Fibring logics: past, present and future

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ABSTRACT. This paper is a guided tour through the theory of fibring as a general mechanism for combining logics. We present the main ideas, constructions and difficulties of fibring, from both a model and a proof-theoretic perspective, and give an outline of soundness, completeness and interpolation preservation results. Along the way, we show how the current algebraic semantics of fibring relates with the original ideas of Dov Gabbay. We also analyze the collapsing problem, the challenges it raises, and discuss a number of future research directions.

1 Introduction

Combination mechanisms are operations that take logics as arguments and produce new logics as a result, including language, deductive calculi and semantics. A thorough understanding of such mechanisms is, of course, interesting in itself and in purely theoretic grounds. Namely, one might be tempted to look at predicate temporal logic as resulting from the combination of first-order logic and propositional temporal logic. But what is more, combined logics can also have a deep practical significance, namely in areas like knowledge representation in artificial intelligence, or the formal specification and verification of algorithms and protocols within software engineering and security. In complex application fields, the need for working with a combination of several different logics at the same time is the rule rather than the exception. Take, for instance, a knowledge representation problem. Depending on the universe of discourse, it may well be necessary to encompass simultaneously temporal, spatial, deontic and probabilistic aspects (e.g., for reasoning about mixed assertions like "with probability greater than 0.99, sometime in the future smoking will be forbidden everywhere"). In general, one needs at least to be able to develop theories with components in different logics or, even better, to work with theories defined in the combination of those logics (where such mixed assertions are allowed and meaningful). For all these reasons, the interest in combining logics has recently been growing, as reflected in the series [Baader and Schulz, 1996, Gabbay and de Rijke, 2000, Kirchner and Ringeissen, 2000, Armando, 2002], the special journal issues [de Rijke and Blackburn, 1996, Gabbay and Pirri, 1997], and the dedicated workshop [Carnielli *et al.*, 2004a]. For an early overview of the issues raised by combining logics, both from a practical and from a theoretical perspective, see also [Blackburn and de Rijke, 1997].

Several mechanisms for combining logics have been introduced and studied. The most abstract mechanisms are the ones that work at the level of the language (no details about how formulas are constructed) and with abstract consequence systems both for derivation and semantics. A good example of this approach is synchronization of logics [Sernadas *et al.*, 1997a, Sernadas *et al.*, 1997b, Caleiro, 2000]. However, the most rewarding mechanisms require a more concrete description of the details of the logics in hand, in particular of their syntax, namely by setting-up with the notion of signature. Temporalization [Finger and Gabbay, 1992, Finger and Gabbay, 1996, Wolter and Zakharyaschev, 2000, Finger and Weiss, 2002] and parameterization [Caleiro *et al.*, 1999] already take some advantage of this added structure. Still, the full combination of signatures that one can get by putting together the symbols that one has in the signatures of all the logics being combined was only first considered in two mechanisms for combining modal logics, namely product [Marx, 1999, Gabbay and Shehtman, 1998, Gabbay and Shehtman, 2000, Gabbay and Shehtman, 2002] and, even more general, the fusion mechanism [Thomason, 1984, Kracht and Wolter, 1991, Kracht and Wolter, 1997, Wolter, 1998, Gabbay *et al.*, 2003]. In short, the fusion of two modal systems leads to a bimodal system including the two original modal operators and common propositional connectives. Several interesting properties of modal logic systems (like soundness, weak completeness, Craig interpolation and decidability) were shown to be preserved by fusion (see [Kracht and Wolter, 1991, Kracht, 1999]).

It is at this point that Dov M. Gabbay proposes fibring [Gabbay, 1996a, Gabbay, 1996b, Gabbay, 1999] as a generalization of such combination mechanisms. Indeed, fibring can be applied beyond the universe of modal systems and captures fusion as a special case. Syntactically, the language of the fibring is freely generated from the combined signature and includes formulas where the symbols of the logics being combined can be intertwined arbitrarily many times. In proof-theoretic terms, as recognized from the very beginning, fibring is relatively easy when the calculi are presented in the same way (homogeneous fibring): the inference rules of the fibring are the union of the inference rules of original calculi, provided that one uses schema variable (for making instantiations of schema variables in a schematic rule of one of the calculi with formulas from the other logic. Herein we shall concentrate just on Hilbert calculi but the fibring of semantic tableaux [Beckert and Gabbay, 1998], or general labelled deductive systems [Rasga *et al.*, 2002, Rasga, 2003] has also been carefully studied. Fibring of calculi presented in different ways (heterogeneous fibring) is a different matter. For instance, what should be the fibring of two calculi when one is presented in a Hilbert style and the other using sequents? An answer to this problem is given in [Cruz-Filipe *et al.*, 2005] using the concept of abstract proof system and observing that all the conventional ways of presenting calculi are particular cases of abstract proof systems. From a semantic point of view, things are even more complex. The original operational idea of Gabbay was based on the assumption that both logics were endowed with a point-based semantics. In this way, a model of the fibring would also be point-based and each point would be closely related to a point in a model of each original logic. As a consequence the same model of the fibring would be related to several models in each of the original logics. As investigated in [Sernadas *et al.*, 1999], the models of the fibring would be obtained via a fixed-point construction corresponding to a universal construction (a colimit in a category with very complex morphisms). It is worthwhile to observe at this point that homogeneous fibring was only investigated in fusion of modal logics. In all other cases, some kind of common structure was defined to cope with logics with quite different semantics (of course, semantic entailment was preserved in the transformation). In [Zanardo *et al.*, 2001], it is shown that if the original point-based semantics is closed for unions then each model of the fibring will point to one model in each original logic. The interesting result is that every point-based semantics can be closed for unions without changing the entailment. With this result, a more abstract structure can be found, as we describe here, by abstracting away the point structure (like modal algebra can be seen as a more abstract structure than general Kripke frames).

Applying general mechanisms for combining logics will be significant only if general preservation results are available. Fibring is no exception. Indeed, fibring raises very nice and often very difficult preservation problems. For example, if it has been established that completeness is preserved by a combination mechanism $\bullet$ and it is known that logic system $\mathcal{L}$ corresponds to $\mathcal{L}' \bullet \mathcal{L}''$, then the completeness of $\mathcal{L}$ follows from the completeness of $\mathcal{L}'$ and $\mathcal{L}''$. This transference problem appears, of course, with respect to any relevant syntactic or semantic property of the logics at hand, possibly including, for instance the metatheorem of deduction, cut elimination, interpolation, decidability, complexity, or the finite model property. In some cases it is possible to identify sufficient conditions for the preservation of a certain property. These sufficient conditions typically involve specific properties of the original logics. No wonder that much effort has been dedicated to establishing preservation results, or finding preservation counterexamples, about different combination mechanisms.

In this paper, we will overview some of the main features and results of the theory of fibring. A more comprehensive survey about fibring can be found in [Caleiro *et al.*, 2005]. For the sake of simplicity we will adopt a simple but expressive basic universe of logic systems encompassing only propositional-based systems endowed with Hilbert calculi and ordered algebraic semantics. Nevertheless, this universe is rich enough to illustrate many interesting properties of fibring and to provide the basis for the combination of systems varying from intuitionistic to many-valued logics (and modal systems as special cases). Along the exposition we will also try to bridge the current status of the theory of fibring with the original ideas underlying Gabbay's initial formulation. Those interested in wider universes of logics (including first-order quantification, higher-order features, non-truth-functional semantics, etc.) where fibring can still be defined should look also at [Sernadas *et al.*, 2000, Sernadas *et al.*, 2002a, Coniglio *et al.*, 2003, Caleiro *et al.*, 2003a, Governatori *et al.*, 2002].

As far as preservation results are concerned, we will concentrate our attention on soundness, completeness and interpolation, along with some necessary metatheoretical properties like the metatheorem of deduction. With respect to completeness, we will focus on finding sufficient conditions for the preservation of strong global completeness, in the lines of [Zanardo *et al.*, 2001, Caleiro, 2000, Caleiro *et al.*, 2001], where further completeness preservation results can also be found. We will also overview the basics of Craig interpolation and identify sufficient conditions for its preservation. The interested reader is referred to [Carnielli and Sernadas, 2004, Carnielli *et al.*, 2004b] for further details. In order to make justice to the title, we will also briefly outline some of the current research being carried out, namely, in connection with the so-called *collapsing problem*. We will review the details of the problem, illustrate it, and explain in broad lines how it can be overcome [Sernadas *et al.*, 2002b, Caleiro and Ramos, 2004, Caleiro and Ramos, 2005].

We proceed as follows. In Section 2 we will introduce our working universe of logic systems. The key ingredients of completeness and interpolation for these logic systems will be explored in Section 3. Section 4 will be devoted to defining and illustrating fibring in this context and to explain the connection between the fibred algebraic semantics being introduced and Gabbay's ideas for point-based fibred semantics using the fibring function. In Section 5 we will concentrate on preservation results for (strong global) completeness and some forms of interpolation. Section 6 will overview and illustrate the semantic collapsing problem and provide an outline of ongoing research aimed at its solution. We will conclude, in Section 7, with some final remarks, references to further work, and ideas for future research. Barring some examples and omitted proofs that can be found in the literature, the presentation will be self contained.

2 Logic systems

A logic system is composed by a signature, a Hilbert calculus and a semantic domain. Both the Hilbert calculus and the semantic domain generate consequence systems. Soundness and completeness have to do with the fact that it is desirable that both consequence systems are the same.

A *signature* C is an $\mathbb{N}$-indexed family of countable sets. The elements of each C_k are called *constructors* of arity k. The signature of classical logic is such that C_0 is a set of propositional constants, $C_1 = \{\sim\}$, $C_2 = \{\supset\}$ and $C_k = \emptyset$ for every $k > 2$. The intuitionistic signature is such that C_0 is a set of propositional constants, $C_1 = \{\neg\}$, $C_2 = \{\Rightarrow, \wedge, \vee\}$ and $C_k = \emptyset$ for every $k > 2$. The modal signature is such that C_0 is a set of propositional constants, $C_1 = \{\neg, \Box\}$, $C_2 = \{\Rightarrow\}$ and $C_k = \emptyset$ for every $k > 2$.

Let $T(C, \Xi)$ be the free algebra over C generated by Ξ. The *language* $L(C)$ is $T(C, \emptyset)$. We shall consider different signatures but we assume fixed once and for all a set Ξ of propositional variables. Fixed Ξ, the *schema language* $T(C, \Xi)$ is denoted by $sL(C)$. Given $\varphi \in sL(C)$ we will use $\text{var}(\varphi)$ to denote the set of propositional variables occurring in φ. When $\text{var}(\varphi) = \{\xi_1, \ldots, \xi_n\}$ and $\sigma : \Xi \to sL(C)$ is a substituition such that $\sigma(\xi_i) = \psi_i$ for $i = 1, \ldots, n$, we may use $\varphi(\psi_1, \ldots, \psi_n)$ to denote $\sigma(\varphi)$.

A (Hilbert) *calculus* is a set of rules for some signature C. A *rule* over C is a pair $r = \langle \Theta, \eta \rangle$ where $\Theta \cup \{\eta\} \subseteq sL(C)$. We shall work only with finitary rules, that is, we assume that the set Θ of premises is finite. We also distinguish between local rules and global rules.

We adopt an algebraic semantics. This means that the basic semantic structure is an algebra of truth values. An *ordered algebra* over C is a tuple $\mathbf{A} = \langle A, \leq, \top, \cdot_{\mathbf{A}} \rangle$ where $\langle A, \leq, \top \rangle$ is a topped partial order and $\langle A, \cdot_{\mathbf{A}} \rangle$ is an algebra over C. The relation $\leq$ allows the comparison between truth-values. We require the existence of a top to represent validity and also global semantic consequence.

A *logic system* is a tuple $\mathcal{L} = \langle C, \mathcal{A}, R_\ell, R_\text{g} \rangle$ where C is a signature, $\mathcal{A}$ is a class of ordered algebras over C (the models of the system) and both R_ℓ and R_g are sets of rules over C called local and global rules, respectively. We further assume that the set of local rules R_ℓ is included in the set R_g of global rules and that all the rules in $R_\text{g} \setminus R_\ell$ have a non-empty set of premises.

As an example, let us consider classical logic. The class of models includes every ordered algebra induced by a Boolean algebra. The local rules are the usual rules of a Hilbert calculus for the negation and implication fragment of classical logic. Finally, there are no extra global rules. Another example is the following intuitionistic system. The class of models includes every ordered algebra induced by a Heyting algebra (with $a \leq b$ iff $a \wedge_{\mathbf{A}} b = a \sqcap b = a$). The local rules are the usual rules of a Hilbert calculus for intuitionistic propositional logic. Finally, there are no extra global rules. A detailed presentation of intuitionistic logic along these lines can be found in [Rybakov, 1997]. Consider also the example of the following modal system. The class of models includes every ordered algebra $\mathbf{B} = \langle \mathcal{B}, \subseteq, W, \cdot_{\mathbf{B}} \rangle$ induced by a general Kripke structure $\langle W, \mathcal{B}, \rho, V \rangle$ as follows:

- $\pi_{\mathbf{A}} = V(\pi)$ for each propositional constant π;
- $\neg_{\mathbf{A}}(b) = W \setminus b$;
- $\Rightarrow_{\mathbf{A}}(b_1, b_2) = (W \setminus b_1) \cup b_2$;
- $\Box_{\mathbf{A}}(b) = \{w \in W : \{w' : w \rho w'\} \subseteq b\}$.

The notion of general Kripke structure was proposed in [van Benthem, 1983] in order to obtain a completeness theorem for modal logic. A more direct approach would be to take as models the ordered algebras induced by modal algebras. The local rules include the classical propositional rules plus the normalization axiom

$$\langle \emptyset, (\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2)) \rangle .$$

The unique extra global rule is the necessitation rule $\langle \{\xi_1\}, \Box\xi_1 \rangle$.

Many other interesting logics (even many-valued ones like Gödel's and Łukasiewicz's — see for instance [Gottwald, 2001]) are also logic systems in the sense given above.

Within the context of a logic system, the denotation $[\![\varphi]\!]^{\alpha}_{\mathbf{A}}$ of a schema formula φ on an ordered algebra $\mathbf{A}$ and for an assignment $\alpha : \Xi \to A$ is easily defined by induction on the structure of φ.

In any given logic system $\mathcal{L} = \langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}} \rangle$ we are able to define the following four consequence operators:

- global entailment: $\Gamma \vDash^{\mathrm{g}}_{\mathcal{L}} \varphi$ iff, for every $\mathbf{A} \in \mathcal{A}$ and $\alpha : \Xi \to A$, if $\top \leq [\![\gamma]\!]^{\alpha}_{\mathbf{A}}$ for each $\gamma \in \Gamma$ then $\top \leq [\![\varphi]\!]^{\alpha}_{\mathbf{A}}$;
- local entailment: $\Gamma \vDash^{\ell}_{\mathcal{L}} \varphi$ iff, for every $\mathbf{A} \in \mathcal{A}$, $\alpha : \Xi \to A$ and $a \in A$, if $a \leq [\![\gamma]\!]^{\alpha}_{\mathbf{A}}$ for each $\gamma \in \Gamma$ then $a \leq [\![\varphi]\!]^{\alpha}_{\mathbf{A}}$;

- global derivation: $\Gamma \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi$ iff φ can be derived from Γ using the rules in R_{g};
- local derivation: $\Gamma \vdash^{\ell}_{\mathcal{L}} \varphi$ iff φ can be derived from Γ and theorems (formulae globally derived from an empty set of assumptions) using only the rules in R_ℓ.

Observe that in the modal system described above we have

$$\{\varphi_1 \Rightarrow \varphi_2\} \vdash^{\mathrm{g}}_{\mathcal{L}} (\Box\varphi_1) \Rightarrow (\Box\varphi_2)$$

but

$$\{\varphi_1 \Rightarrow \varphi_2\} \not\vdash^{\ell}_{\mathcal{L}} (\Box\varphi_1) \Rightarrow (\Box\varphi_2).$$

Semantically the same happens, that is:

$$\{\varphi_1 \Rightarrow \varphi_2\} \vDash^{\mathrm{g}}_{\mathcal{L}} (\Box\varphi_1) \Rightarrow (\Box\varphi_2)$$

but

$$\{\varphi_1 \Rightarrow \varphi_2\} \not\vDash^{\ell}_{\mathcal{L}} (\Box\varphi_1) \Rightarrow (\Box\varphi_2).$$

The distinction between local and global reasoning appeared in the context of modal logic (local means carried out at a single world and global refers to reasoning about all worlds) but can be useful in other universes.

3 Metalogical properties

3.1 Soundness and completeness

Soundness and completeness are two recurrent themes in logic. In the present setting, a logic system $\mathcal{L}$ is said to be *strongly globally sound* when if $\Gamma \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi$ then $\Gamma \vDash^{\mathrm{g}}_{\mathcal{L}} \varphi$. Of course, it is said to be *strongly globally complete* when if $\Gamma \vDash^{\mathrm{g}}_{\mathcal{L}} \varphi$ then $\Gamma \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi$. When we only consider $\Gamma = \emptyset$ we get the corresponding weak notions. Mutatis mutandis, we define the local versions.

Proving that a certain calculus is sound for a given semantics is typically just a matter of showing that each of its rules is fulfilled by the interpretation structures. Completeness, on its turn, is well known to be a much harder problem. Still, it is possible to obtain some general completeness results. A logic system is said to be *full* when $\mathcal{A}$ is composed of all ordered algebras over C that fulfill the rules in both R_ℓ and R_{g}. Therefore, every full logic system is (weakly and strongly, locally and globally) sound. A logic system has *verum* if its language contains a theorem that denotes $\top$ in every model.

A logic system $\mathcal{L}$ is said to be *congruent* when for every Γ closed for global derivation, $c \in C_k$ and $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_k \in sL(C)$:

$$\frac{\begin{array}{c}\Gamma, \varphi_i \vdash^{\ell}_{\mathcal{L}} \psi_i \\ \Gamma, \psi_i \vdash^{\ell}_{\mathcal{L}} \varphi_i\end{array} \quad i = 1, \ldots, k}{\Gamma, c(\varphi_1, \ldots, \varphi_k) \vdash^{\ell}_{\mathcal{L}} c(\psi_1, \ldots, \psi_k)}$$

THEOREM 1. Every full and congruent logic system with verum is strongly globally complete.

The proof is carried out using a Lindenbaum-Tarski construction. A syntactic ordered algebra $\mathbf{A}_\Gamma$ can be built as follows from each Γ closed for $\vdash^{\mathrm{g}}_{\mathcal{L}}$. First we define a congruence relation over $sL(C)$: $\varphi \cong_\Gamma \psi$ iff $\Gamma, \varphi \vdash^{\ell}_{\mathcal{L}} \psi$ and $\Gamma, \psi \vdash^{\ell}_{\mathcal{L}} \varphi$. Then, we choose A to be $sL(C)/\cong_\Gamma$. The partial order is defined as follows: $[\varphi]_\Gamma \leq [\psi]_\Gamma$ iff $\Gamma, \varphi \vdash^{\ell}_{\mathcal{L}} \psi$. The top $\top$ is the equivalence class of the verum. Finally, for each language constructor, $c_{\mathbf{A}_\Gamma}([\varphi_1]_\Gamma, \ldots, [\varphi_k]_\Gamma) = [c(\varphi_1, \ldots, \varphi_k)]_\Gamma$. Clearly, by construction, we infer that $[\![\varphi]\!]^{\lambda\xi.[\xi]_\Gamma}_{\mathbf{A}_\Gamma} = \top$ iff $\varphi \in \Gamma$ and that $\mathbf{A}_\Gamma$ fulfills the rules of the logic system.

Assume that $\Delta \not\vdash^{\mathrm{g}}_{\mathcal{L}} \epsilon$. We have to show $\Delta \not\models^{\mathrm{g}}_{\mathcal{L}} \epsilon$. It is sufficient to find an ordered algebra $\mathbf{A} \in \mathcal{A}$ such that $[\![\delta]\!]^{\lambda\xi.[\xi]_\Gamma}_{\mathbf{A}} = \top$ for each $\delta \in \Delta$ and $[\![\epsilon]\!]^{\lambda\xi.[\xi]_\Gamma}_{\mathbf{A}_\Gamma} \neq \top$. Consider $\Gamma = \Delta^{\vdash^{\mathrm{g}}}$. Then, $\mathbf{A}_\Gamma$ globally satisfies each element of Δ (since $\Delta \subseteq \Gamma$) but $\mathbf{A}_\Gamma$ does not globally satisfy ϵ (since $\epsilon \not\in \Gamma$). This concludes the proof of the completeness theorem.

Observe that any complete logic system can be made full without changing its entailments. And if verum is not present, it can be conservatively added to the language. Note also that through a mild strengthening of the requirements of the theorem we can ensure finitary strong local completeness (see for instance [Sernadas *et al.*, 2002b]). A similar strong (local and global) completeness theorem is obtained in [Zanardo *et al.*, 2001] without extra requirements for local reasoning but assuming a more complex semantics and using a Henkin construction. Other similar completeness results, namely with respect to generalized matrix semantics can be found in [Caleiro, 2000, Caleiro *et al.*, 2001, Caleiro *et al.*, 2003b]. If congruence fails, however, then there is nothing we can do within the scope of the basic theory of fibring outlined here. Still, in that case, we can resort to the theory of non-truth-functional fibring [Caleiro *et al.*, 2003a] where general completeness preservation results also obtain, but using rather different techniques, involving algebraization.

3.2 Interpolation

In its various flavours, *interpolation* is a heritage of the classical results by W. Craig [Craig, 1957] for first-order logic from which several abstractions have emerged, either proof-theoretically (e.g. [Carbone, 1997]) or in (non-constructive) model-theoretical style (e.g. for modal and positive logics [Maksimova, 1997, Maksimova, 2002], but also for intuitionistic logic [Gabbay, 1977], or hybrid logic [Areces *et al.*, 2001, Areces *et al.*, 2003]). Note that general techniques for obtaining interpolation properties are not known. For instance, Craig interpolation fails unexpectedly for all the many-valued logics of Łukasiewicz and Gödel [Krzystek and Zachorowski, 1977, Baaz and Veith, 1999]. Developing constructive proofs of interpolation is still a harder problem.

Moreover, interpolation properties are known to be related with properties of model theory as exemplified by the correspondence between Craig interpolation and joint consistency properties for classical propositional logic. This correspondence is mediated in the classical case by finite algebraizability and by the familiar (global) metatheorem of deduction. In the general case of our deductive calculi, specially due to the peculiarities of local and global deduction, this correspondence opens difficult and challenging problems.

We recast here some forms of interpolation taking into account the distinction between local and global deduction, as well as some general results, following [Carnielli and Sernadas, 2004, Carnielli *et al.*, 2004b]. Whenever a notion applies both to local and global reasoning we will use d instead of ℓ or g. A logic system $\mathcal{L}$ has:

- the *d-Craig interpolation property* (d-CIP) whenever $\Gamma \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi$ and $\mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$ implies that there exists $\Gamma' \subseteq T(C, \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi))$ such that $\Gamma \vdash^{\mathrm{d}}_{\mathcal{L}} \Gamma'$ and $\Gamma' \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi$;
- the *d-extension interpolation property* (d-EIP) whenever $\Gamma, \Psi \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi$ implies that there exists $\Gamma' \subseteq T(C, \mathrm{var}(\Psi) \cup \mathrm{var}(\varphi))$ such that $\Gamma \vdash^{\mathrm{d}}_{\mathcal{L}} \Gamma'$ and $\Gamma', \Psi \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi$;
- the *d-Maehara interpolation property* (d-MIP) whenever $\Gamma, \Psi \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi$ and $T(C, \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi))) \neq \emptyset$ implies that there exists $\Gamma' \subseteq T(C, \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi)))$ such that $\Gamma \vdash^{\mathrm{d}}_{\mathcal{L}} \Gamma'$ and $\Gamma', \Psi \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi$.

As mentioned before, although Łukasiewicz logics do not have the CIP they do enjoy the EIP.

To deal with the peculiarities of local and global deduction we need the following notion. A logic system $\mathcal{L}$ is said to allow *careful-reasoning-by-cases* if whenever $\Gamma \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi$ then there exists Ψ such that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{L}} \Psi$, $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma)$ and $\Psi \vdash^{\ell}_{\mathcal{L}} \varphi$. For instance, modal and first-order logics allow careful-reasoning-by-cases.

THEOREM 2. A logic system $\mathcal{L}$ allowing careful-reasoning-by-cases enjoys the g-CIP whenever it has the ℓ-CIP.

Following the spirit of [Font *et al.*, 2003], we say that a logic system $\mathcal{L}$ enjoys the *d-metatheorem of Modus Ponens* (d-MTMP) with respect to $\Delta \subseteq T(C, \{\xi_1, \xi_2\})$ if

$$\frac{\Gamma \vdash^{\mathrm{d}}_{\mathcal{L}} \Delta(\varphi_1, \varphi_2)}{\Gamma, \varphi_1 \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi_2}.$$

Analogously, $\mathcal{L}$ enjoys the *d-metatheorem of deduction* (d-MTD) with respect to Δ if

$$\frac{\Gamma^{\vdash^{\mathrm{g}}_{\mathcal{L}}}, \varphi_1 \vdash^{\mathrm{d}}_{\mathcal{L}} \varphi_2}{\Gamma^{\vdash^{\mathrm{g}}_{\mathcal{L}}} \vdash^{\mathrm{d}}_{\mathcal{L}} \Delta(\varphi_1, \varphi_2)} .$$

In [Czelakowski and Pigozzi, 1999] it is shown that the g-MIP is equivalent to the conjunction of g-EIP and the g-CIP. The next result, together with the result in [Czelakowski and Pigozzi, 1999], shows that d-MTD and d-EIP are provable from each other.

THEOREM 3. A logic system enjoying d-MTD, d-MTMP and d-CIP has the d-MIP.

4 Fibring

Consider signatures C and C' such that $C'_k \subseteq C_k$ for each $k \in \mathbb{N}$. Given an ordered algebra $\mathbf{A}$ over C, we denote by $\mathbf{A}|_{C'}$ the reduct $\langle A, \leq, \top, \cdot_{\mathbf{A}}|_{C'}\rangle$ of $\mathbf{A}$ by the inclusion (where $\cdot_{\mathbf{A}}|_{C'}$ is the restriction of $\cdot_{\mathbf{A}}$ to C'). Clearly, $\mathbf{A}|_{C'}$ is an ordered algebra over C'.

Given logic systems $\mathcal{L}' = \langle C', \mathcal{A}', R_\ell{}', R_{\mathrm{g}}{}'\rangle$ and $\mathcal{L}'' = \langle C'', \mathcal{A}'', R_\ell{}'', R_{\mathrm{g}}{}''\rangle$, their *fibring* $\mathcal{L}' \odot \mathcal{L}'' = \langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}}\rangle$ is defined as follows:

- $C_k = C'_k \cup C''_k$ for each $k \in \mathbb{N}$;
- $\mathcal{A}$ is the class containing every ordered algebra $\mathbf{A}$ over C such that $\mathbf{A}|_{C'} \in \mathcal{A}'$ and $\mathbf{A}|_{C''} \in \mathcal{A}''$;
- $R_\ell = R_\ell{}' \cup R_\ell{}''$; $R_{\mathrm{g}} = R_{\mathrm{g}}{}' \cup R_{\mathrm{g}}{}''$.

This definition corresponds to the constrained version of fibring (as defined in [Sernadas *et al.*, 1999]) since any symbols common to both logic systems will be shared. Unconstrained fibring can be obtained by making sure that no symbols appear in the intersection of the two signatures. Fibring can be characterized using a universal construction in a suitable category of logic systems (as explored in [Sernadas *et al.*, 1999] where the categorical approach was important in fine tuning the semantics of fibring).

As a first example of fibring, consider the combination of two modal systems while sharing the propositional connectives. This constrained fibring is equivalent to the fusion of the two given modal systems. The result is a bimodal system. The combination of a modal system with a relevance system is similar from the point of view of fibring but beyond the scope of fusion. By sharing the propositional connectives we obtain a logic system with a modal box and a relevance implication. For details about relevance logic see for instance [Dunn, 1986]. Note that, even when no symbols are shared, fibring may impose unexpected interactions between the logical operations from the two given logics. For instance, consider the unconstrained fibring of classical propositional logic and intuitionistic propositional logic. Unexpectedly, in the resulting logic system the intuitionistic implication collapses into classical implication. In short, in the resulting logic system we have two copies of classical logic. This first example of collapsing was first identified in [Gabbay, 1996b, del Cerro and Herzig, 1996]. We will debate this so-called *collapsing problem* later on, in Section 6.

4.1 A bridge to the past

Conceptually, there is a substantial gap between the point based semantics of fibring, as suggested by the fusion of modal logics, and the algebraic semantics we propose here. Let us review this abstraction process. The relationship between the definition of fibring presented above and Gabbay's original idea in [Gabbay, 1996a] certainly deserves a careful analysis. Besides furnishing a historical account of the development of fibring as a methodology for combining logics, we hope that this digression will provide a better overall understanding of the problem of combining logics and the technical difficulties involved.

Since there are no differences with respect to the proof-theoretic dimension of fibring, we shall concentrate just on the semantic aspects. Hence, if $\mathcal{L} = \langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}} \rangle$ is a logic system then we shall dub $\mathcal{I} = \langle C, \mathcal{A} \rangle$ an interpretation system. To get into the level of abstraction considered in [Gabbay, 1996a], we need to restrict ourselves to interpretation systems based on Kripke-like structures (K-structures, for simplicity). A *K-structure* over C is a tuple $\mathbf{W} = \langle W, \cdot_{\mathbf{W}} \rangle$ where W is a non-empty set and $\langle 2^W, \cdot_{\mathbf{W}} \rangle$ is a C-algebra. A *K-interpretation system* is a pair $\mathcal{KI} = \langle C, \mathcal{W} \rangle$ where C is a signature and $\mathcal{W}$ is a class of K-structures over C.

In each K-structure, the set W of *worlds* induces the space of truth values 2^W, ordered by inclusion, whose top element is precisely W. In this way, we can recognize a K-interpretation structure $\mathbf{W}$ as a special way of presenting the ordered structure $\overline{\mathbf{W}} = \langle 2^W, \subseteq, W, \cdot_{\mathbf{W}} \rangle$. Of course, a K-interpretation system $\mathcal{KI} = \langle C, \mathcal{W} \rangle$ can also be seen as a special way of presenting the interpretation system $\overline{\mathcal{KI}} = \langle C, \overline{\mathcal{W}} \rangle$.

It is interesting to note that we can recover the usual Kripke-like notions of local and global reasoning for each $\mathcal{KI}$, if we adopt the corresponding general definitions for $\overline{\mathcal{KI}}$. Given $\mathbf{W} \in \mathcal{W}$, define the *local satisfaction* relation at $w \in W$ by $\mathbf{W}, w \Vdash^\ell \varphi$ if $w \in [\![\varphi]\!]_{\mathbf{W}}$. Analogously, given a global model $\mathbf{W} \in \mathcal{W}$ define the *global satisfaction* by $\mathbf{W} \Vdash^{\mathrm{g}} \varphi$ if $\mathbf{W}, w \Vdash^\ell \varphi$ for every $w \in W$. Then, at $\overline{\mathcal{KI}}$ we have that:

- *globally*: $\Psi \vDash^{\mathrm{g}} \varphi$ if and only if for every $\mathbf{W} \in \mathcal{W}$, if $\mathbf{W} \Vdash^{\mathrm{g}} \psi$ for each

$\psi \in \Psi$ then $\mathbf{W} \Vdash^{\mathrm{g}} \varphi$;

- *locally*: $\Psi \vDash^{\ell} \varphi$ if and only if for every $\mathbf{W} \in \mathcal{W}$ and $w \in W$, if $\mathbf{W}, w \Vdash^{\ell} \psi$ for each $\psi \in \Psi$ then $\mathbf{W}, w \Vdash^{\ell} \varphi$.

Gabbay's original idea of fibred semantics [Gabbay, 1996a, Gabbay, 1996b, Gabbay, 1999] was based on the notion of *fibring function*, and assumed that both logics had a Kripke-like semantics. In this case, the fibring function F would provide, at any moment, a way to map models and worlds from one logic to the other, and back again. Suppose that φ' is a formula and c' a unary constructor of the first logic system, given by $\mathcal{KI}'$, and c'' a unary constructor of the second logic, given by $\mathcal{KI}''$. To evaluate $c'(c''(\varphi'))$ in the combined logic we should proceed as follows.

1. Take a model $\mathbf{W}' = \langle W', \cdot_{\mathbf{W}'} \rangle$ of the first logic.

2. Typically, the satisfaction of $c'(c''(\varphi'))$ at $\mathbf{W}'$ will depend on some condition involving the *unknown* satisfaction of $c''(\varphi')$ at $\mathbf{W}'$.

3. For each world $w' \in W'$, instead of $\mathbf{W}', w' \Vdash'_L c''(\varphi')$, apply the fibring function F to obtain $F(\mathbf{W}', w') = \langle \mathbf{W}'', w'' \rangle$ where $\mathbf{W}'' = \langle W'', \cdot_{\mathbf{W}''} \rangle$ is a model of the second logic and $w'' \in W''$, and use $\mathbf{W}'', w'' \Vdash''_L c''(\varphi')$.

4. Again, the satisfaction of $c''(\varphi')$ at $\mathbf{W}''$ will depend on some condition involving the *unknown* satisfaction of φ' at $\mathbf{W}''$.

5. For each world $u'' \in W''$, instead of $\mathbf{W}'', u'' \Vdash''_L \varphi'$, apply the fibring function F to obtain $F(\mathbf{W}'', u'') = \langle \mathbf{U}', u' \rangle$ where $\mathbf{U}' = \langle U', \cdot_{\mathbf{U}'} \rangle$ is a model of the first logic and $u' \in U'$, and use $\mathbf{U}', u' \Vdash'_L \varphi'$.

The idea behind this procedure is intuitive and appealing, but it is not obvious how to accommodate this operational view into a meaningful definition of fibred model. Things are even less clear if we further require fibring to be a universal construction between K-interpretation systems. In that case, how to characterize the resulting system $\mathcal{KI}' \odot \mathcal{KI}''$? And what is the relevant notion of K-interpretation systems morphism?

The first solution to these questions, proposed in [Sernadas *et al.*, 1999], considered that fibred models could be partitioned, simultaneously, into clouds of disjoint models from each of the logics. The category-theoretical approach was an essential ingredient in fine tuning the notion presented in [Sernadas *et al.*, 1999], although it involved a quite complicate notion of morphism. However, it can be much simplified if we just assume that the classes of K-structures of the logics being combined are already *closed* *for unions*. This simplification was first proposed in [Zanardo *et al.*, 2001], where it was also noted that closing a given K-interpretation system for unions simply does not change its entailment operators. When closing for unions, a K-interpretation structure $\mathbf{W}$ of the resulting system can be seen as being built from a cloud $\{\mathbf{W}_i : i \in I\}$ of pairwise disjoint structures of the original system. Rigorously, $\mathbf{W}$ is defined to be the unique K-structure with $W = \bigcup_{i \in I} W_i$, and $\nu_{\mathbf{W}}(c)(X_1, \ldots, X_k) \cap W_i = \nu_{\mathbf{W}_i}(c)(X_1 \cap W_i, \ldots, X_k \cap W_i)$ for each k-ary constructor in the signature and $X_1, \ldots, X_k \in 2^W$. To make the notion robust with respect to the particular "names" of the worlds in each structure, it is useful to work under the assumption that the K-interpretation systems being combined are *closed under isomorphisms*.

The *fibring of K-interpretation systems* $\mathcal{KI}' = \langle C', \mathcal{W}' \rangle$ *and* $\mathcal{KI}'' = \langle C'', \mathcal{W}'' \rangle$ *closed for unions* is the interpretation system $\mathcal{KI}' \odot \mathcal{KI}'' = \langle C' \cup C'', \mathcal{W} \rangle$ where $\mathcal{W}$ is the class of all K-interpretation structures $\mathbf{W}$ over $C' \cup C''$ that can be built from interpretations structures $\mathbf{W}' \in \mathcal{W}'$ and $\mathbf{W}'' \in \mathcal{W}''$ satisfying:

- $W = W' = W''$;
- if $c \in C'_k \cup C''_k$ and $X_1, \ldots, X_k \in 2^W$ then

$$c_{\mathbf{W}'}(X_1, \ldots, X_k) = c_{\mathbf{W}''}(X_1, \ldots, X_k),$$

by defining $\mathbf{W} = \langle W, \cdot_{\mathbf{W}} \rangle$ as follows:

- for each $c' \in C'_k$ and $X_1, \ldots, X_k \in 2^W$,

$$c'_{\mathbf{W}}(X_1, \ldots, X_k) = c'_{\mathbf{W}'}(X_1, \ldots, X_k);$$

- for each $c'' \in C''_k$ and $X_1, \ldots, X_k \in 2^W$,

$$c''_{\mathbf{W}}(X_1, \ldots, X_k) = c''_{\mathbf{W}''}(X_1, \ldots, X_k).$$

As before, the fibring is defined under the assumption that the common subsignature is shared. In fact, for every shared constructor $c \in C' \cap C''$, the definition above implies that the two clouds of models $\{\mathbf{W}'_i : i \in I\}$ and $\{\mathbf{W}''_j : j \in J\}$, corresponding to each pair of structures $\mathbf{W}'$ and $\mathbf{W}''$ being fibred, agree on their interpretation. Note that, according to Gabbay's operational description, we can recognize the fibring function F associated to the fibred model $\mathbf{W}$ as mapping each pair $\langle \mathbf{W}'_i, w \rangle$ such that $w \in W'_i$ to the pair $\langle \mathbf{W}''_j, w \rangle$ where j is the unique element of J such that $w \in W''_j$, and vice-versa.

To understand the definition of fibring of interpretation systems with algebraic semantics, what we need is an operation on interpretation systems that mimics the closure for unions of K-interpretation systems. Notably, given a set $\{\mathbf{W}_i : i \in I\}$ of K-structures, it is not difficult to conclude that $\overline{\bigcup_{i\in I} \mathbf{W}_i}$ is isomorphic to $\prod_{i\in I} \overline{\mathbf{W}}_i$. Thus, if $\mathcal{KI}$ is closed for unions then it immediately follows that $\overline{\mathcal{KI}}$ is closed for products. Now, it is a small step to ckeck that $\overline{\mathcal{KI}' \odot \mathcal{KI}''}$ and $\overline{\mathcal{KI}'} \odot \overline{\mathcal{KI}''}$ coincide, which justifies our general definition of fibring in the wider setting of interpretation systems. For brevity we omit the (obvious) definition of product of interpretation structures. Still, note that closing a given interpretation system for products also does not change its entailment operators.

This Kripke-like semantic view is nevertheless restrictive. In general, there is no reason to suppose that interesting logics should be endowed with K-interpretation structures. Moreover, general completeness results even for modal logics are only possible if we consider general Kripke structures, or alternatively, modal algebras. Still, by now, it should be easy to bridge the intuitive gap to the broader algebraic setting initially proposed. To this end, the ideas in [Mateus *et al.*, 2004] should be relevant.

5 Transference results

5.1 Soundness and completeness

We now turn our attention to transference results. We start by examining if soundness is preserved by fibring. Then we consider completeness. To this end we have to establish the preservation of other interesting properties, namely the metatheorem of deduction.

THEOREM 4. Soundness is preserved by fibring.

It is straightforward to prove that (strong and weak, global and local) soundness is unconditionally preserved by fibring in the basic universe of logic systems considered here. However, in larger universes things can be more complicated. For instance, when fibring logic systems with quantifiers and using rules with side provisos (like, provided that term θ is free for variable x in formula ξ), soundness is not always preserved [Sernadas *et al.*, 2002a, Coniglio *et al.*, 2003].

Also weak completeness is not always preserved by fibring, as shown in [Zanardo *et al.*, 2001]. Herein we examine in detail if strong global completeness is preserved when fibring basic logic systems as defined above. Adapting the technique originally proposed in [Zanardo *et al.*, 2001], we capitalize on the completeness theorem stated above about such logic systems. That is, when fibring two given logic systems that are full, congruent and with verum (and, therefore, strongly globally complete) we shall try to obtain the strong global completeness of the result by identifying the conditions under which fullness, congruence and verum are preserved by fibring.

LEMMA 5. Fullness is preserved by fibring.

LEMMA 6. The result of fibring has verum provided that at least one of the given logic systems has verum.

However, congruence is not always preserved by fibring. Consider the fibring of two logic systems $\mathcal{L}', \mathcal{L}''$ with the following signatures and rules:

$$C_0' = \{\pi_0, \pi_1, \pi_2\} \quad C_1' = \{c\} \quad C_k' = \emptyset \text{ for } k > 1$$
$$R_\ell{}' = \emptyset \quad R_{\mathrm{g}}{}' = \{\langle\{\xi\}, c(\xi)\rangle\}$$

$$C_0'' = \{\pi_0, \pi_1, \pi_2\} \quad C_k'' = \emptyset \text{ for } k > 0$$
$$R_\ell{}'' = R_{\mathrm{g}}{}'' = \{\langle\{\pi_0, \pi_1\}, \pi_2\rangle, \langle\{\pi_0, \pi_2\}, \pi_1\rangle\}$$

Clearly, both $\mathcal{L}'$ and $\mathcal{L}''$ are congruent, but their fibring $\mathcal{L} = \mathcal{L}' \odot \mathcal{L}''$ is not congruent. Indeed, consider $\Gamma = \{\pi_0\}^{\vdash^{\mathrm{g}}} = \{c^n(\pi_0) : n \geq 0\}$. So, from Γ, π_1 and π_2 are locally interderivable but, from Γ, $c(\pi_1)$ and $c(\pi_2)$ are not locally interderivable.

Fortunately, it is possible to establish a useful sufficient condition for the preservation of congruence by fibring. A logic system $\mathcal{L}$ is said to have *implication* if its signature contains a binary connective $\Rightarrow$ fulfilling the (local versions for $\Delta = \{\xi_1 \Rightarrow \xi_2\}$ of the) metatheorems of Modus Ponens and of deduction, that is

$$\text{(MTMP)} \ \frac{\Gamma \vdash^\ell_{\mathcal{L}} \delta_1 \Rightarrow \delta_2}{\Gamma, \delta_1 \vdash^\ell_{\mathcal{L}} \delta_2} \quad \text{and} \quad \text{(MTD)} \ \frac{\Gamma^{\vdash^{\mathrm{g}}_{\mathcal{L}}}, \delta_1 \vdash^\ell_{\mathcal{L}} \delta_2}{\Gamma^{\vdash^{\mathrm{g}}_{\mathcal{L}}} \vdash^\ell_{\mathcal{L}} \delta_1 \Rightarrow \delta_2} \, .$$

When fibring two logic systems with implication while sharing the implication symbol, it is straightforward to verify that the resulting logic system also has implication. Indeed:

THEOREM 7. The result of fibring has the MTMP for an implication provided that at least one of the given logic systems has the MTMP for that implication.

THEOREM 8. The result of fibring has the MTD for a shared implication provided that both given logic systems have the MTD for that implication.

The latter result is a direct corollary of the following fact:

LEMMA 9. The MTD for $\Rightarrow$ holds in a logic system iff:

- $\vdash^\ell (\xi \Rightarrow \xi)$;
- $\{\xi_1\}^{\vdash^{\mathrm{g}}} \vdash^\ell (\xi_2 \Rightarrow \xi_1)$; and

- $\{(\xi \Rightarrow \gamma_1), \ldots, (\xi \Rightarrow \gamma_k)\}^{\vdash^{\mathrm{g}}} \vdash^{\ell} (\xi \Rightarrow \gamma)$ for each local rule $\langle\{\gamma_1, \ldots, \gamma_k\}, \gamma\rangle$ and ξ that does not occur in the rule.

A logic system is said to have *equivalence* if it has implication and its signature contains a binary connective $\Leftrightarrow$ fulfilling the two metatheorems of biconditionality (relating implication with equivalence) and the metatheorem of substitution of equivalents.

THEOREM 10. A logic system with equivalence is congruent.

When fibring two logic systems with equivalence while sharing the implication symbol as well as the equivalence symbol we obtain a logic system with equivalence. Therefore:

THEOREM 11. The fibring while sharing implication and equivalence of full logic systems with equivalence and verum is strongly globally complete.

This preservation result is quite useful because many widely used logic systems do have equivalence in the sense above.

5.2 Interpolation

Preservation results for Craig interpolation in the context of fusion were obtained in [Kracht and Wolter, 1991]. We will overview some interpolation preservation results in the broader setting of fibring, again following [Carnielli and Sernadas, 2004, Carnielli *et al.*, 2004b].

Since the metatheorems of Modus Ponens and of deduction will play a central role, we first note that the preservation results already obtained for the case of implication extend smoothly to the general case.

THEOREM 12. The d-MTMP and d-MTD are preserved by fibring logic systems with respect to the same shared Δ.

Careful reasoning also transfers along fibring.

THEOREM 13. Careful-reasoning-by-cases is preserved by fibring.

A logic system $\mathcal{L}$ is said to have *conjunction* if its signature contains a binary connective $\wedge$ fulfilling $\{\varphi_1 \wedge \varphi_2\} \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi_1$, $\{\varphi_1 \wedge \varphi_2\} \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi_2$ and $\{\varphi_1, \varphi_2\} \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi_1 \wedge \varphi_2$.

THEOREM 14. The g-CIP is preserved by fibring two logic systems with conjunction, g-MTMP, g-MTMP and an axiom that can be instantiated with any finite number of variables.

For instance in modal logic, the axiom $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$ can be instantiated with any finite number of variables: the instance $((\wedge_{j=1}^{k} \xi_j) \Rightarrow (\xi_2 \Rightarrow (\wedge_{j=1}^{k} \xi_j)))$ includes $\xi_1, \ldots, \xi_k$.

The preservation of ℓ-CIP requires more assumptions on the component logics, namely a refinement of the notion of careful-reasoning-by-cases. A logic system $\mathcal{L}$ allows *localized-careful-reasoning-by-cases* if whenever $\Gamma, \Psi \vdash^{\mathrm{g}}_{\mathcal{L}} \varphi$ with Ψ finite and with a derivation where global rules are only applied to hypotheses in Ψ then there exists a finite $\Omega \in T(C, \mathrm{var}(\Psi))$ such that $\Omega \subseteq \Psi^{\vdash^{\mathrm{g}}_{\mathcal{L}}}$ and $\Omega \vdash^{\ell}_{\mathcal{L}} \varphi$. Both modal and first-order logics have this property.

THEOREM 15. ℓ-CIP is preserved by the fibring logic systems with localized-careful-reasoning-by-cases, conjunction, ℓ-MTMP, ℓ-MTD and an axiom that can be instantiated with any finite number of variables.

THEOREM 16. d-MIP is preserved by fibring logic systems with localized-careful-reasoning-by-cases, conjunction, d-MTMP, d-MTD, and an axiom that can be instanced with any number of variables.

The importance of a general form of metatheorem of deduction is stressed, proving that Craig interpolation implies another form of interpolation proposed by S. Maehara [Maehara, 19601961], thus showing that the mediation of the metatheorem of deduction plays a central role.

6 The collapsing problem and beyond

Despite its many strong points, already illustrated in the previous sections, fibring suffers from an anomaly usually known as "the collapsing problem" [Gabbay, 1996b, del Cerro and Herzig, 1996]. Indeed, since the beginning, it could be noticed that fibring the semantics of classical with intuitionistic logic would collapse into just classical logic.

In this section we will review this concrete example and try to shed some light at the problem in the general case. The analysis will allow us to identify ways to avoid this collapses, and to introduce further related topics of the theory of fibring that need to be addressed in the near future.

Let $\mathcal{L}' = \langle C', \mathcal{A}', R_\ell{}', R_{\mathrm{g}}{}' \rangle$ and $\mathcal{L}'' = \langle C'', \mathcal{A}'', R_\ell{}'', R_{\mathrm{g}}{}'' \rangle$ be, respectively, the logic systems for classical and intuitionistic propositional logics as described in Section 2. In order to ensure that there are absolutely no shared constructors between the two systems we shall assume that their respective sets of propositional constants $\Pi' = C_0'$ and $\Pi'' = C_0''$ are disjoint, that is $\Pi' \cap \Pi'' = \emptyset$.

We are interested in investigating the fibred logic system $\mathcal{L}' \odot \mathcal{L}'' = \langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}} \rangle$. Note that the combined signature is such that $C_0 = \Pi' \cup \Pi''$, $C_1 = \{\sim, \neg\}$, $C_2 = \{\supset, \Rightarrow, \wedge, \vee\}$, and $C_k = \emptyset$ if $k > 2$. For a start, let us have a look at its class of interpretation structures $\mathcal{A}$. From the definition of fibring, as put forth in Section 4, it easily follows that an interpretation structure $\mathbf{A} = \langle A, \leq, \top, \cdot_{\mathbf{A}} \rangle$ over C is in $\mathcal{A}$ if and only if $\mathbf{A}' = \mathbf{A}|_{C'}$ is a Boolean algebra and $\mathbf{A}'' = \mathbf{A}|_{C''}$ is a Heyting algebra. Since the ordered structure $\langle A, \leq, \top \rangle$ is the same in both $\mathbf{A}'$ and $\mathbf{A}''$, it is obvious that the Heyting structure $\mathbf{A}''$ is also Boolean and thus $\neg_{\mathbf{A}} = \sim_{\mathbf{A}}$, $\Rightarrow_{\mathbf{A}} = \supset_{\mathbf{A}}$, and the interpretation of the intuitionistic connectives $\wedge$ and $\vee$ also coincides with the classical conjunction and disjunction operations that can be obtained from $\sim$ and $\supset$ by the usual abbreviations. Therefore, unexpectedly, $\mathcal{L}' \odot \mathcal{L}''$ does not enjoy classical and intuitionistic features simultaneously. Instead, it is just a classical system with two synonymous ways of writing negation and implication.

This was, however, just a semantic analysis. What happens at the deductive level? Certainly the same phenomenon was to be expected if only we could be sure that the completeness of $\mathcal{L}' \odot \mathcal{L}''$ would follow from the completeness of both $\mathcal{L}'$ and $\mathcal{L}''$. However, this is not obvious, specially because fullness (as introduced in Section 3 and used in Section 5) is not guaranteed in either case. If there would also be a collapse at the deductive level then one would have, in particular, $\Gamma \vdash^{\ell}_{\mathcal{L}} \varphi \supset \psi$ if and only if $\Gamma, \varphi \vdash^{\ell}_{\mathcal{L}} \psi$ if and only if $\Gamma \vdash^{\ell}_{\mathcal{L}} \varphi \Rightarrow \psi$. The same would happen globally, of course. But, the fact that both classical and intuitionistic implications per se enjoy the MTD does not imply that either MTD will hold in the fibring (note that this does not contradict Theorem 8 since the two implications are not shared). Indeed, showing that the MTD holds for any of the two implications in $\mathcal{L}$ does not seem possible.

Thus, we have an hint that perhaps the collapse does not happen at the deductive level. But how can we show this? If we focus just on the implications, one possibility would be to show that $\vdash^{\ell}_{\mathcal{L}} ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi$ is not always the case. Note that we know that $\vdash^{\ell}_{\mathcal{L}} ((\varphi \supset \psi) \supset \psi) \supset \psi$ holds because the formula at hand is the well-known *Peirce axiom*. To show that $\vdash^{\ell}_{\mathcal{L}} ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi$ may fail we could try, for instance, to use a semantic argument, but we have already seen that fibred semantics cannot help in this task. Still, it is perhaps a matter of looking beyond fibred models.

Let us consider an interpretation structure obtained by a natural extension of the usual Kripke semantics for intuitionistic logic. Given a partial-order Kripke structure $\langle W, \leq \rangle$, and letting B be the set of all filters (up-closed subsets) of W, any corresponding model $\langle W, \leq, V \rangle$ with $V : \Pi' \cup \Pi'' \to B$ induces an interpretation structure $\mathbf{B} = \langle B, \subseteq, \{W\}, \cdot_{\mathbf{B}}, \rangle$ with:

- $\pi'_{\mathbf{B}} = V(\pi')$ and $\pi''_{\mathbf{B}} = V(\pi'')$;
- $\sim_{\mathbf{B}} (b) = (W \setminus b)^c$;
- $\supset_{\mathbf{B}} (b_1, b_2) = ((W \setminus b_1) \cup b_2)^c$;
- $\neg_{\mathbf{B}}(b) = (W \setminus b)^i$;

- $\Rightarrow_{\mathbf{B}}(b_1, b_2) = ((W \setminus b_1) \cup b_2)^i$;
- $\wedge_{\mathbf{B}}(b_1, b_2) = b_1 \cap b_2$;
- $\vee_{\mathbf{B}}(b_1, b_2) = b_1 \cup b_2$,

where $X^c = \{w \in W : \text{there exists } x \in X \text{ such that } x \leq w\}$ and $X^i = \{w \in W : \{w' : w \leq w'\} \subseteq X\}$, given $X \subseteq W$.

In the particular case when $W = \{u, v\}$ and $u \leq v$, only, we obtain the following 3-valued truth tables:

$\supset$	$\emptyset$	$\{v\}$	$\{u,v\}$
$\emptyset$	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$
$\{v\}$	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$
$\{u,v\}$	$\emptyset$	$\{v\}$	$\{u,v\}$

$\Rightarrow$	$\emptyset$	$\{v\}$	$\{u,v\}$
$\emptyset$	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$
$\{v\}$	$\emptyset$	$\{u,v\}$	$\{u,v\}$
$\{u,v\}$	$\emptyset$	$\{v\}$	$\{u,v\}$

	$\sim$	$\neg$
$\emptyset$	$\{u,v\}$	$\{u,v\}$
$\{v\}$	$\{u,v\}$	$\emptyset$
$\{u,v\}$	$\emptyset$	$\emptyset$

$\wedge$	$\emptyset$	$\{v\}$	$\{u,v\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{v\}$	$\emptyset$	$\{v\}$	$\{v\}$
$\{u,v\}$	$\emptyset$	$\{v\}$	$\{u,v\}$

$\vee$	$\emptyset$	$\{v\}$	$\{u,v\}$
$\emptyset$	$\emptyset$	$\{v\}$	$\{u,v\}$
$\{v\}$	$\{v\}$	$\{v\}$	$\{u,v\}$
$\{u,v\}$	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$

Despite the fact that this interpretation structure is a three-valued Heyting algebra and therefore not a Boolen algebra, it is a simple matter to check that all the axioms and rules of classical and intuitionistic logic are satisfied. Moreover, we can indeed confirm that $((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi$ does not hold in general. Take, for instance, $((\pi_1'' \Rightarrow \pi_2'') \Rightarrow \pi_1'') \Rightarrow \pi_1''$ with $\pi_1'', \pi_2'' \in \Pi''$ and set $V(\pi_1'') = \{v\}$ and $V(\pi_2'') = \emptyset$. It is straightforward to check that in the resulting structure $\mathbf{B}$, $[\![((\pi_1'' \Rightarrow \pi_2'') \Rightarrow \pi_1'') \Rightarrow \pi_1'']\!]_{\mathbf{B}} = \{v\} \neq \{u, v\}$. Similarly, we can show that the two negations do not collapse in this model. Indeed, $(\sim (\sim \varphi)) \supset \varphi$ is valid, but $(\neg(\neg \varphi)) \supset \varphi$ can be falsified. Namely, using the same structure as above, $[\![(\neg(\neg \pi_1'')) \supset \pi_1'']\!]_{\mathbf{B}} = \{v\}$.

This example makes clear that collapsing situations may be due to a limitation in the definition of fibred semantics. Of course, there will always be cases where collapses cannot be avoided, specially if there are shared constructors. However, unwanted collapsing situations can be dealt with if we just enlarge the definition of fibred model in order to encompass combined structures as the one above for classical and intuitionistic logic. What happens here is that whenever the valuation V is taken in such a way that $V(\Pi') \subseteq \{\emptyset, W\}$, it is possible to identify two structures $\mathbf{A}' \in \mathcal{A}'$ and $\mathbf{A}'' \in \mathcal{A}''$ and homomorphisms $h' : \mathbf{A}' \to \mathbf{B}'$ and $h'' : \mathbf{A}'' \to \mathbf{B}''$, of C' and C''-algebras respectively, that preserve the ordered structure in a strict manner. It suffices to consider $\mathbf{A}'$ to be the 2-valued Boolean structure, $\mathbf{A}'' = \mathbf{B}''$, h' injecting top and bottom into W and $\emptyset$, and h'' the identity.

Such a possibility was first considered in [Sernadas *et al.*, 2002b], where *modulated fibring* was introduced and shown to avoid these collapses, by means of a very careful use of adjunctions between the ordered semantic structures. The work on *cryptofibring* [Caleiro and Ramos, 2004], now underway, proposes a structurally simpler alternative to solve the semantic collapse problem by adopting a generalization of fibred semantics using cryptomorphisms [Caleiro and Ramos, 2005]. Cryptomorphisms are precisely the strict homomorphisms that have appeared in the example above. It can be shown that the novel notion encompasses the original definition of fibred model, but admits many more models. Once again the categorial setting is essential to tune up the details of the construction.

The scope of these investigations is, nevertheless, somewhat wider. It directly addresses the question of *conservativeness*. As Gabbay puts it in [Gabbay, 1999], it would be desirable that $\mathcal{L}' \odot \mathcal{L}''$ could be shown to be the least logic system over the combined language that conservatively extends the original logic systems $\mathcal{L}'$ and $\mathcal{L}''$. It is clear that, in general, such a desideratum is unattainable. It may very well happen, namely due to shared constructors, that no conservative extension of the two logic systems exists. Still, we would like to guarantee that we can build it whenever it exists. The relationship between conservativeness and the collapsing problem should be clear: collapses are special cases of failure of conservativeness.

This line of work is not without other difficulties. Namely, if we consider combined structures that are strict homomorphic extensions of structures of the original logic, we may very well get into trouble as far as the associated proof-calculi are concerned. Namely, even if both $\mathcal{L}'$ and $\mathcal{L}''$ are sound, it may happen that we get some enlarged structures that violate the rules. To recover soundness it is essential to get rid of these “bad” structures. But then also conservativeness may be at stake.

7 Final remarks

In this guided tour, we defined fibring in a very simple (yet useful) context and established some prototypical transference results. In this respect note that, concerning conditions for the preservation of weak completeness, it is still an open problem if the ghost symbol technique (used in [Kracht and Wolter, 1991] for proving the preservation of weak completeness by fusion) can be generalized in order to be used for fibring. As already mentioned, fibring can and has been defined and analyzed in much more complex situations. Current research is directed at widening the universe where fibring can be defined and at establishing transference results for other interesting properties. It is clear that in order to encompass, for instance, substructural logics like linear logic, we need a significant revamp of the theory of fibring. A key idea is to replace formula entailment by sequent entailment, using sequents with arbitrary structure. In this direction, it seems worthwhile to look at a logic as a kind of generalized 2-category with formulae as objects, connectives as multimorphisms, and sequents as poly-2-cells which will also raise interesting new problems in the theory of multicategories and of polycategories.

Another topic worth pursuing is the preservation by fibring of algebraizability and related notions, in the sense of [Blok and Pigozzi, 1989, Font *et al.*, 2003], and keeping in mind the results in [Jánossy *et al.*, 1996]. Some preliminary results can be found in [Fernández and Coniglio, 2004]. This question has a deep relationship with the forthcoming development of cryptofibring [Caleiro and Ramos, 2004], and should also take into account the techniques used in [Caleiro *et al.*, 2003a].

Closer to the concerns of the target application area of software engineering, we should mention the effort to bring fibring to the realm of the general theory of logics as institutions [Goguen and Burstall, 1992], namely in connection to the combination of parchments [Mossakowski, 1996, Mossakowski *et al.*, 1997, Mossakowski *et al.*, 1998]. Work along these lines is reported in [Caleiro *et al.*, 2001, Caleiro *et al.*, 2003b] and, more recently, in [Caleiro and Ramos, 2005]. As for new transference results, a particularly interesting open problem is to extend to fibring the results about model checking for combined temporal logics in [Franceschet *et al.*, 2004]. Recent developments in the logic of security protocols, namely concerning the combination of "intruder theories" [Chevalier and Rusinowitch, 2005], suggest that fibring may also become relevant in this application area.

Despite all the work done so far, there still remain many unaddressed challenges to the discipline of combining logics beyond fibring, some of them raised by emerging application areas. The interest in probability logic has recently increased due to the growing importance of probability in cryptography, security, and quantum computation and information. Also in classical software and hardware systems probabilities play an important role, for instance in distributed co-ordination and routing and fault-tolerance problems (see [Rutten *et al.*, 2004]). For other motivations see also [Fox, 2003]. An essential issue in quantum logic is to accommodate the fourth postulate of quantum mechanics stating that when a physical quantity is measured using an observable on a system in a given state, the resulting outcomes are ruled by a probability space. For more details on a quantum logic encompassing the probability aspects of the fourth postulate see [Mateus and Sernadas, 2004a, Mateus and Sernadas, 2004b], and [Mateus and Sernadas, 2005] for a complete axiomatization. The exogenous approach used to build this logic is quite interesting: the models of the quantum logic are superpositions of the models of the underlying logic and so quantum logic is a conservative extension of the underlying logic. Therefore, it seems tempting to define both probabilization and quantization as exogenous operations on an arbitrary base logic. For a detailed discussion of the the exogenous approach see [Mateus *et al.*, 2005].

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