Non-deterministic combination of connectives

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Abstract

Combined connectives arise when combining logics [12] and are also useful for analyzing the common properties of two connectives within a given logic [11]. A non-deterministic semantics and a Hilbert calculus are proposed for the meet-combination of connectives (and other language constructors) of any matrix logic endowed with a Hilbert calculus. The logic enriched with such combined connectives is shown to be a conservative extension of the original logic. It is also proved that both soundness and completeness are preserved. Illustrations are provided for classical propositional logic.

Keywords: combined connectives, non-deterministic logic.

1 Introduction

When combining logics one frequently wants to impose some interaction between connectives.[1] For instance, in the logic $\mathcal{L}$ resulting from the fibring [6] of any two given logics $\mathcal{L}_1$ and $\mathcal{L}_2$ (where one finds all the inference rules from those two logics), if one imposes the sharing of two constructors c_1 and c_2 with the same arity, the resulting *shared constructor* $\langle c_1 c_2 \rangle$ in $\mathcal{L}$ enjoys the logical properties inherited from c_1 together with those inherited from c_2. Such a sharing may easily lead to inconsistency.

One may wonder if it would not be better instead to endow the combined constructor only with the *common* logical properties of the component constructors. As we conjectured in [11], one should expect to be led to a different way of combining logics, at least with respect to the behavior of the combined constructors, hopefully avoiding inconsistency in more situations. This expectancy was fully vindicated in [12].

Herein, we go back to the problem addressed in [11]: provide the means for reasoning about the common properties of any two connectives in a given logic. Given a logic $\mathcal{L}$, an enriched logic $\mathcal{L}^{\times}$ was proposed in [11] for deriving

[1]Connectives are to be taken here in a general sense, including, besides the propositional connectives, other language constructors like modal operators.

such common properties by dealing with ordered pairs of connectives as language constructors. For instance, for reasoning about the common properties of conjunction and disjunction one can analyze in $\mathcal{L}^\times$ the properties either of the combined connective $\lceil\wedge\vee\rceil$ or of $\lceil\vee\wedge\rceil$. Although these two combined connectives were shown to be equivalent to some extent in $\mathcal{L}^\times$, it would be nice to avoid the product semantics given in [11] to such combined connectives.

By adopting instead a non-deterministic semantics for the combination of connectives, we manage herein to collapse $\lceil\wedge\vee\rceil$ and $\lceil\vee\wedge\rceil$, as envisaged. A calculus is also proposed. The resulting logic $\mathcal{L}^\sqcap$ is shown to be sound and complete, as long as the original logic $\mathcal{L}$ is sound and complete. It is also proved that, as envisaged, $\mathcal{L}^\sqcap$ is a conservative extension of $\mathcal{L}$.

For the sake of simplicity, we assume that the given logic $\mathcal{L}$ is of a propositional nature.[2] Furthermore, we assume that $\mathcal{L}$ is endowed with a Hilbert calculus and a matrix semantics. It should be stressed that the proposed logic $\mathcal{L}^\sqcap$ is presented with a Hilbert calculus, but its non-deterministic semantics is outside the universe of matrix semantics.

In Section 2, after recalling the adopted notion of matrix logic with Hilbert calculus, we show how to enrich any such given logic with non-deterministic combinations of constructors. The main results on the enriched logic are established in Section 3. Illustrations are provided for classical propositional logic in Section 4, including a comparison with the results in [11]. Finally, in Section 5 we assess what was achieved and what still lies ahead.

2 Meet-combining constructors

For the purpose of this paper, by a *suitable logic* we mean a triple $\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$ where

- The *signature* Σ is a family $\{\Sigma_n\}_{n\in\mathbb{N}}$ with each Σ_n being a finite set of n-ary language *constructors*. Formulas are built as usual with these constructors. We use L for denoting the set of *formulas*.

- The Hilbert *calculus* Δ is a set of finitary rules of the form

$$\frac{\alpha_1, \ldots, \alpha_m}{\beta}$$

  where formulas $\alpha_1, \ldots, \alpha_m$ are said to be the *premises* of the rule and formula β is said to be its *conclusion*. As expected, the conclusion of a rule without premises is said to be an *axiom*. Derivations are defined as usual for Hilbert calculi. We write

$$\Gamma \vdash \varphi$$

  for stating that there is a derivation of formula φ from set Γ of hypotheses.

[2] That is, we address only logics without variables and binding operators.

- The matrix *semantics* $\mathcal{M}$ is a class of matrices over Σ. Recall that a matrix over Σ is a pair $M = (\mathfrak{A}, D)$ where

  $$\mathfrak{A} = (A, \{\underline{c} : A^n \to A \mid c \in \Sigma_n\}_{n \in \mathbb{N}})$$

  is an algebra over Σ and $D \subseteq A$. The elements of A are known as *truth values* and those of D are the *distinguished* or *designated* ones. Denotation, satisfaction and entailment are as expected for matrix semantics. We write

  $$[\![\varphi]\!]_M$$

  for the denotation of formula φ given by the algebra of matrix M, and we say that M *satisfies* φ, written

  $$M \Vdash \varphi,$$

  if $[\![\varphi]\!]_M \in D$. As usual, given a set Φ of formulas, we write $M \Vdash \Phi$ if $M \Vdash \varphi$ for every $\varphi \in \Phi$. Set Γ of formulas *entails* formula φ, written

  $$\Gamma \vDash \varphi,$$

  if $M \Vdash \varphi$ whenever $M \Vdash \Gamma$ for every $M \in \mathcal{M}$.

In short, by a suitable logic we mean a logic of propositional nature, endowed with a Hilbert calculus and a matrix semantics.

Given such a suitable logic $\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$, the objective now is to define a logic

$$\mathcal{L}^{\sqcap} = (\Sigma^{\sqcap}, \Delta^{\sqcap}, \mathcal{M}^{\sqcap})$$

where, without disturbing the original properties of the constructors in $\mathcal{L}$, one can also reason with and about the envisaged meet-combinations of constructors in the original logic $\mathcal{L}$ by adopting a non-deterministic semantics for those constructors.

Language

The signature $\Sigma^{\sqcap}$ is composed of all possible meet-combinations of constructors in Σ. More concretely,

$$\Sigma^{\sqcap} = \{\Sigma^{\sqcap}_n\}_{n \in \mathbb{N}}$$

with

$$\Sigma^{\sqcap}_n = \{\lceil c_1 c_2 \rceil \mid c_1, c_2 \in \Sigma_n\}.$$

For any constructors $c_1, c_2 \in \Sigma$ of the same arity, $\lceil c_1 c_2 \rceil$ is said to be their meet-combination. Moreover, c_1 and c_2 are said to be the *components* of $\lceil c_1 c_2 \rceil$. Since this paper addresses only such meet-combined constructors, from now on we refer to them simply as combined constructors. As expected, we use $L^{\sqcap}$ for denoting the set of formulas of $\mathcal{L}^{\sqcap}$. Examples are delayed until Section 4.

We look at $\Sigma^{\sqcap}$ as an enrichment of Σ via the embedding

$$\eta : c \mapsto \lceil cc \rceil.$$

Furthermore, in the context of $\Sigma^\sqcap$ we may write c for $\lceil cc \rceil$. In due course, this notational shortcut will be fully vindicated. Accordingly, we take that $L \subset L^\sqcap$.

Given a formula φ over Σ and a formula ψ over $\Sigma^\sqcap$, we say that φ is a *determination* of ψ if $\varphi \sqsubseteq \psi$ where the binary relation $\sqsubseteq \subseteq L \times L^\sqcap$ is inductively defined as follows:

- $c_k \sqsubseteq \lceil c_1 c_2 \rceil$, for each $k = 1, 2$ and $c_1, c_2 \in \Sigma_0$;
- $c_k(\varphi_1, \ldots, \varphi_n) \sqsubseteq \lceil c_1 c_2 \rceil(\psi_1, \ldots, \psi_n)$ whenever $\varphi_1 \sqsubseteq \psi_1, \ldots, \varphi_n \sqsubseteq \psi_n$, for each $k = 1, 2$, $n \in \mathbb{N}^+$ and $c_1, c_2 \in \Sigma_n$.

In short, $\varphi \sqsubseteq \psi$ if φ is obtained by choosing one component of each occurrence of a combined connective in ψ. It should be stressed that, when a combined connective occurs more than once, those choices need not be the same. Observe that $c \sqsubseteq c$ since $c \sqsubseteq \lceil cc \rceil$ which justifies our notation for determination.

Clearly, the determination relation extends the componentship relation to formulas. This extension is needed for setting up in due course the calculus $\Delta^\sqcap$.

For defining the envisaged non-deterministic semantics, it becomes handy to denote by $|\psi|$ the *number of constructors* in a given formula $\psi \in L^\sqcap$. The map $\psi \mapsto |\psi|$ is inductively defined as expected:

- $|\lceil c_1 c_2 \rceil| = 1$, for each $c_1, c_2 \in \Sigma_0$;
- $|\lceil c_1 c_2 \rceil(\psi_1, \ldots, \psi_n)| = 1 + |\psi_1| + \cdots + |\psi_n|$, for each $n \in \mathbb{N}^+$, $c_1, c_2 \in \Sigma_n$ and $\psi_1, \ldots, \psi_n \in L^\sqcap$.

This notation is extended to finite sequences of formulas as follows:

$$|\psi_1 \ldots \psi_n| = |\psi_1| + \cdots + |\psi_n|.$$

Non-deterministic semantics

The non-deterministic semantics $\mathcal{M}^\sqcap$ is not matricial. Each interpretation in $\mathcal{M}^\sqcap$ is a matrix in $\mathcal{M}$ enriched with a choice stream. More concretely, $\mathcal{M}^\sqcap$ is the following class of interpretations

$$\{(M, \omega) \mid M \in \mathcal{M},\ \omega \in \{1, 2\}^{\mathbb{N}}\}.$$

In each interpretation, the matrix M provides the meaning of the original constructors in Σ and the *choice sequence* ω determines the meaning of the combined connectives.

To this end, we need some notation concerning segments of infinite sequences. Given $\omega \in \{1, 2\}^{\mathbb{N}}$, for each $k \in \mathbb{N}$, let

$$\omega_{k:\infty}$$

denote the infinite sequence $\omega_k \omega_{k+1} \ldots$ (the tail of ω starting at element k). Later on, given $k_1 \leq k_2$, we shall also need to write

$$\omega_{k_1:k_2}$$

for denoting the finite sequence $\omega_{k_1} \ldots \omega_{k_2}$ (the segment of ω from element k_1 to element k_2 inclusive).

The *denotation* of a formula given by (M, ω) is inductively defined as follows:

- $[\![\lceil c_1 c_2 \rceil]\!]^{\sqcap}_{M\omega} = \underline{c_{\omega_0}}$;
- $[\![\lceil c_1 c_2 \rceil(\psi_1, \ldots, \psi_n)]\!]^{\sqcap}_{M\omega} = \underline{c_{\omega_0}}([\![\psi_1]\!]^{\sqcap}_{M\omega_{1:\infty}}, \ldots, [\![\psi_n]\!]^{\sqcap}_{M\omega_{1+|\psi_1\cdots\psi_{n-1}|:\infty}})$.

Clearly, the semantics of $\mathcal{L}^{\sqcap}$ is not homomorphic, since the denotation of a combined connective may vary if used more than once in a formula.

Satisfaction is defined as expected. We say that (M, ω) *satisfies* ψ, written

$$M\omega \Vdash^{\sqcap} \psi,$$

if $[\![\psi]\!]^{\sqcap}_{M\omega}$ is a distinguished value of M. Furthermore, we say that M *satisfies* ψ, written

$$M \Vdash^{\sqcap} \psi,$$

if $M\omega \Vdash^{\sqcap} \psi$ for every ω. As usual, we may write $M \Vdash^{\sqcap} \Psi$ for asserting that $M \Vdash^{\sqcap} \psi$ for every $\psi \in \Psi$.

Entailment is made to be global over choices as follows. We say that Θ entails ψ, written

$$\Theta \vDash^{\sqcap} \psi,$$

if, for every M,

$$M \Vdash^{\sqcap} \psi \text{ whenever } M \Vdash^{\sqcap} \Theta.$$

Examples are delayed until Section 4. In due course (in Section 3), we prove that, as envisaged, $\vDash^{\sqcap}$ extends $\vDash$.

Calculus

The Hilbert calculus $\Delta^{\sqcap}$ is an enrichment of Δ (via the embedding $c \mapsto \lceil cc \rceil$), including, besides the rules inherited from Δ, the following rules for dealing with the combined constructors.

For each formula $\psi \in L^{\sqcap}$, the *lifting rule* (in short LFT)

$$\frac{\varphi \quad \text{for each } \varphi \sqsubseteq \psi}{\psi}.$$

This rule is motivated by the idea that each $\lceil c_1 c_2 \rceil$ inherits at least the common properties of c_1 and c_2. Accordingly, ψ should hold whenever all of its determinations hold.

For each formula $\psi \in L^{\sqcap}$, the *co-lifting rule* (in short cLFT)

$$\frac{\psi}{\varphi} \quad \text{provided that } \varphi \sqsubseteq \psi.$$

This rule is motivated by the idea that each $\lceil c_1 c_2 \rceil$ should enjoy at most the common properties of c_1 and c_2. In fact, this rule guarantees more. It guarantees that $\lceil c_1 c_2 \rceil$ enjoys at most the common *original* properties of c_1 and c_2 because, in due course, we show that $\mathcal{L}^{\sqcap}$ is a conservative extension of $\mathcal{L}$.

In the sequel, we write

$$\Theta \vdash^{\sqcap} \psi$$

for stating that ψ is derivable from Θ within $\mathcal{L}^{\sqcap}$. Examples are delayed until Section 4. The following result shows that, as intended, $\vdash^{\sqcap}$ extends $\vdash$.

Proposition 2.1 Let $\Gamma \cup \{\varphi\} \subset L$. Then $\Gamma \vdash^{\sqcap} \varphi$ whenever $\Gamma \vdash \varphi$.

Proof: Taking into account that $\mathcal{L}^{\sqcap}$ contains all the rules of $\mathcal{L}$, the proof is a trivial induction on the length of the derivation of φ from Γ within $\mathcal{L}$. QED

3 Main results

First, assuming that the suitable logic $\mathcal{L}$ is (strongly) sound, we establish the (strong) soundness of $\mathcal{L}^{\sqcap}$. To this end, we need to introduce some notation and prove the relevant technical lemmas, including the fact that $\vDash^{\sqcap}$ extends $\vDash$.

Given a formula ψ of $\mathcal{L}^{\sqcap}$ and a choice sequence ω, we denote by

$$\psi^{\omega}$$

the formula of $\mathcal{L}$ resulting from determining ψ according to ω. This determination map $\psi, \omega \mapsto \psi^{\omega}$ is inductively defined as follows:

- $\lceil c_1 c_2 \rceil^{\omega} = c_{\omega_0}$, for each $c_1, c_2 \in \Sigma_0$;
- $\lceil c_1 c_2 \rceil(\psi_1, \ldots, \psi_n)^{\omega} = c_{\omega_0}(\psi_1^{\omega_{1:\infty}}, \ldots, \psi_n^{\omega_{1+|\psi_1 \cdots \psi_{n-1}|:\infty}})$, for each $n \in \mathbb{N}^+$, $c_1, c_2 \in \Sigma_n$ and $\psi_1, \ldots, \psi_n \in L^{\sqcap}$.

The dual of this notion provides for each given $\varphi \sqsubseteq \psi$ the set

$$\Omega_{\varphi}^{\psi}$$

composed of the choice sequences specified by φ in ψ. This specification map $\varphi, \psi \mapsto \Omega_{\varphi}^{\psi}$ is inductively defined as follows:

- $\Omega_{c_k}^{\lceil c_1 c_2 \rceil} = \{k\omega \mid \omega \in \{1,2\}^{\mathbb{N}}\}$, for each $c_1, c_2 \in \Sigma_0$ and $k = 1, 2$;
- $\Omega_{c_k(\varphi_1,\ldots,\varphi_n)}^{\lceil c_1 c_2 \rceil(\psi_1,\ldots,\psi_n)}$ is the set

  $$\{k\omega^1_{0:|\varphi_1|-1} \cdots \omega^n_{0:|\varphi_n|-1}\omega \mid \omega^j \in \Omega_{\varphi_j}^{\psi_j}, j = 1, \ldots, n \text{ and } \omega \in \{1,2\}^{\mathbb{N}}\},$$

  for each
  $n \in \mathbb{N}^+$, $c_1, c_2 \in \Sigma_n$, $\psi_1, \ldots, \psi_n \in L^{\sqcap}$, $k = 1, 2$ and $\varphi_1 \sqsubseteq \psi_1, \ldots, \varphi_n \sqsubseteq \psi_n$.

Proposition 3.1 Let $\psi \in L^{\sqcap}$, $\varphi \in L$ and $\omega \in \{1,2\}^{\mathbb{N}}$. Then

$$\psi^{\omega} \sqsubseteq \psi \quad \text{and} \quad \varphi^{\omega} = \varphi.$$

Proof: The theses are established by straightforward inductions. QED

Proposition 3.2 Let $\psi \in L^{\sqcap}$, $\omega \in \{1,2\}^{\mathbb{N}}$ and $M \in \mathcal{M}$. Then

$$[\![\psi^{\omega}]\!]_M = [\![\psi]\!]^{\sqcap}_{M\omega}.$$

Moreover,

$$M \Vdash \psi^{\omega} \text{ iff } M\omega \Vdash^{\sqcap} \psi.$$

Proof: The proof of the first assertion follows by induction on the structure of ψ.
(1) ψ is $\lceil c_1 c_2 \rceil$. Then

$$[\![\lceil c_1 c_2 \rceil^\omega]\!]_M = [\![c_{\omega_0}]\!]_M = \underline{c_{\omega_0}} = [\![\lceil c_1 c_2 \rceil]\!]^\sqcap_{M\omega}.$$

(2) ψ is $\lceil c_1 c_2 \rceil(\psi_1, \ldots, \psi_n)$. Then

$$\begin{array}{rcl} [\![\lceil c_1 c_2 \rceil(\psi_1, \ldots, \psi_n)^\omega]\!]_M & = & [\![c_{\omega_0}(\psi_1^{\omega_{1:\infty}}, \ldots, \psi_n^{\omega_{1+|\psi_1 \ldots \psi_{n-1}|:\infty}})]\!]_M \\ & = & \underline{c_{\omega_0}}([\![\psi_1^{\omega_{1:\infty}}]\!]_M, \ldots, [\![\psi_n^{\omega_{1+|\psi_1 \ldots \psi_{n-1}|:\infty}}]\!]_M) \\ & = & \underline{c_{\omega_0}}([\![\psi_1]\!]^\sqcap_{M\omega_{1:\infty}}, \ldots, [\![\psi_n]\!]^\sqcap_{M\omega_{1+|\psi_1 \ldots \psi_{n-1}|:\infty}}) \\ & = & [\![\lceil c_1 c_2 \rceil(\psi_1, \ldots, \psi_n)]\!]^\sqcap_{M\omega}. \end{array}$$

With respect to the second assertion,

$$M \Vdash \psi^\omega \;\text{ iff }\; [\![\psi^\omega]\!]_M \in D \;\text{ iff }\; [\![\psi]\!]^\sqcap_{M\omega} \in D \;\text{ iff }\; M\omega \Vdash^\sqcap \psi.$$

QED

Proposition 3.3 Let $\Gamma \cup \{\varphi\} \subset L$. Then $\Gamma \vDash^\sqcap \varphi$ whenever $\Gamma \vDash \varphi$.

Proof: Let $M \in \mathcal{M}$ such that $M \Vdash^\sqcap \Gamma$. We have to show that $M \Vdash^\sqcap \varphi$, that is, $M\omega \Vdash^\sqcap \varphi$ for every choice sequence ω. Indeed, for an arbitrary choice sequence ω, $M\omega \Vdash^\sqcap \Gamma$ and, so, by Proposition 3.2, $M \Vdash \{\gamma^\omega \mid \gamma \in \Gamma\}$. Thus, by Proposition 3.1, $M \Vdash \Gamma$ and so, since $\Gamma \vDash \varphi$, $M \Vdash \varphi$. Therefore, by Proposition 3.1, $M \Vdash \varphi^\omega$ and so, by Proposition 3.2, $M\omega \Vdash^\sqcap \varphi$. QED

Proposition 3.4 A sound rule in $\mathcal{L}$ is sound in $\mathcal{L}^\sqcap$.

Proof: Consider a sound rule in Δ with premises $\alpha_1, \ldots, \alpha_m$ and conclusion β. Then, $\alpha_1, \ldots, \alpha_m \vDash \beta$ and, so, by Propostion 3.3, $\alpha_1, \ldots, \alpha_m \vDash^\sqcap \beta$. QED

Proposition 3.5 The lifting rule is sound in $\mathcal{L}^\sqcap$.

Proof: We must show that

$$\{\varphi \mid \varphi \sqsubseteq \psi\} \vDash^\sqcap \psi.$$

Let $M \in \mathcal{M}$. Assume that $M \Vdash^\sqcap \{\varphi \mid \varphi \sqsubseteq \psi\}$. Let $\omega \in \{1, 2\}^\mathbb{N}$ be a choice sequence. Then $M \Vdash^\sqcap \psi^\omega$ since $\psi^\omega \sqsubseteq \psi$ by Proposition 3.1. Hence, $M\omega' \Vdash^\sqcap \psi^\omega$ for every $\omega' \in \{1, 2\}^\mathbb{N}$. Therefore, by Proposition 3.2, $M \Vdash (\psi^\omega)^{\omega'}$ for every $\omega' \in \{1, 2\}^\mathbb{N}$. Thus, by Proposition 3.1, $M \Vdash \psi^\omega$. So, by Proposition 3.2, $M\omega \Vdash^\sqcap \psi$. QED

Proposition 3.6 Let $\psi \in L^\sqcap$, $\varphi \in L$ and $\omega, \omega' \in \{1, 2\}^\mathbb{N}$. Assume that $\varphi \sqsubseteq \psi$, $\omega \in \Omega^\psi_\varphi$ and

$$\omega_{0:|\varphi|-1} = \omega'_{0:|\varphi|-1}.$$

Then $\omega' \in \Omega^\psi_\varphi$.

Proof: The proof follows by case analysis.
(1) ψ is $\lceil c_1 c_2 \rceil$. Then $|\psi| = 1$. Since $\omega \in \Omega^{\psi}_{\varphi}$ we conclude that $\varphi = c_{\omega_0}$. On the other hand, $\omega'_0 = \omega_0$ and so $\varphi = c_{\omega'_0}$. Thus, $\omega' \in \Omega^{\psi}_{\varphi}$.
(2) ψ is $\lceil c_1 c_2 \rceil(\psi_1, \dots, \psi_n)$. Then $|\lceil c_1 c_2 \rceil(\psi_1, \dots, \psi_n)| = 1 + |\psi_1| + \dots + |\psi_n|$. Assume that φ is $c_k(\varphi_1, \dots, \varphi_n)$ where $\varphi_j \sqsubseteq \psi_j$ for $j = 1, \dots, n$. Since $\omega \in \Omega^{\psi}_{\varphi}$ we conclude that ω is of the form $k\omega^1_{0:|\varphi_1|-1} \cdots \omega^n_{0:|\varphi_n|-1}\omega''$ where $\omega^j \in \Omega^{\psi_j}_{\varphi_j}$ for $j = 1, \dots, n$ and $\omega'' \in \{1,2\}^{\mathbb{N}}$. Therefore $\omega'_{0:|\varphi|-1} = k\omega^1_{0:|\varphi_1|-1} \cdots \omega^n_{0:|\varphi_n|-1}$ and so $\omega' \in \Omega^{\psi}_{\varphi}$. QED

Proposition 3.7 Let $\psi \in L^{\sqcap}$ and $\varphi \in L$ be such that $\varphi \sqsubseteq \psi$. Then

$$\psi^{\omega} = \varphi$$

for every $\omega \in \Omega^{\psi}_{\varphi}$.

Proof: The proof follows by induction on ψ.
(1) ψ is $\lceil c_1 c_2 \rceil$. Then φ is c_k for some $k = 1, 2$. Let $\omega \in \Omega^{\psi}_{\varphi}$. Hence $\omega_0 = k$. Moreover, by definition, $\psi^{\omega} = c_{\omega_0}$. Thus, $\psi^{\omega} = c_k = \varphi$.
(2) ψ is $\lceil c_1 c_2 \rceil(\psi_1, \dots, \psi_n)$. Let φ be $c_k(\varphi_1, \dots, \varphi_n)$ where $k \in \{1,2\}$ and $\varphi_j \sqsubseteq \psi_j$ for $j = 1, \dots, n$. Let $\omega \in \Omega^{\psi}_{\varphi}$. Then ω is of the form $k\omega^1_{0:|\varphi_1|-1} \cdots \omega^n_{0:|\varphi_n|-1}\omega'$ where $\omega^j \in \Omega^{\psi_j}_{\varphi_j}$ for $j = 1, \dots, n$ and $\omega' \in \{1,2\}^{\mathbb{N}}$. Therefore, by the induction hypothesis, $\psi_j^{\omega^j} = \varphi_j$ for $j = 1, \dots, n$. The thesis follows since ψ^{ω} is

$$c_k(\psi_1^{\omega^1_{0:|\varphi_1|-1} \cdots \omega^n_{0:|\varphi_n|-1}\omega'}, \dots, \psi_n^{\omega^n_{0:|\varphi_n|-1}\omega'})$$

and $\psi_j^{\omega^j_{0:|\varphi_j|-1} \cdots \omega^n_{0:|\varphi_n|-1}\omega'}$ is $\psi_j^{\omega^j} = \varphi_j$ for $j = 1, \dots, n$. QED

Proposition 3.8 The co-lifting rule is sound in $\mathcal{L}^{\sqcap}$.

Proof: We must prove that

$$\psi \vDash^{\sqcap} \varphi$$

for every $\varphi \sqsubseteq \psi$. Let $M \in \mathcal{M}$. Assume that $M \Vdash^{\sqcap} \psi$. Let φ be such that $\varphi \sqsubseteq \psi$ and $\omega \in \{1,2\}^{\mathbb{N}}$. Consider $\omega' \in \Omega^{\psi}_{\varphi}$. Then, $M\omega' \Vdash^{\sqcap} \psi$ since $M \Vdash^{\sqcap} \psi$. So, by Proposition 3.2, $M \Vdash \psi^{\omega'}$. Hence, by Proposition 3.7. $M \Vdash \varphi$. Using Proposition 3.1, $M \Vdash \varphi^{\omega}$ and so, again by Proposition 3.2, $M\omega \Vdash^{\sqcap} \varphi$. QED

Theorem 3.9 (Soundness)
If $\mathcal{L}$ is sound then $\mathcal{L}^{\sqcap}$ is sound.

Proof: Assume that $\mathcal{L}$ is sound. Then, in particular, the rules in Δ are sound in $\mathcal{L}$ and so, by Proposition 3.4, all the rules in Δ are sound in $\mathcal{L}^{\sqcap}$. Moreover, the rules LFT and cLFT are sound thanks to Proposition 3.5 and Proposition 3.8, respectively. QED

With the technical lemmas established towards the soundness result, it is also possible to show that, as envisaged, the combined connectives $\lceil c_1c_2 \rceil$ and $\lceil c_2c_1 \rceil$ are not distinguished at all by the proposed non-deterministic semantics, a major improvement over the limited interchangeability result obtained in [11]. It is in addition possible to prove that, as required, $\mathcal{L}^\sqcap$ is a conservative extension of $\mathcal{L}$.

Theorem 3.10 (Model-theoretic interchangeability)
Let $\psi, \psi' \in L^\sqcap$ be such that ψ' is obtained from ψ by replacing zero or more occurrences of any combined connective $\lceil c_1c_2 \rceil$ by $\lceil c_2c_1 \rceil$. Then

$$\psi \vDash^\sqcap \psi'.$$

Proof: Let $M \in \mathcal{M}$. Assume that $M \Vdash^\sqcap \psi$. Let $\omega \in \{1,2\}^\mathbb{N}$ and $\omega' \in \Omega^{\psi}_{\psi'\omega}$. Then $M\omega' \Vdash^\sqcap \psi$ and so, by Proposition 3.2, $M \Vdash \psi^{\omega'}$. Hence, by Proposition 3.7, $M \Vdash \psi'^{\omega}$. Again, by Proposition 3.2, $M\omega \Vdash^\sqcap \psi'$. QED

Theorem 3.11 (Model-theoretic conservativeness)
For every $\Gamma \cup \{\varphi\} \subset L$,

$$\Gamma \vDash \varphi \text{ whenever } \Gamma \vDash^\sqcap \varphi.$$

Proof: Assume that $\Gamma \vDash^\sqcap \varphi$ and let $M \in \mathcal{M}$ such that $M \Vdash \Gamma$. Let $\omega \in \{1,2\}^\mathbb{N}$. Then, by Proposition 3.1, $M \Vdash \{\gamma^\omega \mid \gamma \in \Gamma\}$. Therefore, by Proposition 3.2, $M\omega \Vdash^\sqcap \Gamma$ and so $M \Vdash^\sqcap \Gamma$. Thus, by hypothesis, $M \Vdash^\sqcap \varphi$. Let $\omega' \in \{1,2\}^\mathbb{N}$. Then $M\omega' \Vdash^\sqcap \varphi$ and, by Proposition 3.2, $M \Vdash \varphi^{\omega'}$. The thesis follows by Proposition 3.1. QED

As an immediate corollary of the conservativeness of the proposed enrichment of $\mathcal{L}$, we obtain the following result on the consistency of $\mathcal{L}^\sqcap$.

Theorem 3.12 (Model-theoretic consistency)
If there is $\varphi \in L$ such that $\nvDash \varphi$, then there is $\psi \in L^\sqcap$ such that $\nvDash^\sqcap \psi$.

Proof: Thanks to Theorem 3.11, just take ψ to be such φ. QED

Observe that if $\mathcal{L}$ is sound and complete, the preservation of soundness allows us to carry over the conservativeness and the consistency results to the proof-theoretic level.

Theorem 3.13 (Proof-theoretic conservativeness)
Assume that $\mathcal{L}$ is sound and complete. Then, for every $\Gamma \cup \{\varphi\} \subset L$,

$$\Gamma \vdash \varphi \text{ whenever } \Gamma \vdash^\sqcap \varphi.$$

Theorem 3.14 (Proof-theoretic consistency)
Assuming that $\mathcal{L}$ is sound and complete, if there is $\varphi \in L$ such that $\nvdash \varphi$, then there is $\psi \in L^\sqcap$ such that $\nvdash^\sqcap \psi$.

Finally, concerning completeness, we show that if $\mathcal{L}$ is (strongly) complete then so is $\mathcal{L}^{\sqcap}$. To this end, we need some technical lemmas.

Proposition 3.15 Assume that $\mathcal{L}$ is complete. Then,

$$\Gamma \vdash^{\sqcap} \varphi \text{ whenever } \Gamma \vDash^{\sqcap} \varphi \quad \text{ for every } \Gamma \cup \{\varphi\} \subset L.$$

Proof: Assume that $\Gamma \vDash^{\sqcap} \varphi$. We start by showing that $\Gamma \vDash \varphi$. Let $M \in \mathcal{M}$. Assume that $M \Vdash \Gamma$. By Proposition 3.1,

$$\Gamma = \{\gamma^{\omega} \mid \gamma \in \Gamma, \omega \in \{1,2\}^{\mathbb{N}}\}.$$

Hence, $M \Vdash \gamma^{\omega}$ for $\gamma \in \Gamma$ and $\omega \in \{1,2\}^{\mathbb{N}}$. Thus, by Proposition 3.2,

$$M\omega \Vdash^{\sqcap} \Gamma$$

for every $\omega \in \{1,2\}^{\mathbb{N}}$. Therefore, by hypothesis, $M\omega \Vdash^{\sqcap} \varphi$. Again by Proposition 3.2, $M \Vdash \varphi^{\omega}$ and so, by Proposition 3.1, $M \Vdash \varphi$. Then, by completeness of $\mathcal{L}$, $\Gamma \vdash \varphi$. Thus, by Proposition 2.1 $\Gamma \vdash^{\sqcap} \varphi$. QED

Proposition 3.16 Assume that

$$\Gamma' \vdash^{\sqcap} \varphi' \text{ whenever } \Gamma' \vDash^{\sqcap} \varphi' \quad \text{ for every } \Gamma' \cup \{\varphi'\} \subset L.$$

Then,

$$\Gamma \vdash^{\sqcap} \psi \text{ whenever } \Gamma \vDash^{\sqcap} \psi \quad \text{ for every } \Gamma \subset L \text{ and } \psi \in L^{\sqcap}.$$

Proof: Assume that $\Gamma \nvdash^{\sqcap} \psi$. Then, taking into account the lifting rule, $\Gamma \nvdash^{\sqcap} \delta$ for some $\delta \sqsubseteq \psi$. Observe that $\Gamma \cup \{\delta\} \subset L$ and so, by the hypothesis, $\Gamma \nvDash^{\sqcap} \delta$. Let $M \in \mathcal{M}$ be such that

$$M \Vdash^{\sqcap} \Gamma \text{ and } M\omega \nVdash^{\sqcap} \delta \text{ for some } \omega \in \{1,2\}^{\mathbb{N}}.$$

Then, by Proposition 3.2, $M \nVdash \delta^{\omega}$ and so $M \nVdash \delta$ by Proposition 3.1. Let $\omega' \in \Omega^{\psi}_{\delta}$. Then, by Proposition 3.7, $\psi^{\omega'} = \delta$. Therefore, $M \nVdash \psi^{\omega'}$ and so, by Proposition 3.2, $M\omega' \nVdash^{\sqcap} \psi$. Thus, $M \nVdash^{\sqcap} \psi$. QED

Proposition 3.17 Assume that

$$\Gamma \vdash^{\sqcap} \psi \text{ whenever } \Gamma \vDash^{\sqcap} \psi \quad \text{ for every } \Gamma \subset L \text{ and } \psi \in L^{\sqcap}.$$

Then,

$$\Theta \vdash^{\sqcap} \psi \text{ whenever } \Theta \vDash^{\sqcap} \psi \quad \text{ for every } \Theta \cup \{\psi\} \in L^{\sqcap}.$$

Proof: Assume that $\Theta \cup \{\psi\} \in L^{\sqcap}$ and $\Theta \nvdash^{\sqcap} \psi$. Then, by the co-lifting rule,

$$\{\gamma \mid \gamma \sqsubseteq \theta, \theta \in \Theta\} \nvdash^{\sqcap} \psi.$$

Therefore, by hypothesis,

$$\{\gamma \mid \gamma \sqsubseteq \theta, \theta \in \Theta\} \nvDash^{\sqcap} \psi.$$

Let $M \in \mathcal{M}$ be such that $M \Vdash^{\sqcap} \gamma$ for each $\gamma \sqsubseteq \theta$, $\theta \in \Theta$ and $M \nVdash^{\sqcap} \psi$. By soundness of the LFT rule, $M \Vdash^{\sqcap} \Theta$ and so $\Theta \nvDash^{\sqcap} \psi$. QED

Theorem 3.18 (Completeness)
If $\mathcal{L}$ is complete then $\mathcal{L}^{\sqcap}$ is complete.

Proof: Assume that $\mathcal{L}$ is complete. Then, by Proposition 3.15,

$$\text{if } \Gamma \vDash^{\sqcap} \varphi \text{ then } \Gamma \vdash^{\sqcap} \varphi$$

for every $\Gamma \cup \{\varphi\} \subset L$. Hence, for every $\Gamma \subset L$ and $\psi \in L^{\sqcap}$,

$$\text{if } \Gamma \vDash^{\sqcap} \psi \text{ then } \Gamma \vdash^{\sqcap} \psi$$

using Proposition 3.16. Thus, thanks to Proposition 3.17,

$$\text{if } \Theta \vDash^{\sqcap} \psi \text{ then } \Theta \vdash^{\sqcap} \psi$$

for every $\Theta \cup \{\psi\} \in L^{\sqcap}$. QED

As an immediate corollary of the preservation of completeness, we are able to carry the interchangeability result (Theorem 3.10) to the proof-theoretic level when enriching a complete logic.

Theorem 3.19 (Proof-theoretic interchangeability)
Let $\psi, \psi' \in L^{\sqcap}$ be such that ψ' is obtained from ψ by replacing zero or more occurrences of any combined connective $\lceil c_1 c_2 \rceil$ by $\lceil c_2 c_1 \rceil$. Then, assuming that $\mathcal{L}$ is complete,

$$\psi \vdash^{\sqcap} \psi'.$$

4 The case of classical propositional logic

For illustrating the proposed calculus and non-deterministic semantics for meet-combined constructors, we choose classical propositional logic (CPL). In fact, $\mathsf{CPL}^{\sqcap}$ is sufficiently rich for providing interesting examples, as well as for showing significant differences between its non-deterministic combination of constructors and the product combination in $\mathsf{CPL}^{\times}$ (as defined in [11]).

We assume that the CPL signature contains the propositional symbols (q_i for $i \in \mathbb{N}$), negation, conjunction, disjunction, implication and equivalence. Moreover, we assume that the CPL calculus includes the tautologies as axioms plus modus ponens (MP). Finally, we assume that the CPL semantics is composed of the matrices induced by valuations. Recall that each valuation $v : \{q_i \mid i \in \mathbb{N}\} \to \{0, 1\}$ canonically induces a matrix M_v with $A_v = \{0, 1\}$, providing the denotation of the propositional symbols imposed by v. Clearly, M_v satisfies precisely the same formulas as v.

Observe that $\mathsf{CPL}^{\sqcap}$ is sound and complete, thanks to the results of the previous section, given the soundness and completeness of CPL as presented above. Furthermore, $\mathsf{CPL}^{\sqcap}$ is a conservative extension of CPL.

We start by looking at the combination of conjunction and disjunction, providing an example of a common property of those connectives and showing that, as required, the property is carried over to their non-deterministic combination in $\mathcal{L}^{\times}$. We also compare it with their product combination. Afterward, we examine the impact of non-determinism on implication and equivalence, concluding that only a mitigated form of the metatheorem of substitution of equivalents holds in $\mathsf{CPL}^{\sqcap}$, contrarily to what happens in $\mathsf{CPL}^{\times}$ where this metatheorem holds in full form. Finally, we establish that the metatheorem of deduction still holds in $\mathsf{CPL}^{\sqcap}$ with provisos similar to those in $\mathsf{CPL}^{\times}$.

1	$q_1 \lceil\wedge\vee\rceil q_2$	HYP
2	$q_1 \wedge q_2$	cLFT : 1
3	$q_1 \vee q_2$	cLFT : 1
4	$(q_1 \wedge q_2) \supset (q_2 \wedge q_1)$	TAUT
5	$(q_1 \vee q_2) \supset (q_2 \vee q_1)$	TAUT
6	$q_2 \wedge q_1$	MP : 2, 4
7	$q_2 \vee q_1$	MP : 3, 5
8	$q_2 \lceil\wedge\vee\rceil q_1$	LFT : 6, 7

Figure 1: Derivation of commutativity of $\lceil\wedge\vee\rceil$ in $\mathsf{CPL}^{\sqcap}$.

Combining conjunction and disjunction

Commutativity, for instance, is a common property of conjunction and disjunction. Thus, we should be able to derive it for $\lceil\wedge\vee\rceil$ and, so, thanks to interchangeability and completeness, also for $\lceil\vee\wedge\rceil$. Indeed, we can build the derivation in Figure 1 for

$$q_1 \lceil\wedge\vee\rceil q_2 \vdash^{\sqcap}_{\mathsf{CPL}} q_2 \lceil\wedge\vee\rceil q_1.$$

In fact, it is straightforward to provide a derivation for

$$\psi_1 \lceil\wedge\vee\rceil \psi_2 \vdash^{\sqcap}_{\mathsf{CPL}} \psi_2 \lceil\wedge\vee\rceil \psi_1$$

given arbitrary $\psi_1, \psi_2 \in L^{\sqcap}_{\mathsf{CPL}}$. For an example of a derivation of this type, involving determinations of arbitrary formulas, see Figure 2.

This desideratum was already achieved for the product combination of conjunction and disjunction in [11]. But therein we had two different but related combinations ($\lceil\wedge\vee\rceil$ and $\lceil\vee\wedge\rceil$), while herein the non-deterministic semantics collapses them, thanks to Theorem 3.10:

$$\begin{cases} q_1 \lceil\wedge\vee\rceil q_2 \vdash^{\sqcap}_{\mathsf{CPL}} q_1 \lceil\vee\wedge\rceil q_2 \\ q_1 \lceil\vee\wedge\rceil q_2 \vdash^{\sqcap}_{\mathsf{CPL}} q_1 \lceil\wedge\vee\rceil q_2. \end{cases}$$

At this point, the reader will ask whether we are able to establish

$$\vdash^{\sqcap}_{\mathsf{CPL}} (q_1 \lceil\wedge\vee\rceil q_2) \equiv (q_1 \lceil\vee\wedge\rceil q_2).$$

The answer is negative, not because of any limitation on the collapsing of the two combined connectives, but for a deeper reason: implication does not behave as might be expected in the presence of non-determinism, as we proceed to analyze.

Impact of non-determinism

Observe that $\psi \supset \psi$ is not always valid in $\mathsf{CPL}^{\sqcap}$. Clearly, if $\varphi \in L_{\mathsf{CPL}}$, then $\varphi \supset \varphi$ is valid in $\mathsf{CPL}^{\sqcap}$. However, if non-deterministic constructors are present in the formula ψ, then $\psi \supset \psi$ is not necessarily valid. For instance, the formula

$$(\mathsf{ff}\lceil\wedge\vee\rceil\mathsf{tt}) \supset (\mathsf{ff}\lceil\wedge\vee\rceil\mathsf{tt})$$

is not valid. Indeed, this formula is not satisfied by an interpretation (M, ω) such that $\omega_1 = 2$ and $\omega_4 = 1$, since such a choice sequence ensures that the $\lceil\wedge\vee\rceil$ on the left is evaluated as $\vee$ and the $\lceil\wedge\vee\rceil$ on the right as $\wedge$.

1	ψ_1	HYP	
2	$\psi_1 \supset \psi_2$	HYP	
3_{φ_1}	φ_1	cLFT : 1	for each $\varphi_1 \sqsubseteq \psi_1$
$4_{\varphi_1\varphi_2}$	$\varphi_1 \supset \varphi_2$	cLFT : 2	for each $\varphi_1 \sqsubseteq \psi_1, \varphi_2 \sqsubseteq \psi_2$
5_{φ_2}	φ_2	MP : $3_{\varphi_1}, 4_{\varphi_1\varphi_2}$	for each $\varphi_2 \sqsubseteq \psi_2$
6	ψ_2	LFT : 5	

Figure 2: Derivation of MP in $\mathsf{CPL}^{\sqcap}$.

Therefore, the presence of non-deterministic constructors in a formula also disturbs the closure for instantiation, even for validity. In particular, the result of instantiating a tautology with non-deterministic formulas is not necessarily valid. For instance,

$$q_1 \supset q_1$$

is valid in $\mathsf{CPL}^{\sqcap}$ while some of its instances (like the one above) are not. As another example, the instance

$$\lceil\mathsf{tt}\mathsf{ff}\rceil \supset (q_2 \supset \lceil\mathsf{tt}\mathsf{ff}\rceil)$$

of $q_1 \supset (q_2 \supset q_1)$ is not valid. On the other hand, for example, every (even non-deterministic) instance of

$$q_1 \supset \mathsf{tt}$$

is valid in $\mathsf{CPL}^{\sqcap}$.

However, the presence of non-deterministic constructors does not affect modus ponens. Indeed, as depicted in Figure 2, MP still holds in general within $\mathsf{CPL}^{\sqcap}$:

$$\psi_1, \psi_1 \supset \psi_2 \vdash^{\sqcap}_{\mathsf{CPL}} \psi_2.$$

Given the impact of non-determinism on implication and, thus, also on equivalence, one should not expect to be able to prove the full version of the metatheorem of of substitution of equivalents in $\mathsf{CPL}^{\sqcap}$. In fact, we are only able to prove the following weak version (substitution only up to interderivability):

Theorem 4.1 (Metatheorem of substitution of equivalents in $\mathsf{CPL}^{\sqcap}$) Let $\psi, \psi', \theta, \theta' \in L^{\sqcap}_{\mathsf{CPL}}$ be such that:

- $\vdash^{\sqcap}_{\mathsf{CPL}} \psi \equiv \psi'$;
- θ' is obtained from θ by replacing zero or more occurrences of ψ by ψ'.

Then:

$$\theta \vdash^{\sqcap}_{\mathsf{CPL}} \theta'.$$

Proof: Assume that $\vdash^{\sqcap}_{\mathsf{CPL}} \psi \equiv \psi'$. The proof follows by induction on the structure of θ.

Base: θ is a propositional symbol. Consider two cases:
(1) θ is ψ. Consider two subcases: (a) θ' is ψ'. Then $\vdash^{\sqcap}_{\mathsf{CPL}} \theta \equiv \theta'$ by hypothesis. Hence $\theta \vdash^{\sqcap}_{\mathsf{CPL}} \theta'$ by MP; (b) θ' is θ. Then $\theta \vdash^{\sqcap}_{\mathsf{CPL}} \theta'$.
(2) θ is not ψ. Then ψ does not occur in θ and, so, θ' is θ. Thus, $\theta \vdash^{\sqcap}_{\mathsf{CPL}} \theta'$.

Step: Let θ be $\lceil c_1 c_2 \rceil(\theta_1, \ldots, \theta_n)$. Consider two cases:
(1) θ is ψ. Consider two subcases: (a) θ' is ψ'. Then $\vdash^{\sqcap}_{\mathsf{CPL}} \theta \equiv \theta'$ by hypothesis. Hence $\theta \vdash^{\sqcap}_{\mathsf{CPL}} \theta'$ by MP; (b) θ' is θ. Then $\theta \vdash^{\sqcap}_{\mathsf{CPL}} \theta'$.
(2) θ is not ψ. Then θ' is $\lceil c_1 c_2 \rceil(\theta'_1, \ldots, \theta'_n)$ where, for $j = 1, \ldots, n$, the formula θ'_j is obtained from θ_j by replacing zero or more occurrences of ψ by ψ'. Assume without loss of generality that n is 1. Observe that,

$$\vdash^{\sqcap}_{\mathsf{CPL}} \theta_1 \equiv \theta'_1$$

by the induction hypothesis. Therefore, by co-lifting,

$$\vdash^{\sqcap}_{\mathsf{CPL}} \varphi_1 \equiv \varphi'_1 \quad \text{for every } \varphi_1 \sqsubseteq \theta_1 \text{ and } \varphi'_1 \sqsubseteq \theta'_1.$$

Hence, by the conservativeness of $\mathsf{CPL}^{\sqcap}$, see Theorem 3.11,

$$\vdash_{\mathsf{CPL}} \varphi_1 \equiv \varphi'_1 \quad \text{for every } \varphi_1 \sqsubseteq \theta_1 \text{ and } \varphi'_1 \sqsubseteq \theta'_1.$$

Thus,

$$\vdash_{\mathsf{CPL}} c_k(\varphi_1) \equiv c_k(\varphi'_1) \quad \text{for every } \varphi_1 \sqsubseteq \theta_1, \varphi'_1 \sqsubseteq \theta'_1 \text{ and } k = 1, 2$$

since the metatheorem of substitution of equivalents holds in CPL. Moreover,

$$\vdash^{\sqcap}_{\mathsf{CPL}} c_k(\varphi_1) \equiv c_k(\varphi'_1) \quad \text{for every } \varphi_1 \sqsubseteq \theta_1, \varphi'_1 \sqsubseteq \theta'_1 \text{ and } k = 1, 2$$

since $\mathsf{CPL}^{\sqcap}$ is an extension of CPL. Hence,

$$c_k(\varphi_1) \vdash^{\sqcap}_{\mathsf{CPL}} c_k(\varphi'_1) \quad \text{for every } \varphi_1 \sqsubseteq \theta_1, \varphi'_1 \sqsubseteq \theta'_1 \text{ and } k = 1, 2$$

by MP, and so

$$\{c_k(\varphi_1) \mid \varphi_1 \sqsubseteq \theta_1, k = 1, 2\} \vdash^{\sqcap}_{\mathsf{CPL}} \{c_k(\varphi'_1) \mid \varphi'_1 \sqsubseteq \theta'_1, k = 1, 2\}.$$

Therefore, $\theta \vdash^{\sqcap}_{\mathsf{CPL}} \theta'$. Indeed, θ is $\lceil c_1 c_2 \rceil(\theta_1)$ and, by co-lifting,

$$\lceil c_1 c_2 \rceil(\theta_1) \vdash^{\sqcap}_{\mathsf{CPL}} \{c_k(\varphi_1) \mid \varphi_1 \sqsubseteq \theta_1, k = 1, 2\}.$$

Furthermore, by lifting,

$$\{c_k(\varphi'_1) \mid \varphi'_1 \sqsubseteq \theta'_1, k = 1, 2\} \vdash^{\sqcap}_{\mathsf{CPL}} \lceil c_1 c_2 \rceil(\theta'_1)$$

and, $\lceil c_1 c_2 \rceil(\theta'_1)$ is θ'. QED

Note that, under the assumptions of Theorem 4.1,

$$\vdash^{\sqcap}_{\mathsf{CPL}} \theta \equiv \theta'$$

does not hold in general. For instance we have

$$\begin{cases} \vdash^{\sqcap}_{\mathsf{CPL}} \mathsf{ff} \equiv \mathsf{ff} \\ \vdash^{\sqcap}_{\mathsf{CPL}} \mathsf{tt} \equiv \mathsf{tt} \end{cases}$$

but

$$\not\vdash^{\sqcap}_{\mathsf{CPL}} (\mathsf{ff} \lceil \wedge \vee \rceil \mathsf{tt}) \equiv (\mathsf{ff} \lceil \wedge \vee \rceil \mathsf{tt})$$

because

$$\not\vdash^{\sqcap}_{\mathsf{CPL}} (\mathsf{ff} \lceil \wedge \vee \rceil \mathsf{tt}) \supset (\mathsf{ff} \lceil \wedge \vee \rceil \mathsf{tt})$$

as we saw above at the semantic level.

Thus, concerning substitution of equivalents, $\mathsf{CPL}^{\times}$ and $\mathsf{CPL}^{\sqcap}$ are quite different. On the other hand, concerning the metatheorem of deduction, the two logics are alike, as we proceed to establish.

Metatheorem of deduction

Towards the metatheorem of deduction in $\mathsf{CPL}^{\sqcap}$ we need the following auxiliary result and to borrow from [11] the notion of essential co-lifting.

Proposition 4.2 Let $\psi \in L^{\sqcap}_{\mathsf{CPL}}$. Then

$$\vdash^{\sqcap}_{\mathsf{CPL}} (\bigwedge_{\varphi \sqsubseteq \psi} \varphi) \supset \psi.$$

Proof: We prove the result semantically making use of the soundness and completeness of CPL and $\mathsf{CPL}^{\sqcap}$. Assume that

$$M_v\, \omega \Vdash^{\sqcap} \bigwedge_{\varphi \sqsubseteq \psi} \varphi.$$

Then, by Proposition 3.2,

$$M_v \Vdash \varphi^{\omega} \quad \text{for each } \varphi \sqsubseteq \psi.$$

Hence, by Proposition 3.1,

$$M_v \Vdash \varphi \quad \text{for each } \varphi \sqsubseteq \psi,$$

and so, by the same proposition,

$$M_v \Vdash \varphi^{\omega'} \quad \text{for each } \omega' \in \{1,2\}^{\mathbb{N}} \text{ and } \varphi \sqsubseteq \psi.$$

Thus, by Proposition 3.2,

$$M_v\, \omega' \Vdash^{\sqcap} \varphi \quad \text{for each } \omega' \in \{1,2\}^{\mathbb{N}} \text{ and } \varphi \sqsubseteq \psi,$$

that is,

$$M_v \Vdash^{\sqcap} \varphi \quad \text{for each } \varphi \sqsubseteq \psi.$$

Then

$$M_v \Vdash^{\sqcap} \psi$$

by Proposition 3.5, and, so, in particular $M_v\, \omega \Vdash^{\sqcap} \psi$. QED

Let $\psi_1 \dots \psi_n$ be a derivation for $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \psi$. Formula ψ_i *depends* on $\theta \in \Theta$ in this derivation if

- either ψ_i is θ;
- or ψ_i is obtained using a rule (either MP or LFT or cLFT) with at least one of the premises depending on θ.

Formula ψ_i is an *essential co-lifting* over a dependent of θ in the derivation $\psi_1 \dots \psi_n$ if

- ψ_i is obtained using cLFT from ψ_j, that is, $\psi_i \sqsubseteq \psi_j$;
- ψ_j depends on θ;
- θ is in $L^{\sqcap}_{\mathsf{CPL}} \setminus L_{\mathsf{CPL}}$.

Theorem 4.3 (Metatheorem of deduction in $\mathsf{CPL}^{\sqcap}$)
Let $\Theta \cup \{\eta, \psi\} \subseteq L^{\sqcap}_{\mathsf{CPL}}$ and $\psi_1 \dots \psi_n$ be a derivation for

$$\Theta, \eta \vdash^{\sqcap}_{\mathsf{CPL}} \psi$$

without essential co-liftings over dependents of η. Then

$$\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi.$$

Proof: The proof is carried out by induction on the length of the given derivation.

Base. There are three cases to consider:

(a) ψ is an axiom of $\mathsf{CPL}^{\sqcap}$. Then ψ is a tautology of CPL. Hence $\vdash_{\mathsf{CPL}} \psi$. Observe that $\vdash_{\mathsf{CPL}} \psi \supset (\delta \supset \psi)$, in particular, for each $\delta \sqsubseteq \eta$. Thus, for each $\delta \sqsubseteq \eta$, $\vdash_{\mathsf{CPL}} \delta \supset \psi$, and, so, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \psi$. Therefore, by LFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi$.

(b) ψ is in Θ. Then $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \psi$ and so, by cLFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \varphi$ for each $\varphi \sqsubseteq \psi$. Recall that $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \varphi \supset (\delta \supset \varphi)$ for each $\delta \sqsubseteq \eta$ and $\varphi \sqsubseteq \psi$. Thus, for each $\delta \sqsubseteq \eta$ and $\varphi \sqsubseteq \psi$, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \varphi$. Therefore, by LFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi$.

(c) ψ is η. Observe that, for each $\delta \sqsubseteq \eta$, $\vdash_{\mathsf{CPL}} \delta \supset \delta$ and, so, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \delta$. Hence, by LFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi$.

Step. There are three additional cases to consider:

(a) ψ is obtained by MP from ψ_{i_1} and ψ_{i_2} with ψ_{i_2} being $\psi_{i_1} \supset \psi$ and $\psi_{i_1}, \psi \in L_{\mathsf{CPL}}$. By the induction hypothesis, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi_{i_1}$ and $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset (\psi_{i_1} \supset \psi)$. Therefore, by cLFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \psi_{i_1}$ and $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset (\psi_{i_1} \supset \psi)$ for every $\delta \sqsubseteq \eta$. Thus,

$$\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \psi, \quad \text{for every } \delta \sqsubseteq \eta.$$

using a tautology and MP. Finally, by LFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi$.

(b) ψ is obtained by LFT from $\psi_{i_1}, \ldots, \psi_{i_m}$. Then $\{\psi_{i_j} \mid j = 1, \ldots, m\} = \{\varphi \mid \varphi \sqsubseteq \psi\}$ and so, for every $\varphi \sqsubseteq \psi$, there is a derivation with length less than n and without essential co-liftings over dependents of η, for $\Theta, \eta \vdash^{\sqcap}_{\mathsf{CPL}} \varphi$. Hence, by the induction hypothesis, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \varphi$ for every $\varphi \sqsubseteq \psi$. Therefore, by cLFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \varphi$ for every $\delta \sqsubseteq \eta$ and $\varphi \sqsubseteq \psi$. Thus,

$$\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset (\bigwedge_{\varphi \sqsubseteq \psi} \varphi), \quad \text{for every } \delta \sqsubseteq \eta.$$

Hence, taking into account Proposition 4.2,

$$\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \psi, \quad \text{for every } \delta \sqsubseteq \eta.$$

Finally, by cLFT and LFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi$.

(c) ψ is obtained by cLFT from ψ_j. Then $\psi \sqsubseteq \psi_j$ and there is a derivation with length less than n and without essential co-liftings over dependents of η, for $\Theta, \eta \vdash^{\sqcap} \psi_j$. Since the co-lifting is non essential there are two cases to consider:

(i) ψ does not depend on η. Then $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \psi$. Hence, by cLFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \varphi$ for every $\varphi \sqsubseteq \psi$. Thus, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \varphi \supset (\delta \supset \varphi)$ for every $\delta \sqsubseteq \eta$ and $\varphi \sqsubseteq \psi$. So, by MP, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \delta \supset \varphi$ for every $\delta \sqsubseteq \eta$ and $\varphi \sqsubseteq \psi$. Therefore, by LFT, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi$.

(ii) η is in L_{CPL}. Observe that, by the induction hypothesis, $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi_j$. Then $\Theta \vdash^{\sqcap}_{\mathsf{CPL}} \eta \supset \psi$ by cLFT. QED

In short, in both $\mathsf{CPL}^{\sqcap}$ (with non-deterministic combination of connectives) and $\mathsf{CPL}^{\times}$ (with product combination of connectives) the metatheorem of deduction holds with provisos on the use of the cLFT rule.

5 Outlook

The non-deterministic semantics proposed herein for combining connectives (and other constructors) was shown to have some advantages over the original semantics that we proposed in [11], namely the collapse of $\lceil c_1 c_2 \rceil$ and $\lceil c_2 c_1 \rceil$, while still endowing them only with the common properties of their components.

However, a price had to be paid, given the complicated nature of non-determinism. Although the calculus is surprisingly simple, the use of the LFT rule requires much more work since a high number of premises has to be established before. This number grows exponentially with the number of non-deterministic constructors in the conclusion.

Furthermore, non-determinism has a deep impact on the logic. For instance, $\psi \supset \psi$ is not any more a valid formula in general.

It is worthwhile to point out that one can easily conceive other ways of introducing non-determinism in the semantics of combined constructors. For instance, we considered the alternative of working with non-deterministic algebras, but the path towards a sound and complete calculus was not so clear.

Actually, the solution adopted herein of making a choice for each use of a non-deterministic constructor in a formula seems much more interesting for applications, namely for reasoning about non-deterministic circuits [13]. Indeed, each non-deterministic gate is expected to behave independently of the other similar gates in the circuit.

For a survey of other approaches to non-determinism in logic see [3], including approaches motivated by paraconsistency [4, 5, 1] and by reasoning about computation errors [2].

Our results on non-deterministically combined constructors endowed with the common properties of their components generalize the pioneering work in [10, 9] on a calculus for the common properties of conjunction and disjunction.

Concerning future research, by replacing the choice sequences by stochastic processes and quantum processes, the results presented herein should be carried over to logics with probabilistic combination of constructors and to logics with quantum superposition of constructors.

Such developments will open the way to much more exciting applications in the field of logics for the verification of faulty hardware [7] and quantum programs [8].

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