Denotational Semantics of Object Specification
Within an Arbitrary Temporal Logic Institution

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Abstract
From an arbitrary temporal logic institution we show how to set-up the corresponding institution of objects. The main properties of the resulting institution are studied and used in establishing the denotational, categorial semantics of several basic object specification constructs, namely aggregation (parallel composition), interconnection, abstraction (interfacing) and monotonic specialization. An isomorphism is established between the category of theories and the category of objects, as a corollary of the Galois correspondence between these concrete categories. The special case of linear temporal logic is analysed in detail in order to show that categorial products do reflect interleaving, and reducts may lead to internal non-determinism.

Key words
Object-orientation, system specification, temporal logic, institution, categorial semantics.

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1. Introduction

The advantages of object-orientation in software engineering in general and system specification in particular are by now well understood [Meyer 88, Wegner 92]. We consider herein the specific problem of specifying communities of concurrent, interacting, reactive objects [Sernadas *et al* 89a]. Therefore, it is natural to adopt temporal logic as a convenient way of specifying such objects in a declarative way [Pnueli 77, Sernadas 80, Sernadas *et al* 89b, Manna and Pnueli 92]. However, the state of the art in this latter domain is yet progressing towards the envisaged degree of compositionality, since the pioneering efforts reported in [Barringer *et al* 84]: how should we combine temporal specifications of isolated components into a meaningful specification of the whole system?

By combining temporal logic and object-orientation we hope to contribute decisively to achieving higher degrees of compositionality. To this end, it is imperative to work within the setting of the institution [Goguen and Burstall 84] of the underlying temporal logic. Indeed, the notion of reduct and the satisfaction condition are essential to any combination mechanism across different alphabets.

In this paper we show how to build from any given temporal institution (satisfying a few reasonable requirements) a suitable semantic domain of objects where it is possible to define categorially the (final) semantics of all basic object constructs (namely, aggregation or parallel composition, interconnection, abstraction or interfacing, and monotonic specialization). This semantic domain is also conveniently presented as an institution (canonically induced by the given temporal institution). The proposed construction is set-theoretic in the sense that each object is identified with the set of its lives (set of life-cycles in the case of linear temporal logic or set of life-trees in the case of branching temporal logic). Previous relevant work in this direction is reported for special cases in [Fiadeiro *et al* 93, Sernadas *et al* 92a].

For the sake of economy of presentation we only consider the propositional fragment of the logic at hand. No problems appear when extending the results to full first order temporal logics as long as the variable domains are rigid and object-independent.

We assume that the reader is conversant with the field of temporal logic specification (for instance at the level of [Goldblatt 87, Emerson 90]) and with institutions (as presented in [Goguen and Burstall 92]). We also use a little bit of category theory assuming that the reader has a working knowledge of the basic notions, up to (co)limits, as found in the introductory chapters of a textbook on the subject, eg [Asperti and Longo 91]. See also [Poigné 92]. But we include in the Appendix a brief on (co)fibrations (following [Barr and Wells 90]) and Galois correspondences (following [Adámek *et al* 90]).

In section 2 we illustrate the use of (linear) temporal logic in specifying communities of objects. We adopt the propositional fragment of OSL, a temporal logic of objects already being used for specifying and verifying systems [Sernadas *et al* 92b]. In section 3 we state the (few) requirements that we need to impose on the base temporal institution and we establish the most interesting properties of theories of that institution. We also extend the notion of satisfaction to sets of interpretation structures. In section 4, objects and their main properties are presented, paving the way for introducing in section 5 the induced object institution. The notion and main properties of the final semantics functor are presented in section 6, including the Galois correspondence between theories and objects. In section 7 the semantics of the main object constructs is discussed, and the special case of linear temporal logic is considered in detail. Future work, alternative approaches and limitations are of course commented in section 8.

2. System specification

At first sight [Sernadas *et al* 89a, Sernadas and Ehrich 91, Sernadas *et al* 92b], an object is an entity with an internal state (reflected in the values of its slots or attributes) that may interact with other objects. While so interacting an object displays some behaviour. That is, depending on its internal state, the object may be ready or not to provide some service. Clearly, the basic unit of interaction is the event: the occurrence of some action. Therefore, the behaviour of an object may be described as a state-dependent menu of enabled actions.

Therefore, the specification of an object should state the set of its (attribute) observations, the set of its actions, and the constraints on its behaviour. As an illustration consider a simple *flip-flop* as an object. The *flip-flop* has a unique attribute status with two possible observations: *on* and off (not on). It allows only two actions: *flip* and *flop*. Its intended behaviour is easily described using propositional linear temporal logic (**X** stands for "next"), as follows:

(a1) $(\blacklozenge \text{flip} \Leftrightarrow \text{on})$;

(a2) $(\blacklozenge \text{flop} \Leftrightarrow (\neg\ \text{on}))$;

(a3) $(\blacktriangledown \text{flip} \Rightarrow (\mathbf{X}\ (\neg\ \text{on})))$;

(a4) $(\blacktriangledown \text{flop} \Rightarrow (\mathbf{X}\ \text{on}))$;

(a5) $(((\neg\ \blacktriangledown \text{flip}) \wedge (\neg\ \blacktriangledown \text{flop})) \Rightarrow (\text{on} \Leftrightarrow (\mathbf{X}\ \text{on})))$.

The propositional symbols are of the form $\blacktriangledown$c or $\blacklozenge$c, where c is an action, or b, where b is an observation. The symbol $\blacktriangledown$c (c happens or c occurs) indicates the occurrence of action c. The symbol $\blacklozenge$c (c is enabled) indicates that action c is possible.

Axioms (a1) and (a2) indicate when actions are enabled. Axioms (a3) and (a4) state the effects of actions on observations. Finally, axiom (a5) indicates that observations do not change if no (local) action occurs.

Such a logic of behaviour with happening ($\blacktriangledown$) distinct from enabled ($\blacklozenge$) is rich enough for describing in practice useful reactive systems. For further examples see [Sernadas *et al* 92b] where a proof system is provided for dealing with object specialization and aggregation. Note that fairness assumptions are easily specified in this logic. For instance, if so specified,

$$((\mathbf{F}\ (\mathbf{G}\ \blacklozenge c)) \Rightarrow (\mathbf{G}\ (\mathbf{F}\ \blacktriangledown c)))$$

states that if action c is continuously enabled from a certain position on then it happens infinitely many times (**F** stands for "sometime" and **G** for "always"). Clearly, we expect the following universal property

$$(\blacktriangledown c \Rightarrow \blacklozenge c)$$

since an action c may happen only when enabled. As we shall see, the proposed semantics ensures the universal validity of this assertion.

For illustrating the specification of interconnections between objects, consider the clone of the flip-flop above with the alphabet on', flip' and flop'. For instance, the following axiom

$(\blacktriangledown \text{flip} \Rightarrow \blacktriangledown \text{flop}')$

states that the happening of a flip in the first flip-flop entails the happening of a flop in the clone, but not vice-versa. Clearly, a full specification logic needs the ability to refer to the identities of the objects at hand: that is of course possible in the first order version of the logic (see [Sernadas *et al* 92b]).

For illustrating the specification of abstractions (interfacing) consider the problem of specifying a "view" of the original flip-flop where flops are made invisible. Within the envisaged categorial setting, this desideratum is easily achieved by stating that the view is the "reduct" of that flip-flop along the obvious inclusion alphabet map (signature morphism).

Finally, monotonic specialization is easily illustrated: more alphabet and/or axioms may be added to the original specification of the flip-flop. However, some care must be taken in order to avoid falling into contradiction (eg if overriding or side effects are envisaged).

3. Base temporal institution

3.1. Requirements

We assume that we are given a base temporal institution $\mathsf{B}=\langle \text{Sig},\text{Sen},\text{Int},\models \rangle$[1], satisfying the following *base requirements*:

- Sig is the category Set⊗Set. Given a signature $\Sigma=\langle\Sigma_{obs},\Sigma_{act}\rangle$, each element $b\in\Sigma_{obs}$ is an *observation symbol* (we are abstracting away attributes and their values) each element $c\in\Sigma_{act}$ is an *action symbol*. Each signature morphism $\sigma:\Sigma\to\Sigma'$ is of the form $\langle\sigma_{obs}:\Sigma_{obs}\to\Sigma'_{obs}, \sigma_{act}:\Sigma_{act}\to\Sigma'_{act}\rangle$.

- Let A_Σ be the least set satisfying the following rules: $b\in A_\Sigma$, provided that $b\in\Sigma_{obs}$; $\blacklozenge c\in A_\Sigma$, provided that $c\in\Sigma_{act}$; $\blacktriangledown c\in A_\Sigma$, provided that $c\in\Sigma_{act}$. Then, Sen(Σ) (the set of *formulae*) is defined inductively on the set A_Σ (of *atomic formulae* over Σ), satisfying the following rules inter alia: $(\neg\gamma)\in \text{Sen}(\Sigma)$ provided that $\gamma\in\text{Sen}(\Sigma)$; $(\gamma1\Rightarrow\gamma2)\in\text{Sen}(\Sigma)$ provided that $\gamma1,\gamma2\in\text{Sen}(\Sigma)$.

- The *translation map* $\text{Sen}(\sigma):\text{Sen}(\Sigma)\to\text{Sen}(\Sigma')$ induced by a signature morphism $\sigma:\Sigma\to\Sigma'$ satisfies the following rules inter alia: $\text{Sen}(\sigma)(b)=\sigma_{obs}(b)$; $\text{Sen}(\sigma)(\blacklozenge c)=\blacklozenge\sigma_{act}(c)$; $\text{Sen}(\sigma)(\blacktriangledown c)=\blacktriangledown\sigma_{act}(c)$; $\text{Sen}(\sigma)(\neg\gamma)=(\neg\text{Sen}(\sigma)(\gamma))$; $\text{Sen}(\sigma)(\gamma1\Rightarrow\gamma2)=(\text{Sen}(\sigma)(\gamma1)\Rightarrow\text{Sen}(\sigma)(\gamma2))$.

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[1] Sig is a category (of *signatures*); Sen is a functor from Sig into Set (providing the *language* for each signature); Int is a functor from Sig into Cat^{op} (providing the category of *interpretation structures* for each signature); and ⊨ is a |Sig|-indexed family of relations with each $\models_\Sigma$ being the *satisfaction relation* between Int(Σ) and Sen(Σ).

• Each interpretation structure λ of $Int(\Sigma)$ is a triple $<W,R,V>$ where: W is a set (of *worlds*); R is a binary relation on W (the *accessibility relation*); $V:A_\Sigma \to 2^W$ (the *valuation map*) is such that $V(\blacktriangledown c) \subseteq V(\blacklozenge c)$ for each $c \in \Sigma_{act}$.

• For each interpretation structure $<W',R',V'>$ of $Int(\Sigma')$ and each signature morphism $\sigma: \Sigma \to \Sigma'$, $Int(\sigma)(<W',R',V'>) = <W',R',V' \circ Sen(\sigma)>$ (the *reduct* of $<W',R',V'>$ with respect to σ).

• For each interpretation structure $\lambda = <W,R,V>$ of $Int(\Sigma)$ and each formula γ of $Sen(\Sigma)$, $\lambda \vDash_\Sigma \gamma$ iff $\lambda \vDash_w \gamma$ for every world $w \in W$, assuming inter alia: $\lambda \vDash_w \alpha$ iff $w \in V(\alpha)$, for each $\alpha \in A_\Sigma$; $\lambda \vDash_w (\neg \gamma)$ iff not $\lambda \vDash_w \gamma$; $\lambda \vDash_w (\gamma 1 \Rightarrow \gamma 2)$ iff $\lambda \vDash_w \gamma 2$ or not $\lambda \vDash_w \gamma 1$.

In the sequel, we denote $Sen(\Sigma)$ by L_Σ, $Sen(\sigma)(\varphi)$ by $\sigma(\varphi)$, $Int(\Sigma)$ by Λ_Σ, $Int(\sigma)(\lambda')$ by $\sigma^{-1}(\lambda')$, and $V' \circ Sen(\sigma)$ by $\sigma^{-1}(V')$. Furthermore, provided that $\Phi \subseteq L_\Sigma$ and $\Phi' \subseteq L_{\Sigma'}$, we denote by $\sigma(\Phi)$ the set $\{\sigma(\varphi) \in L_{\Sigma'}: \varphi \in \Phi\}$, and by $\sigma^{-1}(\Phi')$ the set $\{\varphi \in L_{\Sigma}: \sigma(\varphi) \in \Phi'\}$. We use $\Phi^{\vDash}$ for denoting the semantic closure of Φ: $\{\psi \in L_\Sigma: \Phi \vDash_\Sigma \psi\}$. And, as usual, $\Phi \vDash_\Sigma \psi$ states that Φ *entails* ψ: for every interpretation structure λ over Σ, if $\lambda \vDash_\Sigma \varphi$ for each $\varphi \in \Phi$ then $\lambda \vDash_\Sigma \psi$. Finally, $\lambda \vDash_\Sigma \Phi$ iff $\lambda \vDash_\Sigma \varphi$ for each $\varphi \in \Phi$.

Each valuation map V induces the map $\underline{V}: W \to 2^{A_\Sigma}$ such that $\underline{V}(w) = \{\alpha: w \in V(\alpha)\}$ and vice-versa. In the sequel, we shall use either of them as a matter of convenience. Clearly, $\sigma^{-1}(\underline{V'})$ maps each world w onto the set $\{\alpha: \sigma(\alpha) \in \underline{V'}(w)\}$.

Finally, recall that, according to the definition of institution, the *satisfaction condition* holds in B: $\sigma^{-1}(\lambda') \vDash_\Sigma \varphi$ iff $\lambda' \vDash_{\Sigma'} \sigma(\varphi)$ and, thus, $\sigma^{-1}(\lambda') \vDash_\Sigma \Phi$ iff $\lambda' \vDash_{\Sigma'} \sigma(\Phi)$. As a corollary we obtain the *closure lemma:* $\sigma(\Phi^{\vDash}) \subseteq (\sigma(\Phi))^{\vDash}$.

3.2. Specifications and theories

We recall in this paragraph the main concepts related to theories in B that do not depend on the base requirements, presenting the category of theories as a concrete category.

Definition 3.2.1. A *specification* over Σ is a pair $spec = <\Sigma,\Phi>$ where $\Sigma = Si(spec)$ is a signature and $\Phi \subseteq L_\Sigma$. A *theory* over Σ is a specification $th = <\Sigma,\Phi>$ over Σ such that $\Phi = \Phi^{\vDash}$. □

In a specification (or presentation) $spec = <\Sigma,\Phi>$, the elements of Φ are called *axioms*. In the case of a theory, they are called *theorems*. Clearly, every specification induces a theory by semantic closure.

Definition 3.2.2. A *theory morphism* $f: th \to th'$, where $th = <\Sigma,\Phi>$ and $th' = <\Sigma',\Phi'>$, is a signature morphism $Si(f): Si(th) \to Si(th')$ such that $Si(f)(\Phi) \subseteq \Phi'$. □

Fact/Definition 3.2.3. *Theories and theory morphisms constitute a concrete category* $<Th,Si>$ *over Sig.*
Proof:
The fact that Th is a category in any institution is well known. And it is trivial to verify that the functor $Si: Th \to Sig$ is faithful. □

3.3. Main properties of theories

We now present some useful results concerning theories in B. Note that again they do not depend on the base requirements, though we give illustrations for B.

Proposition 3.3.1. *The category Sig is small cocomplete.* □

The *empty signature* $<\varnothing,\varnothing>$ is initial. We denote by 0_Σ the unique signature morphism $<0_{obs}:\varnothing\to\Sigma_{obs},0_{act}:\varnothing\to\Sigma_{act}>$ from $<\varnothing,\varnothing>$ into Σ. A *coproduct* of signatures $\Sigma\oplus\Sigma'$ is given by $<\Sigma_{obs}\oplus\Sigma'_{obs},\Sigma_{act}\oplus\Sigma'_{act}>$ with injections $inj_\Sigma=<i,j>:\Sigma\to\Sigma\oplus\Sigma'$ and $inj_{\Sigma'}=<i',j'>:\Sigma'\to\Sigma\oplus\Sigma'$, assuming that: $\Sigma_{obs}\oplus\Sigma'_{obs}$ with $i:\Sigma_{obs}\to\Sigma_{obs}\oplus\Sigma'_{obs}$ and $i':\Sigma'_{obs}\to\Sigma_{obs}\oplus\Sigma'_{obs}$ is a coproduct in Set of Σ_{obs} and Σ'_{obs}; and $\Sigma_{act}\oplus\Sigma'_{act}$ with $j:\Sigma_{act}\to\Sigma_{act}\oplus\Sigma'_{act}$ and $j':\Sigma'_{act}\to\Sigma_{act}\oplus\Sigma'_{act}$ is a coproduct in Set of Σ_{act} and Σ'_{act}.

Proposition 3.3.2. *The category Th is cocartesian.* □

Proposition 3.3.3. *The concrete category <Th,Si> is amnestic.*
Proof:
Let $th1=<\Sigma,\Phi1>$ and $th2=<\Sigma,\Phi2>$ be theories over the same signature and $f:th1\to th2$ and $g:th2\to th1$ theory morphisms such that $Si(f)=id_\Sigma$ and $Si(g)=id_\Sigma$. Then, $id_\Sigma(\Phi1)=\Phi1\subseteq\Phi2$ and $id_\Sigma(\Phi2)=\Phi2\subseteq\Phi1$ and so $th1=th2$. □

Lemma 3.3.4. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism, $\Phi\subseteq L_\Sigma$ and $\Phi'\subseteq L_{\Sigma'}$ sets of formulae over Σ and Σ', respectively. Then $\Phi\subseteq\sigma^{-1}(\sigma(\Phi))$ and $\sigma(\sigma^{-1}(\Phi'))\subseteq\Phi'$.
Proof:
Let $\varphi\in\Phi$; then $\sigma(\varphi)\in\sigma(\Phi)$; therefore, $\varphi\in\sigma^{-1}(\sigma(\Phi))$.
Let $\varphi'\in\sigma(\sigma^{-1}(\Phi'))$; then there is $\varphi\in L_\Sigma$ such that $\varphi'=\sigma(\varphi)$ and $\varphi\in\sigma^{-1}(\Phi')$; therefore $\varphi'\in\Phi'$. □

We provide counterexamples showing that these inclusions may be strict. Consider the signatures $\Sigma=<\{a,b,c,d\},\varnothing>$ and $\Sigma'=<\{x,y,z\},\varnothing>$, the signature morphism $\sigma:\Sigma\to\Sigma'$ such that $\sigma_{obs}(a)=x$, $\sigma_{obs}(b)=\sigma_{obs}(c)=y$, $\sigma_{obs}(d)=z$ and the set of formulae $\Phi=\{(a\Rightarrow b),(c\Rightarrow d)\}^{\vDash}$. Then $\sigma(\Phi)=\{(x\Rightarrow y),(y\Rightarrow z)\}^{\vDash}$ and so $(a\Rightarrow d)\in\sigma^{-1}(\sigma(\Phi))$ and $(a\Rightarrow d)\notin\Phi$. Note the role of the non-injectivity of σ.

Consider the signatures $\Sigma=<<\{a,b\},\varnothing>$ and $\Sigma'=<\{x,y,z\},\varnothing>$, the signature morphism $\sigma:\Sigma\to\Sigma'$ such that $\sigma_{obs}(a)=x$, $\sigma_{obs}(b)=y$ and the set of formulae $\Phi'=\{(x\Rightarrow z),(z\Rightarrow y)\}^{\vDash}$ a theory. Then $\sigma^{-1}(\Phi')=\{(a\Rightarrow b)\}^{\vDash}$ and so $(x\Rightarrow z),(z\Rightarrow y)\in\Phi'$ but $(x\Rightarrow z),(z\Rightarrow y)\notin\sigma(\sigma^{-1}(\Phi'))$. Note the role of the non-surjectivity of σ.

Proposition 3.3.5. *The functor Si:Th→Sig is a split cofibration with a unique cocleavage.*
Proof:
(1) Cofibration: For each $\sigma:\Sigma\to\Sigma'$ in Sig and $th=<\Sigma,\Phi>$ in Th, the theory morphism $f:th\to th'$, where $th'=<\Sigma',\sigma(\Phi)^{\vDash}>$, such that $Si(f)=\sigma$ is cocartesian by Si for σ on th. Indeed, f is a theory morphism because $\sigma(\Phi)\subseteq\sigma(\Phi)^{\vDash}$. Moreover, f has the envisaged couniversal property: assume that $g:th\to th''$ is a morphism in Th with $th''=<\Sigma'',\Phi''>$ and $\omega:\Sigma'\to Si(th'')$ is a morphism in Sig such that $\omega\circ\sigma=Si(g)$; the problem is to find a unique $h:th'\to th''$ such that $Si(h)=\omega$ and $h\circ f=g$; we

have only to verify that ω is a theory morphism: since $\omega(\sigma(\Phi))\subseteq\Phi''$ we obtain $\omega(\sigma(\Phi)^{\models})\subseteq\Phi''^{\models}=\Phi''$ invoking the closure lemma.

(2) Uniqueness of cocleavage: We have to show that for each $\sigma:\Sigma\to\Sigma'$ in Sig and th=$<\Sigma,\Phi>$ in Th, the morphism f:th$\to<\Sigma',\sigma(\Phi)^{\models}>$ is the unique cocartesian morphism by Si for σ on th. Assume that there is another cocartesian morphism g:th$\to<\Sigma',\Psi>$ by Si for σ on th. Then, there is an isomorphism h:$<\Sigma',\sigma(\Phi)^{\models}>\to<\Sigma',\Psi>$[2] such that Si(h) $\circ$ $\sigma=\sigma$. Let h^{-1} be the inverse of h. We have $\sigma(\Phi)^{\models}\subseteq\Psi$: if $\sigma(\varphi)\in\sigma(\Phi)$ then $\sigma(\varphi)=\mathrm{Si}(h)(\sigma(\varphi))\in\Psi$ since h is a theory morphism; and, thus, if $\gamma\in\sigma(\Phi)^{\models}$ then $\gamma\in\Psi$ since Ψ is closed. We also have $\Psi\subseteq\sigma(\Phi)^{\models}$: indeed we first show that, for each $\xi'\in\Lambda_{\Sigma'}$, if $\mathrm{Si}(h^{-1})^{-1}(\xi')\models_{\Sigma'}\sigma(\Phi)$ then $\mathrm{Si}(h^{-1})^{-1}(\xi')\models_{\Sigma'}\psi$ for any $\psi\in\Psi$; assume that $\mathrm{Si}(h^{-1})^{-1}(\xi')\models_{\Sigma'}\sigma(\Phi)$; thus[3], $\xi'\models_{\Sigma'}\mathrm{Si}(h^{-1})(\sigma(\Phi))$ and, so[4], $\xi'\models_{\Sigma'}\sigma(\Phi)$; therefore, $\xi'\models_{\Sigma'}\mathrm{Si}(h^{-1})(\psi)$ for any $\psi\in\Psi$, since h^{-1} is a theory morphism; thus[5], $\mathrm{Si}(h^{-1})^{-1}(\xi')\models_{\Sigma'}\psi$ for any $\psi\in\Psi$; we are now able to conclude that $\Psi\subseteq\sigma(\Phi)^{\models}$ because $\mathrm{Si}(h^{-1})^{-1}$ is surjective in $\Lambda_{\Sigma'}$.

(3) Splitting: It is trivial to see that $id_{<\Sigma,\Phi>}$ is cocartesian by Si for $id_\Sigma:\Sigma\to\Sigma$ on $<\Sigma,\Phi>$. On the other hand, let $\sigma1:\Sigma\to\Sigma1$ and $\sigma2:\Sigma1\to\Sigma2$ be signature morphisms, f1:th$\to<\Sigma1,\sigma1(\Phi)>$ the cocartesian morphism by Si for $\sigma1$ on $<\Sigma,\Phi>$, and f2:$<\Sigma1,\sigma1(\Phi)>\to<\Sigma2,\sigma2(\sigma1(\Phi))>$ the cocartesian morphism by Si for $\sigma2$ on $<\Sigma1,\sigma1(\Phi)>$; then, we know that f2 $\circ$ f1: $<\Sigma,\Phi>\to<\Sigma2,\sigma2(\sigma1(\Phi))>$ is cocartesian by Si for $\sigma2\circ\sigma1$ on $<\Sigma,\Phi>$[6]. □

Proposition 3.3.6. *The functor Si:Th→Sig is a fibration.*

Proof:

For each $\sigma:\Sigma\to\Sigma'$ in Sig and th'=$<\Sigma',\Phi'>$ in Th, the theory morphism f:th→th' such that Si(f)=σ and th=$<\Sigma,\Phi>$ where $\Phi=\sigma^{-1}(\Phi')$ is cartesian by Si for σ on th'. We must show that th is a theory and that f is a theory morphism. Indeed, th is a theory: let $\psi\in L_\Sigma$ be such that if $\lambda\models_\Sigma\Phi$ then $\lambda\models_\Sigma\psi$ for each $\lambda\in\Lambda_\Sigma$; we have to show that $\psi\in\Phi$; let $\lambda'\in\Lambda_{\Sigma'}$ be any interpretation structure such that $\lambda'\models_{\Sigma'}\Phi'$; in particular, $\lambda'\models_{\Sigma'}\sigma(\varphi)$ for all $\varphi\in\Phi$; then $\sigma^{-1}(\lambda')\models_\Sigma\varphi$ for all $\varphi\in\Phi$[7] and so $\sigma^{-1}(\lambda')\models_\Sigma\psi$; hence, $\lambda'\models_\Sigma\sigma(\psi)$[8]; so, $\Phi'\models_{\Sigma'}\sigma(\psi)$; since th' is a theory $\sigma(\psi)\in\Phi'$ and so $\psi\in\Phi$. Indeed, f is a theory morphism because $\sigma(\Phi)\subseteq\Phi'$. And we must also show that f has the envisaged universal property: assume that g:th''→th' with th''=$<\Sigma'',\Phi''>$ is a morphism in Th and ω:Si(th'')$\to\Sigma$ is a morphism in Sig such that $\sigma\circ\omega$=Si(g); the problem is to find a unique h:th''→th such that Si(h)=ω and f $\circ$ h=g; we have only to verify that ω is a theory morphism: since $\sigma(\omega(\Phi''))\subseteq\Phi'$, $\sigma(\Phi)\subseteq\Phi'$ and $\sigma(\Phi)$ includes all formulae whose images belong to Φ', we have $\omega(\Phi'')\subseteq\Phi$. □

3.4. Indexing theories

We now extract from the flat category of theories its indexing on signatures (built upon the split cofibration with a unique cocleavage). To this end, we obtain the functor TH:Sig→Cat that maps each signature into the category of all theories over that signature.

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2 See the Appendix: proposition A7.
3 Using the satisfaction condition of the base institution.
4 Since $\mathrm{Si}(h^{-1})\circ\sigma=\sigma$.
5 Using the satisfaction condition of the base institution.
6 See the Appendix: proposition A8.
7 Using the satisfaction condition of the base institution.
8 Using the satisfaction condition of the base institution.

Definition 3.4.1. Let Σ be a signature. The category Th_Σ is the fiber $Si^{-1}(\Sigma)$. The theories of Th_Σ are called *theories* over Σ. The arrows of Th_Σ are called *homomorphisms* between theories over Σ. □

Definition 3.4.2. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism and th a theory over Σ. The *image* σ(th) of th with respect to σ is the target of the cocartesian morphism $\sigma;^{-th}$ for σ on th. □

That is, σ(th) is given by $<\Sigma,\{Sen(\sigma)(\varphi):\varphi\in\Phi\}^{\models}>$.

Fact/Definition 3.4.3. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism. The *image functor* σ: $Th_\Sigma\to Th_{\Sigma'}$ maps each th of Th_Σ onto the image σ(th), and each homomorphism h:th1→th2 of Th_Σ onto the unique homomorphism σ(h):σ(th1)→σ(th2) such that $\sigma;^{-th2}\circ h = \sigma(h)\circ\sigma;^{-th1}$.
Proof:
See A11 in the Appendix. □

Fact/Definition 3.4.4. The *theory functor* TH:Sig→Cat maps each signature Σ onto the fiber Th_Σ, and each signature morphism $\sigma:\Sigma\to\Sigma'$ onto the image functor $\sigma:Th_\Sigma\to Th_{\Sigma'}$.
Proof:
See A11 in the Appendix. □

As expected, by an appropriate flattening construction we recover the category Th from the indexed category $TH^{op}:Sig^{op}\to Cat^{op}$ (cf [Tarlecki *et al* 1989]).

Clearly, we might develop the counterparts to these functors for "reducts" instead of images provided that we verify that the fibration Si is split. But we refrain to do so because we do not need them in the sequel. However, it is useful to introduce:

Definition 3.4.5. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism and th' a theory over Σ'. The *reduct theory* σ^{-1}(th') of th' with respect to σ is the source of the cartesian morphism for σ on th'. □

In the sequel, in order to characterize the (tight) relationship between theories and (some) objects it is also convenient to work with Th^{op} and $Si^{op}:Th^{op}\to Sig^{op}$. Clearly, Si^{op} is also both a cofibration and a fibration (cartesian morphisms become cocartesian and vice-versa).

3.5. Satisfaction vs sets of interpretation structures

In the sequel we need some additional notation and results concerning sets of interpretation structures of the base institution. Provided that $\sigma:\Sigma\to\Sigma'$ is a signature morphism, $\Lambda\subseteq\Lambda_\Sigma$, and $\Lambda'\subseteq\Lambda_{\Sigma'}$, we use: $\Lambda\models_\Sigma\varphi$ iff $\lambda\models_\Sigma\varphi$ for each $\lambda\in\Lambda$; $\sigma^{-1}(\Lambda')=\{\sigma^{-1}(\lambda'): \lambda'\in\Lambda'\}$; and, $\sigma(\Lambda)=\{\lambda': \sigma^{-1}(\lambda')\in\Lambda\}$.

As an illustration of the latter, consider the signatures $\Sigma=<\emptyset,\{a,b\}>$,$\Sigma'=<\emptyset,\{x\}>$, the (unique) signature morphism $\sigma:\Sigma\to\Sigma'$, a set W, a relation R on W, $\underline{V1}:W\to 2^{A\Sigma}$ such that $\underline{V1}(w)=\{\blacklozenge a,\blacklozenge b,\blacktriangledown a,\blacktriangledown b\}$ for each $w\in W$, and $\underline{V2}:W\to 2^{A\Sigma}$ such that $\underline{V2}(w)=\{\blacklozenge a,\blacktriangledown a\}$ for each $w\in W$. Then,

$\sigma(\{<W,R,\underline{V1}>\})=\{<W,R,\underline{V1'}>\}$ where $\underline{V1'}(w)=\{\blacklozenge x,\blacktriangledown x\}$ for each $w\in W$;

$\sigma(\{<W,R,\underline{V2}>\})=\emptyset$.

Proposition 3.5.1. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism, φ a formula of L_Σ, and $\Lambda'\subseteq\Lambda_{\Sigma'}$. Then, $\Lambda'\models_{\Sigma'}\sigma(\varphi)$ iff $\sigma^{-1}(\Lambda')\models_{\Sigma}\varphi$.[9]

Proof:

$\Lambda'\models_{\Sigma'}\sigma(\varphi)$ iff $\lambda'\models_{\Sigma'}\sigma(\varphi)$ for every $\lambda'\in\Lambda'$ iff[10] $\sigma^{-1}(\lambda')\models_{\Sigma}\varphi$ for every $\lambda'\in\Lambda'$ iff[11] $\sigma^{-1}(\lambda')\models_{\Sigma}\varphi$ for every $\sigma^{-1}(\lambda')\in\sigma^{-1}(\Lambda')$ iff[12] $\sigma^{-1}(\Lambda')\models_{\Sigma}\varphi$. □

Proposition 3.5.2. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism, φ a formula of L_Σ, and $\Lambda\subseteq\Lambda_\Sigma$. Then, if $\Lambda\models_{\Sigma}\varphi$ then $\sigma(\Lambda)\models_{\Sigma'}\sigma(\varphi)$.[13]

Proof:

Assume that $\sigma(\Lambda)\not\models\sigma(\varphi)$; then there is an interpretation structure $\lambda'\in\sigma(\Lambda)$ such that $\lambda'\not\models_{\Sigma'}\sigma(\varphi)$; therefore $\sigma^{-1}(\lambda')\not\models_{\Sigma}\varphi$[14]; and so $\Lambda\not\models_{\Sigma}\varphi$ since $\sigma^{-1}(\lambda')\in\Lambda$. □

We provide a counterexample showing that if $\sigma(\Lambda)\models_{\Sigma'}\sigma(\varphi)$ then it may be the case that $\Lambda\not\models_{\Sigma}\varphi$. Consider the signatures $\Sigma=\langle\emptyset,\{a,b,c\}\rangle$ and $\Sigma'=\langle\emptyset,\{x,y\}\rangle$, the signature morphism $\sigma:\Sigma\to\Sigma'$ such that $\sigma_{act}(a)=\sigma_{act}(b)=x$ and $\sigma_{act}(c)=y$, and the set $\Lambda=\{\langle W,R,\underline{V1}\rangle, \langle W,R,\underline{V2}\rangle\}$ where $\underline{V1}(w)=\{\blacklozenge a,\blacktriangledown a\}$ and $\underline{V2}(w)=\{\blacklozenge c,\blacktriangledown c\}$ for each $w\in W$. Its image $\sigma(\Lambda)$ is $\langle\Sigma',\{\langle W,R,\underline{V}'\rangle\}\rangle$ where $\underline{V}'(w)=\{\blacklozenge y,\blacktriangledown y\}$ for each $w\in W$. We have that $\sigma(\Lambda)\models_{\Sigma'}\sigma(\mathbf{F}\blacktriangledown c)$ but $\Lambda\not\models_{\Sigma}(\mathbf{F}\blacktriangledown c)$.

4. Objects over the base institution

We are now ready to introduce the notions of object and object morphism within the setting of the given base institution. The basic idea is to identify an object with its behaviour, that is with a set of lives (interpretation structures) and lift in the obvious way the notion of interpretation structure reduct to sets. However, not all sets of interpretation structures are acceptable as objects as we shall see, since the base logic may have limitations in distinguishing them.

4.1. Preobjects and objects

Definition 4.1.1. A *preobject* over Σ is a pair $ob=\langle\Sigma,\Lambda\rangle$ where $\Sigma=Sg(ob)$ is a signature and $\Lambda\subseteq\Lambda_\Sigma$. □

Definition 4.1.2. Let Σ be a signature $\xi\in\Lambda_\Sigma$, and $\Lambda\subseteq\Lambda_\Sigma$. We say that Λ *entails* ξ (written $\Lambda\models_\Sigma\xi$) iff, for every $\varphi\in L_\Sigma$, if $\Lambda\models_\Sigma\varphi$ then $\xi\models_\Sigma\varphi$. □

Definition 4.1.3. Let Σ be a signature and $\Lambda\subseteq\Lambda_\Sigma$. The *theoretic closure* of Λ is the following set of interpretation structures: $\Lambda^{\models}=\{\xi\in\Lambda_\Sigma: \Lambda\models_\Sigma\xi\}$. □

Definition 4.1.4. An *object* over Σ is a preobject $ob=\langle\Sigma,\Lambda\rangle$ such that $\Lambda=\Lambda^{\models}$. □

Definition 4.1.5. An *object morphism* $f:ob\to ob'$ where $ob=\langle\Sigma,\Lambda\rangle$ and $ob'=\langle\Sigma',\Lambda'\rangle$ is a signature op-morphism $Sg(f):Sg(ob)\to Sg(ob')$ such that $(Sg(f)^{op})^{-1}(\Lambda)\subseteq\Lambda'$. □

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9 We may call this result the *set satisfaction condition* (of the base institution).
10 Using the satisfaction condition of the base institution.
11 Since $\lambda'\in\Lambda'$ iff $\sigma^{-1}(\lambda')\in\sigma^{-1}(\Lambda')$.
12 Note that $\sigma^{-1}:\Lambda_{\Sigma'}\to\Lambda_\Sigma$ is surjective.
13 We may call this result the *set forward satisfaction condition* (of the base institution).
14 Using the satisfaction condition of the base institution.

Fact/Definition 4.1.6. *Objects and object morphisms constitute a concrete category <Ob,Sg> over* Sig^{op}.

Proof:

We have to show that Ob is a category and that $Sg:Ob \to Sig^{op}$ is a faithful functor. We only prove that the composition of two object morphisms is an object morphism since the rest is (also) trivial. Let $ob=(\Sigma,\Lambda)$, $ob'=(\Sigma',\Lambda')$, $ob''=(\Sigma'',\Lambda'')$ be objects and $f:ob \to ob'$ and $g:ob' \to ob''$ be object morphisms; the signature morphism $Sg(f)^{op} \circ Sg(g)^{op}:\Sigma'' \to \Sigma$ is an object morphism since: $(Sg(f)^{op} \circ Sg(g)^{op})^{-1}(\Lambda)=(Sg(g)^{op})^{-1}((Sg(f)^{op})^{-1}(\Lambda)) \subseteq (Sg(g)^{op})^{-1}(\Lambda') \subseteq \Lambda''$. □

Consider the objects $ob=<\Sigma,\Lambda>$ and $ob'=<\Sigma',\Lambda'>$ where

$\Sigma=<\emptyset,\{x,y\}>$,

$\Lambda=\{<W,R,\underline{V}>\}$ where $\underline{V}(w)=\{\blacklozenge x,\blacktriangledown x\}$ for each $w \in W$,

$\Sigma'=<\emptyset,\{a,b,c\}>$,

$\Lambda'=\{<W,R,\underline{V1'}>,<W,R,\underline{V2'}>\}$

where $\underline{V1'}(w)=\{\blacklozenge a,\blacklozenge b,\blacktriangledown a,\blacktriangledown b\}$ and $\underline{V2'}(w)=\{\blacklozenge c,\blacktriangledown c\}$ for each $w \in W$.

Let $\sigma:\Sigma' \to \Sigma$ be the signature morphism such that $\sigma_{act}(a)=x$, $\sigma_{act}(b)=x$, $\sigma_{act}(c)=y$. Since $\sigma^{-1}(<W,R,\underline{V}>)=<W,R,\underline{V1'}>$, σ induces an object morphism from ob onto ob'.

4.2. Main properties of objects

We show that finite products exist in Ob. Actually, Ob is (small) complete but we do not need to consider general limits in this paper. Furthermore, we prove that Ob is amnestic and that Sg is both a cofibration and a fibration.

Lemma 4.2.1. The empty set of interpretation structures is theoretically closed.

Proof:

Assume that $\emptyset \models_\Sigma \xi$ where $\xi=<W,R,\underline{V}> \in \Lambda_\Sigma$. Since, $\emptyset \models_\Sigma \varphi$ for every $\varphi \in L_\Sigma$ then $\xi \models_\Sigma \varphi$ for every $\varphi \in L_\Sigma$ and so $\xi \models_w \varphi$ for every $\varphi \in L_\Sigma$ and every $w \in W$. In particular, consider $\alpha \in A_\Sigma$. Two cases are possible: (i) $\alpha \in \underline{V}(w)$ and then $\xi \not\models_w (\neg \alpha)$; (ii) $\alpha \notin \underline{V}(w)$ and then $\xi \not\models_w \alpha$ In both cases we reach a contradiction and so no ξ is entailed by $\emptyset$. □

Lemma 4.2.2. Let Σ and Σ' be signatures, $\Lambda \subseteq \Lambda_\Sigma$ and $\Lambda' \subseteq \Lambda'_{\Sigma'}$ theoretically closed sets of interpretation structures. Then, $\Theta=\{\theta \in \Lambda_{\Sigma\oplus\Sigma'}: inj_\Sigma^{-1}(\theta) \in \Lambda, inj_{\Sigma'}^{-1}(\theta) \in \Lambda'\}$ is theoretically closed.

Proof:

Assume that $\Theta \models_{\Sigma\oplus\Sigma'} \xi$. We have to show that $\xi \in \Theta$, ie, $inj_\Sigma^{-1}(\xi) \in \Lambda$ and $inj_{\Sigma'}^{-1}(\xi) \in \Lambda'$. We only show that $inj_\Sigma^{-1}(\xi) \in \Lambda$ (the other part is similar). Since Λ is theoretically closed, it is enough to verify that $\Lambda \models_\Sigma inj_\Sigma^{-1}(\xi)$. Let $\varphi \in L_\Sigma$ be any formula such that $\Lambda \models_\Sigma \varphi$. We show that $inj_\Sigma^{-1}(\xi) \models_\Sigma \varphi$: $inj_\Sigma(\Lambda) \models_{\Sigma\oplus\Sigma'} inj_\Sigma(\varphi)$[15]; hence, $\Theta \models_{\Sigma\oplus\Sigma'} inj_\Sigma(\varphi)$[16]; therefore, since $\Theta \models_{\Sigma\oplus\Sigma'} \xi$, $\xi \models_{\Sigma\oplus\Sigma'} inj_\Sigma(\varphi)$; finally, $inj_\Sigma^{-1}(\xi) \models_\Sigma \varphi$[17]. □

Proposition 4.2.3. *The category Ob is cartesian.*

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[15] Using the set forward satisfaction condition of the base institution.

[16] Since $\Theta \subseteq inj_\Sigma(\Lambda)$.

[17] Using the satisfaction condition of the base institution.

Proof:

(1) Ob has terminal object:

The obvious candidate is $<<\varnothing,\varnothing>,\varnothing>$ and for each object $ob=<\Sigma,\Lambda>$ the unique morphism $1_{ob}:ob\rightarrow<<\varnothing,\varnothing>,\varnothing>$ is induced by the empty signature morphism $0_\Sigma:<\varnothing,\varnothing>\rightarrow\Sigma$.

(2) Ob has binary products:

Assuming that $ob=<\Sigma,\Lambda>$ and $ob'=<\Sigma',\Lambda'>$ are objects, the binary product object is

$ob\otimes ob'=<\Sigma\oplus\Sigma',\{\theta\in\Lambda_{\Sigma\oplus\Sigma'}: inj_\Sigma{}^{-1}(\theta)\in\Lambda,\ inj_{\Sigma'}{}^{-1}(\theta)\in\Lambda'\}>$

(where $inj_\Sigma:\Sigma\rightarrow\Sigma\oplus\Sigma'$ and $inj_{\Sigma'}:\Sigma'\rightarrow\Sigma\oplus\Sigma'$ are the injection signature morphisms) endowed with the projections:

$proj_{ob}:ob\otimes ob'\rightarrow ob \quad proj_{ob'}:ob\otimes ob'\rightarrow ob'$

such that

$Sg(proj_{ob})^{op}=inj_\Sigma \quad Sg(proj_{ob'})^{op}=inj_{\Sigma'}$.

Moreover, the candidate does have the envisaged universal property: assuming that $ob''=<\Sigma'',\Lambda''>$ is an object and $g:ob''\rightarrow ob$ and $g':ob''\rightarrow ob'$ are object morphisms, it is straightforward to verify that there is a unique object morphism $k:ob''\rightarrow ob\otimes ob'$ such that $g=proj_{ob}\circ k$ and $g'=proj_{ob'}\circ k$; k is induced by the unique signature morphism $\sigma:\Sigma\oplus\Sigma'\rightarrow\Sigma''$ such that $Sg(g)^{op}=\sigma\circ inj_\Sigma$ and $Sg(g')^{op}=\sigma\circ inj_{\Sigma'}$. □

Proposition 4.2.4. *The category Ob is amnestic.*

Proof:

Let $ob1=<\Sigma,\Lambda1>$ and $ob2=<\Sigma,\Phi2>$ be objects over the same signature and $f:ob1\rightarrow ob2$ and $g:ob2\rightarrow ob1$ object morphisms such that $Si(f)=id_\Sigma$ and $Si(g)=id_\Sigma$. Then, $id_\Sigma(\Lambda1)=\Lambda1\subseteq\Lambda2$ and $id_\Sigma(\Lambda2)=\Lambda2\subseteq\Lambda1$ and so ob1=ob2. □

Lemma 4.2.5. Let $\sigma:\Sigma\rightarrow\Sigma'$ be a signature morphism, $\Lambda\subseteq\Lambda_\Sigma$, and $\Lambda'\subseteq\Lambda_{\Sigma'}$ sets of interpretation structures over Σ and Σ', respectively. Then $\sigma^{-1}(\sigma(\Lambda))\subseteq\Lambda$ and $\Lambda'\subseteq\sigma(\sigma^{-1}(\Lambda'))$.

Proof:

If $\lambda\in\sigma^{-1}(\sigma(\Lambda))$ then there is $\lambda'\in\sigma(\Lambda)$ such that $\sigma^{-1}(\lambda')=\lambda$; hence, $\sigma^{-1}(\lambda')\in\Lambda$.

If $\lambda'\in\Lambda'$ then $\sigma^{-1}(\lambda')\in\sigma^{-1}(\Lambda')$ and so $\lambda'\in\sigma(\sigma^{-1}(\Lambda'))$. □

We provide counterexamples showing that these inclusions may be strict. Let W a set and R a relation on W.

Consider the signatures $\Sigma=<\varnothing,\{a,b\}>$ and $\Sigma'=<\varnothing,\{x\}>$ and the (unique) signature morphism $\sigma:\Sigma\rightarrow\Sigma'$. Consider also the following subset of Λ_Σ: $\Lambda=\{<W,R,\underline{V}>\}$ where $\underline{V}(w)=\{\blacklozenge a,\blacktriangledown a\}$ for each $w\in W$. Then, $\sigma(\Lambda)=\varnothing$ and $\sigma^{-1}(\sigma(\Lambda))=\varnothing$. Therefore, we do not have $\Lambda\subseteq\sigma^{-1}(\sigma(\Lambda))$. Note the role of the non-injectivity of σ.

Consider the signatures $\Sigma=<\varnothing,\{a\}>$ and $\Sigma'=<\varnothing,\{x,y\}>$ and the signature morphism $\sigma:\Sigma\rightarrow\Sigma'$ such that $\sigma_{act}(a)=x$. Consider also the following subset of $\Lambda_{\Sigma'}$: $\Lambda'=\{<W,R,\underline{V'}>\}$ where $\underline{V'}(w)=\{\blacklozenge x,\blacktriangledown x\}$ for each $w\in W$. Then, $\sigma^{-1}(\Lambda')=\{<W,R,\underline{V}>\}$ where $\underline{V}(w)=\{\blacklozenge a,\blacktriangledown a\}$ for each $w\in W$.

For instance, $<W,R,\underline{V2'}>\in\sigma(\sigma^{-1}(\Lambda'))$ where $\underline{V2'}(w)=\{\blacklozenge x,\blacktriangledown x,\blacklozenge y,\blacktriangledown y\}$ for each $w\in W'$, but $<W,R,\underline{V2'}>\notin\Lambda'$. Note the role of the non-surjectivity of σ.

Lemma 4.2.6. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism, $\Lambda'\subseteq\Lambda_{\Sigma'}$ and $\xi\in\Lambda_\Sigma$. If $\sigma^{-1}(\Lambda')\vDash_\Sigma\xi$ then there is $\xi'\in\Lambda_{\Sigma'}$ such that $\xi=\sigma^{-1}(\xi')$.
Proof:
Assume that $\sigma^{-1}(\Lambda')\vDash_\Sigma\xi$ where $\xi=<W,R,\underline{V}>$ and that there is no $\xi'\in\Lambda_{\Sigma'}$ such that $\xi=\sigma^{-1}(\xi')$. Then there is an atomic formula $\alpha\in A_\Sigma$ and a world $w\in W$ such that $\alpha\in\underline{V}(w)$ but $\beta\notin\underline{V}(w)$ where $\beta\in A_\Sigma$ and $\sigma(\alpha)=\sigma(\beta)$. Therefore $\sigma^{-1}(\Lambda')\vDash_\Sigma(\alpha\Leftrightarrow\beta)$ but $\xi\nvDash_\Sigma(\alpha\Leftrightarrow\beta)$[18]. And so $\sigma^{-1}(\Lambda')\nvDash_\Sigma\xi$. □

Definition 4.2.7. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism. The *range* of σ is the signature $\sigma(\Sigma)$ such that

$\sigma(\Sigma)_{obs}=\{b'\in\Sigma'_{obs}$: there is $b\in\Sigma_{obs}$ such that $\sigma_{obs}(b)=b'\}$,

$\sigma(\Sigma)_{act}=\{c'\in\Sigma'_{act}$: there is $c\in\Sigma_{act}$ such that $\sigma_{act}(c)=c'\}$.

In the sequel, $i:\sigma(\Sigma)\to\Sigma'$ is the inclusion signature morphism from $\sigma(\Sigma)$ into Σ', and $j:(\Sigma'\setminus\sigma(\Sigma))\to\Sigma'$ is the inclusion signature morphism from $(\Sigma'\setminus\sigma(\Sigma))$ into Σ'. □

Lemma 4.2.8. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism and $\Lambda'\subseteq\Lambda_{\Sigma'}$ a theoretically closed set of interpretation structures. Then, $i^{-1}(\Lambda')$ is theoretically closed.
Proof:
Assume that $i^{-1}(\Lambda')\vDash_{\sigma(\Sigma)}\lambda'$ where $\lambda'=<W',R',\underline{V}'>$. We have to show that $\lambda'\in i^{-1}(\Lambda')$. Consider an interpretation structure $\xi'=<W',R',\underline{V}>\in\Lambda_{\Sigma'}$ such that $i^{-1}(\xi')=\lambda'$ and $j^{-1}(\xi')\in j^{-1}(\Lambda')$. Consider also an interpretation structure $\theta'=<W',R',\underline{V}''>\in\Lambda'$ such that $j^{-1}(\xi')=j^{-1}(\theta')$. We show that if $\Lambda'\vDash_{\Sigma'}\varphi'$ then $\xi'\vDash_{\Sigma'}\varphi'$ for every $\varphi'\in L_{\Sigma'}$: (i) if $\varphi'\in L_{\Sigma'\setminus\sigma(\Sigma)}$ and $\Lambda'\vDash_{\Sigma'}\varphi'$ then $\theta'\vDash_{\Sigma'}\varphi'$, and so $\xi'\vDash_{\Sigma'}\varphi'$; (ii) if $\varphi'=\sigma(\varphi)$ for some $\varphi\in L_\Sigma$ and $\Lambda'\vDash_{\Sigma'}\sigma(\varphi)$ then $\Lambda'\vDash_{\Sigma'}i(\sigma(\varphi))$; thus, $i^{-1}(\Lambda')\vDash_{\sigma(\Sigma)}\sigma(\varphi)$[19], and so $\lambda'\vDash_{\sigma(\Sigma)}\sigma(\varphi)$; hence, $i^{-1}(\xi')\vDash_{\sigma(\Sigma)}\sigma(\varphi)$; thus, $\xi'\vDash_{\Sigma'}i(\sigma(\varphi))$[20] and so $\xi'\vDash_{\Sigma'}\sigma(\varphi)$. Hence, $\Lambda'\vDash_{\Sigma'}\xi'$ and so $\xi'\in\Lambda'$ and $i^{-1}(\xi')\in i^{-1}(\Lambda')$. Therefore, $\lambda'\in i^{-1}(\Lambda')$. □

Lemma 4.2.9. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism and $\Lambda'\subseteq\Lambda_{\Sigma'}$ a theoretically closed set of interpretation structures. Then, $\sigma^{-1}(\Lambda')$ is theoretically closed.
Proof:
Assume that $\sigma^{-1}(\Lambda')\vDash_\Sigma\sigma^{-1}(\xi')$; hence, if $\sigma^{-1}(\Lambda')\vDash_\Sigma\varphi$ then $\sigma^{-1}(\xi')\vDash_\Sigma\varphi$ for every $\varphi\in L_\Sigma$. That is[21], if $\Lambda'\vDash_{\Sigma'}\sigma(\varphi)$ then $\xi'\vDash_{\Sigma'}\sigma(\varphi)$ for every $\varphi\in L_\Sigma$[22]. Hence, if $i^{-1}(\Lambda')\vDash_{\sigma(\Sigma)}\sigma(\varphi)$ then $i^{-1}(\xi')\vDash_{\sigma(\Sigma)}\sigma(\varphi)$ for every $\varphi\in L_\Sigma$. Therefore $i^{-1}(\xi')\in i^{-1}(\Lambda')$. Finally, $\sigma^{-1}(\xi')\in\sigma^{-1}(\Lambda')$ because $\sigma^{-1}(\xi')=\sigma^{-1}(i^{-1}(\xi'))$ and $\sigma^{-1}(\Lambda')=\sigma^{-1}(i^{-1}(\Lambda'))$. □

Proposition 4.2.10. *The functor $Sg:Ob\to Sig^{op}$ is a split cofibration with a unique cocleavage.*
Proof:
(1) Cofibration: For each $\sigma:\Sigma\to\Sigma'$ in Sig and $ob'=<\Sigma',\Lambda'>$ in Ob, the object morphism $f:ob'\to ob$, where $ob=<\Sigma,\Lambda>$ with $\Lambda=\sigma^{-1}(\Lambda')$, such that $Sg(f)=\sigma^{op}$ is cocartesian by Sg for σ^{op} on ob'. Indeed, ob is an object and f is a morphism because $\sigma^{-1}(\Lambda')\subseteq\Lambda$. Moreover, f has the envisaged couniversal property: assume that $g:ob'\to ob''$, where $ob''=<\Sigma'',\Lambda''>$, is a morphism in Ob and $\omega:\Sigma''\to\Sigma$ is a

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[18] Equivalence is introduced in the language of the base institution as an abbreviation: $(\gamma1\Leftrightarrow\gamma2)=(\neg((\gamma1\Rightarrow\gamma2)\Rightarrow(\neg(\gamma2\Rightarrow\gamma1))))$.
[19] Using the set satisfaction condition of the base institution.
[20] Using the satisfaction condition of the base institution.
[21] Using the set satisfaction condition and the satisfaction condition of the base institution.
[22] Using the satisfaction condition of the induced institution.

morphism in Sig such that $\omega^{op} \circ \sigma^{op}$=Sg(g); the problem is to find a unique h:ob→ob'' such that Sg(h)=ω^{op} and h $\circ$ f=g; we have only to verify that ω induces an object morphism: since $\omega^{-1} \circ \sigma^{-1}(\Lambda') \subseteq \Lambda''$ and $\sigma^{-1}(\Lambda')=\Lambda$ we obtain $\omega^{-1}(\Lambda) \subseteq \Lambda''$.

(2) Uniqueness of cocleavage: We have to show that for each σ:Σ→Σ' in Sig and ob'=<Σ',Λ'> in Ob, the morphism f:ob'→<Σ,$\sigma^{-1}(\Lambda')$> is the unique cocartesian morphism by Sg for σ on ob'. Assume that there is another cocartesian morphism g:ob'→<Σ,Ξ> by Sg for σ on ob'. Then, there is an isomorphism h:<Σ,$\sigma^{-1}(\Lambda')$>→<Σ,Ξ>[23] such that σ $\circ$ Sg^{op}(h)=σ. Thus, if Sg^{op}(h)(a)=b then σ(a)=σ(b), for all $a \in \Sigma_{ob} \cup \Sigma_{act}$ (note that it may be the case that a≠b). We show that λ∈ Ξ whenever $\lambda \in \sigma^{-1}(\Lambda')$: let <W,R,$\underline{V}$>$\in \sigma^{-1}(\Lambda')$; then, <W,R,$\underline{V}$>=$(Sg(h)^{op})^{-1}$(<W,R,$\underline{V}$>) since, for $\alpha,\beta \in A_\Sigma$, if σ(α)=σ(β) then α∈ $\underline{V}$(w) iff β∈ $\underline{V}$(w) for every w∈ W; therefore, <W,R,$\underline{V}$>∈ Ξ since h is an object morphism. A similar reasoning on h^{-1} allows us to conclude that $\lambda \in \sigma^{-1}(\Lambda')$ whenever λ∈ Ξ. Thus, $\sigma^{-1}(\Lambda')$=Ξ.

(3) Splitting: Similar to the proof that Si is a split cofibration. □

Lemma 4.2.11. Let σ:Σ→Σ' be a signature morphism and $\Lambda \subseteq \Lambda_\Sigma$ a theoretically closed set of interpretation structures. Then, σ(Λ) is theoretically closed.

Proof:

Assume that σ(Λ) is not theoretically closed. Then there is $\xi' \in \Lambda_{\Sigma'}$ such that $\sigma(\Lambda) \vDash_{\Sigma'} \xi'$ but $\xi' \notin \sigma(\Lambda)$ and so $\sigma^{-1}(\xi') \notin \Lambda$. Therefore, there is $\varphi \in L_\Sigma$ such that $\Lambda \vdash_\Sigma \varphi$ but $\sigma^{-1}(\xi') \nvDash_\Sigma \varphi$. Then, $\sigma(\Lambda) \vDash_{\Sigma'} \sigma(\varphi)$[24] and $\xi' \nvDash_{\Sigma'} \sigma(\varphi)$[25]. Finally, we have $\sigma(\Lambda) \nvDash_{\Sigma'} \xi'$. □

Proposition 4.2.12. *The functor Sg:Ob→*Sig^{op} *is a fibration.*

Proof:

For each σ:Σ→Σ' in Sig and ob=<Σ,Λ> in Ob, the object morphism f:ob'→ob, where ob'=<Σ',Λ'> with Λ'=σ(Λ), such that Sg(f)=σ^{op} is cartesian by Sg for σ^{op} on ob. Indeed, ob' is an object and f is a morphism since $\sigma^{-1}(\sigma(\Lambda)) \subseteq \Lambda$. Moreover, f has the envisaged universal property: assume that g:ob''→ob, where ob''=<Σ'',Λ''>, is a morphism in Ob and ω:Σ'→Σ'' is a morphism in Sig such that $\sigma^{op} \circ \omega^{op}$=Sg(g); the problem is to find a unique h:ob''→ob' such that Sg(h)=ω^{op} and f $\circ$ h=g; we have only to verify that ω induces an object morphism: $\omega^{-1}(\Lambda'') \subseteq \Lambda'$ because if $\lambda' \in \omega^{-1}(\Lambda'')$ then $\sigma^{-1}(\lambda') \in \Lambda$ (since $\sigma^{-1} \circ \omega^{-1}(\Lambda'') \subseteq \Lambda$), and thus $\lambda' \in \{\lambda' \in \Lambda_{\Sigma'} : \sigma^{-1}(\lambda') \in \Lambda\} = \sigma(\Lambda) = \Lambda'$. □

5. Induced object institution

We are now ready to set-up the envisaged institution of objects where objects play the role of models. The object institution induced by B is the quadruple O_B=<Sig,Sen,Obj,⊫> where Sig and Sen are as in B, and the functor Obj: Sig→Cat^{op} and the family ⊫ of satisfaction relations are defined below. The functor Obj is defined capitalizing on the fact that the functor Sg is a split cofibration (following a general construction as presented in [Barr and Wells 90]: see the Appendix): it maps each each signature onto the category of objects over that signature.

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[23] See the Appendix: proposition A7.
[24] Using the set forward satisfaction condition of the base institution.
[25] Using the satisfaction condition of the base institution.

Since B and O_B share the category Sig and the functor Sen, they also share the language and the category of theories. Therefore, all the results and constructions of sections 3.3 and 3.4 are retained, since they do not depend on the semantic components of the institutions.

5.1. Interpretation over objects

Definition 5.1.1. Let Σ be a signature. The category Ob_Σ is the fiber $Sg^{-1}(\Sigma)$. The objects of Ob_Σ are called *interpretation structures* over Σ. The arrows of Ob_Σ are called *homomorphisms* between interpretation structures over Σ. □

Definition 5.1.2. Let $\sigma:\Sigma\rightarrow\Sigma'$ be a signature morphism and ob' an interpretation structure over Σ'. The *reduct object* of ob' with respect to σ (denoted by $\sigma^{-1}(ob')$) is the target of the cocartesian morphism $\underline{\sigma}\_;^{op}{}_{ob'}$ for σ^{op} on ob'. □

Clearly, as expected, the signature morphism $\sigma:\Sigma\rightarrow\Sigma'$ leads to an object morphism from $ob'=<\Sigma',\Lambda'>$ into $ob=<\Sigma,\Lambda>$ iff the set of interpretation structures of the reduct $\sigma^{-1}(ob')$ is a subset of Λ.

Fact/Definition 5.1.3. Let $\sigma:\Sigma\rightarrow\Sigma'$ be a signature morphism. The *reduct functor* $\sigma^{-1}:Ob_{\Sigma'}\rightarrow Ob_\Sigma$ maps each ob' of $Ob_{\Sigma'}$ onto the reduct $\sigma^{-1}(ob')$ and each homomorphism $h':ob1'\rightarrow ob2'$ of $Ob_{\Sigma'}$ onto the unique homomorphism $\sigma^{-1}(h'):\sigma^{-1}(ob1')\rightarrow\sigma^{-1}(ob2')$ such that $\sigma^{-1}(h')\circ\underline{\sigma}\_;^{op}{}_{ob1'} = \underline{\sigma}\_;^{op}{}_{ob2'}\circ h'$.
Proof:
Consider the contravariant version of A11 in the Appendix. □

Fact/Definition 5.1.4. The *object interpretation functor* $Obj:Sig\rightarrow Cat^{op}$ maps each signature Σ onto the fiber Ob_Σ, and each signature morphism $\sigma:\Sigma\rightarrow\Sigma'$ onto the opposite of the reduct functor $\sigma^{-1}:Ob_{\Sigma'}\rightarrow Ob_\Sigma$.
Proof:
Consider the contravariant version of A11 in the Appendix. □

In this way we established an indexed category (cf [Tarlecki *et al* 1989]) $Obj^{op}:Sig^{op}\rightarrow Cat$. By an appropriate flattening construction we should recover the original category Ob. However we shall not dwell on this issue.

Clearly, we might develop the counterparts to these functors for "images" instead of reducts provided that we verify that the fibration Sg is split. But we refrain to do so because we do not need them in the sequel. However, it is useful to introduce:

Definition 5.1.5. Let $\sigma:\Sigma\rightarrow\Sigma'$ be a signature morphism and ob an interpretation structure over Σ. The *image object* of ob with respect to σ (denoted by $\sigma(ob)$) is the source of the cartesian morphism for σ^{op} on ob. □

5.2. Satisfaction by objects

Definition 5.2.1. Let Σ be a signature, φ a formula of L_Σ, and ob=$<\Sigma,\Lambda>$ an object. We say that ob *satisfies* φ (written ob$\Vdash_\Sigma\varphi$) iff $\Lambda\vDash_\Sigma\varphi$. □

Proposition 5.2.2. *Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism, φ a formula of L_Σ, and ob'=$<\Sigma',\Lambda'>$ an object. Then, ob'$\Vdash_{\Sigma'}\sigma(\varphi)$ iff σ^{-1}(ob')$\Vdash_\Sigma\varphi$.*
Proof:
Corollary of proposition 3.5.1. □

Proposition 5.2.3.*The quadruple O_B=<Sig,Sen,Obj,$\Vdash$> is an institution.* □

Proposition 5.2.4. *Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism, φ a formula of L_Σ, and ob=$<\Sigma,\Lambda>$ an object. Then, if ob$\Vdash_\Sigma\varphi$ then σ(ob)$\Vdash_{\Sigma'}\sigma(\varphi)$.*
Proof:
Corollary of proposition 3.5.2. □

5.3. Loose semantics

Although we envisage a final semantics mapping each theory onto the "maximal" object that satisfies the theory, we start by introducing the traditional loose semantics of theories within the setting of the object institution, since we need in the sequel to be able to talk about all object models of a theory and all theories entailed by an object.

Fact/Definition 5.3.1. The *loose semantics functor* Mod:Th$\to$Cat^{op} maps each theory th onto the full subcategory Mod(th) of Obj(Σ) whose elements satisfy th, and each theory morphism f:th$\to$th' onto Mod(f):Mod(th)$\to$Mod(th') corresponding to the restriction of the op-functor Obj(Si(f)):Obj(Si(th))$\to$Obj(Si(th')) to Mod(th').
Proof:
Let th=$<\Sigma,\Phi>$ and th'=$<\Sigma',\Phi'>$ be theories, f:th$\to$th' a theory morphism and ob'=$<\Sigma',\Lambda'>$ an object. Recall that Obj(Si(f)):Obj(Σ)$\to$Obj(Σ') is the opposite of the reduct functor (Si(f))$^{-1}$:Obj(Σ')$\to$Obj(Σ). We just verify that if ob'$\in$Mod(th') then (Obj(Si(f)))^{op}(ob')$\in$Mod(th). Indeed, if ob'$\in$Mod(th') then: $\Lambda'\vDash_{\Sigma'}\varphi'$ for each $\varphi'\in\Phi'$ and since Si(f)(φ)$\in\Phi'$ then $\Lambda'\vDash_{\Sigma'}$Si(f)(φ) for each $\varphi\in\Phi$. Hence, (Si(f))$^{-1}$(Λ')$\vDash_\Sigma\varphi$[26] for each $\varphi\in\Phi$. Therefore, we obtain Obj(Si(f))^{op}(ob')= (Si(f))$^{-1}$(ob')$\in$Mod(th). □

Conversely, we might be interested in extracting from an object the category of all theories that are satisfied by it. Taking advantage of the indexing of theories by signatures introduced above it is straightforward to establish:

Fact/Definition 5.3.2. The *loose theory functor* Lth:Ob$\to$Cat^{op} maps each object ob onto the full subcategory Lth(ob) of TH(Σ) whose elements are satisfied by ob, and each object morphism f:ob$\to$ob' onto Lth(f):Lth(ob)$\to$Lth(ob') corresponding to the restriction of the op-functor TH(Sg(f)^{op})^{op}:TH(Sg(ob))$\to$TH(Sg(ob')) to Lth(ob').

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[26] Using the set satisfaction condition for the base instititution.

Proof:
Let ob=<Σ,Λ> and ob'=<Σ',Λ'> be objects, f:ob→ob' an object morphism and th'=<Σ',Φ'> a theory. Recall that $TH(Sg(f)^{op}):TH(\Sigma')\to TH(\Sigma)$ is the image functor $(Sg(f)^{op}):TH(\Sigma')\to TH(\Sigma)$. We just verify that if th'∈Lth(ob') then $TH(Sg(f)^{op})(th')\in Lth(ob)$. Indeed, if th'∈Lth(ob') then: $\Lambda'\vDash_{\Sigma'}\varphi'$, for each φ'∈Φ' and since $(Sg(f)^{op})^{-1}(\Lambda)\subseteq\Lambda'$ then $(Sg(f)^{op})^{-1}(\Lambda)\vDash_{\Sigma'}\varphi'$, for each φ'∈Φ'. Hence, $\Lambda\vDash_{\Sigma}(Sg(f)^{op})(\varphi')$[27], for each φ'∈Φ'. Therefore, $TH(Sg(f))^{op}(th')=Sg(f)^{op}(th')\in Lth(ob)$. □

6. Final semantics

We are now ready to present the envisaged final semantics functor that maps each theory onto the object composed of all lives that satisfy (in the base institution) the theory. We also present the functor that maps each object to the theory composed of all formulae satisfied (in the base institution) by the object. As might be expected, these two functors are the reverse of each other and, therefore, establish an useful isomorphism between the category of theories and the category of objects.

6.1. Maximal object model

Proposition 6.1.1. Let th=<Σ,Φ> be a theory and $\Omega_{th}=\{\lambda\in\Lambda_\Sigma: \lambda\vDash_\Sigma\Phi\}$. Then, $<\Sigma,\Omega_{th}>$ is the final object in the category Mod(th).
Proof:
Theoretical closure: we have to verify that $\Omega_{th}=(\Omega_{th})^{\vDash}$; the left to right inclusion is trivial; the opposite inclusion is obtained as follows: if $\Omega_{th}\vDash_\Sigma\xi$ where $\xi\in\Lambda_\Sigma$ then $\xi\vDash_\Sigma\varphi$ for all φ∈Φ since $\Omega_{th}\vDash_\Sigma\varphi$ for all φ∈Φ and, thus, $\xi\in\Omega_{th}$. Finality: let <Σ,Λ> be any object in the category Mod(th); we have a (unique) object homomorphism $f:<\Sigma,\Lambda>\to<\Sigma,\Omega_{th})>$ since $\Lambda\subseteq\Omega_{th}$. □

Let ob=<Σ,Λ>∈Mod(th) for some theory th=<Σ,Φ>. We denote by $1_{ob}:ob\to<\Sigma,\Omega_{th}>$ the unique homomorphism that exists in Mod(th) from ob into the final object in Mod(th).

Fact/Definition 6.1.2. *The final semantics functor* $[\![\_]\!]:Th^{op}\to Ob$ *maps each theory th=<Σ,Φ> onto the final object of Mod(th) and each theory morphism f:th→th' onto the morphism* $[\![f]\!]=1_{(Si(f))^{-1}([\![th']\!])}\circ \underline{Si(f)\quad};^{op}_{[\![th']\!]}:[\![th']\!]\to[\![th]\!]$. □

That is, ⟦th⟧ is the set of all interpretation structures that satisfy all axioms of th; and ⟦f⟧ is given by composing the cocartesian morphism for $Si(f)^{op}$ on ⟦th'⟧

$\underline{Si(f)\quad};^{op}_{[\![th']\!]}:[\![th']\!]\to(Si(f))^{-1}([\![th']\!])$

with the unique object homomorphism

$1_{(Si(f))^{-1}([\![th']\!])}:(Si(f))^{-1}([\![th']\!])\to[\![th]\!]$

from the reduct $(Si(f))^{-1}([\![th']\!])$ into the final object $[\![th]\!]=<\Sigma,\Omega_{th}>$.

[27] Using the set satisfaction condition for the base instititution.

6.2. Maximal entailed theory

Conversely, we are also interested in associating to each object the theory containing all formulae satisfied by it, ie the largest theory satisfied by it.

Proposition 6.2.1. Let ob=$<\Sigma,\Lambda>$ be an object and $\Omega_{ob}=\{\varphi\in L_\Sigma : \Lambda\vDash_\Sigma\varphi\}$. Then, $<\Sigma,\Omega_{ob}>$ is the final theory in the category Lth(ob).
Proof:
Semantic closure: we have to verify that $\Omega_{ob}=(\Omega_{ob})^{\vDash}$; the left to right inclusion is trivial; the opposite inclusion is obtained as follows: if $\Omega_{ob}\vDash_\Sigma\psi$ where $\psi\in L_\Sigma$ then $\Lambda\vDash_\Sigma\psi$ since $\Lambda\vDash_\Sigma\varphi$ for all $\varphi\in\Omega_{ob}$ and, thus, $\psi\in\Omega_{ob}$. Finality: let $<\Sigma,\Lambda>$ be any object in the category Mod(th); we have a (unique) object homomorphism f:$<\Sigma,\Lambda>\to<\Sigma,\Omega_{th}>$) since $\Lambda\subseteq\Omega_{th}$. □

Let th=$<\Sigma,\Phi>\in$ Lth(ob) for some object ob=$<\Sigma,\Lambda>$. We denote by 1_{th}:th$\to<\Sigma,\Omega_{ob}>$ the unique homomorphism that exists in Lth(ob) from th into the final object in Lth(ob).

Fact/Definition 6.2.2. *The final theory functor* $\ulcorner\_\urcorner:Ob\to Th^{op}$ *maps each object* $ob=<\Sigma,\Lambda>$ *onto the final theory of Lth(ob) and each object morphism* $f:ob\to ob'$ *onto the opposite of the theory morphism* $\ulcorner f\urcorner=1_{Sg(f)^{op}(\ulcorner ob'\urcorner)}\circ Sg(f)^{op};\overline{\qquad\quad}^{\ulcorner ob'\urcorner}:\ulcorner ob'\urcorner\to\ulcorner ob\urcorner$. □

That is, $\ulcorner$ob$\urcorner$ contains all formulae satisfied by ob; and $\ulcorner$f$\urcorner$ is given by composing the cocartesian morphism for Sg(f)^{op} on $\ulcorner$ob'$\urcorner$ with the unique theory homomorphism from the image theory Sg(f)^{op}($\ulcorner$ob'$\urcorner$) into the final theory $\ulcorner$ob$\urcorner$.

6.3. Main properties

Lemma 6.3.1. Let th=$<\Sigma,\Phi>$ be a theory. Then $\ulcorner[\![th]\!]\urcorner$=th.
Proof:
Let $\ulcorner[\![th]\!]\urcorner=<\Sigma,\Psi>$ and $[\![th]\!]=<\Sigma,\Omega_{th}>$. Clearly, $\Phi\subseteq\Psi$. Moreover, $\Psi\subseteq\Phi$. Indeed, we show that $\Phi\vDash_\Sigma\psi$ for any $\psi\in\Psi$: assume that $\xi\vDash_\Sigma\Phi$; then $\xi\in\Omega_{th}$; on the other hand, $\Omega_{th}\vDash_\Sigma\psi$ and so $\xi\vDash_\Sigma\psi$; therefore $\Phi\vDash_\Sigma\psi$ for any $\psi\in\Psi$. And so, $\psi\in\Phi$ for every $\psi\in\Psi$. □

Lemma 6.3.2. Let ob=$<\Sigma,\Lambda>$ be an object. Then $[\![\ulcorner ob\urcorner]\!]$=ob.
Proof:
Let $[\![\ulcorner ob\urcorner]\!]=<\Sigma,\Xi>$ and $\ulcorner ob\urcorner=<\Sigma,\Omega_{ob}>$. Clearly, $\Lambda\subseteq\Xi$. Moreover, $\Xi\subseteq\Lambda$. Indeed, we show that $\Lambda\vDash_\Sigma\xi$ for every $\xi\in\Xi$: $\Lambda\vDash_\Sigma\varphi$ iff $\varphi\in\Omega_{ob}$; $\xi\vDash_\Sigma\varphi$ for every $\varphi\in\Omega_{ob}$; hence $\Lambda\vDash_\Sigma\xi$. Therefore, $\xi\in\Lambda$ for every $\xi\in\Xi$. □

Proposition 6.3.3. *The pair of functors* $<[\![\_]\!]\,,\ \ulcorner\_\urcorner>$ *is a Galois correspondence.*
Proof:
Let ob=$<\Sigma,\Lambda>$ be an object and th=$<\Sigma',\Phi'>$ a theory. Let $\sigma^{op}:\Sigma'\to\Sigma$ be a signature op-morphism. We show that σ^{op} induces a theory op-morphism from th into $\ulcorner$ob$\urcorner$ iff σ^{op} induces an object morphism from $[\![th]\!]$ into ob. That is, we show that $\sigma(\Omega_{ob})\subseteq\Phi'$ iff $\sigma^{-1}(\Omega_{th})\subseteq\Lambda$.
(1) Assume that $\sigma(\Omega_{ob})\subseteq\Phi'$. Then, since $\Omega_{th}\vDash_{\Sigma'}\varphi'$ for every $\varphi'\in\Phi'$, we conclude that $\Omega_{th}\vDash_{\Sigma'}\sigma(\varphi)$ for every $\varphi\in\Omega_{ob}$. Therefore, $\sigma^{-1}(\Omega_{th})\vDash_\Sigma\varphi$ for every $\varphi\in\Omega_{ob}$. Hence, $\sigma^{-1}(\Omega_{th})\subseteq\Xi$, where $[\![\ulcorner ob\urcorner]\!]=<\Sigma,\Xi>$. Therefore, $\sigma^{-1}(\Omega_{th})\subseteq\Lambda$.

(2) Assume that $\sigma^{-1}(\Omega_{th})\subseteq\Lambda$. Let $\varphi\in\Omega_{ob}$. Then, $\Lambda\vDash_{\Sigma}\varphi$ and so $\sigma^{-1}(\Omega_{th})\vDash_{\Sigma}\varphi$. Thus, $\Omega_{th}\vDash_{\Sigma'}\sigma(\varphi)$ and so $\sigma(\varphi)\in\Psi$, where $\ulcorner\llbracket th\rrbracket\urcorner=<\Sigma',\Psi>$. Then, $\sigma(\varphi)\in\Phi'$. □

Corollary 6.3.4. Let ob=$<\Sigma,\Lambda>$ be an object and th=$<\Sigma,\Phi>$ a theory such that $\ulcorner ob\urcorner=<\Sigma,\Gamma>$ and $\llbracket th\rrbracket=<\Sigma,\Xi>$, respectively. Then, $\Xi\subseteq\Lambda$ iff $\Gamma\subseteq\Phi$. □

Proposition 6.3.5. *The pair of functors* $<\llbracket\_\rrbracket\ ,\ \ulcorner\_\urcorner>$ *constitute a concrete isomorphism over* Sig^{op}.

Proof:

Since (Th,Si) and (Ob,Sg) are amnestic then by proposition A16 of the Appendix we can conclude that $\llbracket\_\rrbracket^*:\llbracket|Th^{op}|\rrbracket\to\ulcorner|Ob|\urcorner$ and $\ulcorner\_\urcorner^*:\ulcorner|Ob|\urcorner\to\llbracket|Th^{op}|\rrbracket$ constitute a concrete isomorphism. But $\llbracket|Th^{op}|\rrbracket=Ob$ and $\ulcorner|Ob|\urcorner=Th^{op}$ and so $\llbracket\_\rrbracket^*=\llbracket\_\rrbracket$ and $\ulcorner\_\urcorner^*=\ulcorner\_\urcorner$. □

Proposition 6.3.6. *The functor* $\llbracket\_\rrbracket:Th^{op}\to Ob$ *preserves limits and colimits, as well as cartesian and cocartesian morphisms.*

Proof:

Clearly, an isomorphism preserves (co)limits. Concerning (co)cartesian morphisms see proposition A13 of the Appendix. □

Corollary 6.3.7. Let $\sigma:\Sigma\to\Sigma'$ be a signature morphism and th' theory over Σ'. Then, we have that $\llbracket\sigma^{-1}(th')\rrbracket=\sigma^{-1}(\llbracket th'\rrbracket)$. □

Corollary 6.3.8. Let th and th' be theories. Then, $\llbracket th\oplus th'\rrbracket=\llbracket th\rrbracket\otimes\llbracket th'\rrbracket$. □

7. Application to denotational semantics

7.1. Main object constructs

The specification spec=$<\Sigma,\Phi>$ of a primitive object induces the theory th=$<\Sigma,\Phi^{\vDash}>$. The denotation of the theory is the final object $\llbracket th\rrbracket$ given by the functor $\llbracket\_\rrbracket$.

The aggregation of two objects specified by spec=$<\Sigma,\Phi>$ and spec'=$<\Sigma',\Phi'>$, inducing the theories th=$<\Sigma,\Phi^{\vDash}>$ and th'=$<\Sigma',\Phi'^{\vDash}>$, is defined by the coproduct theory th$\oplus$th'= $<\Sigma\oplus\Sigma',(inj_{\Sigma}(\Phi)\cup inj_{\Sigma'}(\Phi'))^{\vDash}>$ corresponding to the coproduct of th and th'. Its denotation is the final object $\llbracket th\oplus th'\rrbracket=\llbracket th\rrbracket\otimes\llbracket th'\rrbracket$.

The interconnection of two objects specified by spec=$<\Sigma,\Phi>$, spec'=$<\Sigma',\Phi'>$ and a set Ψ of interaction axioms over $\Sigma\oplus\Sigma'$ (eg ($\blacktriangledown c\Rightarrow\blacktriangledown c'$) for c *calls* c' and ($\blacktriangledown c\Rightarrow b'$) for c *queries* b') is defined by the theory $<\Sigma\oplus\Sigma',(inj_{\Sigma}(\Phi)\cup inj_{\Sigma'}(\Phi')\cup\Psi)^{\vDash}>$ corresponding to the coproduct of th=$<\Sigma,\Phi^{\vDash}>$ and th'=$<\Sigma',\Phi'^{\vDash}>$ enriched with the interaction axioms. Its denotation is the "constrained product" of $\llbracket th\rrbracket$ and $\llbracket th'\rrbracket$ that also satisfies the interaction axioms.

The view given by an injective hiding signature morphism $\sigma:\Sigma'\to\Sigma$ of an object specified by spec=$<\Sigma,\Phi>$ is defined by the theory $\sigma^{-1}(th)=<\Sigma',\sigma^{-1}(\Phi^{\vDash})>$. Its denotation is, as expected, $\llbracket\sigma^{-1}(th)\rrbracket=\sigma^{-1}(\llbracket th\rrbracket)$.

The (monotonic) specialization of an object specified by spec=<Σ,Φ> is specified by spec'=<ΣUΣ1,ΦUΦ1>. The denotation of the resulting theory is the "constrained" object that also satisfies the additional axioms.

7.2. Working with linear temporal logic

We illustrate in some detail the proposed general construction when the base institution is the (propositional) linear temporal logic institution L. This institution is established by the following specific requirements (besides the base requirements stated in section 3):

• Given a signature Σ, L_Σ also satisfies the following rules: $✳\in L_\Sigma$; $(\mathbf{X}\gamma)\in L_\Sigma$, provided that $\gamma\in L_\Sigma$; $(\mathbf{G}\gamma)\in L_\Sigma$, provided that $\gamma\in L_\Sigma$; $(\gamma_1\mathbf{U}\gamma_2)\in L_\Sigma$, provided that both $\gamma_1,\gamma_2\in L_\Sigma$. As usual, **X** is the "next" operator, **G** is the "always in the future" operator, and **U** is the "until" operator. Furthermore, ✳ is the "initial instant" symbol.

• Each interpretation structure λ, called a *life-cycle*, is of the form $<\mathbb{N}_\circ,s,V>$, where s is the successor relation on natural numbers. For each $n\in\mathbb{N}_\circ$, $\underline{V}(n)$ is called the *situation at position* n. Note that this institution is "uniframe" in the sense that all interpretation structures are built over the same frame (in this case $<\mathbb{N}_\circ,s>$). Therefore, in the sequel, we identify each $\lambda=<\mathbb{N}_\circ,s,V>$ with the sequence $\underline{V}$ and write $\lambda:\mathbb{N}_\circ\rightarrow 2^{A_\Sigma}$.

• Let λ be a life-cycle and $n\in\mathbb{N}_\circ$. Then: $\lambda\models_n ✳$ iff n=0; $\lambda\models_n(\mathbf{X}\gamma)$ iff $\lambda\models_{n+1}\gamma$; $\lambda\models_n(\mathbf{G}\gamma)$ iff $\lambda\models_m\gamma$ for every m>n; $\lambda\models_n(\gamma_1\mathbf{U}\gamma_2)$ iff $\lambda\models_m\gamma_2$ for some m>n and $\lambda\models_i\gamma_1$ for every i such that n<i<m.

In the case of L, each object ob is a pair <Σ,Λ> where Λ is a set of life-cycles. A morphism from ob to ob' states that each life-cycle of ob, when reducted by σ:Sg(ob')→Sg(ob) to the signature of ob', must be a life-cycle of ob'.

7.3. Products for linear temporal logic

We examine now the *products* in the category Ob built upon L. We show that for "reasonable" objects the product reflects "fair interleaving" of the components. Such reasonable objects are those objects that are open to embedding in any environment. Namely, they do not assume that they will be able to execute actions before the environment does something else. In order to formalize this concept we need to develop the notion of (fair) delay.

For k>0 we denote by $[\lambda[_k$ the prefix $\lambda_0...\lambda_{k-1}$ of length k of λ. By convention $[\lambda[_0$ is the empty sequence. Given $s\subseteq A_\Sigma$ we denote by $s^{-\nabla}$ the set given by $s\cap(\{b:b\in\Sigma_{obs}\}\cup\{◆c:c\in\Sigma_{act}\})$.

Definition 7.3.1. Let Σ be a signature and k an ordinal ≤ω. The *one-step delay at* k *map* $\Delta^k:\Lambda_\Sigma\rightarrow\Lambda_\Sigma$ is defined as follows: for $k\in\mathbb{N}_\circ$, $\Delta^k(\lambda)_n=\lambda_n$ for every n<k, $\Delta^k(\lambda)_n=\lambda_{n-1}$ for every n>k, and $\Delta^k(\lambda)_k=(\lambda_k)^{-\nabla}$; and $\Delta^\omega(\lambda)=\lambda$. □

Definition 7.3.2. A *delay-point sequence* η is a sequence of ordinals such that: (i) for each $i\in\mathbb{N}_\circ$ $\eta_i\leq\omega$; (ii) for each $i,j\in\mathbb{N}_\circ$ such that i<j either $\eta_i<\eta_j$ or $\eta_i=\eta_j=\omega$; (iii) for each $i\in\mathbb{N}_\circ$ there is $j\in\mathbb{N}_\circ$ such that either $\eta_j\in\mathbb{N}_\circ$ and $\eta_j-\eta_i>j-i$ or $\eta_j=\omega$. □

Condition (iii) in the previous definition is called the *diligence condition*[28].

Definition 7.3.3. Given a life-cycle λ over a signature Σ, a sequence η of delay points *induces* the sequence $\delta^{\eta}(\lambda)$ of life-cycles over Σ such that: $\delta^{\eta}(\lambda)^0=\lambda$, and $\delta^{\eta}(\lambda)^{i+1}=\Delta^{\eta_i}(\delta^{\eta}(\lambda)^i)$. □

Definition 7.3.4. The *delay relation* $\infty\subseteq\Lambda_{\Sigma}^2$ is defined as follows: $\lambda\infty\lambda'$ iff there is a delay-point sequence η such that for each $n\in\mathbb{N}_\circ$ there is $i\in\mathbb{N}_\circ$ such that $[\lambda'[_n=[\delta^{\eta}(\lambda)^i[_n$. □

For instance, $\lambda\infty\lambda'$, assuming that $\lambda_n=\{◆c,▼c\}$ for $n\in\mathbb{N}_\circ$, and $\lambda'_n=\{◆c,▼c\}$ for n odd and $\lambda'_n=\{◆c\}$ for n even. Indeed, consider the sequence η such that $\eta_i=i\times 2$ for every $i\in\mathbb{N}_\circ$ which trivially satisfies the diligence condition. The sequence $\delta^{\eta}(\lambda)$ is as follows for i>0:

$(\delta^{\eta}(\lambda)^i)_n=\{◆c\}$ for $n<\eta_{i-1}+1$ and n even,
$(\delta^{\eta}(\lambda)^i)_n=\{◆c,▼c\}$ for $n\leq\eta_{i-1}+1$ and n odd,
$(\delta^{\eta}(\lambda)^i)_n=\{◆c,▼c\}$ for $n>\eta_{i-1}+1$.

On the other hand, if $\lambda''_n=\{◆c\}$ for $n\in\mathbb{N}_\circ$ then $\lambda\not\infty\lambda''$ because the sequence $\eta=\{i\}_{i\in\mathbb{N}_\circ}$ violates the diligence condition. The life-cycle λ'' would be an "unfair" delay of λ since action c would never happen. Note that if $\lambda\infty\lambda'$ and the delay-point sequence is η such that there is $i\in\mathbb{N}_\circ$ such that $\eta_j=\omega$ for all $j\geq i$ then λ' has a finite number of delays relatively to λ.

Proposition 7.3.5. *The delay relation is reflexive and transitive.*
Proof:
(a) Reflexivity: choose $\eta_i=\omega$ for every $i\in\mathbb{N}_\circ$.
(b) Transitivity: assume that $\lambda\infty\lambda'$ and $\lambda'\infty\lambda''$ with delay-point sequences η and η', respectively; consider the sequence $\eta;\eta'$ obtained by merging η' with the sequence ν such that $\nu_n=\eta_n+\max\{k+1: \eta'_k<(\eta_n+k+1)\}$ if $\eta_n<\omega$, and $\nu_n=\omega$ otherwise; that is, each delay-point of η is moved to the right depending on the delays introduced by η'; it is trivial to see that $\eta;\eta'$ satisfies requirements (i) and (ii) on delay-point sequences; verifying that $\eta;\eta'$ satisfies requirement (iii) is more difficult; the thesis is obtained by showing that each point (of $\mathbb{N}_\circ$) where η does not introduce a delay is preserved by $\eta;\eta'$ as a no-delay-point, but suitably moved to the right (like a delay-point: a similar formula is applied). □

Definition 7.3.6. Let Λ be a set of life-cycles over a signature Σ. The *closure for delays* of Λ is the following set of life-cycles over Σ: $\Lambda^{\varnothing}=\{\lambda'\in\Lambda_{\Sigma}: \lambda\infty\lambda'$ for some $\lambda\in\Lambda\}$. □

Definition 7.3.7. An object ob=<Σ,Λ> is said to be *closed for delays* iff $\Lambda=\Lambda^{\varnothing}$. □

Consider the object $<<\varnothing,\{a\}>,\{\lambda\}^{\varnothing}>$ where the life-cycle λ is such that $\lambda_n=\{◆a,▼a\}$ for every $n\in\mathbb{N}_\circ$. Then:

$<<\varnothing,\{a\}>,\{\lambda\}^{\varnothing}>\models_{<\varnothing,\{a\}>}◆a$
$<<\varnothing,\{a\}>,\{\lambda\}^{\varnothing}>\models_{<\varnothing,\{a\}>}(\mathbf{G}◆a)$
$<<\varnothing,\{a\}>,\{\lambda\}^{\varnothing}>\models_{<\varnothing,\{a\}>}(\mathbf{F}▼a)$

Note that the diligence condition is essential for the satisfaction of (**F**▼a) by $<<\varnothing,\{a\}>,\{\lambda\}^{\varnothing}>$.

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[28] The terminology is borrowed from [Manna and Pnueli 92]. "Diligent" objects allow delays of their actions but not for ever.

The following results show that the relevant constructions (reducts, images and products) respect the closure for delays.

Proposition 7.3.8. Let $\sigma:\Sigma\rightarrow\Sigma'$ be a signature morphism, $\Lambda'\subseteq\Lambda_{\Sigma'}$ and $\Lambda'=(\Lambda')^{\varnothing}$. Then, $\sigma^{-1}(\Lambda')$ is closed for delays.

Proof:

Note first that if $\lambda=\sigma^{-1}(\lambda')$ for some $\lambda'\in\Lambda'$ then $\Delta^k(\lambda)=\sigma^{-1}(\Delta^k(\lambda'))$ and thus $\Delta^k(\lambda)\in\sigma^{-1}(\Lambda')$ since $\Delta^k(\lambda')\in\Lambda'$. Now assume that $\lambda\in\sigma^{-1}(\Lambda')$, $\theta\in\Lambda_\Sigma$ and $\lambda\infty\theta$. We have to show that $\theta\in\sigma^{-1}(\Lambda')$. Since $\lambda\infty\theta$ there is a delay-point sequence η such that for each $n\in\mathbb{N}_\circ$ there is $i\in\mathbb{N}_\circ$ such that $[\theta[_n=[\delta^\eta(\lambda)^i[_n$. Moreover, there is $\lambda'\in\Lambda'$ such that $\lambda=\sigma^{-1}(\lambda')$. Thus, $\delta^\eta(\lambda)^i=\sigma^{-1}(\delta^\eta(\lambda')^i)\in\sigma^{-1}(\Lambda')$ for each $i\in\mathbb{N}_\circ$. Hence, there is $\theta'\in\Lambda_{\Sigma'}$ such that $\theta=\sigma^{-1}(\theta')$. Indeed, otherwise for some $n\in\mathbb{N}_\circ$ there would be no $s'\subseteq A_{\Sigma'}$ such that $\theta_n=\sigma^{-1}(s')$, which is absurd because for each $n\in\mathbb{N}_\circ$ there is $i\in\mathbb{N}_\circ$ such that $\theta_n=(\delta^\eta(\lambda)^i)_n$. Now we show that $\lambda'\infty\theta'$. Consider again the delay-point sequence η. It is straightforward to verify that for each $n\in\mathbb{N}_\circ$ there is $i\in\mathbb{N}_\circ$ such that $[\theta'[_n=[\delta^\eta(\lambda')^i[_n$. Thus $\theta'\in\Lambda'$ and so $\theta\in\sigma^{-1}(\Lambda')$. □

Proposition 7.3.9. Let $\sigma:\Sigma\rightarrow\Sigma'$ be a signature morphism, $\Lambda\subseteq\Lambda_\Sigma$ and $\Lambda=\Lambda^{\varnothing}$. Then, $\sigma(\Lambda)$ is closed for delays.

Proof:

Assume that $\lambda'\in\sigma(\Lambda)$, $\theta'\in\Lambda_{\Sigma'}$ and $\lambda'\infty\theta'$. We have to show that $\theta'\in\sigma(\Lambda)$. Since $\lambda'\infty\theta'$ there is a delay-point sequence η such that for each $n\in\mathbb{N}_\circ$ there is $i\in\mathbb{N}_\circ$ such that $[\theta'[_n=[\delta^\eta(\lambda')^i[_n$. It is straightforward to verify that $\sigma^{-1}(\lambda')\infty\sigma^{-1}(\theta')$ using the delay-point sequence η. Moreover, $\sigma^{-1}(\theta')\in\Lambda$ since $\sigma^{-1}(\lambda')\in\Lambda$. Therefore, $\theta'\in\sigma(\Lambda)$. □

Proposition 7.3.10. Let ob and ob' be objects closed for delays. Then, ob⊗ob' is closed for delays.

Proof:

Let ob=$<\Sigma,\Lambda>$, ob'=$<\Sigma',\Lambda'>$, $\theta\in\Lambda\otimes\Lambda'$ and $\theta\infty\theta'$. Then, using the same delay-point sequence, we obtain $inj_\Sigma{}^{-1}(\theta)\infty inj_\Sigma{}^{-1}(\theta')$ and $inj_{\Sigma'}{}^{-1}(\theta)\infty inj_{\Sigma'}{}^{-1}(\theta')$. □

Observation 7.3.11. Let ob and ob' be objects closed for delays. Then, the set of life-cyles of ob⊗ob' is the set of fair interleavings of life-cycles of ob and ob'.

As an illustration consider the following objects and their product: ob=$<<\varnothing,\{u\}>,\{\lambda\}^{\varnothing}>$, ob'=$<<\varnothing,\{v\}>,\{\xi\}^{\varnothing}>$, where $\lambda_n=\{\blacklozenge u,\blacktriangledown u\}$ and $\xi_n=\{\blacklozenge v,\blacktriangledown v\}$ for each $n\in\mathbb{N}_\circ$. It is easy to verify that any fair interleaving of λ and ξ is a life-cycle of the product ob⊗ob'. By fair interleaving we mean an interleaving of actions u and v where there is an infinite number of u's and also of v's.

If we dropped the diligence condition, we would obtain all interleavings of u's and v's including those life-cycles where action u does not occur or occurs a finite number of times (and similarly for action v). Clearly, the diligence condition ensures fairness in the aggregation (parallel composition) of objects.

It should be obvious that the product of two objects that are not closed for delays may not include the envisaged interleavings. For instance the product of $<\Sigma,\{\lambda\}>$ and $<\Sigma',\{\xi\}>$ contains a unique life-cycle: total synchronization.

7.4. Reducts for linear temporal logic

It is also worthwhile to examine the *reducts* in the category Ob built upon L. We explain how the reduct may be used for "hiding" observations/actions. Moreover, we show with an example that it may lead to "internal non-determinism".

Consider the object ob=<<{x},{c}>,$\{\lambda\}^{\varnothing}$> where λ_n={x,◆c,▼c} for n even and λ_n={◆c,▼c} for n odd. Note that ob is "deterministic" in the sense that when a given event (set of actions) happens if we know the "value" of x (true or false) then we know the next situation: from true to false and from false to true if {c} happens; and from true to true and from false to false if ∅ happens.

Consider now the reduct induced by the hiding signature morphism σ:<{x},∅>→<{x},{c}> that hides the action c. Then, σ^{-1}(ob)=<<{x},∅>,$\{\xi\}^{\varnothing}$> where ξ_n={x} for n even and ξ_n=∅ for n odd. The resulting object is "non-deterministic" since when the event ∅ happens (no action occurs) the value of x may remain unchanged or it may change (for no apparent reason).

In conclusion:

Observation 7.4.1. Reducts used for hiding may introduce internal non-determinism.

7.5. Working with other temporal logics

In some cases it might be worthwhile to work with multilinear temporal logic or a full branching temporal logic. It is up to the specifier the choice of the most convenient base logic. The notion of object is then canonically induced. And the categorial semantics of the object constructs are instantiated to the chosen base institution.

Such non-linear temporal logics are quite powerful. For instance they provide the means for expressing a deep relationship between the enabling and the occurring of an action. Beyond

(▼c ⇒ ◆c)

already expressible in the linear temporal language, we may want to impose

(◆c ⇒ (**M** ▼c))

stating (in the multilinear temporal logic) that if action c is enabled in position n of a given trajectory t then there is a trajectory t' identical to t up to n and with the same observations and enablings at n where c occurs at n. In short, informally, if c is enabled then c happens in an alternative trajectory.

Clearly, an object is not any more a set of life-cycles, but instead a set of sets of life-cycles in the case of a multilinear base or a set of trees in the case of a branching base. Note that in these cases the base institution is not any more "uniframe": different interpretation structures may have different frames, that is, they may differ in the set of worlds and/or in the accessibility relation.

Maybe more interesting would be to consider a base institution that would lead to objects and their morphisms such that "full concurrency" is reflected by products. Starting with the temporal logic proposed in [Reisig 92], the induced notion of object would be a set of concurrent runs and it seems that the envisaged "full concurrency" in products would be achieved.

8. Concluding remarks

We have shown how to set-up the "power model" institution induced by a given temporal institution. Clearly, the proposed construction and its properties do not depend on the fact that the base logic is modal. However, the investigation of the special case is well justified by the envisaged domain of application: denotational semantics of temporal specifications of objects. It remains to be seen if it might be useful in other cases (eg starting with the equational institution).

We have also shown how the resulting institution of objects can be used for providing categorial semantics for the basic object constructs, such as aggregation, interconnection, abstraction and (monotonic) specialization. This semantics is rich enough for analysing liveness properties, external and internal non-determinism, and interleaving even if we adopt a simple linear temporal logic as the base institution. Furthermore, it is conceivable that even full concurrency can be analysed, provided that a suitable temporal logic coping with concurrency (such as the logic proposed in [Reisig 92]) is chosen as the starting point.

Clearly, an interesting line of research concerns the semantics of non-monotonic mechanisms, since these are so pervasive in object-oriented formalisms. Anyway, experience clearly indicates that non-monotonic features should be added to a well understood monotonic framework. We hope that the proposed object institution is suitable for this purpose.

Reification is another important construct to be studied within this context. Obviously, it imposes a more general notion of object morphism: for instance, the reification of actions implies a change of frame.

Another important research issue concerns the semantics of dynamic reconfiguration constructs: objects are created, deleted, dynamically specialized, and their interconnections are changed. It is not clear at this stage how suitable are traditional temporal languages for this purpose. And in any case much work is needed in the semantic front.

We are also working in establishing a suitable categorial link (adjunction) between the proposed denotational semantics of objects and an operational semantics (such as the one described in [Sernadas *et al* 92a]).

Finally, it is worthwhile to note that the proposed construction of the object institution from a given institution of temporal logic is purely set-theoretic. Each object appears as a *set* of models of the base institution. An interesting alternative would be to postulate that an object is a *sheaf* [Goguen 92][29]. Relevant work in this direction is also reported in [Ehrich *et al* 91]. It is simple to develop this idea in the case of a "uniframe" base (institution whose models are all over the same frame, such as the linear temporal logic). Then the unique frame is used to set-up the base category for the envisaged sheaves (objects) and an object morphism is a (traditional) sheaf morphism. The relationships between the set-theoretic construction and the sheaf-theoretic one should be easy to establish. In the case of a "multiframe" base institution the technical challenge

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[29] Or a presheaf if we are to accept as objects agents with unbounded liveness requirements.

is much greater since it is essential to work with sheaves over different bases and an object morphism will be in general a sheaf "cryptomorphism" or "heteromorphism" (after a suitable Grothendieck construction for setting-up the envisaged flat category of multi-base sheaves).

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Appendix

Fiber

Definition A1. Let $F:C \to X$ be a functor and x an element of X. The *fiber* C_x of C over x is the subcategory of C whose elements are mapped by F onto x and whose arrows are mapped onto the identity morphism at x (id_x).

Concrete category

Definition A2. Let X be a category. A *concrete category* over X is a pair (C,U) where C is a category and $U:C \to X$ is a faithful functor.

Definition A3. Let (C,U) be a concrete category over X. We say that a morphism $m:U(c) \to U(c')$ in X *induces* a morphism in C iff there is morphism $h:c \to c'$ in C such that U(h)=m. If there is such a morphism it is unique.

Definition A4. A concrete category (C,U) over X is said to be *amnestic* iff C_x is a partial order for each element x of X.

Definition A5. Let (C,U) and (D,V) be concrete categories over X. A *concrete functor* from (C,U) into (D,V) is a functor $F:C \to D$ such that $U=V \circ F$.

Cofibration (and fibration)

Definition A6. Let $F:C \to X$ be a functor, $m:x \to x'$ a morphism in X and c and element of C such that F(c)=x. A morphism $f:c \to c'$ in C is said to be *cocartesian* by F for m on c iff F(f)=m and for each pair of morphisms $g:c \to c''$ in C and $n:x' \to F(c'')$ in X such that $n \circ m=F(g)$ there is a unique morphism $h:c' \to c''$ in C such that F(h)=n and $h \circ f=g$. We say that c' is the *target* of the cocartesian morphism f. □

Proposition A7. Let $F:C \to X$ be a functor, $m:x \to x'$ a morphism in X and F(c)=x. If f1 and f2 are cocartesian morphisms by F for m on c then their targets are isomorphic.

Proof:
Straightforward: use the couniversal property of f1 on f2 and itself (and vice-versa). □

Proposition A8. Let $F:C \to X$ be a functor, $m1:x \to x1$ and $m2:x1 \to x2$ morphisms in X, and $F(c)=x$. If $f1:c \to c1$ is cocartesian by F for m1 on c and $f2:c1 \to c2$ is cocartesian by F for m2 on c1 then $f2 \circ f1:c \to c2$ is cocartesian by F on $m2 \circ m1$ for c.
Proof:
Straightforward: Assume that $m1:x \to x1$ and $m2:x1 \to x2$ are morphisms in X, $f1:c \to c1$ is the cocartesian morphism by F for m1 on x, and $f2:c1 \to c2$ is the cocartesian morphism by F for m2 on c1, we show that $f2 \circ f1:c \to c2$ is the cocartesian morphism by F for $m2 \circ m1$ on c: let $g:c \to c3$ be any morphism in C and $n2:x2 \to x3$ be any morphism in X such that $F(c3)=x3$ and $n2 \circ (m2 \circ m1)=F(g)$; consider the morphism $n1:x1 \to x3$ such that $n1=n2 \circ m2$; thus, $F(g)=n1 \circ m1$; hence, there is a unique morphism $h1:c1 \to c3$ such that $F(h1)=n1$ and $h1 \circ f1=g$; on the other hand, there is a unique morphism $h2:c2 \to c3$ such that $F(h2)=n2$ and $h2 \circ f2=h1$; finally, we may conclude that there is a unique such that $F(h2)=n2$ and $h2 \circ (f2 \circ f1)=g$. □

Definition A9. A functor $F:C \to X$ is said to be a *cofibration* iff there is a cocartesian morphism by F for m on c, for every morphism $m:x \to x'$ in X and every c of C such that $F(c)=x$. □

Definition A10. Let $F:C \to X$ be a cofibration. A *cocleavage* k for F maps each pair $<m:F(c) \to x',c>$ onto a cocartesian morphism by F for m on c. A cocleavage is said to be a *splitting* of the cofibration iff

$k(id_{F(c)},c)=id_c$;

$k(m' \circ m:F(c) \to x'',c)=k(m':F(c') \to x'',c') \circ k(m:F(c) \to F(c'),c)$. □

Fact/Definition A11. Let $F:C \to X$ be a cofibration. Any splitting cocleavage k for F induces the functors:

- for each $m:x \to x'$ in X, the *transformation functor* on m: $T_m:C_x \to C_{x'}$ such that $T_m(c)$ is the target of the cocartesian morphism $k(<m:F(c) \to x',c>)$, and $T_m(h:c1 \to c2)$ is the unique morphism h' in $C_{x'}$ such that $h' \circ k(<m:F(c1) \to x',c1>)=k(<m:F(c2) \to x',c2>) \circ h$;
- the *indexing functor*: $I:X \to Cat$ such that I(x) is the fiber C_x and $I(m:x \to x')$ is T_m.

Proof:
See pages 255-256 of [Barr and Wells 90]. Note that the splitting property is essential only for establishing the indexing functor. □

Note that if the cofibration functor is $F:C \to X^{op}$, then $T_m:C_{x'} \to C_x$ and $I:X \to Cat^{op}$.

By duality we introduce the notion of *fibration* and obtain the corresponding results.

Concrete isomorphism

Definition A12. Assume that $H:C \to D$ and $K:D \to C$ establish an isomorphism in Cat and that $F:C \to X$ and $G:D \to X$ are functors. The isomorphism is said to be *concrete* with respect to F and G iff $G \circ H=F$ and $F \circ K=G$.

Proposition A13. Assume that $H:C \to D$ and $K:D \to C$ is a concrete isomorphism in Cat with respect to $F:C \to X$ and $G:D \to X$. Then H preserves cocartesian and cartesian morphisms.

Proof:

• Let $m:x \to x'$ be a morphism in X and c and element of C such that $F(c)=x$. Assume that $f:c \to c'$ is cocartesian by F for $m:x \to x'$ on c. We want to show that $H(f):H(c) \to H(c')$ is cocartesian by G for $m:x \to x'$ on $H(c)$. Clearly, $G(H(f))=m$. We have to verify that this morphism satisfies the envisaged couniversal property. Let $g:H(c) \to d''$ be a morphism in D, and $n:x' \to G(d'')$ a morphism in X such that $n \circ m=G(g)$. Consider the morphism $K(g):c \to K(d'')$ in C. Clearly, $n \circ m=F(K(g))=G(g)$. By the couniversal property of $f:c \to c'$, there is a unique morphism $h:c' \to K(d'')$ such that $h \circ f=K(g)$. Consider the morphism $H(h):H(c') \to d''$ in D. Then, $H(h) \circ H(f)=H(h \circ f)=H(K(g))=g$ and, clearly, $G(H(h))=n$. Moreover, $H(h)$ is the unique morphism such that $G(H(h))=n$ and $H(h) \circ H(f)=g$. Otherwise, we would be able to violate the couniversal property of $f:c \to c'$ mapping back via K.

• Preservation of cartesian morphisms follows by duality. □

Galois correspondence

Definition A14. Let (C,U) and (D,V) be concrete categories over X and $H:C \to D$ and $K:D \to C$ concrete functors over X. The pair <H,K> is said to be a *Galois correspondence* iff, for any c of C and d of D, a morphism $f:U(c) \to V(d)$ in X induces a morphism in C from c into K(d) iff f induces a morphism in D from H(c) into d. □

Definition A15. Let $H:C \to D$ be a functor. We denote by $H(|C|)$ the full subcategory of D containing all elements of D in the range of H.

Proposition A16. Let (C,U) and (D,V) be amnestic concrete categories over X and $\langle H:C \to D, K:D \to C\rangle$ a Galois correspondence. The restrictions of H and K to $H(|C|)$ and $K(|D|)$, respectively, are concrete isomorphisms: $H^*:H(|C|) \to K(|D|)$ and $K^*:K(|D|) \to H(|C|)$ are inverse to each other.

Proof:

See page 83 of [Adamek *et al* 90]. □

References

[Adámek *et al* 90]
J.Adámek, H.Herrlich and G.Strecker, *Abstract and Concrete Categories*, Wiley, 1990.

[Asperti and Longo 91]
A.Asperti and G.Longo, *Categories, Types, and Structures*, MIT Press, 1991.

[Barr and Wells 90]
M.Barr and C.Wells, *Category Theory for Computing Science*, Prentice-Hall, 1990.

[Barringer *et al* 84]
H.Barringer, R.Kuiper and A.Pnueli, "Now You May Compose Temporal Logic Specifications", in *Procs. of 16th ACM Symp. on Theory of Computing*, 1984, 51-63.

[Ehrich *et al* 91]

H.-D.Ehrich, J.A.Goguen and A.Sernadas, "A Categorial Theory of Objects as Observed Processes", in J.deBakker, W.-P.deRoever and G.Rozenberg (eds), *Foundations of Object-oriented Languages*, Springer-Verlag, 1991, 203-228.

[Emerson 90]

E.Emerson, "Temporal and Modal Logic", in J.Leeuwen (ed), *Handbook of Theoretical Computer Science VOL B*, 1990, 995-1072.

[Fiadeiro *et al* 93]

J.Fiadeiro, J.F.Costa, A.Sernadas and T.Maibaum, "Object Semantics of Temporal Logic Specification", in M.Bidoit and C.Choppy (eds), *Recent Trends in Data Type Specification: 8th Workshop on Specification of Abstract Data Types - Selected Papers*, Springer-Verlag, 1993, 236-253.

[Goguen 92]

J.A.Goguen, "Sheaf Semantics for Concurrent Interacting Objects", *Mathematical Structures in Computer Science*, 2(2), 1992, 159-191.

[Goguen and Burstall 84]

J.A.Goguen and R.M.Burstall, "Introducing Institutions", in E.Clarke (ed), *Logics of Programming Workshop*, Springer-Verlag, 1984, 221-256.

[Goguen and Burstall 92]

J.A.Goguen and R.M.Burstall, "Institutions: Abstract Model Theory for Specification and Programming", *Journal of the Association for Computer Machinery*, 39(1), 1992, 95-146.

[Goldblatt 87]

R.Goldblatt, *Logics of Time and Computation*, CSLI, 1987.

[Manna and Pnueli 92]

Z.Manna and A.Pnueli, *The Temporal Logic of Reactive and Concurrent Systems*, Springer-Verlag, 1992.

[Meyer 88]

B.Meyer, *Object-Oriented Software Construction*, Prentice-Hall, 1988.

[Pnueli 77]

A.Pnueli, "The Temporal Logic of Programs", in *Procs. of the 18th IEEE Symp. on Foundations of Computer Science*, 1977, 46-57.

[Poigné 92]

A.Poigné, "Basic Category Theory", in S.Abramsky, D.Gabbay and T.Maibaum (eds), *Handbook of Logic in Computer Science VOL 1*, Oxford University Press, 1992, 413-640.

[Reisig 92]

W.Reisig, "Elements of a Temporal Logic Coping with Concurrency", TUM, Technical Report 342/23/92, 1992.

[Sernadas 80]

A.Sernadas, "Temporal Aspects of Logical Procedure Definition", *Information Systems*, 5, 1980, 167-187.

[Sernadas and Ehrich 91]

A.Sernadas and H.-D.Ehrich, "What is an Object, After All?", in R.Meersman, W.Kent and S.Khosla (eds), *Object Oriented Databases: Analysis, Design and Construction*, North-Holland, 1991, 39-69.

[Sernadas *et al* 89a]

A.Sernadas, J.Fiadeiro, C.Sernadas and H.-D.-Ehrich, "Basic Building Blocks of Information Systems", in E.Falkenberg and P.Lindgreen (eds), *Information System Concepts: An In-depth Analysis*, North-Holland, 1989, 225-246.

[Sernadas *et al* 89b]

A.Sernadas, J.Fiadeiro, C.Sernadas and H.-D.-Ehrich, "Abstract Object Types: A Temporal Perspective", in B.Banieqbal, H.Barringer and A.Pnueli (eds), *Temporal Logic in Specification*, Springer-Verlag, 1989, 324-350.

[Sernadas *et al* 92a]

A.Sernadas, J.F.Costa and C.Sernadas, "An Institution of Object Behaviour", ADT-COMPASS'92, to be published.

[Sernadas *et al* 92b]

A.Sernadas, C.Sernadas and J.F.Costa, "Object Specification Logic", INESC/DMIST Research Report, June 1992, submitted.

[Tarlecki *et al* 89]

A.Tarlecki, R.M.Burstall and J.A.Goguen, "Indexed Categories as a Tool for the Semantics of Computation", Oxford University Computing Laboratory, Technical Monograph PRG-77, 1989.

[Wegner 92]

P.Wegner, "Dimensions of Object-oriented Modeling", *IEEE Computer*, 1992, 12-21.