(\varphi) : \varphi \in \Phi\}^{\vDash_{I_{\Sigma'}}}\rangle$. Then, $\Omega\, h_{I_\sigma} : Y_{I_\Sigma} \to Y_{I_{\Sigma'}}$ is a site morphism.*

Proposition 29. *Let* $\overline{\Omega\, h_{I_\sigma}} : \overline{Th}^{op}_{I_\Sigma} \to \overline{Th}^{op}_{I_{\Sigma'}}$ *be the unique frame (site) morphism such that* $q_{I_\Sigma}; \overline{\Omega\, h_{I_\sigma}} = \Omega\, h_{I_\sigma}; q_{I_{\Sigma'}}$. *Let* $\mathrm{pt}\, h_{I_\sigma} : Int_I(\Sigma') \to Int_I(\Sigma)$ *be such that* $\mathrm{pt}\, h_{I_\sigma}(x') = Int_I(\sigma)(x')$. *Then the pair* $h_{I_\sigma} = \langle \mathrm{pt}\, h_{I_\sigma}, \overline{\Omega\, h_{I_\sigma}} \rangle : D_{I_{\Sigma'}} \to D_{I_\Sigma}$ *is a topological system morphism.*

Theorem 30. *Every institution* I *induces the topological institution* $Top(I)$ *such that:*

- $\mathrm{sg}Top(I) = Sig_I$;
- $\mathrm{sy}Top(I)(\Sigma) = D_{I_\Sigma}$;
- $\mathrm{sy}Top(I)(\sigma) = h_{I_\sigma}$.

We denote by EqT the topological institution $Top(EqI)$.

3.3 Adjunction

We conclude this section by showing that the map $I \mapsto Top(I)$ can be extended to a functor with a left adjoint. We adopt the definition of institution morphism given in [5] and denote the category of institutions by In.

Note that for every institution morphism $m : I \to I'$ and signature Σ in Sig_I, the map $m_{\alpha_\Sigma} : Sen_{I'}(m_s(\Sigma)) \to Sen_I(\Sigma)$ can be extended to $m_{\alpha_\Sigma} : Th_{I'_{m_s(\Sigma)}} \to Th_{I_\Sigma}$ as follows: $m_{\alpha_\Sigma}(\langle m_s(\Sigma), \Phi' \rangle) = \langle \Sigma, \{m_{\alpha_\Sigma}(\varphi') : \varphi' \in \Phi'\}^{\models_{I_\Sigma}} \rangle$. Furthermore, consider the site $Y_{I_\Sigma} = \langle Th^{op}_{I_\Sigma}, cv_{I_\Sigma} \rangle$ where $cv_{I_\Sigma}(m_{\alpha_\Sigma}(th')) = \{m_{\alpha_\Sigma}(S') : S' \in cv_{I'_{m_s(\Sigma)}}(th')\}$. Then, m_{α_Σ} is a site morphism.

Lemma 31. *Every institution morphism* $m : I \to I'$ *induces a topological institution morphism* $Top(m) : Top(I) \to Top(I')$ *defined as follows:*

- $\mathrm{sg}Top(m) = m_s$;
- $\mathrm{pt}\ \mathrm{sy}Top(m)_\Sigma = m_{\beta_\Sigma}$;
- $\Omega\, \mathrm{sy}Top(m)_\Sigma$ *is the unique* $\overline{m}_{\alpha_\Sigma}$ *such that* $q_{I'_{\mathrm{sg}Top(\Sigma)}}; \overline{m}_{\alpha_\Sigma} = m_{\alpha_\Sigma}; q_{I_\Sigma}$.

So in order to get a topological institution out of an institution we have to set up the frame of theories, by selecting the appropriate coverage and then making the ideal completion as we did in subsection 3.2, and then to extend the satisfaction relation to this frame. We have thus established the envisaged functor $Top : In \to TIn$. Conversely:

Lemma 32. *Every topological institution* T *induces the institution* $Geo(T)$ *such that:*

- $Sig_{Geo(T)} = \mathrm{sg}T$;
- $Sen_{Geo(T)}(\Sigma) = |\Omega\, \mathrm{sy}T(\Sigma)|$ *and* $Sen_{Geo(T)}(\sigma) = \Omega\, \mathrm{sy}T(\sigma)$;
- $Int_{Geo(T)}(\Sigma) = \mathrm{pt}\ \mathrm{sy}T(\Sigma)$ *and* $Int_{Geo(T)}(\sigma) = \mathrm{pt}\ \mathrm{sy}T(\sigma)$;
- $\Vdash_{Geo(T)} = \Vdash_T$.

The main step for getting an institution out of a topological institution is to recognize that each open will be a formula. All the other components are then straightforward.

Lemma 33. *Every topological institution morphism $\tau : T \to T'$ induces an institution morphism $Geo(\tau) : Geo(T) \to Geo(T')$ defined as follows:*

- $Geo(\tau)_s = \mathrm{sg}\tau$;
- $Geo(\tau)_{\alpha_\Sigma} = \Omega\,\mathrm{sy}\tau_\Sigma$;
- $Geo(\tau)_{\beta_\Sigma} = \mathrm{pt}\ \mathrm{sy}\tau_\Sigma$.

We have thus established the functor $Geo : TIn \to In$. Finally:

Theorem 34. *Geo is left adjoint to Top.*

Proof. The counit $\epsilon : Top; Geo \to id_{In}$ is as follows: for each institution I, $\epsilon_{I_s} = id_{Sig_I}$; $\epsilon_{I_{\alpha_\Sigma}}(\varphi) = q_{I_\Sigma}(\langle \Sigma, \{\varphi\}^{\vDash_{I_\Sigma}}\rangle)$; $\epsilon_{I_\beta} = id_{Int_I}$. Indeed, given a topological institution T and an institution morphism $m : Geo(T) \to I$, we show that there is a unique topological institution morphism $\tau : T \to Top(I)$ such that $Geo(\tau); \epsilon_I = m$. (1) Consider the map $h_\Sigma : |Th^{op}_{Geo(T)_\Sigma}| \to |\Omega\,\mathrm{sy}T(\Sigma)|$ such that $h_\Sigma(\langle \Sigma, A\rangle) = \bigwedge A$. We show that h_Σ is a site morphism. We have that h_Σ is a meet-semilattice morphism. We verify that h_Σ preserves the coverage. Let $m_{\alpha_\Sigma}(S) \in cv_{Geo(T)_\Sigma}(m_{\alpha_\Sigma}(th))$. Then, $m_{\alpha_\Sigma}(s) \leq_{Th^{op}_{Geo(T)_\Sigma}} m_{\alpha_\Sigma}(th)$ for every $s \in S$. Therefore, the set of formulae of $m_{\alpha_\Sigma}(th)$ is included in the set of formulae of $m_{\alpha_\Sigma}(s)$ for every $s \in S$. Hence, $h_\Sigma(m_{\alpha_\Sigma}(s)) \leq_{\Omega\,\mathrm{sy}T(\Sigma)} h_\Sigma(m_{\alpha_\Sigma}(th))$ for every $s \in S$ and so $m_{\alpha_\Sigma}(th)$ is an upper bound of $m_{\alpha_\Sigma}(S)$. Assume that there is $a' \in \Omega\,\mathrm{sy}T(\Sigma)$ such that $h_\Sigma(m_{\alpha_\Sigma}(s)) \leq_{\Omega\,\mathrm{sy}T(\Sigma)} a'$ for every $s \in S$ and $a' \leq_{\Omega\,\mathrm{sy}T(\Sigma)} h_\Sigma(m_{\alpha_\Sigma}(th))$. Assume that $x \Vdash_{Geo(T)_\Sigma} m_{\alpha_\Sigma}(s)$. Then, $x \Vdash_{\mathrm{sy}T(\Sigma)} h_\Sigma(m_{\alpha_\Sigma}(s))$, $x \Vdash_{\mathrm{sy}T(\Sigma)} a'$ and so $x \Vdash_{Geo(T)_\Sigma} \langle \Sigma, \{a'\}^{\vDash_{Geo(T)_\Sigma}}\rangle$. Hence, $m_{\alpha_\Sigma}(s) \leq_{Geo(T)_\Sigma} \langle \Sigma, \{a'\}^{\vDash_{Geo(T)_\Sigma}}\rangle$. Similarly, $\langle \Sigma, \{a'\}^{\vDash_{Geo(T)_\Sigma}}\rangle \leq_{Geo(T)_\Sigma} m_{\alpha_\Sigma}(th)$. So $m_{\alpha_\Sigma}(th)$ would not be the join of $m_{\alpha_\Sigma}(S)$. Therefore, h_Σ is a site morphism.

(2) Consider the pair $\mathrm{sy}\tau_\Sigma = \langle m_{\beta_\Sigma}, \Omega\,\mathrm{sy}\tau_\Sigma\rangle$ where $\Omega\,\mathrm{sy}\tau_\Sigma$ is the unique frame (site) morphism such that $q_{I_{m_s(\Sigma)}}; \Omega\,\mathrm{sy}\tau_\Sigma = m_{\alpha_\Sigma}; h_\Sigma$. We show that $\mathrm{sy}\tau_\Sigma$ is a topological system morphism: $x \Vdash_{\mathrm{sy}T(\Sigma)} \bigvee \Omega\,\mathrm{sy}\tau_\Sigma(q_{I_{m_s(\Sigma)}}(S))$ iff $x \Vdash_{\mathrm{sy}T(\Sigma)} h_\Sigma(m_{\alpha_\Sigma}(s))$ for some $s \in S$ iff $x \Vdash_{\mathrm{sy}T(\Sigma)} \bigwedge\{a : a \in m_{\alpha_\Sigma}(s)\}$ for some $s \in S$ iff $x \Vdash_{Geo(T)_\Sigma} \langle \Sigma, \{m_{\alpha_\Sigma}(\varphi) : \varphi \in s\}^{\vDash_{Geo(T)_\Sigma}}\rangle$ for some $s \in S$ iff $m_{\beta_\Sigma}(x) \Vdash_{I_{m_s(\Sigma)}} s$ for some $s \in S$ iff $m_{\beta_\Sigma}(x) \Vdash_{\mathrm{sy}Top(I)_{m_s(\Sigma)}} q_{I_{m_s(\Sigma)}}(s)$ for some $s \in S$ iff $m_{\beta_\Sigma}(x) \Vdash_{\mathrm{sy}Top(I)_{m_s(\Sigma)}} \bigvee q_{I_{m_s(\Sigma)}}(S)$.

(3) We show that $Geo(\tau); \epsilon_I = m$. Let $A = \{m_{\alpha_\Sigma}(\psi) : \psi \in \{\varphi\}^{\vDash_{I_{m_s(\Sigma)}}}\}^{\vDash_{Geo(T)_\Sigma}}$. Then we have: $Geo(\tau)_{\alpha_\Sigma}(\epsilon_{I_{\alpha_{m_s(\Sigma)}}}(\varphi)) = Geo(\tau)_{\alpha_\Sigma}(q_{I_{m_s(\Sigma)}}(\langle m_s(\Sigma), \{\varphi\}^{\vDash_{I_{m_s(\Sigma)}}}\rangle)) = h_\Sigma(\langle \Sigma, A\rangle) = \bigwedge A = m_{\alpha_\Sigma}(\varphi)$. The equality of the other components is immediate. And it is straightforward to check that τ is the unique morphism fulfilling this condition. □

The sentence counterpart of the institution morphism ϵ_Σ for signature Σ assigns to each formula, for that signature, the ideal for the theory presented by that formula. The other components are just identities.

4 Presenting topological institutions

Although topological institutions provide the right level of abstraction for working with theories and their models, we do need a mechanism for presenting the envisaged frame of theories. As expected such a mechanism may involve formulae, axioms and inference rules. Actually, we introduce the notion of topological institution presentation directly based on the notion of frame presentation. And only afterwards we recognize how deductive systems can be used for presenting topological institutions.

4.1 Frame presentations

Frames can be presented as (uni-sorted) algebras as long as we accept operations of unbounded arity. But with such operations the general method of finding an initial model by setting up the algebra of congruence classes of terms fails because the class of all freely generated terms is a proper class. Fortunately, in the specific case of frames there is a solution using the coverage technique already described: first generate a meet-semilattice using the traditional congruence method; then find the suitable coverage and produce by ideal completion the envisaged frame; the resulting frame is canonical in the sense of having the universal property of a reflection. Besides reviewing the method as explained in [14], we introduce the notion of frame presentation morphism and show how it extends uniquely to a frame (site) morphism.

Definition 35. A *frame presentation* is a pair $pres = \langle \Sigma_{pres}, Req_{pres} \rangle$ where:

- the signature Σ_{pres} contains the following operation symbols:
  - $\mathbf{t}$ of arity 0,
  - $\wedge$ of arity 2,
  - $\bigvee$ of unbounded arity,
  - g of arity 0, for each $g \in G_{pres}$ (the set of proper symbols known as *generators*);
- Req_{pres} is a set (of equational *proper requirements*) such that:
  - each $e \in Req_{pres}$ is a pair $\langle e_1, e_2 \rangle$ where both e_1 and e_2 are of the form: $\bigvee \{\wedge_{i=1}^{n_k} g_{k_i} : k \in K\}$ where K is a set of indexes.

Each requirement $\langle e_1, e_2 \rangle$ is to be understood as stating that the left element e_1 is equal to the right element e_2. Note that any requirement of the form $y \leq z$ can be rewritten as $\langle y \wedge z, y \rangle$.

Definition 36. A frame presentation $pres = \langle \Sigma_{pres}, Req_{pres} \rangle$ is said to be *simple* iff in each requirement $\langle e_1, e_2 \rangle \in Req_{pres}$ the member e_2 is a generator.

Without loss of generality we may assume that we work only with simple frame presentations. Indeed, any frame presentation can be reduced to a simple frame presentation by enriching the set of generators with an additional element $z_{\langle e_1, e_2 \rangle}$ for each offending requirement $\langle e_1, e_2 \rangle$ and replacing it by the two following requirements: $\langle e_1, z_{\langle e_1, e_2 \rangle} \rangle$ and $\langle e_2, z_{\langle e_1, e_2 \rangle} \rangle$. From any frame presentation *pres*, we extract a meet-semilattice finitary equational presentation $\mathrm{m}pres = \langle \Sigma_{\mathrm{m}pres}, Req_{\mathrm{m}pres} \rangle$, where $\Sigma_{\mathrm{m}pres}$ is Σ_{pres} minus $\bigvee$ and $Req_{\mathrm{m}pres}$ contains:

- the $\wedge$-laws: commutativity, associativity, idempotence and $\mathbf{t}$-neutrality;
- the proper requirements: $g \wedge (\wedge_{i=1}^{n_k} g_{k_i}) = \wedge_{i=1}^{n_k} g_{k_i}$ for each $k \in K$ such that $\langle \bigvee \{\wedge_{i=1}^{n_k} g_{k_i} : k \in K\}, g \rangle$ is in Req_{pres}.

Definition 37. The meet-semilattice M_{pres} presented by $pres$ is induced by the $\Sigma_{\mathrm{m}pres}$-algebra $T_{\mathrm{m}pres}$ with $a \leq a'$ iff $a \wedge a' = a$.

The existence of the initial model $T_{\mathrm{m}pres}$ of m$pres$ is a well known result of universal algebra extensively used in algebraic specification of abstract data types [1].

Definition 38. The map $\mathrm{pcv}_{pres} : |M_{pres}| \to \wp\wp|M_{pres}|$, called in the sequel the *precoverage* induced by $pres$, is established as follows. For each $a \in |M_{pres}|$, $\mathrm{pcv}_{pres}(a)$ is the least set satisfying $\{\wedge_{i=1}^{n_k} g_{k_i} : k \in K\} \in \mathrm{pcv}_{pres}(a)$ whenever $\langle \bigvee\{\wedge_{i=1}^{n_k} g_{k_i} : k \in K\}, a\rangle \in Req_{pres}$.

Theorem 39. *The precoverage* pcv_{pres} *generates a coverage* cv_{pres} *on* $|M_{pres}|$ *as follows: for each* $a \in |M_{pres}|$, $\mathrm{cv}_{pres}(a)$ *is the least set satisfying* $\{a' \wedge s : s \in S\} \in \mathrm{cv}_{pres}(a')$ *whenever* $a' \leq a$ *and* $S \in \mathrm{pcv}_{pres}(a)$*. Therefore,* $Y_{pres} = \langle M_{pres}, \mathrm{cv}_{pres}\rangle$ *is a site.*

Definition 40. The *frame presented* by $pres$ is $F_{pres} = \mathrm{Idl}\langle M_{pres}, \mathrm{cv}_{pres}\rangle$.

We denote by FS_{pres} the frame site corresponding to the frame F_{pres} and by q_{pres} the $FrSit$-reflection for the site $\langle M_{pres}, \mathrm{cv}_{pres}\rangle$.

Definition 41. *A frame presentation morphism* $h : pres \to pres'$ *is a map* $h : G_{pres} \to G_{pres'}$ *such that if* $\langle \bigvee\{\wedge_{i=1}^{n_k} g_{k_i} : k \in K\}, g\rangle \in Req_{pres}$ *then* $\langle \bigvee\{\wedge_{i=1}^{n_k} h(g_{k_i}) : k \in K\}, h(g)\rangle \in Req_{pres'}$.

Frame presentations and their morphisms constitute a category: FrP. As expected, frame presentation morphisms induce morphisms between the presented frames [12]:

Theorem 42. *A frame presentation morphism* $h : pres \to pres'$ *induces the site morphism* $h^* : Y_{pres} \to Y_{pres'}$ *defined as follows:* $h^*([\bigwedge_{i=1}^{n} g_i]) = [\bigwedge_{i=1}^{n} h(g_i)]$*. Then* $\overline{h}^* : F_{pres} \to F_{pres'}$ *is the unique frame (site) morphism such that* $q_{pres}; \overline{h}^* = h^*; q_{pres'}$.

4.2 Deductive systems present topological institutions

Definition 43. A *topological institution presentation* U is a pair $\langle \mathrm{sg}U, \mathrm{fp}U\rangle$ where

- $\mathrm{sg}U$ is a category (of signatures);
- $\mathrm{fp}U : \mathrm{sg}U \to FrP$ is a functor.

Definition 44. A topological institution T is said to *comply* with a topological institution presentation U iff

- $\mathrm{sg}T = \mathrm{sg}U$;
- for each signature Σ, $\mathrm{sy}T(\Sigma) \in |TSy_{F_{\mathrm{fp}U(\Sigma)}}|$;
- for each signature morphism $\sigma : \Sigma \to \Sigma'$, $\Omega\,\mathrm{sy}T(\sigma) = \overline{\mathrm{fp}U(\sigma)}^*$.

Definition 45. A *deductive system* Δ is a quadruple $\langle \mathrm{sg}\Delta, \mathrm{sen}\Delta, \mathrm{ax}\Delta, \mathrm{ru}\Delta \rangle$ where

- $\mathrm{sg}\Delta$ is a category (of signatures);
- $\mathrm{sen}\Delta : \mathrm{sg}\Delta \to Set$ is a functor (the formula functor);
- $\mathrm{ax}\Delta = \{\mathrm{ax}\Delta_\Sigma\}_{\Sigma \in |\mathrm{sg}\Delta|}$ where:
  - $\mathrm{ax}\Delta_\Sigma \subseteq \mathrm{sen}\Delta(\Sigma)$;
  - $\mathrm{sen}\Delta(\sigma)(\alpha) \in \mathrm{ax}\Delta_{\Sigma'}$ for each $\sigma : \Sigma \to \Sigma'$ and $\alpha \in \mathrm{ax}\Delta_\Sigma$;
- $\mathrm{ru}\Delta = \{\mathrm{ru}\Delta_\Sigma\}_{\Sigma \in |\mathrm{sg}\Delta|}$ where[11]:
  - $\mathrm{ru}\Delta_\Sigma \subseteq \wp^f \mathrm{sen}\Delta(\Sigma) \times \mathrm{sen}\Delta(\Sigma)$;
  - $\langle \{\mathrm{sen}\Delta(\sigma)(\varphi) : \varphi \in p_\rho\}, \mathrm{sen}\Delta(\sigma)(c_\rho)\rangle \in \mathrm{ru}\Delta_{\Sigma'}$ for each $\sigma : \Sigma \to \Sigma'$ and $\rho = \langle p_\rho, c_\rho \rangle \in \mathrm{ru}\Delta_\Sigma$.

The elements in $\mathrm{ax}\Delta_\Sigma$ are called *axioms* over Σ and the elements in $\mathrm{ru}\Delta_\Sigma$ are called *inference rules* over Σ. For each $\rho \in \mathrm{ru}\Delta_\Sigma$ we say the elements in p_ρ are *premises* and that c_ρ is the *conclusion*.

Theorem 46. *Every deductive system Δ induces a topological institution presentation U_Δ as follows:*

- $\mathrm{sg}U_\Delta = \mathrm{sg}\Delta$*;*
- *for each signature Σ, $\mathrm{fp}U_\Delta(\Sigma)$ is the frame presentation $pres_{\Delta,\Sigma}$ such that*
  - $G_{pres_{\Delta,\Sigma}} = \mathrm{sen}\Delta(\Sigma)$*;*
  - $Req_{pres_{\Delta,\Sigma}}$ *contains:*
    - $\langle \alpha \wedge \varphi, \varphi \rangle$ *for each* $\alpha \in \mathrm{ax}\Delta_\Sigma, \varphi \in \mathrm{sen}\Delta(\Sigma)$*;*
    - $\langle (\wedge_{\varphi \in p_\rho} \varphi) \wedge c_\rho, (\wedge_{\varphi \in p_\rho} \varphi) \rangle$ *for each* $\rho \in \mathrm{ru}\Delta_\Sigma$*;*
- *for each signature morphism* $\sigma : \Sigma \to \Sigma'$, $\mathrm{fp}U_\Delta(\sigma) = \mathrm{sen}\Delta(\sigma)$.

In $M_{pres_{\Delta,\Sigma}}$ we have:

- $[\alpha] = [\alpha']$ for every $\alpha, \alpha' \in \mathrm{ax}\Delta_\Sigma$ meaning that the theories generated by the axioms are the same;
- $[(\wedge_{\varphi \in p_\rho} \varphi) \wedge c_\rho] = [(\wedge_{\varphi \in p_\rho} \varphi)]$ which leads to $[c_\rho] \subseteq [(\wedge_{\varphi \in p_\rho} \varphi)]$ meaning that the theory generated by the conclusion of an inference rule must be included in the theory generated by the premises of that rule.

We consider now an example of a "finitary" topological institution. This terminology arises from the fact that every theory is finitely observable provided that the generator theories are finitely observable.

Example 4. In the equational case, assume that we adopt the deductive system in [1], which we call $Eq\Delta$. Then, we would have the following requirements inter alia assuming a given equational signature Σ:

- $e = (t = t) \wedge e$ where e is any equation and t is any term;
- $(t = t') \wedge (o(t) = o(t')) = (t = t')$ where t, t' are terms of the same sort, say s and o is an operation symbol with argument sort s.

Therefore, in $M_{pres_{Eq\Delta,\Sigma}}$ we have:

[11] We denote by $\wp^f \mathrm{sen}\Delta(\Sigma)$ the set of all finite subsets of $\mathrm{sen}\Delta(\Sigma)$.

- $[e] = [(t = t) \wedge e]$ which leads to $[t = t] \subseteq [e]$ meaning that the theory generated by $(t = t)$ is included in the theory generated by e;
- $[(t = t') \wedge (o(t) = o(t'))] = [(t = t')]$ which leads to $[(o(t) = o(t'))] \subseteq [t = t']$ meaning that the theory generated by $(o(t) = o(t'))$ is included in the theory generated by $(t = t')$.

We denote by EqT^f the topological institution complying with the presentation $U_{Eq\Delta}$, and such that:

- pt sy$EqT^f(\Sigma)$ is the class of Σ-algebras;
- for each signature $\Sigma \in |\text{sg}EqT^f|$, the satisfaction relation $\Vdash_{\text{pt sy}EqT^f(\Sigma)}$ is as follows:
$$x \Vdash_{\text{pt sy}EqT^f(\Sigma)} ith \text{ iff } \overline{sat}_{\text{pt sy}EqT^f(\Sigma)}(ith) = \mathbf{t}$$
where $\overline{sat}_{\text{pt sy}EqT^f(\Sigma)}$ the unique frame (site) morphism extending the site morphism
$$sat_{\text{pt sy}EqT^f(\Sigma)} : Y_{\text{fp}U_{Eq\Delta}} \rightarrow S\mathbf{2}$$
such that
$$sat_{\text{pt sy}EqT^f(\Sigma)} = \lambda th.\ \text{if } x \Vdash_\Sigma th \text{ then } \mathbf{t} \text{ else } \mathbf{f}.$$

We should stress that when we extend the satisfaction to theories in the resulting frame (as we did in the example above) we make sure that satisfaction does respect joins while still respecting finite meets. However, in general, infinite meets will not be respected by the extended satisfaction.

4.3 Applications

Finitary topological institutions promise to be useful in monitoring and querying. We examine herein how to set up a weak version of a temporal logic suitable for monitoring. In database monitoring applications [8] we would like to be able to characterize the properties of the system that can be monitored. Namely, an invariant property $(\mathbf{G}\varphi)$ cannot be monitored because it cannot be positively established in finite time. On the contrary, a reactivity property $(\mathbf{F}\varphi)$ can be positively established in finite time.

Therefore, we set up the envisaged logic as a finitary topological institution as follows. We first adopt a propositional deductive system enriched with the next operator $\mathbf{X}$. To this end, we enrich the language (if φ is a formula so is $(\mathbf{X}\varphi)$) and accept the necessary axioms on $\mathbf{X}$ (such as $((\mathbf{X}(\varphi \Rightarrow \varphi')) \Rightarrow ((\mathbf{X}\varphi) \Rightarrow (\mathbf{X}\varphi'))))$. Let us call this deductive system $TM\Delta$. Then, we generate the topological institution presentation $U_{TM\Delta}$. Finally, we choose the (finitary) topological institution TMT complying with $U_{TM\Delta}$ as follows. For each Σ, pt sy$TMT(\Sigma)$ is the set of sequences $\lambda : \mathbb{N} \rightarrow \wp\Sigma$. Satisfaction is defined on the generator theories as expected. First we define satisfaction by a sequence at a point:

- $\lambda \Vdash_n p$ iff $p \in \lambda(n)$;
- $\lambda \Vdash_n (\varphi \Rightarrow \varphi')$ iff $\lambda \Vdash_n \varphi'$ or $\lambda \nVdash_n \varphi$;
- $\lambda \Vdash_n (\neg\varphi)$ iff $\lambda \nVdash_n \varphi$;
- $\lambda \Vdash_n (\mathbf{X}\varphi)$ iff $\lambda \Vdash_{n+1} \varphi$.

Then, we have $\lambda \Vdash \varphi$ iff $\lambda \Vdash_0 \varphi$. It is straightforward to verify the compliance with $U_{TM\Delta}$. Finally, satisfaction is extended to all theories by universal construction as we did in Example 5 above.

Recall that each generator theory is presented by a single formula of the proposed language (using only propositional symbols and connectives as well as the next operator). Clearly, all generator theories can be monitored in the sense that they can be positively observed in finite time.

Therefore, it is easy to see that *all* theories of the proposed topological institution can be monitored. Indeed, arbitrary joins can be monitored. For instance, the reactivity property $(\mathbf{F}\varphi)$ is represented by the theory $\bigvee\{[(\mathbf{X}^n\varphi)] : n \in I\!N\}$. And its semantics is as envisaged. On the other hand, we cannot find in the proposed topological institution a theory corresponding to an invariant property such as $(\mathbf{G}\varphi)$ since the theory $\bigwedge\{[(\mathbf{X}^n\varphi)] : n \in I\!N\}$ does not have the same semantics. But we can find theories corresponding to bounded invariant properties (invariants up to a certain finite time).

In short, a temporal property can be monitored iff we can find a theory of TMT with the same semantics.

5 Conclusion and Outlook

The concept of institution as proposed by Goguen and Burstall in [4, 5] is now widely accepted. But, by recognizing that theories are the key components of institutions and that formulae are just presentation details, we arrived at the new concept of topological institution: a signature-indexed family of topological systems. Topological systems as proposed by Vickers in [14] provided the starting point. We believe that this is indeed the "natural" notion of institution to be built around theories. Indeed, the obvious alternative of, given an institution I, replacing the functor Sen by a functor Th' mapping each signature to the complete lattice of theories over that signature leads to a concept with undesirable properties. Namely, the satisfaction relations (suitably extended to theories as usual) do not respect in general the joins. Furthermore, if this requirement is forced by canonical addition of more theories (using the coverage technique), then meets will not be respected in general by satisfaction. Only finite meets are guaranteed to be respected. That is, by forcing we arrive at a frame structure.

On the way to the envisaged concept, we showed how to extract a topological institution out of a given institution, using the completion technique proposed by Johnstone in [6]. Already in the setting of a topological institution, we recovered the results on composition of theories and provided a characterization of the relationship between theories and models. We also established an adjunction between topological institutions and institutions. We are now investigating the existence of an adjunction when considering the more general notion of institution morphism presented in [9]. Although theories are the key components and not formulae, we may need to work with the latter for presenting the former. To this end, we introduced a general technique for presenting topological institutions adopting an algebraic view on deductive systems. We applied this technique for setting up logics with finitely observable theories and examined their usefulness in monitoring appplications. Other applications of topological concepts in the area of behavior specification are already reported in the literature. See for instance [7]. It remains to bring in the institutional approach that may help with compositionality.

The main limitation of this paper concerns the fact that we disregarded homomorphisms between interpretation structures. That is, we work only with "elementary" institutions where the codomain of the semantic functor is Set^{op}. However, it is clear what should be done towards endowing topological institutions with a category of points (instead of a simple set of points) for each signature. We leave for future work this extension, as well as initiality and liberality issues. It is clear that the concept of "determinate" theory will play an important role. The "dual" view of considering suitable sets of models as opens and taking formulae as points is another interesting topic for future work. We already explored this dual view in the context of a temporal logic institution in order to provide a categorial semantics of object behavior [11].

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