On combined connectives

A. Sernadas C. Sernadas J. Rasga
Dep. Mathematics, Instituto Superior Técnico, TU Lisbon
SQIG, Instituto de Telecomunicações, Portugal
{acs,css,jfr}@math.ist.utl.pt

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Abstract

Combined connectives arise in combined logics. In fibrings, such combined connectives are known as shared connectives and inherit the logical properties of each component. A new way of combining connectives (and other language constructors of propositional nature) is proposed by inheriting only the common logical properties of the components. A sound and complete calculus is provided for reasoning about the latter. The calculus is shown to be a conservative extension of the original calculus. Examples are provided contributing to a better understanding of what are the common properties of any two constructors, say disjunction and conjunction.

Keywords: combined connectives, common properties of connectives, combined logics.

1 Introduction

When combining logics one may want to put together two logics with no intended interaction (so called unconstrained or free combination), but frequently one wants to impose some interaction between connectives[1] (so called constrained or synchronized combination). For instance, fusion [7] can be seen as a form of constrained fibring [6] of two modal logics by imposing the sharing of the matching pairs of propositional connectives while keeping the two modalities apart.

In the logic $\mathcal{L}$ resulting from the fibring of any two given logics $\mathcal{L}_1$ and $\mathcal{L}_2$, one finds all the inference rules from those two logics. Therefore, in the case of the sharing of two constructors c_1 and c_2 with the same arity, the *shared constructor* $\langle c_1 c_2\rangle$ in the resulting logic enjoys all logical properties inherited from c_1 and c_2. For instance, if one shares a classical negation $\neg_1$ and an intuitionistic negation $\neg_2$, the resulting shared negation $\langle \neg_1 \neg_2\rangle$ is classical. As expected, such a sharing may easily lead to inconsistency. For example, if one shares conjunction $\wedge_1$ and disjunction $\vee_2$, since the resulting shared connective $\langle \wedge_1 \vee_2\rangle$ inherits the logical properties of conjunction and those of disjunction,

[1] In a general sense, including, besides the propositional connectives, also modal operators and other language constructors.

we can infer φ from ψ for any formulas φ, ψ in $\mathcal{L}$:

1	ψ	Hypothesis;
2	$\psi \langle\wedge_1\vee_2\rangle \varphi$	Disjunction introduction;
3	φ	Conjunction elimination.

In short, the sharing of conjunction and disjunction turns out to be the tonk connective. For details on the latter see for instance [10, 2].

One may wonder if it would not be better instead to endow the combined constructor only with the *common* logical properties of the component constructors. In this way we would be led to a different approach to combining logics, at least with respect to the behavior of the combined constructors. Such a combination mechanism would have the advantage over fibring of avoiding inconsistency in more situations. For instance, if one combines conjunction $\wedge_1$ with disjunction $\vee_2$ in this sense, the resulting *meet-combined connective* $\lceil\wedge_1\vee_2\rceil$ would only have the logical properties that hold for conjunction and also for disjunction. Examples of such logical properties are:

$$\begin{aligned} \varphi \wedge_1 \psi &\vdash \varphi \lceil\wedge_1\vee_2\rceil \psi; \\ \varphi \lceil\wedge_1\vee_2\rceil \psi &\vdash \varphi \vee_2 \psi. \end{aligned}$$

Inconsistency, as obtained in the derivation above, does not arise because the converse of the latter does not hold.

In this paper we abstract away from the setting of combined logics and focus our attention on meet-combining language constructors in any given logic of propositional nature[2], assumed to be endowed with a Hilbert calculus and matrix semantics. We have in mind contributing to a better understanding of what are the logical properties common to two any given constructors (of the same arity). In Section 2 we provide the means for enriching any given logic (language, calculus and semantics) with meet-combined constructors. The additional inference rules for the meet-combined constructors are chosen in order to ensure that they inherit the common properties of their components and only those common properties. Each matrix of the enriched logic is just the product of a matrix of the original logic with itself. In Section 3 we show the soundness and the completeness of the proposed calculus with regard to the proposed product semantics, as well as the conservative nature of the enrichment and, as an immediate corollary, the preservation of consistency. The particular case of classical propositional logic is addressed in Section 4, including the properties of some interesting examples of meet-combined connectives. Section 4 also includes a preliminary study of the enriched logic, namely concerning the metatheorems of deduction and of substitution of equivalents. Finally, in Section 5 we assess what was achieved and speculate on possible applications to the field of combined logics.

[2]That is, with no binding operators.

2 Meet-combining constructors

For the purposes of this paper, by a *logic* we mean a triple $\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$ where:

- The *signature* Σ is a family $\{\Sigma_n\}_{n\in\mathbb{N}}$ with each Σ_n being a finite set of n-ary language *constructors*. Formulas are built as usual with these constructors and the schema variables in $\Xi = \{\xi_k \mid k \in \mathbb{N}\}$. We use L and $L(\Xi)$ for denoting the set of *concrete formulas*[3] and the set of all *formulas*, respectively. If a formula contains schema variables we may emphasize the fact by saying that it is a *schema formula*.

- The Hilbert *calculus* Δ is a set of finitary rules of the form

$$\frac{\alpha_1, \ldots, \alpha_n}{\beta}$$

  where formulas $\alpha_1, \ldots, \alpha_n$ are said to be the *premises* of the rule and formula β is said to be its *conclusion*. Derivations are defined as usual for Hilbert calculi. We write

$$\Gamma \vdash \varphi$$

  for stating that there is a derivation of formula φ from set Γ of hypotheses.

- The matrix *semantics* $\mathcal{M}$ is a class of matrices over Σ. Recall that a matrix over Σ is a pair $M = (\mathfrak{A}, D)$ where

$$\mathfrak{A} = (A, \{\underline{c} : A^n \to A \mid c \in \Sigma_n\}_{n\in\mathbb{N}})$$

  is an algebra over Σ and $D \subseteq A$. The elements of A are known as *truth values* and those of D are the *distinguished* or *designated* ones. Denotation, satisfaction and entailment are as expected for matrix semantics. We write

$$[\![\varphi]\!]_{\mathfrak{A}\rho}$$

  for the denotation of formula φ by algebra $\mathfrak{A}$ at assignment $\rho : \Xi \to A$. Furthermore, if φ is concrete then we may write $[\![\varphi]\!]_{\mathfrak{A}}$ for $[\![\varphi]\!]_{\mathfrak{A}\rho}$ since the denotation is independent of the assignment. Matrix M and assignment ρ *satisfy* formula φ, written

$$M, \rho \Vdash \varphi,$$

  if $[\![\varphi]\!]_{\mathfrak{A}\rho} \in D$. Set Γ of formulas *entails* formula φ, written

$$\Gamma \vDash \varphi,$$

  if $M, \rho \Vdash \varphi$ whenever $M, \rho \Vdash \Gamma$.

[3]Formulas without schema variables.

In short, by a logic we mean a system of propositional nature, endowed with a Hilbert calculus and a matrix semantics. In Section 4 we address the special case of classical propositional logic, but the results in this paper are applicable to a wide class of propositional logics, including intuitionistic and modal logics among others.

Given such a logic $\mathcal{L} = (\Sigma, \Delta, \mathcal{M})$, the objective now is to define a logic

$$\mathcal{L}^\times = (\Sigma^\times, \Delta^\times, \mathcal{M}^\times)$$

where one can also reason with and about the envisaged meet-combinations of constructors in the original logic $\mathcal{L}$, without disturbing the original properties of the constructors in $\mathcal{L}$.

Language

The signature $\Sigma^\times$ is composed of all possible meet-combinations of constructors in Σ. More concretely,

$$\Sigma^\times = \{\Sigma_n^\times\}_{n\in\mathbb{N}}$$

with

$$\Sigma_n^\times = \{\lceil c_1 c_2 \rceil \mid c_1, c_2 \in \Sigma_n\}.$$

For any constructors $c_1, c_2 \in \Sigma$ of the same arity, $\lceil c_1 c_2 \rceil$ is said to be their meet-combination. Moreover, c_1 and c_2 are the first component and the second component of $\lceil c_1 c_2 \rceil$, respectively. Since this paper addresses only such meet-combined constructors, from now on we refer to them simply as combined constructors. As expected, we use $L^\times$ and $L^\times(\Xi)$ for denoting the set of concrete formulas and the set of all formulas over $\Sigma^\times$, respectively.

We look at $\Sigma^\times$ as an enrichment of Σ via the embedding $\eta : c \mapsto \lceil cc \rceil$. Furthermore, in the context of $\Sigma^\times$ we may write c for $\lceil cc \rceil$. In due course this shortcut will be fully vindicated. Accordingly, we take that $L \subset L^\times$ and $L(\Xi) \subset L^\times(\Xi)$.

Given a formula φ over $\Sigma^\times$ and combined constructor $\lceil c_1 c_2 \rceil$, for $k = 1, 2$, we denote by

$$\varphi|_k^{\lceil c_1 c_2 \rceil}$$

the formula obtained from φ by replacing every occurrence of the combined constructor $\lceil c_1 c_2 \rceil$ by constructor c_k. In addition, we denote by

$$\varphi|_k^{\lceil\cdot\cdot\rceil}$$

the formula obtained from φ by replacing every occurrence of each combined constructor by its k-th component.

Calculus

The calculus $\Delta^\times$ is an enrichment of Δ (via the embedding $c \mapsto \lceil cc \rceil$) with the following additional rules for dealing with the combined constructors.

For each formula $\varphi \in L^\times(\Xi)$, the *lifting rule* (in short LFT)

$$\frac{\varphi|_1^{\lceil\cdot\cdot\rceil}, \varphi|_2^{\lceil\cdot\cdot\rceil}}{\varphi}.$$

This rule is motivated by the idea that $\lceil c_1 c_2 \rceil$ inherits the common properties of c_1 and c_2.

For each concrete formula $\varphi \in L^\times$ and $k = 1, 2$, the *co-lifting rule* (in short cLFT)

$$\frac{\varphi}{\varphi|_k^{\lceil\cdot\cdot\rceil}}.$$

This rule is motivated by the idea that $\lceil c_1 c_2 \rceil$ should enjoy only the common properties of c_1 and c_2. In fact, this rule guarantees more. It guarantees that $\lceil c_1 c_2 \rceil$ enjoys only the common *original* properties of c_1 and c_2 because, in due course, we show that $\mathcal{L}^\times$ is a conservative extension of $\mathcal{L}$.

One may wonder why cLFT applies only to concrete formulas. In Section 4 we provide a counterexample illustrating that extending it to schema formulas would put soundness at stake. Nevertheless, we mention *en passant* that we could allow schema formulas while preserving soundness by imposing as a proviso on their instantiations that they should only be replaceable by formulas in $\mathcal{L}$. We refrain to do so because not much would be gained in exchange for the added complexity of the calculus.

Our objective in defining the meet-combination of two constructors c_1 and c_2 of $\mathcal{L}$ is to obtain a constructor precisely with the logical properties of c_1 and c_2. However, we end up with two such meet-combinations: $\lceil c_1 c_2 \rceil$ and $\lceil c_2 c_1 \rceil$. As one might expect, these meet-combined constructors are related:

Theorem 2.1 (Exchangeability of components)
Let $\varphi \in L^\times$ be a concrete formula with no meet-combined constructors barring $\lceil c_1 c_2 \rceil$ and φ' be the concrete formula obtained from φ by replacing every occurrence of $\lceil c_1 c_2 \rceil$ by $\lceil c_2 c_1 \rceil$. Then, φ and φ' are interderivable in $\mathcal{L}^\times$.

Proof: Apply first cLFT and then LFT, in each direction. QED

In Section 4 we provide a counterexample showing that the assumption that φ contains no meet-combined constructors barring $\lceil c_1 c_2 \rceil$ is essential.

Semantics

The semantics $\mathcal{M}^\times$ is the class of matrices over $\Sigma^\times$

$$\{M^\times \mid M \in \mathcal{M}\}$$

such that each

$$M^\times = (\mathfrak{A}^\times, D^\times)$$

where

- $\mathfrak{A}^\times = (A^\times, \{\underline{\lceil c_1 c_2 \rceil} : (A^\times)^n \to A^\times \mid \lceil c_1 c_2 \rceil \in \Sigma_n^\times\}_{n \in \mathbb{N}})$ with

- $A^\times = A^2$;
- $\underline{\lceil c_1 c_2 \rceil}((a_1, b_1), \dots, (a_n, b_n)) = (\underline{c_1}(a_1, \dots, a_n), \underline{c_2}(b_1, \dots, b_n))$;

- $D^\times = D^2$.

Clearly, each $M^\times \in \mathcal{M}^\times$ has the nature of a product $(M \times M)$ but we refrain from exploring this fact since it is not essential in what follows. Nevertheless we may refer to $\mathcal{M}^\times$ as being the product semantics for the meet-combination of constructors of $\mathcal{L}$.

In the sequel, we use $\Vdash_\times$ and $\vDash_\times$ for satisfaction and entailment in $\mathcal{L}^\times$. Furthermore, given $M \in \mathcal{M}$ and an assignment $\rho : \Xi \to A^\times$ over $M^\times$, we denote by ρ_1 and ρ_2 the unique assignments over M such that $\rho(\xi) = (\rho_1(\xi), \rho_2(\xi))$.

3 Main results

Assuming that $\mathcal{L}$ is strongly sound, we establish the strong soundness of $\mathcal{L}^\times$ (Theorem 3.9). To this end, we start by proving some relevant technical lemmas.

Proposition 3.1 Let $M \in \mathcal{M}$, ρ an assignment over $M^\times$ and φ a formula in $L(\Xi)$. Then

$$[\![\varphi]\!]_{\mathfrak{A}^\times \rho} = ([\![\varphi]\!]_{\mathfrak{A}\rho_1}, [\![\varphi]\!]_{\mathfrak{A}\rho_2}).$$

Moreover

$$M^\times, \rho \Vdash_\times \varphi \quad \text{iff} \quad M, \rho_1 \Vdash \varphi \text{ and } M, \rho_2 \Vdash \varphi.$$

Proof: The proof follows by induction on the structure of φ.
(a) φ is ξ. Then $[\![\varphi]\!]_{\mathfrak{A}^\times \rho} = \rho(\xi) = (\rho_1(\xi), \rho_2(\xi)) = ([\![\varphi]\!]_{\mathfrak{A}\rho_1}, [\![\varphi]\!]_{\mathfrak{A}\rho_2})$.
(b) φ is $c(\varphi_1, \dots, \varphi_n)$. Then

$$\begin{aligned}
[\![\varphi]\!]_{\mathfrak{A}^\times \rho} &= \underline{\lceil cc \rceil}([\![\varphi_1]\!]_{\mathfrak{A}^\times \rho}, \dots, [\![\varphi_n]\!]_{\mathfrak{A}^\times \rho}) \\
&= \underline{\lceil cc \rceil}(([\![\varphi_1]\!]_{\mathfrak{A}\rho_1}, [\![\varphi_1]\!]_{\mathfrak{A}\rho_2}), \dots, ([\![\varphi_n]\!]_{\mathfrak{A}\rho_1}, [\![\varphi_n]\!]_{\mathfrak{A}\rho_2})) \\
&= (\underline{c}([\![\varphi_1]\!]_{\mathfrak{A}\rho_1}, \dots, [\![\varphi_n]\!]_{\mathfrak{A}\rho_1}), \underline{c}([\![\varphi_1]\!]_{\mathfrak{A}\rho_2}, \dots, [\![\varphi_n]\!]_{\mathfrak{A}\rho_2})) \\
&= ([\![\varphi]\!]_{\mathfrak{A}\rho_1}, [\![\varphi]\!]_{\mathfrak{A}\rho_2}).
\end{aligned}$$

Observe that $M^\times, \rho \Vdash_\times \varphi$ iff $[\![\varphi]\!]_{\mathfrak{A}^\times \rho} \in D^\times$ iff $([\![\varphi]\!]_{\mathfrak{A}\rho_1}, [\![\varphi]\!]_{\mathfrak{A}\rho_2}) \in D^\times$ iff $[\![\varphi]\!]_{\mathfrak{A}\rho_1}, [\![\varphi]\!]_{\mathfrak{A}\rho_2} \in D$ iff $M, \rho_1 \Vdash \varphi$ and $M, \rho_2 \Vdash \varphi$. QED

Proposition 3.2 A sound rule in $\mathcal{L}$ is sound in $\mathcal{L}^\times$.

Proof: Let $M \in \mathcal{M}$ and r a rule in Δ with premises $\alpha_1, \dots, \alpha_n$ and conclusion β. Assume that $M^\times, \rho \Vdash \alpha_j$ for $j = 1, \dots, n$ and that r is sound in $\mathcal{L}$. Then, by Proposition 3.1, $M, \rho_1 \Vdash \alpha_j$ and $M, \rho_2 \Vdash \alpha_j$ for $j = 1, \dots, n$ and so, by soundness of r in $\mathcal{L}$, $M, \rho_1 \Vdash \beta$ and $M, \rho_2 \Vdash \beta$. Again, by Proposition 3.1, $M^\times, \rho \Vdash \beta$. Hence, rule r is sound in $\mathcal{L}^\times$. QED

Proposition 3.3 Let $M \in \mathcal{M}$, ρ be an assignment over $M^\times$ and φ a formula in $L^\times(\Xi)$. Then

$$[\![\varphi]\!]_{\mathfrak{A}^\times \rho} = \left(([\![\varphi|_1^{\lceil c_1 c_2 \rceil}]\!]_{\mathfrak{A}^\times \rho})_1, ([\![\varphi|_2^{\lceil c_1 c_2 \rceil}]\!]_{\mathfrak{A}^\times \rho})_2 \right).$$

Proof: The proof is carried out by induction on φ.

(a) φ is ξ. Then

$$[\![\xi]\!]_{\mathfrak{A}\times\rho} = \rho(\xi) = (\rho(\xi)_1, \rho(\xi)_2) = (([\![\xi|_1^{\lceil c_1c_2\rceil}]\!]_{\mathfrak{A}\times\rho})_1, [\![\xi|_2^{\lceil c_1c_2\rceil}]\!]_{\mathfrak{A}\times\rho})_2).$$

(b) φ is $\lceil c_1'c_2'\rceil(\varphi_1,\ldots,\varphi_n)$. Assume that

$$[\![\varphi_k]\!]_{\mathfrak{A}\times\rho} = (a_k, b_k) \quad \text{for } k = 1,\ldots,n.$$

By the induction hypothesis

$$([\![\varphi_k|_1^{\lceil c_1c_2\rceil}]\!]_{\mathfrak{A}\times\rho})_1 = a_k,\ ([\![\varphi_k|_2^{\lceil c_1c_2\rceil}]\!]_{\mathfrak{A}\times\rho})_2 = b_k \quad \text{for } k = 1,\ldots,n.$$

Let $([\![\varphi_k|_1^{\lceil c_1c_2\rceil}]\!]_{\mathfrak{A}\times\rho})_2 = a_k'$ and $([\![\varphi_k|_2^{\lceil c_1c_2\rceil}]\!]_{\mathfrak{A}\times\rho})_1 = b_k'$.

(i) $c_1' \neq c_1$ or $c_2' \neq c_2$. Then

$$\begin{array}{lcl}
[\![\lceil c_1'c_2'\rceil(\varphi_1,\ldots,\varphi_n)]\!]_{\mathfrak{A}\times\rho} & = & (\underline{c_1'}(a_1,\ldots,a_n), \underline{c_2'}(b_1,\ldots,b_n)) \\
[\![\lceil c_1'c_2'\rceil(\varphi_1|_1^{\lceil c_1c_2\rceil},\ldots,\varphi_n|_1^{\lceil c_1c_2\rceil})]\!]_{\mathfrak{A}\times\rho} & = & (\underline{c_1'}(a_1,\ldots,a_n), \underline{c_2'}(a_1',\ldots,a_n')) \\
[\![\lceil c_1'c_2'\rceil(\varphi_1|_2^{\lceil c_1c_2\rceil},\ldots,\varphi_n|_2^{\lceil c_1c_2\rceil})]\!]_{\mathfrak{A}\times\rho} & = & (\underline{c_1'}(b_1',\ldots,b_n'), \underline{c_2'}(b_1,\ldots,b_n))
\end{array}$$

and so the thesis follows.

(ii) $c_1' = c_1$ and $c_2' = c_2$. Then

$$\begin{array}{lcl}
[\![\lceil c_1c_2\rceil(\varphi_1,\ldots,\varphi_n)]\!]_{\mathfrak{A}\times\rho} & = & (\underline{c_1}(a_1,\ldots,a_n), \underline{c_2}(b_1,\ldots,b_n)) \\
[\![c_1(\varphi_1|_1^{\lceil c_1c_2\rceil},\ldots,\varphi_n|_1^{\lceil c_1c_2\rceil})]\!]_{\mathfrak{A}\times\rho} & = & (\underline{c_1}(a_1,\ldots,a_n), \underline{c_1}(a_1,\ldots,a_n)) \\
[\![c_2(\varphi_1|_2^{\lceil c_1c_2\rceil},\ldots,\varphi_n|_2^{\lceil c_1c_2\rceil})]\!]_{\mathfrak{A}\times\rho} & = & (\underline{c_2}(b_1,\ldots,b_n), \underline{c_2}(b_1,\ldots,b_n))
\end{array}$$

and the thesis follows. QED

Proposition 3.4 Let $M \in \mathcal{M}$, ρ be an assignment over $M^\times$ and φ a formula in $L^\times(\Xi)$. Then

$$[\![\varphi]\!]_{\mathfrak{A}\times\rho} = \left(([\![\varphi|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_1, ([\![\varphi|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_2\right).$$

Proof: Assume that the combined constructors that occur in φ are $\lceil c_kd_k\rceil$ for $k = 1,\ldots,n$. We prove the result by induction on n.

(a) the case $n = 1$ is a consequence of Proposition 3.3.

(b) $n > 0$. The induction hypothesis states that

$$[\![\varphi|_1^{\lceil c_nd_n\rceil}]\!]_{\mathfrak{A}\times\rho} = \left(([\![\varphi|_1^{\lceil c_nd_n\rceil}|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_1, ([\![\varphi|_1^{\lceil c_nd_n\rceil}|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_2\right)$$

and

$$[\![\varphi|_2^{\lceil c_nd_n\rceil}]\!]_{\mathfrak{A}\times\rho} = \left(([\![\varphi|_2^{\lceil c_nd_n\rceil}|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_1, ([\![\varphi|_2^{\lceil c_nd_n\rceil}|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_2\right).$$

On the other hand, by the same proposition,

$$[\![\varphi]\!]_{\mathfrak{A}\times\rho} = \left(([\![\varphi|_1^{\lceil c_nd_n\rceil}]\!]_{\mathfrak{A}\times\rho})_1, ([\![\varphi|_2^{\lceil c_nd_n\rceil}]\!]_{\mathfrak{A}\times\rho})_2\right)$$

and so

$$[\![\varphi]\!]_{\mathfrak{A}\times\rho} = \left(([\![\varphi|_1^{\lceil c_nd_n\rceil}|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_1, ([\![\varphi|_2^{\lceil c_nd_n\rceil}|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}\times\rho})_2\right)$$

and the thesis follows. QED

Proposition 3.5 The lifting rule LFT is sound in $\mathcal{L}^\times$.

Proof: Let $M \in \mathcal{M}$ and ρ an assignment over $M^\times$. Assume that:

$$M^\times, \rho \Vdash_\times \varphi|_1^{\lceil\cdot\cdot\rceil} \text{ and } M^\times, \rho \Vdash_\times \varphi|_2^{\lceil\cdot\cdot\rceil}.$$

Then $[\![\varphi|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho}, [\![\varphi|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho} \in D^\times$ and so

$$([\![\varphi|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho})_1, ([\![\varphi|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho})_2 \in D.$$

Thus, by Proposition 3.4, $[\![\varphi]\!]_{\mathfrak{A}^\times\rho} \in D^\times$ and, consequently, $M^\times, \rho \Vdash_\times \varphi$. QED

Proposition 3.6 Let $M \in \mathcal{M}$ and φ a concrete formula in L. Then,

$$([\![\varphi]\!]_{\mathfrak{A}^\times})_1 = ([\![\varphi]\!]_{\mathfrak{A}^\times})_2.$$

Proof: The proof follows by a straightforward induction on the structure of φ.
(a) φ is p. Then $([\![\varphi]\!]_{\mathfrak{A}^\times})_1 = \underline{p} = ([\![\varphi]\!]_{\mathfrak{A}^\times})_2$.
(b) φ is $c(\varphi_1, \ldots, \varphi_n)$. Then, by the induction hypothesis, for $j = 1, \ldots, n$,

$$([\![\varphi_j]\!]_{\mathfrak{A}^\times})_1 = ([\![\varphi_j]\!]_{\mathfrak{A}^\times})_2.$$

Hence,

$$\begin{aligned} ([\![\varphi]\!]_{\mathfrak{A}^\times})_1 &= \underline{c}(([\![\varphi_1]\!]_{\mathfrak{A}^\times})_1, \ldots, ([\![\varphi_n]\!]_{\mathfrak{A}^\times})_1) \\ &= \underline{c}(([\![\varphi_1]\!]_{\mathfrak{A}^\times})_2, \ldots, ([\![\varphi_n]\!]_{\mathfrak{A}^\times})_2) \\ &= ([\![\varphi]\!]_{\mathfrak{A}^\times})_2. \end{aligned}$$

QED

Proposition 3.7 The co-lifting rule cLFT is sound in $\mathcal{L}^\times$.

Proof: Let $M \in \mathcal{M}$, ρ an assignment over $M^\times$ and φ a concrete formula in $L^\times$. Assume that $M^\times, \rho \Vdash_\times \varphi$. Then, $[\![\varphi]\!]_{\mathfrak{A}^\times\rho} \in D^\times$. By Proposition 3.4, $([\![\varphi|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho})_1, ([\![\varphi|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho})_2 \in D$. Therefore, using Proposition 3.6,

$$[\![\varphi|_1^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho}, [\![\varphi|_2^{\lceil\cdot\cdot\rceil}]\!]_{\mathfrak{A}^\times\rho} \in D^\times.$$

Thus, $M^\times, \rho \Vdash_\times \varphi|_1^{\lceil\cdot\cdot\rceil}$ and $M^\times, \rho \Vdash_\times \varphi|_2^{\lceil\cdot\cdot\rceil}$. QED

Proposition 3.8 Entailment $\vDash_\times$ is closed for substitution.

Proof: Let $\sigma : \Xi \to L^\times(\Xi)$ be a substitution. Given a formula $\psi \in L^\times(\Xi)$, we denote by $\sigma(\psi)$ the formula in $L^\times(\Xi)$ obtained from ψ by simultaneously replacing each schema variable ξ by $\sigma(\xi)$.

We start by showing, by induction on $\psi \in L^\times(\Xi)$, that

$$[\![\psi]\!]_{\mathfrak{A}^\times\rho_\sigma} = [\![\sigma(\psi)]\!]_{\mathfrak{A}^\times\rho}$$

for every $M \in \mathcal{M}$ and assignment ρ over $M^\times$, where ρ_σ is an assignment over $M^\times$ such that $\rho_\sigma(\xi) = [\![\sigma(\xi)]\!]_{\mathfrak{A}^\times\rho}$ for every $\xi \in \Xi$.
(a) ψ is $\xi \in \Xi$. Then

$$[\![\xi]\!]_{\mathfrak{A}^\times\rho_\sigma} = \rho_\sigma(\xi) = [\![\sigma(\xi)]\!]_{\mathfrak{A}^\times\rho}.$$

(b) ψ is $\lceil c_1c_2\rceil(\varphi_1,\dots,\varphi_n)$. Observe that, for $k = 1, 2$,

$$\begin{array}{rcl}([\![\lceil c_1c_2\rceil(\varphi_1,\dots,\varphi_n)]\!]_{\mathfrak{A}^\times\rho_\sigma})_k & = & \underline{c_k}(([\![\varphi_1]\!]_{\mathfrak{A}^\times\rho_\sigma})_k,\dots,([\![\varphi_n]\!]_{\mathfrak{A}^\times\rho_\sigma})_k) \\ & = & \underline{c_k}(([\![\sigma(\varphi_1)]\!]_{\mathfrak{A}^\times\rho})_k,\dots,([\![\sigma(\varphi_n)]\!]_{\mathfrak{A}^\times\rho})_k).\end{array}$$

Hence

$$\begin{array}{rcl}[\![\lceil c_1c_2\rceil(\varphi_1,\dots,\varphi_n)]\!]_{\mathfrak{A}^\times\rho_\sigma} & = & \underline{\lceil c_1c_2\rceil}([\![\sigma(\varphi_1)]\!]_{\mathfrak{A}^\times\rho},\dots,[\![\sigma(\varphi_n)]\!]_{\mathfrak{A}^\times\rho}) \\ & = & [\![\lceil c_1c_2\rceil(\sigma(\varphi_1),\dots,\sigma(\varphi_n))]\!]_{\mathfrak{A}^\times\rho} \\ & = & [\![\sigma(\lceil c_1c_2\rceil(\varphi_1,\dots,\varphi_n))]\!]_{\mathfrak{A}^\times\rho}.\end{array}$$

Now we can prove

$$\text{if } \Gamma \vDash_\times \varphi \ \text{ then } \ \sigma(\Gamma) \vDash_\times \sigma(\varphi).$$

Assume that $\Gamma \vDash_\times \varphi$. Let $M \in \mathcal{M}$ and ρ an assignment over $M^\times$. Assume that $M^\times, \rho \Vdash_\times \sigma(\gamma)$ for every $\gamma \in \Gamma$. Hence $[\![\sigma(\gamma)]\!]_{\mathfrak{A}^\times\rho} \in D^\times$ for every $\gamma \in \Gamma$. Thus, by the lemma, $[\![\gamma]\!]_{\mathfrak{A}^\times\rho_\sigma} \in D^\times$ for every $\gamma \in \Gamma$ and so $M^\times, \rho_\sigma \Vdash_\times \Gamma$. By the hypothesis, $M^\times, \rho_\sigma \Vdash_\times \varphi$. Similarly, we obtain that $M^\times, \rho \Vdash_\times \sigma(\varphi)$. QED

Theorem 3.9 (Soundness)
If $\mathcal{L}$ is sound then $\mathcal{L}^\times$ is sound.

Proof: Assume that $\mathcal{L}$ is sound. Then, in particular, all the rules in Δ are sound in $\mathcal{L}$ and so, by Proposition 3.2, all the rules in Δ are sound in $\mathcal{L}^\times$. Moreover, the rules LFT and cLFT are sound thanks to Proposition 3.5 and Proposition 3.7, respectively. Therefore, $\mathcal{L}^\times$ is sound using also Proposition 3.8. QED

The task now is to show that if $\mathcal{L}$ is complete then so is $\mathcal{L}^\times$. However, we are able to establish this result only for concrete formulas (Theorem 3.13). Thus, it becomes handy to say that a logic is *concretely complete* if it is complete with respect to concrete formulas. We start by proving the relevant lemmas.

Proposition 3.10 If $\mathcal{L}$ is concretely complete then, for every $\Gamma \cup \{\varphi\} \subset L$,

$$\text{if } \Gamma \nvdash_\times \varphi \text{ then } \Gamma \nvDash_\times \varphi.$$

Proof: Assume that $\Gamma \nvdash_\times \varphi$. Then $\Gamma \nvdash \varphi$ and, since $\mathcal{L}$ is concretely complete, then $\Gamma \nvDash \varphi$. Hence, there is $M \in \mathcal{M}$ such that $M \Vdash \gamma$ for every $\gamma \in \Gamma$ and $M \nVdash \varphi$. That is, there is $M \in \mathcal{M}$ such that

$$[\![\gamma]\!]_{\mathfrak{A}} \in D \text{ for every } \gamma \in \Gamma \quad \text{and} \quad [\![\varphi]\!]_{\mathfrak{A}} \notin D$$

By Proposition 3.1,

$$[\![\gamma]\!]_{\mathfrak{A}^\times} = ([\![\gamma]\!]_{\mathfrak{A}}, [\![\gamma]\!]_{\mathfrak{A}}) \in D^\times \text{ for every } \gamma \in \Gamma \quad \text{and} \quad [\![\varphi]\!]_{\mathfrak{A}^\times} = ([\![\varphi]\!]_{\mathfrak{A}}, [\![\varphi]\!]_{\mathfrak{A}}) \notin D^\times$$

and so the thesis $\Gamma \nvDash_\times \varphi$ follows. QED

Proposition 3.11 Assume that

$$\text{if } \Gamma' \not\vdash_\times \varphi' \text{ then } \Gamma' \not\vDash_\times \varphi'$$

for every $\Gamma' \cup \{\varphi'\} \subset L$. Then, for every $\Gamma \subset L$ and $\varphi \in L^\times$,

$$\text{if } \Gamma \not\vdash_\times \varphi \text{ then } \Gamma \not\vDash_\times \varphi.$$

Proof: Assume that $\Gamma \not\vdash_\times \varphi$. Then, taking into account the lifting rule LFT,

$$\text{either} \quad \Gamma \not\vdash_\times \varphi|^{\lceil\cdot\cdot\rceil}_{1} \quad \text{or} \quad \Gamma \not\vdash_\times \varphi|^{\lceil\cdot\cdot\rceil}_{2}.$$

Assume, without loss of generality, that $\Gamma \not\vdash_\times \varphi|^{\lceil\cdot\cdot\rceil}_{1}$. Then, by the hypothesis,

$$\Gamma \not\vDash_\times \varphi|^{\lceil\cdot\cdot\rceil}_{1}.$$

That is, there is $M \in \mathcal{M}$ such that

$$M^\times \Vdash_\times \gamma, \;\; \text{for every } \gamma \in \Gamma \quad \text{and} \quad M^\times \nVdash_\times \varphi|^{\lceil\cdot\cdot\rceil}_{1}.$$

Thus, $\llbracket \varphi|^{\lceil\cdot\cdot\rceil}_{1} \rrbracket_{\mathfrak{A}^\times} \notin D^\times$ and, moreover, $(\llbracket \varphi|^{\lceil\cdot\cdot\rceil}_{1} \rrbracket_{\mathfrak{A}^\times})_1, (\llbracket \varphi|^{\lceil\cdot\cdot\rceil}_{1} \rrbracket_{\mathfrak{A}^\times})_2 \notin D$ since $\varphi|^{\lceil\cdot\cdot\rceil}_{1}$ does not have combined constructors. By Proposition 3.4

$$\llbracket \varphi \rrbracket_{\mathfrak{A}^\times} = \left((\llbracket \varphi|^{\lceil\cdot\cdot\rceil}_{1} \rrbracket_{\mathfrak{A}^\times})_1, (\llbracket \varphi|^{\lceil\cdot\cdot\rceil}_{2} \rrbracket_{\mathfrak{A}^\times})_2 \right).$$

Hence, $\llbracket \varphi \rrbracket_{\mathfrak{A}^\times} \notin D^\times$ and so $M^\times \nVdash_\times \varphi$. Therefore, $\Gamma \not\vDash_\times \varphi$. QED

Proposition 3.12 Assume that

$$\text{if } \Gamma' \not\vdash_\times \varphi' \text{ then } \Gamma' \not\vDash_\times \varphi'$$

for every $\Gamma' \subset L$ and $\varphi' \in L^\times$. Then

$$\text{if } \Gamma \not\vdash_\times \varphi \text{ then } \Gamma \not\vDash_\times \varphi$$

for every $\Gamma \cup \{\varphi\} \in L^\times$.

Proof: Assume that $\Gamma \cup \{\varphi\} \in L^\times$ and $\Gamma \not\vdash_\times \varphi$. Then

$$\Gamma|^{\lceil\cdot\cdot\rceil}_{1} \cup \Gamma|^{\lceil\cdot\cdot\rceil}_{2} \not\vdash_\times \varphi$$

taking into account the co-lifting rule cLFT, where $\Gamma|^{\lceil\cdot\cdot\rceil}_{k} = \{\gamma|^{\lceil\cdot\cdot\rceil}_{k} : \gamma \in \Gamma\}$ for $k = 1, 2$. Since no combined constructors occur in both $\Gamma|^{\lceil\cdot\cdot\rceil}_{1}$ and $\Gamma|^{\lceil\cdot\cdot\rceil}_{2}$, we can use the hypothesis to conclude that

$$\Gamma|^{\lceil\cdot\cdot\rceil}_{1} \cup \Gamma|^{\lceil\cdot\cdot\rceil}_{2} \not\vDash_\times \varphi.$$

That is, there is $M \in \mathcal{M}$ such that

$$M^\times \Vdash_\times \gamma', \;\; \text{for every } \gamma' \in \Gamma|^{\lceil\cdot\cdot\rceil}_{1} \cup \Gamma|^{\lceil\cdot\cdot\rceil}_{2} \quad \text{and} \quad M^\times \nVdash_\times \varphi.$$

Hence, $[\![\gamma']\!]_{\mathfrak{A}^\times} \in D^\times$ for every $\gamma' \in \Gamma|_1^{[\cdot,\cdot]} \cup \Gamma|_2^{[\cdot,\cdot]}$ and $[\![\varphi]\!]_{\mathfrak{A}^\times} \notin D^\times$. Therefore,

$$([\![\gamma']\!]_{\mathfrak{A}^\times})_1 = ([\![\gamma']\!]_{\mathfrak{A}^\times})_2 \in D$$

for every $\gamma' \in \Gamma|_1^{[\cdot,\cdot]} \cup \Gamma|_2^{[\cdot,\cdot]}$. Let $\gamma \in \Gamma$. By Proposition 3.4,

$$[\![\gamma]\!]_{\mathfrak{A}^\times \rho} = \left(([\![\gamma|_1^{[\cdot,\cdot]}]\!]_{\mathfrak{A}^\times})_1, ([\![\gamma|_2^{[\cdot,\cdot]}]\!]_{\mathfrak{A}^\times})_2\right).$$

Thus, $[\![\gamma]\!]_{\mathfrak{A}^\times} \in D^\times$ and so $M^\times \Vdash \Gamma$. Therefore, $\Gamma \not\models_\times \varphi$. QED

Theorem 3.13 (Concrete completeness)
If $\mathcal{L}$ is concretely complete then $\mathcal{L}^\times$ is concretely complete.

Proof: Assume that $\mathcal{L}$ is concretely complete. Then, by Proposition 3.10,

$$\text{if } \Gamma \not\vdash_\times \varphi \text{ then } \Gamma \not\models_\times \varphi$$

for every $\Gamma \cup \{\varphi\} \subset L$. Hence, for every $\Gamma \subset L$ and $\varphi \in L^\times$,

$$\text{if } \Gamma \not\vdash_\times \varphi \text{ then } \Gamma \not\models_\times \varphi$$

using Proposition 3.11. Thus, thanks to Proposition 3.12,

$$\text{if } \Gamma \not\vdash_\times \varphi \text{ then } \Gamma \not\models_\times \varphi$$

for every $\Gamma \cup \{\varphi\} \in L^\times$. QED

The completeness result was established only for concrete formulas. There is no hope of extending the completeness result to schema formulas (at least without constraining their instantiation only to concrete formulas of $\mathcal{L}$) given the concrete nature of cLFT.

Finally, we check if $\mathcal{L}^\times$ is a conservative extension of $\mathcal{L}$. In this case we are able to establish the envisaged result also for schema formulas since it does not depend on cLFT.

Theorem 3.14 (Conservativeness)
For every $\Gamma \cup \{\varphi\} \subset L(\Xi)$, if $\Gamma \models_\times \varphi$ then $\Gamma \models \varphi$.

Proof: Assume that $\Gamma \models_\times \varphi$ and let $M \in \mathcal{M}$ and ρ an assignment over M such that $M, \rho \Vdash \Gamma$. Hence $[\![\gamma]\!]_{\mathfrak{A}\rho} \in D$ for every $\gamma \in \Gamma$. Denote by $\rho^\times$ the unique assignment over $M^\times$ such that $(\rho^\times)_1 = (\rho^\times)_2 = \rho$. Then, by Proposition 3.1, $[\![\gamma]\!]_{\mathfrak{A}^\times \rho^\times} \in D^\times$ for every $\gamma \in \Gamma$. Therefore, $[\![\varphi]\!]_{\mathfrak{A}^\times \rho^\times} \in D^\times$ and so by the same proposition, $[\![\varphi]\!]_{\mathfrak{A}\rho} \in D$. That is $M, \rho \Vdash \varphi$. QED

It is worthwhile to mention again that, thanks to the conservativeness result above, cLFT guarantees that each meet-combined constructor enjoys only the common properties of its components in the original logic. Furthermore, this conservativeness result leads directly to one of our major goals: the preservation of consistency.

Theorem 3.15 (Consistency)
If $\mathcal{L}$ is consistent then so is $\mathcal{L}^\times$.

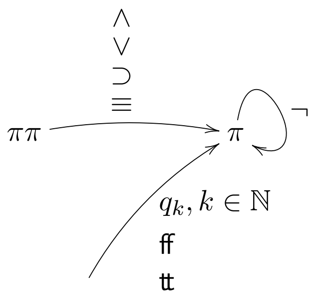

Figure 1: CPL signature.

4 The case of classical propositional logic

For illustrating the proposed calculus and product semantics for meet-combined constructors, we choose classical propositional logic (CPL), endowed with the signature in Figure 1 (where π is the assertions sort and each q_k is a propositional symbol). Indeed, $\mathsf{CPL}^\times$ is sufficiently rich to provide interesting examples. We assume that the CPL calculus includes the tautologies as axioms plus modus ponens (MP). Furthermore, we assume that the CPL semantics is composed of the matrices induced by valuations. Recall that each valuation $v : \{q_k : k \in \mathbb{N}\} \to \{0,1\}$ canonically induces a matrix M_v with $A_v = \{0,1\}$, with the denotation of the propositional symbols imposed by v, satisfying precisely the same formulas for each assignment. We start by analyzing some interesting meet-combinations and only afterward prove that the metatheorem of deduction (albeit with a proviso concerning the use of cLFT) and the metatheorem of substitution of equivalents still hold in $\mathsf{CPL}^\times$.

Examples and counterexamples

Commutativity, for instance, is a common property of conjunction and disjunction. Thus, we should be able to derive it for $\lceil\wedge\vee\rceil$. Indeed, for arbitrary $\varphi, \psi \in L^\times$, we can build the derivation in Figure 2 for

$$\varphi \lceil\wedge\vee\rceil \psi \vdash_\times \psi \lceil\wedge\vee\rceil \varphi.$$

One may wonder if we can extend this result to schema formulas. In fact we have

$$\varphi \lceil\wedge\vee\rceil \psi \vDash_\times \psi \lceil\wedge\vee\rceil \varphi$$

even when φ and ψ contain schema variables. But, we are not able to recover in this case the result with the calculus (recall that completeness was established only for concrete formulas).

Observe that φ, ψ must be concrete in the derivation in Figure 2 because cLFT is used. One might be tempted to relax the cLFT rule to schematic formulas. However, as mentioned before, this would not be sound. Indeed,

$$\xi_1 \lceil\wedge\vee\rceil \xi_2 \nvDash_\times \xi_1 \wedge \xi_2.$$

1	$\varphi \lceil\wedge\vee\rceil \psi$	HYP
2	$\varphi\|^{\lceil\cdot\cdot\rceil}_1 \wedge \psi\|^{\lceil\cdot\cdot\rceil}_1$	cLFT : 1
3	$\varphi\|^{\lceil\cdot\cdot\rceil}_2 \vee \psi\|^{\lceil\cdot\cdot\rceil}_2$	cLFT : 1
4	$(\varphi\|^{\lceil\cdot\cdot\rceil}_1 \wedge \psi\|^{\lceil\cdot\cdot\rceil}_1) \supset (\psi\|^{\lceil\cdot\cdot\rceil}_1 \wedge \varphi\|^{\lceil\cdot\cdot\rceil}_1)$	TAUT
5	$(\varphi\|^{\lceil\cdot\cdot\rceil}_2 \vee \psi\|^{\lceil\cdot\cdot\rceil}_2) \supset (\psi\|^{\lceil\cdot\cdot\rceil}_2 \vee \varphi\|^{\lceil\cdot\cdot\rceil}_2)$	TAUT
6	$\psi\|^{\lceil\cdot\cdot\rceil}_1 \wedge \varphi\|^{\lceil\cdot\cdot\rceil}_1$	MP : 2, 4
7	$\psi\|^{\lceil\cdot\cdot\rceil}_2 \vee \varphi\|^{\lceil\cdot\cdot\rceil}_2$	MP : 3, 5
8	$\psi \lceil\wedge\vee\rceil \varphi$	LFT : 6, 7

Figure 2: Derivation of commutativity of $\lceil\wedge\vee\rceil$.

For instance, consider the matrix M_v induced by a valuation v and the assignment ρ over $M_v^\times$ such that $\rho(\xi_1) = (1, 0)$ and $\rho(\xi_2) = (1, 1)$. Then,

$$[\![\xi_1 \lceil\wedge\vee\rceil \xi_2]\!]_{\mathfrak{A}_v^\times \rho} = \underline{\lceil\wedge\vee\rceil}((1, 0), (1, 1)) = (1, 1) \in D_v^\times$$

and

$$[\![\xi_1 \wedge \xi_2]\!]_{\mathfrak{A}_v^\times \rho} = \underline{\lceil\wedge\wedge\rceil}((1, 0), (1, 1)) = (1, 0) \notin D_v^\times.$$

By the way, making use of the mixed truth values, we can also provide now a counterexample showing that exchangeability of components does not hold in general when the formula contains other combined connectives. In fact,

$$\lceil\mathtt{tt}\mathtt{ff}\rceil \lceil\wedge\vee\rceil \lceil\mathtt{tt}\mathtt{tt}\rceil \not\models_\times \lceil\mathtt{tt}\mathtt{ff}\rceil \lceil\vee\wedge\rceil \lceil\mathtt{tt}\mathtt{tt}\rceil .$$

Consider again the matrix M_v above. Then,

$$[\![\lceil\mathtt{tt}\mathtt{ff}\rceil \lceil\wedge\vee\rceil \lceil\mathtt{tt}\mathtt{tt}\rceil]\!]_{\mathfrak{A}_v^\times} = \underline{\lceil\wedge\vee\rceil}((1, 0), (1, 1)) = (1, 1) \in D_v^\times$$

and

$$[\![\lceil\mathtt{tt}\mathtt{ff}\rceil \lceil\vee\wedge\rceil \lceil\mathtt{tt}\mathtt{tt}\rceil]\!]_{\mathfrak{A}_v^\times} = \underline{\lceil\vee\wedge\rceil}((1, 0), (1, 1)) = (1, 0) \notin D_v^\times.$$

In short, the hypothesis on φ in Theorem 2.1 is essential.

Observe also that the mixed truth values raise new problems regarding the internalization of entailment by implication. For instance, we have

$$\mathtt{tt} \lceil\wedge\vee\rceil \mathtt{ff} \models_\times \mathtt{ff}.$$

However, we have

$$\not\models_\times (\mathtt{tt} \lceil\wedge\vee\rceil \mathtt{ff}) \supset \mathtt{ff}.$$

Indeed, for every valuation v, $[\![(\mathtt{tt} \lceil\wedge\vee\rceil \mathtt{ff}) \supset \mathtt{ff}]\!]_{\mathfrak{A}_v^\times} = (1, 0) \notin D^\times$. This example shows that the metatheorem of deduction will need some constraining in $\mathsf{CPL}^\times$. We show below that it holds as long as no essential usage of cLFT is made in the derivation.

As a final remark concerning $\lceil\wedge\vee\rceil$, since conjunction entails disjunction, we have:

$$\varphi \lceil\wedge\vee\rceil \psi \ \dashv\vdash_\times \ \varphi \wedge \psi \text{ for } \varphi, \psi \in L.$$

In general, the same will happen whenever we meet-combine two constructors c_1 and c_2 such that

$$c_1(\varphi_1, \ldots, \varphi_n) \vdash c_2(\varphi_1, \ldots, \varphi_n).$$

The combined constructor $\lceil c_1 c_2 \rceil$ and c_1 will be weakly inter-derivable in the enriched logic:

$$\lceil c_1 c_2 \rceil(\varphi_1, \ldots, \varphi_n) \ \dashv\vdash_\times \ c_1(\varphi_1, \ldots, \varphi_n) \text{ for } \varphi_1, \ldots, \varphi_n \in L.$$

In particular, $\lceil\equiv\supset\rceil$ and $\equiv$ are weakly inter-derivable since $\varphi \equiv \psi \vdash \varphi \supset \psi$. Moreover, $\lceil \mathsf{ff}\mathsf{tt} \rceil$ and ff are also weakly inter-derivable since $\mathsf{ff} \vdash \mathsf{tt}$.

Furthermore, in $\mathsf{CPL}^\times$ we have the following weak inter-derivability

$$\lceil c_1 c_2 \rceil(\varphi_1, \ldots, \varphi_n) \ \dashv\vdash_\times \ c_1(\varphi_1, \ldots, \varphi_n) \wedge c_2(\varphi_1, \ldots, \varphi_n)$$

for any pair of connectives of arity n and $\varphi_1, \ldots, \varphi_n \in L$. Observe that this relationship is quite weak because it holds only for arguments in L and, more importantly, does not correspond to a valid equivalence.

Nevertheless, one should wonder if the connective '$c_1 c_2$' introduced in CPL by the following abbreviation would not do the trick:

$$\text{`}c_1 c_2\text{'}(\varphi_1, \ldots, \varphi_n) \text{ stands for } c_1(\varphi_1, \ldots, \varphi_n) \wedge c_2(\varphi_1, \ldots, \varphi_n).$$

If so, our desideratum could be achieved within CPL itself by just using a few abbreviations. However, as one might expect, these mixed connectives introduced by abbreviation do not have the envisaged properties, namely they do not comply with LFT and cLFT.

Indeed, LFT is violated as illustrated by the following example. Take φ to be

$$(\mathsf{tt} \lceil\vee\supset\rceil \mathsf{ff}) \lceil\vee\supset\rceil \mathsf{ff},$$

that is,

$$(\mathsf{tt} \text{`}\vee\supset\text{'} \mathsf{ff}) \text{`}\vee\supset\text{'} \mathsf{ff}.$$

Then, $\varphi|_1^{\lceil\cdot\cdot\rceil}$ is $(\mathsf{tt} \vee \mathsf{ff}) \vee \mathsf{ff}$ and, so, true, Moreover, $\varphi|_2^{\lceil\cdot\cdot\rceil}$ is $(\mathsf{tt} \supset \mathsf{ff}) \supset \mathsf{ff}$ and, so, also true. However, φ is

$$(((\mathsf{tt} \vee \mathsf{ff}) \wedge (\mathsf{tt} \supset \mathsf{ff})) \vee \mathsf{ff}) \wedge (((\mathsf{tt} \vee \mathsf{ff}) \wedge (\mathsf{tt} \supset \mathsf{ff})) \supset \mathsf{ff})$$

and, so, false.

Furthermore, cLFT is also violated. Take φ to be

$$\neg \lceil \mathsf{tt}\mathsf{ff} \rceil,$$

that is,

$$\neg \text{`}\mathsf{tt}\mathsf{ff}\text{'}.$$

Clearly, φ is $\neg(\mathtt{tt} \wedge \mathtt{ff})$ and, so, true. However, $\varphi|_1^{\lceil\cdot\cdot\rceil}$ is $\neg\,\mathtt{tt}$ and, thus, false.

After looking at several combinations with a component stronger than the other, we now turn our attention to the combination of disjunction and implication. When the second argument is true then both disjunction and implication are true, an obvious example of a property shared by the two connectives. More concretely,

$$\begin{cases} \psi \vdash \varphi \vee \psi \\ \psi \vdash \varphi \supset \psi \end{cases}$$

and, so, we expect this property to hold also for $\lceil\vee\supset\rceil$. Indeed,

$$\psi \vdash_{\times} \varphi \lceil\vee\supset\rceil \psi$$

follows by LFT. For another example, recall that

$$\begin{cases} \varphi \supset \psi \vdash (\neg\,\varphi) \vee \psi \\ \varphi \vee \psi \vdash \varphi \vee \psi \end{cases}$$

that can be rewritten as

$$\begin{cases} \varphi \supset \psi \vdash (\neg\,\varphi) \vee \psi \\ \varphi \vee \psi \vdash (\neg^2\,\varphi) \vee \psi \end{cases}$$

making good use of the unary derived connective $\neg^2$ introduced by abbreviation as follows:

$$\neg^2\,\varphi \text{ stands for } \neg\,\neg\,\varphi.$$

Thus, we can easily produce a derivation for

$$\varphi \lceil\vee\supset\rceil \psi \;\vdash_{\times}\; (\lceil\neg\neg^2\rceil\,\varphi) \vee \psi.$$

Metatheorems

The examples above show that the nature of $\mathsf{CPL}^{\times}$ is quite different from classical logic, notwithstanding the fact that it is a conservative extension of CPL. Indeed, the presence of the combined constructors and the existence of mixed truth values in the models have far reaching consequences.

In particular, the metatheorem of deduction does not hold in general. We proceed to establish that it does as long as no essential use is made of co-liftings. To this end we need some auxiliary notions and the following technical lemma on promoting the lifting rule to a valid implication.

Proposition 4.1 Let $\varphi \in L^{\times}$. Then

$$\vdash_{\times} (\varphi|_1^{\lceil\cdot\cdot\rceil} \wedge \varphi|_2^{\lceil\cdot\cdot\rceil}) \supset \varphi.$$

Proof: The result follows from

$$\vDash_{\times} (\varphi|_1^{\lceil\cdot\cdot\rceil} \wedge \varphi|_2^{\lceil\cdot\cdot\rceil}) \supset \varphi$$

using Theorem 3.13. Indeed, for $k = 1, 2$, observe that

$$([\![(\varphi|^{\lceil\cdot\cdot\rceil}_{1} \wedge \varphi|^{\lceil\cdot\cdot\rceil}_{2}) \supset \varphi]\!]_{\mathfrak{A}^{\times}_{v}})_k = \underline{\supset}(\underline{\wedge}(([\![\varphi|^{\lceil\cdot\cdot\rceil}_{1}]\!]_{\mathfrak{A}^{\times}_{v}})_k, ([\![\varphi|^{\lceil\cdot\cdot\rceil}_{2}]\!]_{\mathfrak{A}^{\times}_{v}})_k), ([\![\varphi]\!]_{\mathfrak{A}^{\times}_{v}})_k)$$

for every valuation v. Moreover, by Proposition 3.4,

$$([\![\varphi]\!]_{\mathfrak{A}^{\times}_{v}})_k = ([\![\varphi|^{\lceil\cdot\cdot\rceil}_{1}]\!]_{\mathfrak{A}^{\times}_{v}})_k.$$

Consider two cases: (i) $([\![\varphi|^{\lceil\cdot\cdot\rceil}_{1}]\!]_{\mathfrak{A}^{\times}_{v}})_k = 0$: then the antecedent of the implication is false and so the implication is true; (ii) $([\![\varphi|^{\lceil\cdot\cdot\rceil}_{1}]\!]_{\mathfrak{A}^{\times}_{v}})_k = 1$: then the consequent of the implication is true and so the implication is true. QED

Let $\psi_1 \dots \psi_n$ be a derivation for $\Gamma \vdash_{\times} \varphi$. Formula ψ_i *depends* on $\gamma \in \Gamma$ in this derivation if

- either ψ_i is γ;
- or ψ_i is obtained using a rule (either MP or LFT or cLFT) with at least one of the premises depending on γ.

Formula ψ_i is an *essential co-lifting* over a dependent of γ in the derivation $\psi_1 \dots \psi_n$ if

- $\psi_i = \psi_j|^{\lceil\cdot\cdot\rceil}_{k}$ is obtained using cLFT from ψ_j;
- ψ_j depends on γ;
- γ has combined connectives.

Theorem 4.2 (Metatheorem of deduction in $\mathsf{CPL}^{\times}$)
Let $\Gamma \cup \{\eta, \varphi\} \subseteq L^{\times}$ and $\psi_1 \dots \psi_n$ be a derivation for $\Gamma, \eta \vdash_{\times} \varphi$ without essential co-liftings over dependents of η. Then $\Gamma \vdash_{\times} \eta \supset \varphi$.

Proof: The proof is carried out by induction on n. We consider the step for the new rules.
Step: (a) φ is obtained by LFT from $\varphi|^{\lceil\cdot\cdot\rceil}_{1}$ and $\varphi|^{\lceil\cdot\cdot\rceil}_{2}$. Then $\Gamma, \eta \vdash_{\times} \varphi|^{\lceil\cdot\cdot\rceil}_{1}$ and $\Gamma, \eta \vdash_{\times} \varphi|^{\lceil\cdot\cdot\rceil}_{1}$ with derivations with length less than n and without essential co-liftings over dependents of η. Observe that, by the induction hypothesis, $\Gamma \vdash_{\times} \eta \supset (\varphi|^{\lceil\cdot\cdot\rceil}_{1})$ and $\Gamma \vdash_{\times} \eta \supset (\varphi|^{\lceil\cdot\cdot\rceil}_{2})$. Therefore,

$$\Gamma \vdash_{\times} \eta \supset (\varphi|^{\lceil\cdot\cdot\rceil}_{1} \wedge \varphi|^{\lceil\cdot\cdot\rceil}_{2})$$

and so, by Proposition 4.1, $\Gamma \vdash_{\times} \eta \supset \varphi$.

(b) φ is $\psi|^{\lceil\cdot\cdot\rceil}_{k}$ and is obtained by cLFT from ψ. Assume, without loss of generality, that $k = 1$. Then $\Gamma, \eta \vdash_{\times} \psi$ with a derivation with length less than n and without essential co-liftings over dependents of η. Since the co-lifting in non essential there are two cases to consider:
(i) ψ does not depend on η. Then $\Gamma \vdash_{\times} \psi$ and so $\Gamma \vdash_{\times} \varphi$. Since $\Gamma \vdash_{\times} \varphi \supset (\eta \supset \varphi)$, the thesis follows by MP.
(ii) there is no combined connective occurring in η. Observe that, by the induction hypothesis, $\Gamma \vdash_{\times} \eta \supset \psi$. Then $\Gamma \vdash_{\times} \eta \supset \psi|^{\lceil\cdot\cdot\rceil}_{1}$ by cLFT. Furthermore, $\Gamma \vdash_{\times} \eta \supset \varphi$. QED

Observe that the absence of essential co-liftings is required. Indeed, for instance, an essential co-lifting is used in the derivation of

$$\mathtt{tt}\,\lceil\wedge\vee\rceil\, q_0 \vdash_\times q_0$$

provided in Figure 3.

1	$\mathtt{tt}\,\lceil\wedge\vee\rceil\, q_0$	HYP
2	$\mathtt{tt} \wedge q_0$	cLFT : 1
3	$(\mathtt{tt} \wedge q_0) \supset q_0$	TAUT
4	q_0	MP : 2, 3

Figure 3: Derivation of $\mathtt{tt}\,\lceil\wedge\vee\rceil\, q_0 \vdash_\times q_0$.

On the other hand, $\not\vDash_\times (\mathtt{tt}\,\lceil\wedge\vee\rceil\, q_0) \supset q_0$. Just consider the valuation v such that $v(q_0) = 0$. Then $[\![(\mathtt{tt}\,\lceil\wedge\vee\rceil\, q_0) \supset q_0]\!]_{\mathfrak{A}_v^\times} = (1, 0) \not\in D^\times$. Hence, by the soundness of $\mathsf{CPL}^\times$ (thanks to the soundness of CPL and Theorem 3.9),

$$\not\vdash_\times (\mathtt{tt}\,\lceil\wedge\vee\rceil\, q_0) \supset q_0.$$

This should not be surprising since $(\mathtt{tt}\,\theta\, q_0) \supset q_0$, as a predicate on θ, does not state a common logical property of conjunction and disjunction. Indeed, it is a property of conjunction since we do have $\vdash (\mathtt{tt} \wedge q_0) \supset q_0$, but it is not a property of disjunction since $\not\vdash (\mathtt{tt} \vee q_0) \supset q_0$.

Given the impact of the combined connectives on the MTD, one may wonder if any metatheorem survives as it is. As an example of a metatheorem that remains untouched in $\mathsf{CPL}^\times$, we show that substitution of equivalents still holds with no additional conditions.

Theorem 4.3 (Metatheorem of substitution of equivalents in $\mathsf{CPL}^\times$) Let $\gamma, \gamma', \varphi \in L^\times$ be such that $\vDash_\times \gamma \equiv \gamma'$. Assume that γ is a subformula of φ. Then

$$\vDash_\times \varphi \equiv \varphi'$$

where φ' is obtained from φ by substituting zero or more occurrences of γ by γ'.

Proof: Assume that $\vDash_\times \gamma \equiv \gamma'$. The proof that $\vDash_\times \varphi \equiv \varphi'$ where φ' is obtained from φ by substituting zero or more occurrences of γ by γ' follows by induction on the structure of φ. We only consider the step where φ is $\lceil c_1 c_2 \rceil(\varphi_1, \ldots, \varphi_n)$. By the induction hypothesis, for $j = 1, \ldots, n$,

$$\vDash_\times \varphi_j \equiv \varphi'_j$$

where φ'_j is obtained from φ_j is such a way that φ' is $\lceil c_1 c_2 \rceil(\varphi'_1, \ldots, \varphi'_n)$. Observe that

$$\vDash_\times \varphi_j|_k^{\lceil\cdot\cdot\rceil} \equiv \varphi'_j|_k^{\lceil\cdot\cdot\rceil}$$

for $j = 1, \ldots, n$ and $k = 1, 2$ by Proposition 3.7. Moreover, using Proposition 3.1,

$$\vDash \varphi_j|_k^{\lceil\cdot\cdot\rceil} \equiv \varphi'_j|_k^{\lceil\cdot\cdot\rceil}$$

for $j = 1, \ldots, n$ and $k = 1, 2$. Furthermore, for $k = 1, 2$,

$$\vDash c_k(\varphi_1|_k^{\lceil\cdot\cdot\rceil}, \ldots, \varphi_n|_k^{\lceil\cdot\cdot\rceil}) \equiv c_k(\varphi'_1|_k^{\lceil\cdot\cdot\rceil}, \ldots, \varphi'_n|_k^{\lceil\cdot\cdot\rceil})$$

since CPL enjoys the metatheorem of substitution of equivalents. Again, by Proposition 3.1, for $k = 1, 2$,

$$\vDash_\times c_k(\varphi_1|_k^{\lceil\cdot\cdot\rceil}, \ldots, \varphi_n|_k^{\lceil\cdot\cdot\rceil}) \equiv c_k(\varphi'_1|_k^{\lceil\cdot\cdot\rceil}, \ldots, \varphi'_n|_k^{\lceil\cdot\cdot\rceil}).$$

That is,

$$\vDash_\times (\lceil c_1 c_2 \rceil(\varphi_1, \ldots, \varphi_n) \equiv \lceil c_1 c_2 \rceil(\varphi'_1, \ldots, \varphi'_n))|_k^{\lceil\cdot\cdot\rceil}$$

for $k = 1, 2$. Hence, by Proposition 3.5,

$$\vDash_\times \lceil c_1 c_2 \rceil(\varphi_1, \ldots, \varphi_n) \equiv \lceil c_1 c_2 \rceil(\varphi'_1, \ldots, \varphi'_n),$$

as required. QED

5 Outlook

While investigating the nature of shared connectives (and other language constructors) in combined logics and the reasons why they lead frequently to inconsistency, we came up with the idea of considering combined constructors inheriting only the logical properties common to their components. More concretely, we attempted to define such a meet-combined constructor $\lceil c_1 c_2 \rceil$ having the common properties and only those common properties of constructors c_1 and c_2. That is, we wanted to enrich the original calculus with the constructor $\lceil c_1 c_2 \rceil$ specified by the following rules:

$$\frac{\varphi|_{c_1}^{\lceil c_1 c_2 \rceil}, \varphi|_{c_2}^{\lceil c_1 c_2 \rceil}}{\varphi} \qquad \frac{\varphi}{\varphi|_{c_1}^{\lceil c_1 c_2 \rceil}} \qquad \frac{\varphi}{\varphi|_{c_2}^{\lceil c_1 c_2 \rceil}}$$

The problem was to check if we would so obtain a conservative extension of the original calculus and, so, if the rules above would ensure the inheritance by $\lceil c_1 c_2 \rceil$ of precisely the common properties from its components.

To this end, we developed a semantics for the enriched calculus (actually allowing all possible meet-combinations of constructors of the original calculus). More concretely, assuming that the original logic is sound and complete with respect to a matrix semantics, we were able to endow the enriched calculus with a matrix semantics for which the enriched calculus is still sound and complete. The conservative and consistency preserving nature of the enrichment followed easily.

For assessing what was achieved we looked with some detail into the case of classical propositional logic. In particular we looked at the combined connective with the common logical properties of disjunction and conjunction, at the combined connective with the common logical properties of falsum and verum, and at the combined connective with common logical properties of disjunction and implication. It immediately became clear that in the enriched calculus the metatheorem of deduction would need some constraining. In fact, we were able to show that the metatheorem of deduction holds with the proviso that no essential use is made of the co-lifting rule.

To our knowledge not much work has been done on the idea of combining connectives (and other language constructors) outside the field of combined logics. A related idea should be mentioned. In [3] new connectives are considered with partial semantics. For instance, a new connective is proposed which is defined only for the pairs of truth values where conjunction and disjunction agree.

This first step in the investigation of combined connectives and other constructors is expected to be followed in several directions. Even within the adopted setting (language of propositional nature, Hilbert calculus and matrix semantics) many other logics can be considered, namely modal and intuitionistic logics. However, in order to address other logics the work should be carried over to other kinds of semantics, such as non-deterministic matrices [1], possible-translations semantics [5], abstract valuations [4], and graph-theoretic interpretations [9], as well as to other kinds of deduction systems, such as sequent calculi. Furthermore, at some point the attempt should be made to come out of the realm of propositional-based logics and address logics with variables and binding operators.

Returning to our original motivation in the field of combined logics, we intend to investigate the possibility of defining a combination mechanism including the constructors from the original logics plus their meet-combinations. In this way we hope to reach more conservative way of combining logics than, say, fibring. This development seems to be the most promising field of application for the idea of combining connectives [8].

Finally, regarding the combination of constructors one should investigate and compare different ways of combining them. So far, we know two ways: either by meet-combination (as proposed in this paper) or by sharing (as used in fibring [6, 11]). This may help in comparing fibring with the envisaged new way of combining logics using meet-combinations of constructors. In addition, it is conceivable that a third way of combining constructors exists, namely specified by the following rules (assuming that the original logic has some kind of disjunction):

$$\frac{\varphi|^{\lceil c_1 c_2 \rceil}_{c_1}}{\varphi} \qquad \frac{\varphi|^{\lceil c_1 c_2 \rceil}_{c_2}}{\varphi} \qquad \frac{\varphi}{\varphi|^{\lceil c_1 c_2 \rceil}_{c_1} \vee \varphi|^{\lceil c_1 c_2 \rceil}_{c_2}}$$

At this stage it is not clear what should be the semantics of such *join-combinations* and how far they are from sharings in fibrings.

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