Truth-values as labels: a general recipe for labelled deduction

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Abstract

We introduce a general recipe for presenting non-classical logics in a modular and uniform way as labelled deduction systems. Our recipe is based on a labelling mechanism where labels are general entities that are present, in one way or another, in all logics, namely truth-values. More specifically, the main idea underlying our approach is the use of algebras of truth-values, whose operators reflect the semantics we have in mind, as the labelling algebras of our labelled deduction systems. The "truth-values as labels" approach allows us to give generalized systems for multiple-valued logics within the same formalism: since we can take multiple-valued logics as meaning not only finitely or infinitely many-valued logics but also power-set logics, i.e. logics for which the denotation of a formula can be seen as a set of worlds, our recipe allows us to capture also logics such as modal, intuitionistic and relevance logics, thus providing a first step towards the fibring of these logics with many-valued ones.

1 Introduction

1.1 Context

Labelled deduction is an approach to presenting different logics in a uniform and natural way as Gentzen-style deduction systems, such as natural deduction, sequent or tableaux systems; see, for instance, [3, 4, 11, 12, 15, 25, 31]. It has been applied, for example, to formalize and reason about dynamic "state-oriented" properties, such as knowledge, belief, time, space, and resources, and thereby formalize deduction systems for a wide range of non-classical logics, such as modal, temporal, intuitionistic, relevance and other substructural logics. The intuition behind labelled deduction is that the *labelling* (sometimes also called prefixing, annotating or subscripting) allows one to explicitly encode additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the logic one wants to capture. To illustrate this, take the simple, standard example of modal logics, where the additional information encoded into the syntax usually comes from the underlying Kripke semantics: instead of considering a modal formula φ, we can consider the *labelled formula* $w{:}\varphi$, which intuitively means that φ holds at the world denoted by the world variable w within the underlying Kripke structure (i.e. model). In other words, the labelled formula $w{:}\varphi$ indicates that the modal formula φ is to be evaluated at w. We can also use labels to specify at the syntactic level the way in which the different worlds are related in the Kripke structures; for example, we can use the formula wRv to specify that the world denoted by v is accessible from that denoted by w. As discussed in, among others, [4, 25, 31], a modal labelled natural deduction system over this extended language is then obtained by giving inference rules for deriving labelled formulae, introducing or eliminating formula constructors such as implication $\sqsupset$ and modal necessity $\Box$, and by defining a suitable *labelling algebra*, which governs the inferences of formulae *about* labels, such as wRv.

Labelled deduction systems are *modular* for families of logics, such as the family of normal modal logics, in that to capture logics in the family we only need to vary appropriately the labelling algebra, while leaving the language and the rules for the formula constructors unchanged. Labelled deduction systems are also *uniform*, in that the same philosophy and technique can be applied for different, unrelated logic families. More specifically, changes in the labelling, i.e. in how formulae are labelled and with what labels (as we might need labels that are structurally more complex than the simple equivalents of Kripke worlds), together with changes in the language and rules, allow for the formalization of systems for non-classical logics other than the modal ones.

For instance, [4, 25, 31] also show that the approach where labelling is driven by an underlying Kripke semantics can be followed also to formalize labelled deduction systems for relevance logics. A related semantically-driven labelling approach is employed in [11] to present substructural logics in terms of tableaux-systems where structured labels capture the underlying substructural algebras, and other similar labelled Gentzen-style systems are discussed in [12, 15].

An approach that is closely related to labelled deduction is that of hybrid logic, e.g. [1, 5, 17]. There, the operators on world variables are $\downarrow$ and @, and among the labelled formulae there are $\downarrow w.\varphi$ (indicating that the world variable w has to be bound to the current world) and $@w.\varphi$ (indicating that the evaluation of φ has to be shifted to w) where w is a world variable and φ is a modal formula. Modal logic constructors can be also applied to labels providing a very rich language where, for instance, irreflexivity can be described by the formula $\downarrow w.\Box\neg w$. Hence, hybrid logic is more expressive than (pure) modal logic where irreflexivity is not expressible. Note that [4, 31] also show how to extend the labelling language to express non-axiomatizable properties such as irreflexivity, and how this extension is reflected in the proof-theoretical and computational properties of the resulting systems. Also related to our work is the uniform deductive treatment given to logics based on lattice-like algebraic structures as in [29, 30].

Deduction systems for hybrid logic can be formalized either by extending the standard Hilbert system for the modal logic K with rules for dealing with the new operators on labels, or by directly expressing the properties of the formula constructors and of the label operators by means of tableaux rules.

Labels can also be employed to give Gentzen-style systems for many-valued logics; in this case, labels are used to represent the set of truth-values of the particular logic, which can be either the unit interval $[0,1]$ on the rational numbers or a finite set of rational numbers of the form $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, e.g. the set $\{0, 0.5, 1\}$ that is used in 3-valued Gödel logic. For examples of many-valued labelled deduction systems, mostly tableaux systems, see [2, 8, 10, 19, 20]. In these systems, the labelled formula $b \approx \varphi$ intuitively means either that the formula φ has truth-value b, or that φ has one of the values in b when b is a set of truth-values as in the "sets-as-signs" approach. The inference rules and the labelling algebra of a system then essentially mirror the truth-tables of the formula constructors of the corresponding logic.

1.2 Contributions

We here introduce a general recipe for presenting non-classical logics in a modular and uniform way as labelled deduction systems. Our systems are in the style of natural deduction (even though, as we discuss below, not all the systems that we consider here are "natural" in the sense of Prawitz [23]; cf. the discussions in Section 3 and Section 4), but, mutatis mutandis, the recipe can be applied also to formalize labelled sequent or tableaux systems. [21] provides a first step in this direction.

Our recipe is based on a labelling mechanism where labels are general entities that are present, in one way or another, in all logics, namely: we use *truth-values* as labels. More specifically, the main idea underlying our approach is the use of algebras of truth-values, whose operators reflect the semantics we have in mind, as the labelling algebras of our labelled deduction systems. Moreover, more logics can be considered, i.e. we can present a larger number of non-classical logics as labelled deduction systems possessing all the good uniformity and modularity features of labelled deduction.

Our "truth-values as labels" approach allows us to provide generalized systems for *multiple-valued logics*. More specifically, our approach generalizes previous work, including our own, on labelled deduction systems where labels represent worlds in the underlying Kripke structures, and this generalization is illustrated by the following observation: since we can take multiple-valued logics as meaning not only finitely or infinitely many-valued logics but also *power-set logics*, i.e. logics for which the denotation of a formula can be seen as a set of worlds, our framework allows us to give deduction systems also for logics such as modal, intuitionistic and relevance ones. In a nutshell, the novelty of our approach with respect to previous approaches based on labelling is that we can capture, i.e. give deduction systems for, all these different logics *within the same formalism*. It is interesting to note that this also provides a first, large, step towards the *fibring* [6, 16] of these logics with many-valued ones; we have begun investigating in this direction as part of our research program on fibring of logics [25, 27, 28, 32], and we will illustrate this further in Section 5.

The fact that the labels constitute an algebra of truth-values means that we can have operations on truth-values, and that formula constructors can be associated with these operations. To this end, the syntax of our systems defines operators that build labels as terms of truth-values, and uses these "complex" labels to build two kinds of formulae: (i) *labelled formulae*, which are built by prefixing "standard", unlabelled, formulae with a label (and with an infix operator $\approx$ or $\prec$), and (ii) *truth-value formulae*, which are equalities and inequalities between labels. For labels $\delta, \delta_1, \delta_2$ and a formula φ, the semantic intuition behind these formulae is as follows: $\delta \approx \varphi$ means that the value of φ is equal to that of the truth-value term δ, $\delta \prec \varphi$ means that the value of φ is greater than or equal to that of δ, $\delta_1 = \delta_2$ means that the value of δ_2 is equal to that of δ_1, and $\delta_1 \leq \delta_2$ means that the value of δ_2 is greater than or equal to that of δ_1.

A system for a particular logic comprises then (i) inference rules that define how these formulae can be derived, e.g. "basic" rules expressing the properties of $\approx$, $\prec$, $=$ and $\leq$, (ii) rules defining how the formula constructors are introduced and eliminated and how this is reflected in the associated label operators, and (iii) rules defining properties of these operators.

Different logics have different algebras of truth-values. Therefore, the syntactic labelling algebras of truth-values of the corresponding systems reflect this difference. Even for the same logic, different semantics lead to different labelled deduction systems because they differ on the algebra of truth-values. In order to understand better the issue, and assess what we gain and how we can recover the previous view of having worlds as labels, let us consider again modal logic. We start by observing that there are at least three well known ways of defining the semantics of modal logic: *Kripke frames*, *general Kripke frames* and *modal algebras* [7].

When having (general) Kripke semantics in mind, the intuition of $\delta \prec \varphi$ is that the interpretation of φ, a set of worlds, includes at least the set of worlds in δ. Of course, $\top \prec \varphi$ indicates that the formula φ is valid, i.e. that it is true in every world. We can recover worlds as labels as a particular case when we work with $\delta \approx \varphi$ and want δ to be (semantically) interpreted by a singleton. Recall that a general Kripke frame is a triple $\langle W, \rho, B\rangle$ where the pair $\langle W, \rho\rangle$ is a Kripke frame with W a non-empty set of worlds and ρ a binary accessibility relation between worlds, and B is a set of subsets of W that should be closed for the denotations of the constructors in the logic. Hence, the set of truth-values B can include fewer sets than the powerset of worlds considered in pure Kripke semantics; general Kripke semantics restricts the truth-values to the elements of B. In both cases, we can define an algebra of truth-values with a unary operator $\Box^{\circ}$, where intuitively $\Box^{\circ}\delta$ returns the set of all worlds that are accessible from the worlds of δ. The deduction rules for $\Box$ are easy to establish if we observe that $\delta \prec \Box\varphi$ iff $\Box^{\circ}\delta \prec \varphi$. When looking into labels in a modal algebra perspective, we can define algebra for the truth-values with a unary operator $\Box^{v}$ and include as rules: $\Box^{v}(\delta_1 \sqcap \delta_2) = (\Box^{v}\delta_1) \sqcap (\Box^{v}\delta_2)$, $\Box^{v}\top = \top$ and $\varphi^{v} \approx \varphi$ (the first two reflecting the properties of the $\Box$ in modal algebra and the third relating truth-values with formulae).

Irreflexivity is expressed very easily in the modal labelled deduction systems that encompass the three semantics described above. The same happens with other properties that are not expressible in (pure) modal logic. Moreover, deductions in all systems are written in the same way: for instance we say that necessitation is globally derivable iff $\top \prec \Box\varphi$ whenever $\top \prec \varphi$ where $\top$ represents the greatest value in a Kripke frame, or in a general Kripke frame (where it represents the set W), or in a modal algebra (where it represents the top element).

Finally, but not less importantly, considering truth-values as labels also allows us to clarify some of the more subtle aspects of Hilbert systems with proof and derivation rules, such as in the Hilbert systems for modal logic. For instance, necessitation is a proof rule (corresponding to global semantics) and can be presented as "$\top \prec \Box\varphi$ whenever $\top \prec \varphi$", and modus ponens is a derivation rule (corresponding to local semantics) and can be presented as "$\delta \prec \psi$ whenever $\delta \prec \varphi$ and $\delta \prec \varphi \sqsupset \psi$", where δ is any truth-value variable.

1.3 Organization

The remainder of this paper is organized as follows. In Section 2, we introduce the basic concepts for employing truth-values as labels, namely we introduce the signature, language, and the semantics of our labelled deduction systems. As exemplified in the intuitive explanations above, the semantics of our systems is given by structures interpreting both formulae and labels as truth-values, and checking if the relationship between them complies with the labelling, e.g. for the labelled formula $\delta \approx \varphi$ if the value of φ is indeed that of δ. Rather than simply proving soundness and completeness for a particular system, we then establish general results on soundness and completeness in the context of our recipe, and afterwards analyze relationships between labelled deduction systems and Hilbert systems. We conclude the section with a recipe for building labelled deduction systems for given logics.

In Section 3, we discuss labelled deduction systems for logics endowed with a pure algebraic semantics (meaning that the properties of the operators are expressed via equations), and in Section 4, we consider well-behaved logics and discuss general properties of operations that provide algebraic semantics of constructors.

The first general property that we consider in Section 4 is the universal property, which includes constructors like different forms of implication (classic, intuitionistic and relevant), conjunction and necessitation. We also briefly illustrate how universal quantification can be included in this category of operators, although in this paper we do not further elaborate on first-order logics (which will be a subject of future work). The last property analyzed is the unary co-universal property, which is satisfied by different forms of negation. Finally ending the section we present a labelled logic system that puts together and illustrates connectives for all the properties introduced.

The results in Section 4 demonstrate that our labelled deduction rules (and thus, a fortiori, our systems) are not just uniform but also fully modular. For instance, we first give the rules for implication in general and then obtain different forms of implication by adding new rules to the general ones, and a similar modularity holds for the other formula constructors.

Finally, in Section 5, we conclude by briefly referring to immediate applications of our results.

2 Labelled deduction systems

The objective of this section is to introduce the main ideas underlying our labelled deduction systems in their generality, i.e. without any particular logic in mind. We start, in Subsection 2.1, by defining the signature and the language, namely the algebra of labels and the composed formulae. Afterwards, in Subsection 2.2, we introduce deduction rules and deductions. In Subsection 2.3, we give the semantics of our systems, and in Subsection 2.4 we prove the main general result on completeness. Finally, in Subsection 2.5 we give guidelines for defining semantically-driven labelled deduction systems for given logics.

2.1 Signature and language

In order to use truth-values as labels, we need to "promote" truth-values (usually a semantic concept) to the syntax. This will give us the possibility of comparing a truth-value with the value of a formula, and also to compare two truth-values. Therefore, besides for unlabelled formulae, we also consider *composed formulae*, which are of two kinds: (i) *labelled formulae*, which we obtain by prefixing "standard", unlabelled, formulae with a label (and with an infix operator $\approx$ or $\prec$), and (ii) *truth-value formulae*, which are equalities and inequalities between labels. To define these formulae, we first introduce the concept of signature.

We assume fixed once and for all two sets Ξ^f and Ξ^v of *schema variables* for formulae and truth-values. A *truth-value signature* is a triple $\Sigma = \langle C^f, C^v, s\rangle$ where $C^f = \{C^f_k\}_{k\in\mathbb{N}}$ and $C^v = \{C^v_k\}_{k\in\mathbb{N}}$ are families of sets and $s \in C^v_0$ is a distinguished element (essential for validity). More specifically, the elements of C^f_k are the *formula constructors of arity* k, the elements of C^v_k are the *truth-value operators of arity* k, and s is a *distinguished truth-value*. Hence C^f_0 can be empty but C^v_0 is always not empty.[1]

The problem of defining a truth-value signature for a labelled deduction system can be stated in the following way. Assume that we have a (algebraic) semantics for a certain logic. That is, we have a pair $\langle C, \mathbb{A}\rangle$ where $C = \{C_k\}_{k\in\mathbb{N}}$ is a family of sets of constructors and $\mathbb{A}$ is a class of structures (usually a class of algebras). The objective is then to define for that logic a labelled deduction system $\mathcal{D}(\langle C, \mathbb{A}\rangle)$ reflecting, from a deductive point of view, the semantics. The system $\mathcal{D}(\langle C, \mathbb{A}\rangle)$ can be defined as the pair $\langle \Sigma, R\rangle$, where Σ is a truth-value signature and R is a set of *inference* (or *deduction*) *rules*.

The first step is to choose the signature Σ. Of course C^f should be C. Therefore the problem at this point is the choice of C^v. As a recipe we can say that an operator must be introduced in C^v for each operation in the structures used to give the denotation of a formula constructor. Therefore C^v should be a "syntactic algebra" for truth-values. Let us illustrate this by means of an example.

Example 2.1 Consider modal logic as presented, for instance, in [7]. That is, assume that we are given $\langle C, \mathbb{A}\rangle$ where $C_1 = \{\neg, \Box\}$, $C_2 = \{\sqsupset\}$, all the other C_k are empty and $\mathbb{A}$ is a class of Kripke frames of the form $\langle W, \rho\rangle$ where W is a non-empty set (the set of possible worlds) and $\rho \subseteq W^2$ is a binary relation (the accessibility relation).

We can look at each subset of W as being a truth-value. Each formula φ in a particular Kripke structure based on $\langle W, \rho\rangle$ has the truth-value $b \subseteq W$ iff φ is true in exactly the worlds of b. A formula is valid in a structure if in that structure it has W as truth-value.

We can then define a truth-value signature for a modal labelled deduction system $\mathcal{D}(\langle C, \mathbb{A}\rangle)$ as follows: $C^f = C$, $C^v_0 = \{\top\}$, $C^v_1 = \{\Box^\circ\}$, $C^v_2 = \{\sqsupset^\circ\}$, s is $\top$ and all the other sets are empty. The operator $\Box^\circ$ is the syntactic counterpart of the accessibility relation of Kripke frames, $\sqsupset^\circ$ is needed in order to define the rules of $\sqsupset$, and $\top$ is the syntactic counterpart of W, used for stating that a formula is valid. Note that there may be other truth-value signatures for modal labelled logic systems. $\triangle$

The set of *(schema) formulae* is inductively defined using the constructors in C^f and the schema variables in Ξ^f, whereas the set of *non-schema formulae* is inductively defined using C^f. More specifically, the set of formulae is defined as follows: elements of C^f_0 and of Ξ^f are formulae, and $c(\varphi_1, \ldots, \varphi_k)$ is a formula for $c \in C^f_k$ and formulae $\varphi_1, \ldots, \varphi_k$. Observe that schema variables are different from the elements in C^f_0 because schema variables can be instantiated whereas constants in C^F_0 cannot.

The set of *(schema) truth-value terms* is inductively defined in a similar way using the operators in C^v and the schema variables in Ξ^v, whereas the set of *non-schema truth-value terms* is inductively defined using C^v.

[1] Note that in this paper we consider only one distinguished truth-value, which is sufficient for the examples that we discuss here, but our approach generalizes straightforwardly to signatures with more than one distinguished element. In such a case, we would have to change the notions of proof and derivation that we introduce below, which is not difficult to do, but makes the notation and the development more cumbersome.

Note that schema variables act as place-holders and so may be substituted, in the schema formulae or schema terms in which they appear, by other schema formulae or schema truth-value terms. This feature (i) is essential when combining logic systems, which is a very important issue that we want to pursue in the future in the context of the labelled systems introduced in this paper, and (ii) allows one to present a deduction system in a more compact form.

We consider *(schema) composed formulae*, which are of two kinds:

- for comparing truth-values, we consider *truth-value formulae*, which are equalities $\beta_1 = \beta_2$ and inequalities $\beta_1 \leq \beta_2$ where β_1, β_2 are truth-value terms;
- for comparing a truth-value with the value of a formula, we consider *labelled formulae*, which are of the form $\beta \approx \varphi$ and $\beta \prec \varphi$ where β is a truth-value term and φ is a formula.

We define in a similar manner *non-schema composed formulae.*

The semantic intuition behind these formulae is as follows: $\beta \approx \varphi$ means that the value of φ is equal to that of the truth-value term β, $\beta \prec \varphi$ means that the value of φ is greater than or equal to that of β, $\beta_1 = \beta_2$ means that the value of β_2 is equal to that of β_1, and $\beta_1 \leq \beta_2$ means that the value of β_2 is greater than or equal to that of β_1.

As notation, we use φ and γ for schema formulae, ξ for schema formula variables, β and δ for schema truth-value variables, and η and μ for composed formulae.

2.2 Deduction

A labelled deduction system allows us to deduce composed formulae from sets of composed formulae, and thus to globally and locally derive formulae from sets of formulae. A deduction system is composed of a number of deduction rules. A n-ary *deduction rule*, for $n \in \mathbb{N}$, is a pair $\langle\{\langle\Theta_1, \eta_1\rangle, \ldots, \langle\Theta_n, \eta_n\rangle\}, \eta\rangle$ where $\Theta_1, \ldots, \Theta_n$ are sets of composed formulae and $\eta, \eta_1, \ldots, \eta_n$ are composed formulae. Each $\langle\Theta_i, \eta_i\rangle$ is said to be a *premise* of the rule and η is said to be the *conclusion.* A rule whose set of premises is empty is said to be an *axiom.* We will display rules also graphically as

$$\frac{\Theta_1 \,/\, \eta_1 \quad \cdots \quad \Theta_n \,/\, \eta_n}{\eta}$$

and we will omit the reference to the empty components.

Definition 2.2 A *labelled deduction system* $\mathcal{D}$ is a pair $\langle\Sigma, R\rangle$, where Σ is a truth-value signature and R is a set of rules including at least the following ones:

$$\frac{}{\delta = \delta} =_r, \qquad \frac{\delta_1 \leq \delta_2 \quad \delta_2 \leq \delta_3}{\delta_1 \leq \delta_3} \leq_t, \qquad \frac{\delta_1 = \delta_2 \quad \delta_2 = \delta_3}{\delta_1 = \delta_3} =_t, \qquad \frac{\delta_2 = \delta_1}{\delta_1 = \delta_2} =_s,$$

$$\frac{\delta_1 \leq \delta_2 \quad \delta_2 \leq \delta_1}{\delta_1 = \delta_2} =_I, \qquad \frac{\delta_1 = \delta_2}{\delta_1 \leq \delta_2} =_E, \qquad \frac{\delta_1 = \delta'_1 \ \cdots \ \delta_k = \delta'_k}{o(\delta_1, \ldots, \delta_k) = o(\delta'_1, \ldots, \delta'_k)} =^o_{cong},$$

$$\frac{\delta_1 \leq \delta_2 \quad \delta_2 \prec \xi}{\delta_1 \prec \xi} \prec_I, \qquad \frac{\delta_1 \prec \xi \quad \delta_2 \approx \xi}{\delta_1 \leq \delta_2} \prec_E, \qquad \frac{\delta_1 = \delta_2 \quad \delta_2 \approx \xi}{\delta_1 \approx \xi} \approx_I, \qquad \frac{\delta \approx \xi}{\delta \prec \xi} \approx_E,$$

for any o in C^v_k and natural k, and where $\delta, \delta_1, \ldots, \delta_k, \delta'_1, \ldots, \delta'_k \in \Xi^v$ and $\xi \in \Xi^f$. △

Labelled deduction systems for particular logics are obtained by fixing a particular signature Σ and adding rules to the ones in Definition 2.2. In fact, the rules in Definition 2.2 establish minimal properties on labelling, common to all our labelled deduction systems. The rules $=_I$ and $=_E$ introduce and eliminate $=$, and $=_s$, $=_t$ and $=_r$, together with $=^o_{cong}$, for every o in C^v_k and natural k, express that $=$ constitutes a congruence relation. The rules $\prec_I$, $\prec_E$ and $\approx_I$ define conditions that $\prec$ and $\approx$ should obey, relating labelled formulae to truth-value formulae and vice-versa. Similarly to $=_E$, rule $\approx_E$ relates $\approx$ to $\prec$ (intuitively because if the value of ξ is equal to δ, i.e. $\delta \approx \xi$ holds, then a fortiori it is greater than or equal to it, i.e. $\delta \prec \xi$ holds).

Example 2.3 Consider the following positive modal labelled deduction system $\langle \Sigma, R\rangle$ where Σ is the truth-value signature with $C^f_0 = \{\mathbf{t}\}$, $C^f_1 = \{\Box\}$, $C^f_2 = \{\wedge, \vee, \sqsupset\}$, $C^v_0 = \{\top\}$, $C^v_1 = \{\Box^\circ\}$, $C^v_2 = \{\sqsupset^\circ\}$, all the other sets are empty, s is $\top$, and R is obtained by adding the following rules to the set of rules which are given in Definition 2.2 (and which are common to all deduction systems):

$$\frac{\Box^\circ\delta \prec \xi}{\delta \prec \Box\xi}\ \Box_I \qquad \frac{\delta \prec \Box\xi}{\Box^\circ\delta \prec \xi}\ \Box_E \qquad \frac{b_1 \leq b_2}{\Box^\circ(b_1) \leq \Box^\circ(b_2)}\ mon_{\Box^\circ}$$

$$\frac{\delta \prec \xi_1}{\delta \prec \xi_1 \vee \xi_2}\ \vee^1_I \qquad \frac{\delta \prec \xi_2}{\delta \prec \xi_1 \vee \xi_2}\ \vee^2_I \qquad \frac{\delta \prec \xi_1 \vee \xi_2 \quad \delta \prec \xi_1 \,/\, \eta \quad \delta \prec \xi_2 \,/\, \eta}{\eta}\ \vee_E$$

$$\frac{}{\delta \leq \top}\ \top \qquad \frac{}{\top \prec \mathbf{t}}\ \mathbf{t}$$

$$\frac{\delta \prec \xi_1 \quad \delta \prec \xi_2}{\delta \prec \xi_1 \wedge \xi_2}\ \wedge_I \qquad \frac{\delta \prec \xi_1 \wedge \xi_2}{\delta \prec \xi_1}\ \wedge_{E^1} \qquad \frac{\delta \prec \delta_1 \wedge \xi_2}{\delta \prec \xi_2}\ \wedge_{E^2}$$

$$\frac{\delta_1 \prec \xi_1 \,/\, \sqsupset^\circ(\delta, \delta_1) \prec \xi_2}{\delta \prec \xi_1 \sqsupset \xi_2}\ \sqsupset_I \qquad \frac{\delta \prec \xi_1 \sqsupset \xi_2 \quad \delta_1 \prec \xi_1}{\sqsupset^\circ(\delta, \delta_1) \prec \xi_2}\ \sqsupset_E$$

$$\frac{\delta_1 \prec \xi}{\sqsupset^\circ(\delta_1, \delta_2) \prec \xi}\ \sqsupset^\circ{}_{I_1} \qquad \frac{\delta_2 \prec \xi}{\sqsupset^\circ(\delta_1, \delta_2) \prec \xi}\ \sqsupset^\circ{}_{I_2} \qquad \frac{\sqsupset^\circ(\delta_1, \delta_2) \prec \xi \quad \delta \leq \delta_1 \quad \delta \leq \delta_2}{\delta \prec \xi}\ \sqsupset^\circ{}_E\,.$$

The meaning of the deduction rules $\Box_I$ and $\Box_E$ is quite intuitive: the formula $\Box\xi$ is true at least in the set of worlds δ iff the formula ξ is true in the set of all the worlds that can be seen by worlds in δ, which is represented by $\Box^\circ\delta$. The rule $mon_{\Box^\circ}$ expresses the monotony of $\Box^\circ$. The rule $\top$ indicates that the distinguished value $\top$ is the greatest truth-value and the rule $\mathbf{t}$ that $\mathbf{t}$ is its formula counterpart. The rules for $\wedge$ and $\vee$ express the expected properties of conjunction and disjunction, respectively. Finally, the rules $\sqsupset_I$ and $\sqsupset_E$ express that implication $\sqsupset$ and $\sqsupset^\circ$ constitute an adjoint pair, and $\sqsupset^\circ{}_{I_1}$, $\sqsupset^\circ{}_{I_2}$ and $\sqsupset^\circ{}_E$ define the properties of $\sqsupset^\circ$. $\triangle$

In order to formalize deductions, we first need to define substitutions. A *substitution* σ over a signature Σ is a map that assigns to each schema formula variable a formula, and to each schema truth-value variable a truth-value term. A substitution σ can be canonically extended to formulae and terms. Given a composed formula ψ and a substitution σ, we denote by $\psi\sigma$ the composed formula that results from ψ by the simultaneous substitution of each ξ by $\sigma(\xi)$ and each δ by $\sigma(\delta)$. And we denote by $\Psi\sigma$ the set of all $\psi\sigma$ with $\psi \in \Psi$.

A composed formula μ is *derived* (or, synonymously, *deduced*) from a set Ψ of composed formulae in $\mathcal{D}$, written $\Psi \vdash^{\mathcal{D}} \mu$, iff there is a sequence of pairs $\langle \Psi_1, \mu_1\rangle, \ldots, \langle \Psi_n, \mu_n\rangle$ where $\mu_1, \ldots, \mu_n$ are composed formulae and $\Psi_1, \ldots, \Psi_n$ are sets of composed formulae such that $\Psi_n \subseteq \Psi$, μ_n is μ and for each $i = 1, \ldots, n$

- either $\mu_i \in \Psi_i$,
- or there are a rule $\langle\{\langle\Theta_1, \eta_1\rangle, \dots, \langle\Theta_k, \eta_k\rangle\}, \eta\rangle$, a substitution σ and natural numbers $i_1, \dots, i_k$ in $\{1, \dots, i-1\}$ such that:
  - μ_i is $\eta\sigma$,
  - $\Psi_{i_r} = \Psi_i \cup \{\Theta_r\sigma\}$ and $\mu_{i_r} = \eta_r\sigma$, for each r in $1, \dots, k$.

As the examples below will help illustrate, the sets Ψ_i can be interpreted as hypotheses and, for the sake of readability, we will not include them explicitly in deductions, but rather indicate after the deductions the points where the sets of hypotheses changed. Observe that, among others, from the rules in Definition 2.2 it is possible to derive in any labelled deduction system the reflexivity of $\leq$.

Given $\mathcal{D}$, we can also define global and local derivations of formulae from sets of formulae, corresponding, from a semantic point of view, to global and local entailments, respectively. For a set of formulae Γ and a formula φ we say that:

- φ is *globally derivable* from Γ in $\mathcal{D}$, written $\Gamma \vdash_g^{\mathcal{D}} \varphi$, iff $\{s \prec \gamma \mid \gamma \in \Gamma\} \vdash^{\mathcal{D}} s \prec \varphi$, where s is the distinguished truth-value element in $\mathcal{D}$,
- φ is *locally derivable* from Γ in $\mathcal{D}$, written $\Gamma \vdash_l^{\mathcal{D}} \varphi$ iff $\{\delta \prec \gamma \mid \gamma \in \Gamma\} \vdash^{\mathcal{D}} \delta \prec \varphi$, for any δ in Ξ^v.

Of course, if $\Gamma \vdash_l^{\mathcal{D}} \varphi$ then also $\Gamma \vdash_g^{\mathcal{D}} \varphi$ but not the other way around.

Example 2.4 Consider the labelled deduction system $\mathcal{D}(\langle C, \mathbb{A}\rangle)$ for modal logic introduced in Example 2.3. We can, for instance, show $\varphi_1 \sqsupset \varphi_2 \vdash_g^{\mathcal{D}} (\Box(\varphi_1)) \sqsupset (\Box(\varphi_2))$ and $(\Box(\varphi_1)) \vdash_l^{\mathcal{D}} (\Box(\varphi_2)) \sqsupset (\Box(\varphi_1))$. Moreover, the following proof

1	$\Box^\circ(\top) \leq \top$	$\top$
2	$\top \prec \xi$	*hyp*
3	$\Box^\circ(\top) \prec \xi$	$\prec_I$ 1, 2
4	$\top \prec \Box(\xi)$	$\Box_I$ 3

with $\Psi_4 = \{\top \prec \xi\}$, shows that $\top \prec \Box(\xi)$ follows from $\top \prec \xi$, which thus provides a labelled equivalent in $\mathcal{D}$ of the necessitation rule (see also the discussion on Hilbert systems below). $\triangle$

Note that we chose not to adopt the usual tree-notation for deductions, as the examples we consider result in trees that are too wide to fit the page. Instead, we write deductions in a tabular notation, listing, for each formula in the deduction, the rule we have used to obtain it (and from which formulae and hypotheses).

We conclude this subsection by exploring the relationships between Hilbert systems and labelled deduction systems so that we can conclude when a labelled deduction system globally/locally derives at least the same schema formulae as the corresponding Hilbert system. We first briefly review what is a Hilbert system.

In general, see for instance [32], a Hilbert system is a triple $H = \langle C, \mathit{Prf}, \mathit{Der}\rangle$ where C is a family of sets of formula constructors (organized by arities and possibly with several constructors for each arity), Prf is a set of proof rules and Der is a set of derivation rules such that $\mathit{Der} \subseteq \mathit{Prf}$. A proof rule, and similarly a derivation rule, is a pair composed by a finite set of formulae (the premises) and a formula (the conclusion). Intuitively, proof rules correspond to global entailment and derivation rules to local entailment.

A proof of a formula φ from a set of formulae Γ in a Hilbert system H, written $\Gamma \vdash_p^H \varphi$, is a sequence of formulae obtained by instantiating the proof rules. A derivation of a formula φ from a set of formulae Γ in H, written $\Gamma \vdash_d^H \varphi$, is a sequence of formulae where each formula is an element of Γ, an instance of a derivation rule or a theorem (obtained using the proof rules without hypotheses).

We can compare a Hilbert system H for a logic $\langle C, \mathbb{A}\rangle$ with a labelled deduction system $\mathcal{D}(\langle C, \mathbb{A}\rangle)$ for the same logic. In particular, the Hilbert system is *sound with respect to the labelled deduction system* iff $\Gamma \vdash_g^{\mathcal{D}(\langle C,\mathbb{A}\rangle)} \varphi$ for every proof rule $\langle \Gamma, \varphi\rangle$ in H and $\Gamma \vdash_l^{\mathcal{D}(\langle C,\mathbb{A}\rangle)} \varphi$ for every derivation rule $\langle \Gamma, \varphi\rangle$ in H. It is easily proved by a straightforward induction that in the presence of soundness we have:

- $\Gamma' \vdash_g^{\mathcal{D}(\langle C,\mathbb{A}\rangle)} \varphi'$ whenever $\Gamma' \vdash_p^H \varphi'$,
- $\Gamma' \vdash_l^{\mathcal{D}(\langle C,\mathbb{A}\rangle)} \varphi'$ whenever $\Gamma' \vdash_d^H \varphi'$.

It is also easy to see that the labelled deduction system that we introduced in Example 2.3 above is sound with respect to the Hilbert system $H(K)$ for the modal logic K.

2.3 Semantics

A structure for a truth-value signature is an algebra providing the interpretation of formula constructors and truth-value operators.

Definition 2.5 A *structure* m for a signature Σ is a tuple $\langle B_m, \leq_m, \cdot_m\rangle$ where $\langle B_m, \leq_m\rangle$ is a partial order, and $\langle B_m, \cdot_m\rangle$ is an algebra for $C^f \cup C^v$. △

The elements of the set B_m are the truth-values, $\leq_m$ allows for the comparison of truth-values, and $\cdot_m$ is a family of maps $c_m{:}B_m^k \to B_m$ for each $c \in C_k^f \cup C_k^v$ and natural k. In the sequel, we will sometimes omit the reference to the arity of the constructors and operators in order to make the notation lighter, and sometimes we will even omit the reference to m. Also with the same purpose we use the vectorial notation $\vec{b_i} = b_1, \ldots, b_i$, $\vec{w_i} = \{w_1\}, \ldots, \{w_i\}$ and $\vec{w_i} \in \vec{b_i}$ whenever $w_j \in b_j$ for $j = 1, \ldots, i$. Moreover, we may confuse an operator with its denotation. An *interpretation system* is a pair $\mathcal{I} = \langle \Sigma, M\rangle$ where Σ is a signature and M is a class of structures for Σ.

Example 2.6 Consider the modal truth-value signature of Example 2.1 based on $\langle C, \mathbb{A}\rangle$ where C is the usual signature for modal logic and $\mathbb{A}$ is a class of Kripke frames of the form $\langle W, \rho\rangle$ where W is a non-empty set (the set of possible worlds) and $\rho \subseteq W^2$ is a binary relation (the accessibility relation). An interpretation system $I(\langle C, \mathbb{A}\rangle)$ induced by $\langle C, \mathbb{A}\rangle$ is composed of structures of the form $m_{\langle W,\rho\rangle}$, one for each Kripke frame $\langle W, \rho\rangle \in \mathbb{A}$, such that $B_m = \wp W$, $\leq_m$ is set inclusion and $\cdot_m$ is defined as follows: $\neg_m(b) = W \setminus b$, $\Box_m(b) = \{w \in W \mid w\rho v \text{ implies } v \in b\}$, $\Box^\circ{}_m(d) = \{v \in W \mid w\rho v \text{ for some } w \in d\}$, $\sqsupset_m (b_1, b_2) = (W \setminus b_1) \cup b_2$, $\sqsupset^\circ (b_1, b_2) = b_1 \sqcap b_2$, $\top_m = W$, and where we use m for $m_{\langle W,\rho\rangle}$. △

We now proceed with the denotation of (schema) formulae and (schema) terms. For this purpose, we need the straightforward notion of assignment. An *assignment* α over a structure $m = \langle B_m, \leq_m, \cdot_m \rangle$ is a map that assigns to each schema formula variable and to each schema truth-value variable a value in B_m. Given a structure m and an assignment α, the *interpretation of formulae* is a map $[\![.]\!]_\alpha^m : L(\langle C^f, \Xi^f \rangle) \to B_m$ inductively defined as follows:

- $[\![c]\!]_\alpha^m = c_m$;
- $[\![\xi]\!]_\alpha^m = \alpha(\xi)$;
- $[\![c(\varphi_1, \ldots, \varphi_k)]\!]_\alpha^m = c_m([\![\varphi_1]\!]_\alpha^m, \ldots, [\![\varphi_k]\!]_\alpha^m)$.

The interpretation of truth-value terms is defined in a similar way.

Now we can define satisfaction of composed formulae. A structure m and an assignment α *satisfy*

- $\beta \approx \varphi$ iff $[\![\beta]\!]_\alpha^m = [\![\varphi]\!]_\alpha^m$,
- $\beta \prec \varphi$ iff $[\![\beta]\!]_\alpha^m \leq [\![\varphi]\!]_\alpha^m$,
- $\beta_1 * \beta_2$ iff $[\![\beta_1]\!]_\alpha^m * [\![\beta_2]\!]_\alpha^m$, with $* \in \{=, \leq\}$.

As usual, we write

$$m, \alpha \Vdash \psi$$

when the structure m and the assignment α satisfy the composed formula ψ, and we write

$$m, \alpha \Vdash \Psi$$

whenever $m, \alpha \Vdash \psi$ for every $\psi \in \Psi$. We say in this case that m is a structure for ψ and Ψ. Finally, we write

$$m, \alpha \Vdash \langle \Theta, \eta \rangle$$

iff $m, \alpha \Vdash \eta$ whenever $m, \alpha \Vdash \Theta$. We can now define the entailments. A composed formula μ is *entailed* by a set of composed formulae Ψ, in symbols

$$\Psi \vDash \mu$$

iff $m, \alpha \Vdash \mu$ whenever $m, \alpha \Vdash \Psi$ for every structure m and assignment α. A formula φ is *g-entailed* (globally entailed) by the set of formulae Γ, denoted by

$$\Gamma \vDash_g \varphi$$

iff $\{s \prec \gamma \mid \gamma \in \Gamma\} \vDash s \prec \varphi$. A formula φ is *l-entailed* (locally entailed) by the set of formulae Γ, denoted by

$$\Gamma \vDash_l \varphi$$

iff $\{\delta \prec \gamma \mid \gamma \in \Gamma\} \vDash \delta \prec \varphi$ for an arbitrary δ in Ξ^v.

2.4 Soundness and completeness

Rather than focussing on a particular labelled deduction system and showing its soundness and completeness with respect to the corresponding structures, we will now introduce the concept of logic system. Then we analyze soundness and completeness for a general labelled deduction system in the context of our recipe for labelled deduction. To this end, we introduce and work with exhaustive labelled deduction systems, i.e. systems where each formula constructor c is associated with a truth-value operation symbol c^v of the same arity.

Definition 2.7 An *exhaustive* truth-value signature is a pair $\langle \Sigma, \cdot^v \rangle$, where Σ is a truth-value signature and $\cdot^v$ is a pair composed of a family of maps from C^f to C^v and a one to one and onto map from Ξ^f to Ξ^v. In the following when referring to an exhaustive truth-value signature we omit the explicit reference to $\cdot^v$.

An *exhaustive* labelled deduction system $\langle \Sigma, R \rangle$ is a labelled deduction system where Σ is an exhaustive truth-value signature, R includes the additional rule

$$\frac{}{\xi^v \approx \xi}\ v$$

and for each truth-value term β there is a formula φ with $\vdash \beta = \varphi^v$, where we denote by φ^v the truth-value term that corresponds by $\cdot^v$ to the formula φ. Note that by ξ^v in rule v we do not intend to represent the application of $\cdot^v$ to ξ but the representation of the image by $\cdot^v$ of the instance of ξ in a derivation, when this rule is used in that derivation. The same applies to all the uses of $\cdot^v$ in rules.

An *exhaustive* interpretation system $\langle \Sigma, M \rangle$ is an interpretation system where the signature is an exhaustive truth-value signature, and $c_m = {c^v}_m$ for each connective c in C^f and structure m in M. $\triangle$

It is straightforward to show that in an exhaustive labelled deduction system, the following are derived rules:

$$\frac{\delta = \xi^v}{\delta \approx \xi} \approx_{I_2}, \qquad \frac{\delta \approx \xi}{\delta = \xi^v} \approx_{E_2}, \qquad \frac{\delta \leq \xi^v}{\delta \prec \xi} \prec_{I_2}, \qquad \frac{\delta \prec \xi}{\delta \leq \xi^v} \prec_{E_2}.$$

Given a labelled formula $\delta \approx \varphi$, we denote by $(\delta \approx \varphi)^v$ the truth-value formula $\delta = \varphi^v$. Similarly for $\delta \prec \varphi$, where $\prec$ corresponds to $\leq$.

Remark 2.8 Observe that our focusing on exhaustive systems is not a restriction since in many cases labelled deduction systems can be transformed into exhaustive ones: given a system $\mathcal{D}$ it is possible to enrich its signature and deduction rules so that we get an exhaustive labelled deduction system $\mathcal{D}^v$. This is the case for instance for the positive modal labelled deduction system presented in Example 2.3. It is straightforward to show, by a simple induction on the length of derivations, that $\Theta \vdash^{\mathcal{D}^v} \mu$ whenever $\Theta \vdash^{\mathcal{D}} \mu$. $\triangle$

We now introduce logic systems. A *logic system* is a triple $\langle \Sigma, R, M \rangle$ where $\langle \Sigma, R \rangle$ is an exhaustive labelled deduction system and $\langle \Sigma, M \rangle$ is an exhaustive interpretation system.

Finally, we define when a logic system is sound and complete. Let $\Psi \cup \{\mu\}$ be a set of composed formulae and $\Gamma \cup \{\varphi\}$ a set of (schema) formulae. A logic system $\mathcal{L}$ is

- *sound* iff $\Psi \vDash \mu$ whenever $\Psi \vdash \mu$;

- *g-sound* iff $\Gamma \vDash_g \varphi$ whenever $\Gamma \vdash_g \varphi$;
- *l-sound* iff $\Gamma \vDash_l \varphi$ whenever $\Gamma \vdash_l \varphi$;
- *complete* iff $\Psi \vdash \mu$ whenever $\Psi \vDash \mu$;
- *g-complete* iff $\Gamma \vdash_g \varphi$ whenever $\Gamma \vDash_g \varphi$;
- *l-complete* iff $\Gamma \vdash_l \varphi$ whenever $\Gamma \vDash_l \varphi$.

A technique to prove soundness is to prove soundness of the deduction rules. A structure m *validates* a deduction rule $\langle\{\langle\Theta_1, \eta_1\rangle, \ldots, \langle\Theta_k, \eta_k\rangle\}, \eta\rangle$ iff for every assignment α over m, we have $m, \alpha \Vdash \eta$ whenever $m, \alpha \Vdash \langle\Theta_i, \eta_i\rangle$ for each $i = 1, \ldots, k$. A logic system is sound with respect to the rules iff every structure m validates the deduction rules.

For completeness, we consider the logic system

$$LS^*(\mathcal{D}) = \langle\Sigma, R, M\rangle$$

where M includes all structures for the signature Σ that validate the deduction rules. The proof of completeness follows, with the necessary changes and adaptions, a standard approach [7] by defining the Lindenbaum-Tarski structure for each maximal consistent set and showing that this structure validates the rules.

More specifically, completeness is proved by showing that if a composed formula μ is not a consequence of a set of composed formulae Θ, then μ is not entailed by Θ. In order to show this we consider a set Θ' maximal consistent with respect to μ, extending Θ. Then the result follows since the Lindenbaum-Tarski algebra for Θ', (i) satisfies Θ' and so Θ and not satisfies μ, and (ii) is a structure in $LS^*(\mathcal{D})$ since it validates the deduction rules of R, and M includes all structures for the signature Σ that validate the deduction rules, see Lemma 2.9.

We say that a set Θ of composed formulae is *consistent with respect to a composed formula* μ iff $\Theta \nvdash \mu$, that it is *maximally consistent with respect to a composed formula* μ if it is consistent with respect to μ and no proper extension of it is consistent with respect to μ, and that it is *maximally consistent* iff it is a maximally consistent set with respect to some composed formula μ. Note that we did not define maximality of Θ by relying on its consistency, i.e. that $\beta \approx \perp$ does not follow from Θ, but rather *with respect to* a composed formula μ. This happens since the signature Σ of the logic system that we are considering might not contain a falsum $\perp$ (indeed, we might be considering a purely positive logic in which negation is not present at all). Before stating the completeness theorem, observe that every consistent set of composed formulae can be extended to a maximally consistent set.

Lemma 2.9 Every set of composed formulae consistent with respect to a composed formula has a structure in the context of an exhaustive labelled deduction system.

Proof Let $\mathcal{D} = \langle\Sigma, R\rangle$ be an exhaustive labelled deduction system. We divide the proof in three parts. The first part is the definition of the candidate structure (the Lindenbaum-Tarski algebra, as we call it) for each maximal consistent set of composed formulae. The second part is related to the satisfaction in the structure. The third part is showing that the structure validates the deduction rules. Then the lemma follows by noting that every consistent set with respect to a composed formulae can be extended to a maximal consistent set with respect to that same composed formula.

Part I: definition of the Lindenbaum-Tarski algebra. Let Ψ be a maximal set of composed formulae consistent with respect to a composed formula and closed under deduction. Consider the following equivalence relation on truth-value terms:

$$\beta =_\Psi \beta' \text{ iff } \Psi \vdash \beta = \beta'.$$

This relation is also congruent (see Definition 2.2 and rule $=^o_{cong}$, for any o in C^v_k and natural k): $o(\beta_1, \ldots, \beta_k) =_\Psi o(\beta'_1, \ldots, \beta'_k)$ whenever $\beta_i =_\Psi \beta'_i$ for every o in C^v_k and terms β_i, β'_i with $i = 1, \ldots, k$. Let $[\beta]_{=_\Psi}$ be the equivalence class of β, that is the set of all β' such that $\beta =_\Psi \beta'$.

The Lindenbaum-Tarski algebra $\lambda\tau_\Psi = \langle B, \leq, \cdot_{\lambda\tau_\Psi} \rangle$ for Ψ is defined as follows:

- the truth-value set B is the set of all equivalence classes $[\beta]_{=_\Psi}$,
- the comparison relation $\leq$ is such that $[\beta]_{=_\Psi} \leq [\beta']_{=_\Psi}$ iff $\Psi \vdash \beta \leq \beta'$,
- the denotations are such that $c_{\lambda\tau_\Psi}([\beta_1]_{=_\Psi}, \ldots, [\beta_k]_{=_\Psi}) = [c^v(\beta_1, \ldots, \beta_k)]_{=_\Psi}$, for $c \in C^f_k$, and $o_{\lambda\tau_\Psi}([\beta_1]_{=_\Psi}, \ldots, [\beta_k]_{=_\Psi}) = [o(\beta_1, \ldots, \beta_k)]_{=_\Psi}$, for $o \in C^v_k$.

It remains to show that the comparison relation is well defined. Assume that $\beta'_i \in [\beta_i]_{=_\Psi}$ with $i = 1, 2$ and $\beta_1 \leq \beta_2$. We must verify that $\beta'_1 \leq \beta'_2$. Assume that $\Psi \vdash \beta_1 \leq \beta_2$, $\Psi \vdash \beta_1 = \beta'_1$ and $\Psi \vdash \beta_2 = \beta'_2$ with deductions Π, Π_1 and Π_2, respectively. Then a deduction for $\Psi \vdash \beta'_1 \leq \beta'_2$ is as follows:

$$\begin{array}{lll}
1 & \beta'_1 = \beta_1 & \Pi_1 \\
2 & \beta'_1 \leq \beta_1 & =_E 1 \\
3 & \beta_2 = \beta'_2 & \Pi_2 \\
4 & \beta_2 \leq \beta'_2 & =_E 3 \\
5 & \beta_1 \leq \beta_2 & \Pi \\
6 & \beta'_1 \leq \beta_2 & \leq_t 2, 5 \\
7 & \beta'_1 \leq \beta'_2 & \leq_t 6, 4
\end{array}$$

Part II: some results on satisfaction. For any assignment α over $\lambda\tau_\Psi$ and substitution σ_α over Σ such that $[\sigma_\alpha(\delta)]_{=_\Psi} = \alpha(\delta)$ and $[\sigma_\alpha(\xi)^v]_{=_\Psi} = \alpha(\xi)$ we have:

1. $\beta \approx \varphi \in \Psi$ iff $[\beta]_{=_\Psi} = [\varphi^v]_{=_\Psi}$;
2. $\beta \prec \varphi \in \Psi$ iff $[\beta]_{=_\Psi} \leq [\varphi^v]_{=_\Psi}$;
3. $[\![\varphi]\!]^\alpha_{\lambda\tau_\Psi} = [\varphi\sigma_\alpha]_{=_\Psi}$;
4. $[\![\beta]\!]^\alpha_{\lambda\tau_\Psi} = [\beta\sigma_\alpha]_{=_\Psi}$;
5. $\lambda\tau_\Psi, \alpha \Vdash \eta$ iff $\eta\sigma_\alpha \in \Psi$;
6. $\lambda\tau_\Psi, \alpha \Vdash \langle \Theta, \eta \rangle$ iff $\Psi \cup \Theta\sigma_\alpha \vdash \eta\sigma_\alpha$.

For 1, assume that $\Psi \vdash \beta = \varphi^v$ with deduction Π. Then, consider the following deduction

$$\begin{array}{lll} 1 & \beta = \varphi^v & \Pi \\ 2 & \varphi^v \approx \varphi & v \\ 3 & \beta \approx \varphi & \approx_I 1,2 \end{array}$$

that shows that $\Psi \vdash \beta \approx \varphi$ and so $\beta \approx \varphi \in \Psi$ since Ψ is closed under deduction. Assume now that $\Psi \vdash \beta \approx \varphi$ with deduction Π''. Then we conclude by considering the following deduction:

$$\begin{array}{llr} 1 & \beta \approx \varphi & \Pi'' \\ 2 & \beta \prec \varphi & \approx_E 1 \\ 3 & \varphi^v \approx \varphi & v \\ 4 & \beta \leq \varphi^v & \prec_E 2,3 \\ 5 & \varphi^v \prec \varphi & \approx_E 3 \\ 6 & \varphi^v \leq \beta & \prec_E 5,1 \\ 7 & \varphi^v = \beta & \prec_E 5,1 \end{array}$$

Clause 2 is similar. Clauses 3 is proved by induction on the structure of φ and clause 4 is proved by induction on the structure of β. For 5, suppose that η is $\beta \approx \varphi$. Then $\lambda\tau_\Psi, \alpha \Vdash \beta \approx \varphi$ iff $[\![\beta]\!]^\alpha_{\lambda\tau_\Psi} = [\![\varphi]\!]^\alpha_{\lambda\tau_\Psi}$ iff $[\beta\sigma_\alpha]_{=_\Psi} = [\varphi{\sigma_\alpha}^v]_{=_\Psi}$ iff $\beta\sigma_\alpha \approx \varphi\sigma_\alpha \in \Psi$. The proof is analogous for the other cases. For 6, observe that $\lambda\tau_\Psi, \alpha \Vdash \langle\Theta, \eta\rangle$ iff if $\lambda\tau_\Psi, \alpha \Vdash \Theta$ then $\lambda\tau_\Psi, \alpha \Vdash \eta$. So, by 5, iff if $\Theta\sigma_\alpha \subseteq \Psi$ then $\eta\sigma_\alpha \in \Psi$. Now, for any assignment α, we have two cases: (i) if indeed $\Theta\sigma_\alpha \subseteq \Psi$ then $\eta\sigma_\alpha \in \Psi$ and so $\Psi \cup \Theta\sigma_\alpha \vdash \eta\sigma_\alpha$, (ii) otherwise, since Ψ is maximal then $\Psi \cup \Theta\sigma_\alpha \vdash \eta\sigma_\alpha$.

Part III: the Lindenbaum-Tarski algebra defined in part I validates the rules. Let $\langle\{\langle\Theta_1, \eta_1\rangle, \ldots, \langle\Theta_k, \eta_k\rangle\}, \eta\rangle$ be a deduction rule, α be an assignment over $\lambda\tau_\Psi$, and σ_α a substitution as defined in part II (note that it is possible to consider such a substitution since we are in the context of an exhaustive labelled deduction system). Assume $\lambda\tau_\Psi, \alpha \Vdash \langle\Theta_i, \eta_i\rangle$ for every $i = 1, \ldots, k$. Then, by clause 6 of part II, $\Psi \cup \Theta_i \vdash \eta_i\sigma_\alpha$ for $i = 1, \ldots, k$. Thus $\Psi \vdash \eta\sigma_\alpha$, and since Ψ is closed under deduction then $\eta\sigma_\alpha \in \Psi$. Using clause 5 of part II, we conclude that $\lambda\tau_\Psi, \alpha \Vdash \eta$. QED

Theorem 2.10 The exhaustive logic system $LS^*(\mathcal{D})$ is complete.

Proof We proceed by contraposition and assume that $\Psi_0 \nvdash \mu$. Then $\mu \notin \Psi$ where Ψ is an extension of Ψ_0 maximal with respect to μ. So, using the results of part II of Lemma 2.9, we have that $\lambda\tau_\Psi, \iota \Vdash \eta$ for any η in Ψ_0, and $\lambda\tau_\Psi, \iota \nVdash \mu$, where ι is the canonical assignment. Since $\lambda\tau_\Psi \in M$, by part III of Lemma 2.9, then $\Psi_0 \nvDash \mu$. QED

Corollary 2.11 The exhaustive logic system $LS^*(\mathcal{D})$ is l-complete and g-complete.

Proof For g-completeness, assume that $\Gamma_0 \nvdash_g \varphi$. Then, $\{s \prec \gamma \mid \gamma \in \Gamma_0\} \nvdash s \prec \varphi$. So, using Theorem 2.10, we have that $\{s \prec \gamma \mid \gamma \in \Gamma_0\} \nvDash s \prec \varphi$, and so $\Gamma_0 \nvDash_g \varphi$. For l-completeness, assume that $\Gamma_0 \nvdash_l \varphi$. Then, there is a $\delta \in \Xi^v$ such that $\{\delta \prec \gamma \mid \gamma \in \Gamma_0\} \nvdash \delta \prec \varphi$. So, using Theorem 2.10, we have $\{\delta \prec \gamma \mid \gamma \in \Gamma_0\} \nvDash \delta \prec \varphi$, and we can conclude $\Gamma_0 \nvDash_l \varphi$. QED

2.5 Building a deduction system for your logic

We now give some guidelines for setting-up a labelled deduction system for a particular logic endowed with an algebraic semantics. Assume that we know the algebraic semantics $\langle C, \mathbb{A}\rangle$ of a logic. We can then define a logic system $LS(\langle C, \mathbb{A}\rangle) = \langle \Sigma, R, M\rangle$ in the following way:

- $\Sigma = \langle C^f, C^v\rangle$ is an exhaustive truth-value signature where $C^f = C$ and C^v is a "syntactic algebra" for truth-values, including, for each $c \in C^f$, the operators that represent at the syntactic level all the operations in $\mathbb{A}$ relevant for the denotation of c;
- $\langle \Sigma, R\rangle$ is an exhaustive labelled deduction system where R, besides for the general rules in Definition 2.2, should also include rules that describe from a syntactic point of view the denotations of the symbols of C in $\mathbb{A}$ along with rules for the operators in C^v reflecting, from a syntactic point of view, the properties of the algebras in $\mathbb{A}$;
- $\langle \Sigma, R\rangle$ is an exhaustive interpretation system where M is the class of structures determined by $\mathbb{A}$ that extend the algebras in $\mathbb{A}$ to the operators in C^v in the expected way.

Note that we can speak of the interpretation system and of the labelled deduction system induced by $\langle C, \mathbb{A}\rangle$, which we denote by $\mathcal{I}(\langle C, \mathbb{A}\rangle)$ and $\mathcal{D}(\langle C, \mathbb{A}\rangle)$, respectively. Analogously we define the logic system $LS(\mathcal{I})$, induced by the exhaustive interpretation system $\mathcal{I}$. Observe that in this case the only component really induced is the set of deduction rules of the system, which we denote by $\mathcal{D}(\mathcal{I})$.

We consider two kinds of logic systems induced by an algebraic semantics of a logic. The "direct" one, say $LS(\langle C, \mathbb{A}\rangle)$, is such that the connectives in C^v are the syntactic counterparts of the connectives considered in the algebraic semantics $\langle C, \mathbb{A}\rangle$, and R includes as rules all the axioms of the class of algebras. This is the perspective that we adopt in Section 3.

Another possibility is to define a logic system, called "well behaved", where the deduction rules express, in most of the cases in a natural deduction way, the algebraic properties of the operations in the algebras. This is the perspective that we adopt in Section 4.

As an example consider intuitionistic logic. That is, consider its algebraic semantics, $\langle C, \mathbb{A}\rangle$, which is such that in C we have the usual intuitionistic constructors and in $\mathbb{A}$ we have the class of Heyting algebras. We can consider two logic systems for $\langle C, \mathbb{A}\rangle$. The "direct" one is such that $C^v_1 = \{\bot\}$ and $C^v_2 = \{\sqcap, \sqcup, \sqsupset\}$ (where $\sqcap$, $\sqcup$ and $\sqsupset$ are the syntactic counterparts of the meet, join and relative complement, respectively) and R includes as rules all the axioms of Heyting algebras. The "well behaved" one is obtained from the examination in-depth of the algebraic properties of the operations in the Heyting algebras. For instance, by seeing that relative complements are adjoint situations.

The guidelines we propose assume that the original logic should have an algebraic semantics. Observe that most logics seems to be presented in a non-algebraic form when indeed they are defined over power-set algebras. For instance, modal logic, when endowed with Kripke semantics, is presented in terms of sets and relations. But it is not so difficult to think of each Kripke frame $\langle W, \rho\rangle$ as generating a power-set algebra where the carrier set is $\wp W$ and where ρ can be used to provide the denotations of the modal operators. This perspective also stresses that each subset of W can be seen as a truth-value. Hence, in the power-set algebra, the denotation of a formula is a subset of W, that is a truth-value. In this case, the labels in composed formulae represent from a syntactic point of view a set of worlds; note that world-based labelled deduction systems are of this form where the label represents a singleton (just one world at a time).

We can straightforwardly generalize these observations to argue that our guidelines apply also to other *power-set logics*, i.e. logics that like modal logics are defined over power-set algebras (so that the denotation of a formula can be seen as a set of worlds), such as intuitionistic and relevance logics.

Finally observe that there are cases where the same algebraic semantics, may induce logic systems of several kinds. Moreover, there are cases of logics with distinct algebraic semantics and so with possibly different kinds of induced logic systems. These situations have the advantage of making possible the choice of the system better suited to the needs in presence.

3 Logics with an algebraic semantics

Several logics are endowed with an algebraic semantics. For instance, propositional logic has an algebraic semantics based on Boolean algebras, and modal logic also has an algebraic semantics based on modal algebras (Boolean algebras with a new operator $\Box^\circ$ corresponding to the necessity operator $\Box$).

Assume that we have an algebraic semantics for a certain logic. That is, we have a pair $\langle C, \mathbb{A}\rangle$ where $C = \{C_k\}_{k\in\mathbb{N}}$ is a family of sets of constructors and $\mathbb{A}$ is a class of algebras for the signature at hand. We want to build an exhaustive labelled deduction system $\mathcal{D}(\langle C, \mathbb{A}\rangle) = \langle \Sigma, R\rangle$ for that logic reflecting from a deductive point of view the algebraic semantics. Herein we discuss the case where we want to take the algebraic semantics in a very "pure" way to the deduction system. The first step is to choose the exhaustive labelled signature Σ. Of course C^f should be C and C^v is a family of sets of operators such there is a $o \in C^v_k$ for each operator in the algebra involved in the denotation of a constructor in C^f. Then, we represent each equation in the algebra by an adequate deduction rule.

Example 3.1 Consider the following exhaustive labelled deduction system $\langle \Sigma, R\rangle$, induced by $\langle C, \mathbb{A}\rangle$, where C is the usual signature for modal logic and $\mathbb{A}$ is the class of all modal algebras (see [7]), such that Σ is the exhaustive truth-value signature with $C^f = C$, $C^v_0 = \{\top, \mathbf{f}^v\}$, $C^v_1 = \{\Box^v\}$, $C^v_2 = \{\sqsupset^v, \wedge^v, \vee^v\}$, s is $\top$ and all the other sets are empty, and R, besides the rules mentioned in Definition 2.2 common to all deduction systems, and the exhaustive rule introduced in Definition 2.7, includes the following deduction rules:

- rules indicating that conjunctions and disjunctions are binary meets and binary joins respectively:

$$\frac{\delta \leq \delta_1 \quad \delta \leq \delta_2}{\delta \leq \delta_1 \wedge^v \delta_2}\ \wedge^v_I\,, \qquad \frac{\delta \leq \delta_1 \wedge^v \delta_2}{\delta \leq \delta_1}\ \wedge^v_{E^1}\,, \qquad \frac{\delta \leq \delta_1 \wedge^v \delta_2}{\delta \leq \delta_2}\ \wedge^v_{E^2}\,,$$

$$\frac{\delta_1 \leq \delta \quad \delta_2 \leq \delta}{\delta_1 \vee^v \delta_2 \leq \delta}\ \vee^v_I\,, \qquad \frac{\delta_1 \vee^v \delta_2 \leq \delta}{\delta_1 \leq \delta}\ \vee^v_{E^1}\,, \qquad \frac{\delta_1 \vee^v \delta_2 \leq \delta}{\delta_2 \leq \delta}\ \vee^v_{E^2}\,,$$

- rules indicating that implication is a form of 2-residuation:

$$\frac{\delta_1 \leq \delta_2}{\top \leq \delta_1 \sqsupset^v \delta_2}\sqsupset^v{}_I\,, \qquad \frac{\top \leq \delta_1 \sqsupset^v \delta_2}{\delta_1 \leq \delta_2}\sqsupset^v{}_E\,,$$

- rules expressing the properties of bottom,

$$\frac{}{\mathbf{f}^v \leq \delta}\,\mathbf{f}^v{}_I\,, \qquad \frac{}{(\delta \sqsupset^v \mathbf{f}^v) \sqsupset^v \mathbf{f}^v \leq \delta}\sqsupset^v\mathbf{f}^v{}_c\,,$$

- and rules indicating that necessitation behaves as in modal algebras and that $\top$ is the greatest element:

$$\frac{}{\top \leq \Box^v(\delta_1 \sqsupset^v \delta_2) \sqsupset^v ((\Box^v \delta_1) \sqsupset^v (\Box^v \delta_2))}\,N\,, \qquad \frac{}{\top \leq \Box^v \top}\,G\,, \qquad \frac{}{\delta \leq \top}\,\top\,.$$

The following example deduction shows that necessity distributes over conjunction

1	$\varphi_1^v \wedge^v \varphi_2^v = \varphi_1^v \wedge^v \varphi_2^v$	$=_r$
2	$\varphi_1^v \wedge^v \varphi_2^v \leq \varphi_1^v \wedge^v \varphi_2^v$	$=_E$ 1
3	$\varphi_1^v \wedge^v \varphi_2^v \leq \varphi_1^v$	$\wedge^v{}_{E^1}$ 2
4	$\top \leq (\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v$	$\sqsupset^v_I$ 3
5	$(\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v \leq \top$	$\top$
6	$\top = (\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v$	$=_I$ 5, 4
7	$\Box^v(\top) = \Box^v((\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v)$	$=^{\Box}_{cong}$ 6
a	$\top \leq \Box^v(\top)$	G
b	$\Box^v(\top) \leq \top$	$\top$
10	$\top = \Box^v(\top)$	$=_t$ 8, 9
11	$\top = \Box^v((\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v)$	$=_t$ 10, 7
12	$\top \leq \Box^v((\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v)$	$=_E$ 11
13	$\top \leq \Box^v((\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v) \sqsupset^v (\Box^v(\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \Box^v(\varphi_1^v))$	N
14	$\Box^v((\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \varphi_1^v) \leq \Box^v(\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \Box^v(\varphi_1^v)$	$\sqsupset^v_E$ 13
15	$\top \leq \Box^v(\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v \Box^v(\varphi_1^v)$	$\leq_t$ 12, 14
16	$\Box^v(\varphi_1^v \wedge^v \varphi_2^v) \leq \Box^v(\varphi_1^v)$	$\sqsupset^v_E$ 15
17	$\Box^v(\varphi_1^v \wedge^v \varphi_2^v) \leq \Box^v(\varphi_2^v)$	$*$
18	$\Box^v(\varphi_1^v \wedge^v \varphi_2^v) \leq \Box^v(\varphi_1^v) \wedge^v \Box^v(\varphi_2^v)$	$\wedge^v{}_I$ 16, 17
19	$\top \leq \Box^v(\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v (\Box^v(\varphi_1^v) \wedge^v \Box^v(\varphi_2^v))$	$\sqsupset^v_I$ 18
20	$\Box^v(\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v (\Box^v(\varphi_1^v) \wedge^v \Box^v(\varphi_2^v)) \approx \Box(\varphi_1 \wedge \varphi_2) \sqsupset (\Box(\varphi_1) \wedge \Box(\varphi_2))$	v
21	$\Box^v(\varphi_1^v \wedge^v \varphi_2^v) \sqsupset^v (\Box^v(\varphi_1^v) \wedge^v \Box^v(\varphi_2^v)) \prec \Box(\varphi_1 \wedge \varphi_2) \sqsupset (\Box(\varphi_1) \wedge \Box(\varphi_2))$	$\approx_E$ 20
22	$\top \prec \Box(\varphi_1 \wedge \varphi_2) \sqsupset (\Box(\varphi_1) \wedge \Box(\varphi_2))$	$\prec_I$ 19, 21

where * indicates that that sub-deduction is similar to the sub-deduction for $\Box^v(\varphi_1^v \wedge^v \varphi_2^v) \leq \Box^v(\varphi_1^v)$. $\triangle$

There are other kinds of exhaustive labelled deduction systems, that may possibly be induced by the algebraic semantics for a logic, perhaps more intuitive and simple, like for instance natural labelled deduction systems. A *natural* labelled deduction system is a labelled deduction system where for each $c \in C_k^f$ there are two kinds of rules in R: (i) *introduction rules* for c whose conclusion is $\delta \prec c(\xi_1, \ldots, \xi_k)$, and (ii) *elimination rules* for c, where one of the premisses is $\delta \prec c(\xi_1, \ldots, \xi_k)$ where $\xi_1, \ldots, \xi_k \in \Xi^f$ and $\delta \in \Xi^v$. The labelled deduction system in Example 3.1 is not natural in this sense, but in Section 4 we will capture, among others, the algebraic properties of the constructors of modal logic in a natural labelled deduction system.

4 Well-behaved logics

In this section, we explore semantic properties of constructors and establish categories of constructors that can be presented in an exhaustive labelled deduction system by similar deduction rules, i.e. rules of the same general form.

Assuming that we have an algebraic semantics for a logic, given by an interpretation system $\mathcal{I} = \langle \Sigma, M \rangle$, the idea is to obtain an exhaustive natural labelled deduction system $\mathcal{D}(\mathcal{I})$ by exploring, and exploiting, the algebraic properties of the constructors — we call *well-behaved* the logics whose constructors have some of these particular algebraic properties. For instance, if we have an interpretation system for intuitionistic logic, we can use relative complements to induce an exhaustive labelled deduction system where implication fulfills the properties of intuitionistic implication.

The generality of the analysis we will carry out is based on the fact that several constructors share similar algebraic properties, and we can thus treat them in a similar fashion by giving uniform deduction rules for them. In the following two subsections, we concentrate on the following properties: the universal property (covering different forms of implication, conjunction and necessitation), and the unary co-universal property (that allows us to capture different forms of negation).

4.1 The universal property

The classical, intuitionistic and relevance implication, necessitation and conjunction are connectives that satisfy the universal property. To illustrate this, we begin by characterizing universality at the semantic level.

Definition 4.1 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system. The constructor $c \in C^f_k$ has the *n-universal property* iff there is $c^\circ \in C^v_n$ with $1 \leq n \leq k$ such that for every $m \in M$ where $m = \langle B, \leq, \cdot \rangle$ and $b, b_1, \ldots, b_k \in B$,

$$b \leq c^v_m(\vec{b}_{k-1}, b_k) \quad \text{iff} \quad c^\circ_m(b, \vec{b}_{n-1}) \leq b_j \text{ for } j = n, \ldots, k. \qquad \triangle$$

As an example, consider intuitionistic implication $\sqsupset$ and its truth-value operator counterpart $\sqsupset^v$. Taking $\sqsupset^\circ_m$ as the binary meet $\sqcap$, we have $b \leq (b_1 \sqsupset^v_m b_2)$ iff $(b \sqcap b_1) \leq b_2$.

Note that if $c \in C^f_k$ has the n-universal property then for every algebra $m \in M$, the inequality

$$c^\circ_m(c^v_m(\vec{b}_{k-1}, b_k), \vec{b}_{n-1}) \leq b_j$$

holds for $j = n, \ldots, k$. Therefore $c^v_m(\vec{b}_{k-1}, b_k)$ is the greatest element of B such that $c^\circ_m(b, \vec{b}_{n-1}) \leq b_j$ for $j = n, \ldots, k$. In other words, c has the n-universal property iff the set $\{b \in B \mid c^\circ_m(b, \vec{b}_{n-1}) \leq b_j, j = n, \ldots, k\}$ has a maximum for every $\vec{b}_{n-1} \in B^{n-1}$ and algebra $m \in M$.

Note also that the n-universal property always refers to b_k. It is possible to have other forms of the universal property where instead of b_k we have another b_i for some $i = 1, \ldots, k-1$. Another possibility consists in having a universal property over more than one argument. Here we will not discuss this further, but leave the investigation of these further possibilities for future work.

Example 4.2 Assume that $\langle \Sigma, M \rangle$ is an exhaustive interpretation system for intuitionistic logic where each m in M is a Heyting algebra. Then, for each m in M, $\wedge_m$ and $\sqsupset^v{}_m$, are relative complements, meaning that $b \leq \sqsupset^v{}_m(b_1, b_2)$ iff $b \wedge_m b_1 \leq b_2$. Hence, we can say that $\sqsupset$ has the 2-universal property. (Note that the fact that residuation of implication is related to Galois connections was already recognized in [13]). $\triangle$

We now present a list with the operators that satisfy the universal property:

- implication in systems with semantics based on Heyting algebras has the 2-universal property, see Example 4.2,
- implication in systems with semantics based on relevance algebras has the 2-universal property, see Example 4.6,
- conjunction in systems with semantics based on Heyting algebras has the 1-universal property, see Example 4.8,
- necessitation in systems with semantics based on a class of power-set algebras over the class of Kripke models has the 1-universal property, see Example 4.16,
- universal quantification in systems with semantics based on a class of power-set algebras of assignments has the 1-universal property, see Example 4.18.

4.1.1 Deduction system

We can characterize the deduction rules, for a constructor with the universal property, that must be part of an exhaustive labelled deduction system induced by an exhaustive interpretation system.

Definition 4.3 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system and $c \in C^f_k$ a constructor with the n-universal property. Then $\mathcal{D}(\mathcal{I})$ includes the following *introduction* and *elimination rules* for c

$$\frac{\vec{\delta}_{n-1} \prec \vec{\xi}_{n-1} \, / \, c^\circ(\delta, \vec{\delta}_{n-1}) \prec \xi_j \ , \ j = n, \ldots, k}{\delta \prec c(\xi_1, \ldots, \xi_k)} \, c_I$$

$$\frac{\delta \prec c(\xi_1, \ldots, \xi_k) \quad \vec{\delta}_{n-1} \prec \vec{\xi}_{n-1}}{c^\circ(\delta, \vec{\delta}_{n-1}) \prec \xi_i} \, c_{E^i} \ , \ i = n, \ldots, k$$

where the vector $\vec{\delta}_{n-1} \prec \vec{\xi}_{n-1}$ stands for the $n-1$ composed formulae $\delta_1 \prec \xi_1, \ldots, \delta_{n-1} \prec \xi_{n-1}$, and where $c^\circ(\delta, \vec{\delta}_{n-1})$ stands for $c^\circ(\delta, \delta_1, \ldots, \delta_{n-1})$. $\triangle$

The introduction rule, for example, states that $\delta \prec c(\xi_1, \ldots, \xi_k)$ whenever $c^\circ(\delta, \vec{\delta}_{n-1}) \prec \xi_j$ for $j = n, \ldots, k$ assuming that $\vec{\delta}_{n-1} \prec \vec{\xi}_{n-1}$.

We now introduce well-known constructors satisfying the universal property, and we present the rules associated with that constructors in the labelled deduction systems induced by interpretation systems with those constructors. We start with a very general notion of interpretation system with implication.

Definition 4.4 An exhaustive interpretation system $\mathcal{I}$ *has implication* iff $\sqsupset \in C^f_2$ and $\sqsupset$ satisfies the 2-universal property. Then, according to Definition 4.3, $\mathcal{D}(\mathcal{I})$ includes the following two rules for the introduction and elimination of $\sqsupset$:

$$\frac{\delta_1 \prec \xi_1 \,/\, \sqsupset^\circ(\delta,\delta_1) \prec \xi_2}{\delta \prec \xi_1 \sqsupset \xi_2}\ \sqsupset_I \qquad \frac{\delta \prec \xi_1 \sqsupset \xi_2 \quad \delta_1 \prec \xi_1}{\sqsupset^\circ(\delta,\delta_1) \prec \xi_2}\ \sqsupset_E\ .$$

△

Observe that different kinds of implication are captured by imposing further properties on $\sqsupset^\circ$ as the examples below will illustrate.

Example 4.5 An exhaustive interpretation system $\mathcal{I} = \langle \Sigma, M\rangle$ has an *intuitionistic implication* iff $\mathcal{I}$ has an implication $\sqsupset$, M is a class of Heyting algebras, and for every m in M, $\sqsupset^\circ_m$ is the meet. Then $\mathcal{D}(\mathcal{I})$ includes additionally the rules

$$\frac{\delta_1 \prec \xi}{\sqsupset^\circ(\delta_1,\delta_2) \prec \xi}\ {\sqsupset^\circ}_{I_1},\quad \frac{\delta_2 \prec \xi}{\sqsupset^\circ(\delta_1,\delta_2) \prec \xi}\ {\sqsupset^\circ}_{I_2} \quad \text{and} \quad \frac{\sqsupset^\circ(\delta_1,\delta_2) \prec \xi \quad \delta \leq \delta_1 \quad \delta \leq \delta_2}{\delta \prec \xi}\ {\sqsupset^\circ}_E\,,$$

which impose, in a syntactic way, that $\sqsupset^\circ$ is to be interpreted as the meet. △

In the previous example note that the introduction rule states that $\sqsupset^\circ$ is a lower bound and the elimination rule states that it is the greatest lower bound.

Example 4.6 An exhaustive interpretation system $\mathcal{I} = \langle \Sigma, M\rangle$ has a *relevant implication* iff $\mathcal{I}$ has an implication $\sqsupset$ and M is the class of relevance algebras (see, for instance, [13]). Note that there is $\circ \in C^f_2$ such that ${\circ^v}_m(b_1, b_2)$ is the fusion of b_1 and b_2 for every $m \in M$, and $\sqsupset^\circ$ is $\circ^v$. Moreover, note that there is t is in C^f_1 such that t^v is the distinguished truth-value 1 of relevance logic. Then $\mathcal{D}(\mathcal{I})$ includes additionally the rules

$$\frac{\sqsupset^\circ(\delta_2,\delta_1) \prec \xi}{\sqsupset^\circ(\delta_1,\delta_2) \prec \xi}\ \sqsupset^\circ_c\,, \qquad \frac{}{\delta \leq \sqsupset^\circ(\delta,\delta)}\ \sqsupset^\circ_i\,,$$

$$\frac{\sqsupset^\circ(\sqsupset^\circ(\delta_1,\delta_2),\delta_3) \prec \xi}{\sqsupset^\circ(\delta_1,\sqsupset^\circ(\delta_2,\delta_3)) \prec \xi}\ \sqsupset^\circ_{a_1}\,, \qquad \frac{\sqsupset^\circ(\delta_1,\sqsupset^\circ(\delta_2,\delta_3)) \prec \xi}{\sqsupset^\circ(\sqsupset^\circ(\delta_1,\delta_2),\delta_3) \prec \xi}\ \sqsupset^\circ_{a_2}\,,$$

$$\frac{\sqsupset^\circ(1,\delta) \prec \xi}{\delta \prec \xi}\ \sqsupset^\circ_{1_1}\,, \qquad \frac{\delta \prec \xi}{\sqsupset^\circ(1,\delta) \prec \xi}\ \sqsupset^\circ_{1_2}\,.$$

△

Observe that the rules $\sqsupset^\circ_c$, $\sqsupset^\circ_{a_1}$ and $\sqsupset^\circ_{a_2}$ state that $\sqsupset^\circ$ is a commutative and associative operator. So, as a direct consequence of rule $\sqsupset^\circ_i$, to prove $\delta \prec \xi$ it is sufficient to prove $\sqsupset^\circ(\delta,\delta) \prec \xi$. Finally, rules $\sqsupset^\circ_{1_1}$ and $\sqsupset^\circ_{1_2}$ state that the distinguished truth-value 1 is neutral with respect to $\sqsupset^\circ$.

Definition 4.7 An exhaustive interpretation system $\mathcal{I} = \langle \Sigma, M\rangle$ *has conjunction* iff $\wedge \in C^f_k$ and $\wedge$ has the 1-universal property. Then, according to Definition 4.3, $\mathcal{D}(\mathcal{I})$ includes the following introduction and elimination rules:

$$\frac{\wedge^\circ(\delta) \prec \xi_1 \quad \wedge^\circ(\delta) \prec \xi_2}{\delta \prec \xi_1 \wedge \xi_2}\ \wedge_I\,, \qquad \frac{\delta \prec \xi_1 \wedge \xi_2}{\wedge^\circ(\delta) \prec \xi_1}\ \wedge_{E^1}\,, \qquad \frac{\delta \prec \delta_1 \wedge \xi_2}{\wedge^\circ(\delta) \prec \xi_2}\ \wedge_{E^2}\,.$$

△

Example 4.8 An exhaustive interpretation system $\mathcal{I} = \langle \Sigma, M\rangle$ has a *intuitionistic conjunction* iff $\mathcal{I}$ has a conjunction $\wedge$, M is the class of Heyting algebras, and for every m in M, $\wedge^\circ_m(b) = b$. Then $\mathcal{D}(\mathcal{I})$ includes additionally the following rule expressing that conjunction is "local":

$$\frac{}{\delta = \wedge^\circ(\delta)}\ \wedge^\circ .$$

Analogously for a *relevant conjunction.* △

In the sequel we consider exhaustive truth-value signatures *with true*, i.e., exhaustive truth-value signatures such that $\mathbf{t} \in C_0^f$, $\top \in C_0^v$ and $\mathbf{t}^v$ is $\top$. We also consider exhaustive interpretation systems *with true*, i.e. exhaustive interpretation systems where Σ is a exhaustive truth-value signature with true and for every structure $m = \langle B_m, \leq_m, \cdot_m \rangle$ in the interpretation system, $\top_m = \mathbf{t}_m$ and for any b in B_m, $b \leq \top_m$. Moreover, we need exhaustive labelled deduction systems *with true*, i.e., exhaustive labelled deduction systems with a exhaustive truth-value signature with true, and whose set of rules include the rule

$$\frac{}{\delta \leq \top}\ \top \quad \text{and} \quad \frac{}{\top \prec \mathbf{t}}\ \mathbf{t}.$$

Note that it is possible to derive the rules

$$\frac{\sqsupset^\circ(\top, \delta) \prec \xi}{\delta \prec \xi}\ \sqsupset^\circ_{\top_E} \quad \text{and} \quad \frac{\delta \prec \xi}{\sqsupset^\circ(\top, \delta) \prec \xi}\ \sqsupset^\circ_{\top_I}$$

in the context of a labelled deduction system with truth and with an intuitionistic implication.

There are certain deductions that can be done when considering constructors of arity k with the k-universal property.

Proposition 4.9 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system with true and $c \in C_k^f$ a constructor with the k-universal property. Then $\vdash_g^{\mathcal{D}(\mathcal{I})} c(\vec{\xi}_{k-1}, \mathbf{t})$.

Proof The proposition is a straightforward consequence of $c^\circ(\top, \vec{\delta}_{k-1}) \prec \mathbf{t}$ where δ_i is ξ_i^v for $i = 1, \ldots, k-1$. QED

Proposition 4.10 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system with truth, intuitionistic conjunction and implication, and $c \in C_k^f$ a constructor with the k-universal property. Then, in the context of $\mathcal{D}(\mathcal{I})$, c distributes over conjunction, and c has (global) substitution of equivalents.

Proof For the distribution of c over $\wedge$, we show that $c(\vec{\xi}_{k-1}, (\xi_1 \wedge \xi_2)) \vdash_l^{\mathcal{D}(\mathcal{I})} c(\vec{\xi}_{k-1}, \xi_1) \wedge c(\vec{\xi}_{k-1}, \xi_2)$ as follows:

1	$\delta \prec c(\vec{\xi}_{k-1}, (\xi_1 \wedge \xi_2))$	*hyp*
2	$(\vec{\xi}_{k-1})^v \approx \vec{\xi}_{k-1}$	v
3	$(\vec{\xi}_{k-1})^v \prec \vec{\xi}_{k-1}$	$\approx_E 2$
4	$c^\circ(\delta, (\vec{\xi}_{k-1})^v) \prec \xi_1 \wedge \xi_2$	$c_E 1, 3$
5	$c^\circ(\delta, (\vec{\xi}_{k-1})^v) \prec \xi_1$	$\wedge_{E_1} 4$
6	$c^\circ(\delta, (\vec{\xi}_{k-1})^v) \prec \xi_2$	$\wedge_{E_2} 4$
7	$\delta \prec c(\vec{\xi}_{k-1}, \xi_1)$	$c_I 5$
8	$\delta \prec c(\vec{\xi}_{k-1}, \xi_2)$	$c_I 6$
9	$\delta \prec c(\vec{\xi}_{k-1}, \xi_1) \wedge c(\vec{\xi}_{k-1}, \xi_2)$	$\wedge_I 7, 8$

with $\Psi_8 = \{\delta \prec c(\vec{\xi}_{k-1}, (\xi_1 \wedge \xi_2))\}$. The converse direction is shown similarly. That c has substitution of equivalents, i.e. that $\xi_1 \sqsupset \xi_2 \vdash_g^{\mathcal{D}(\mathcal{I})} c(\vec{\xi}_{k-1}, \xi_1) \sqsupset c(\vec{\xi}_{k-1}, \xi_2)$, can be shown as follows:

$$
\begin{array}{lll}
1 & \delta \prec c(\vec{\xi}_{k-1}, \xi_1) & hyp \\
2 & \vec{\delta}_{k-1} \prec \vec{\xi}_{k-1} & hyp \\
3 & c^\circ(\delta, \vec{\delta}_{k-1}) \prec \xi_1 & c_{E^k}\ 1,2 \\
4 & \top \prec \xi_1 \sqsupset \xi_2 & hyp \\
5 & \sqsupset^\circ(\top, c^\circ(\delta, \vec{\delta}_{k-1})) \prec \xi_2 & \sqsupset_E\ 4,3 \\
6 & c^\circ(\delta, (\vec{\xi}_{k-1})^v) \leq \top & \top \\
7 & c^\circ(\delta, \vec{\delta}_{k-1}) = c^\circ(\delta, \vec{\delta}_{k-1}) & =_r \\
8 & c^\circ(\delta, \vec{\delta}_{k-1}) \leq c^\circ(\delta, \vec{\delta}_{k-1}) & =_E\ 7 \\
9 & c^\circ(\delta, \vec{\delta}_{k-1}) \prec \xi_2 & \sqsupset^\circ{}_E\ 5,6,8 \\
10 & \delta \prec c(\vec{\xi}_{k-1}, \xi_2) & c_I\ 9 \\
11 & \sqsupset^\circ(\top, \delta) \prec c(\vec{\xi}_{k-1}, \xi_2) & \sqsupset^\circ{}_{I_2}\ 10 \\
12 & \top \prec c(\vec{\xi}_{k-1}, \xi_1) \sqsupset c(\vec{\xi}_{k-1}, \xi_2) & \sqsupset_I\ 11
\end{array}
$$

with $\Psi_{12} = \{\top \prec \xi_1 \sqsupset \xi_2\}$, $\Psi_{11} = \Psi_{12} \cup \{\delta \prec c(\vec{\xi}_{k-1}, \xi_1)\}$, and $\Psi_9 = \Psi_{11} \cup \{\vec{\delta}_{k-1} \prec \vec{\xi}_{k-1}\}$.
QED

The following result showing that if a constructor $c \in C_1^f$ has the 1-universal property then it satisfies the axioms of modal algebras, is a direct corollary of Propositions 4.9 and 4.10.

Corollary 4.11 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system with truth, intuitionistic conjunction and implication, such that $c \in C_1^f$ is a constructor with the 1-universal property. Then, $\vdash_g^{\mathcal{D}(\mathcal{I})} c(\mathbf{t})$ and $\vdash_g^{\mathcal{D}(\mathcal{I})} c(\xi_1 \wedge \xi_2) \sqsupset ((c(\xi_1)) \wedge (c(\xi_2)))$. $\triangle$

4.1.2 Semantic characterization

We now characterize the algebras for constructors of arity k with the k-universal property via *adjunctions* As a particular case, we will be able to identify the class of modal algebras where necessitation has the 1-universal property, i.e., where $\Box^v$ has a left adjoint. In the following, in the context of an exhaustive interpretation system $\mathcal{I} = \langle \Sigma, R \rangle$ with a constructor c of arity k with the k-universal property, given m in M and fixed $\vec{b}_{k-1}$ in ${B_m}^{k-1}$, we consider the map

$$g_{m,\vec{b}_{k-1}} = \lambda d \in B_m.\, c_m^v(\vec{b}_{k-1}, d)$$

that assign to each truth-value d the value of $c_m^v(\vec{b}_{k-1}, d)$.

Proposition 4.12 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system. The constructor $c \in C_k^f$ has the k-universal property iff each $g_{m,\vec{b}_{k-1}}$ has a left adjoint for every $\vec{b}_{k-1}$ and structure $m \in M$.

Proof For the right-to-left direction, assume that for every $m \in M$, each $g_{m,\vec{b}_{k-1}} = \lambda d.c^v_m(\vec{b}_{k-1}, d)$ has a left adjoint $f_{m,\vec{b}_{k-1}}$. So $b \leq g_{m,\vec{b}_{k-1}}(d)$ iff $f_{m,\vec{b}_{k-1}}(b) \leq d$, and thus $c^\circ_m(b, \vec{b}_{k-1}) = f_{m,\vec{b}_{k-1}}(b)$ for every $b \in B_m$.

For the left-to-right direction, assume that $c \in C^f_k$ has the k-universal property. Then, by definition of the k-universal property, it has a left adjoint $f_{m,\vec{b}_{k-1}}$ such that $f_{m,\vec{b}_{k-1}}(b) = c^\circ_m(b, \vec{b}_{k-1})$. QED

The following proposition introduces inequalities relating a truth-value with the image resulting from the applications of $g_{m,\vec{b}_{k-1}}$ and $f_{m,\vec{b}_{k-1}}$. These inequalities are consequences of the fact that $g_{m,\vec{b}_{k-1}}$ is right adjoint to $f_{m,\vec{b}_{k-1}}$.

Corollary 4.13 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system and $c \in C^f_k$ a constructor with the k-universal property. Then for every $b, d \in B_m$ the following inequalities hold:

$$b \leq c^v_m(\vec{b}_{k-1}, c^\circ_m(b, \vec{b}_{k-1})) \text{ and } c^\circ_m(c^v_m(\vec{b}_{k-1}, d), \vec{b}_{k-1}) \leq d.$$

Moreover, if m is a meet semilattice then

$$c^v_m(\vec{b}_{k-1}, b_1 \sqcap b_2) = c^v_m(\vec{b}_{k-1}, b_1) \sqcap c^v_m(\vec{b}_{k-1}, b_2). \qquad \triangle$$

Note that the latter identity is the semantic justification for the property in Proposition 4.10 concerning the distribution of the universal connective over the conjuntion.

Example 4.14 (Residuation) Observe that in the context of an exhaustive interpretation systems $\langle \Sigma, M \rangle$ with an intuitionistic or a relevant implication $\sqsupset$ (see Example 4.5 and Example 4.6, respectively) such that $\sqsupset^\circ_m (s_m, b) = b$ for any m in M and b in B_m, we have the following property

$$s_m \leq \sqsupset^v_m (b_1, b_2) \text{ iff } b_1 \leq b_2,$$

which is usually called *s-residuation.* $\triangle$

4.1.3 Power-set semantics

We now identify classes of a specific kind of algebras, named (general) power-set algebras, for a generic constructor $c \in C^f_k$, where the semantic interpretation of c^v leads to c having the k-universal property. Let W be a nonempty set, B be contained in $\wp W$, $\rho \subseteq W^{k+1}$ be a relation for some $k \geq 1$, and $\cdot$ be the interpretation of the connectives. A *general power-set algebra* $m(\langle W, B, \rho, \cdot \rangle)$ over $\langle W, B, \rho, \cdot \rangle$ is the structure $\langle B_m, \leq_m, \cdot_m \rangle$ where $B_m = B$, $\leq_m$ is the subset relation, and $\cdot_m = \cdot$. A *power-set algebra* m is a general power-set algebra m over $\langle W, \wp W, \rho, \cdot \rangle$. A class of (general) power-set algebras over a class of (general) Kripke models $\mathcal{W}$ is a class where each element is a (general) power-set algebra over an element of $\mathcal{W}$.

Proposition 4.15 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system where M is a class of general power-set algebras over a class $\mathcal{W}$ of general Kripke models where $\{w\} \in B$ for every w in W and $\langle W, B, \rho, \cdot \rangle$ in $\mathcal{W}$. Then $c \in C^f_k$ has the k-universal property provided that for every $m(\langle W, B, \rho, \cdot \rangle) \in M$, $\vec{w}_{k-1} \in W^{k-1}$, $\vec{b}_{k-1} \in B^{k-1}_m$, and $d \in B_m$:

1. $c^v_m(\vec{b}_{k-1}, d) = g_{m,\vec{b}_{k-1}}(d) = \bigcup_{\vec{w}_{k-1}\in\vec{b}_{k-1}} g_{m,\vec{w}_{k-1}}(d)$,
2. $c^\circ_m(b, \vec{b}_{k-1}) = f_{m,\vec{b}_{k-1}}(b) = \bigcup_{\vec{w}_{k-1}\in\vec{b}_{k-1}} f_{m,\vec{w}_{k-1}}(b)$,
3. $f_{m,\vec{b}_{k-1}}(b) \in B_m$,

whenever

- $g_{m,\vec{w}_{k-1}}(d) = \{w \in W \mid \rho\, w\vec{w}_{k-1}v \text{ implies } v \in d\}$,
- $f_{m,\vec{w}_{k-1}}(b) = \{v \in W \mid \rho\, w\vec{w}_{k-1}v \text{ for some } w \in b\}$.

Proof Note that statement 3 is a pre-requisite for the k-universal property. For each $m \in M$, using Proposition 4.12, it is enough to show that each $g_{m,\vec{b}_{k-1}}$ is right adjoint of $f_{m,\vec{b}_{k-1}}$. We start by showing that

$$b \leq g_{m,\vec{w}_{k-1}}(d) \text{ iff } f_{m,\vec{w}_{k-1}}(b) \leq d.$$

For the left-to-right direction, assume that $b \leq g_{m,\vec{w}_{k-1}}(d)$ and let $v \in f_{m,\vec{w}_{k-1}}(b)$. Then $\rho\, w\vec{w}_{k-1}v$ for some $w \in b$, hence $\rho\, w\vec{w}_{k-1}v$ for some $w \in g_{m,\vec{w}_{k-1}}(d)$, and so $v \in d$. For the right-to-left direction, assume now that $f_{m,\vec{w}_{k-1}}(b) \leq d$ and let $w \in b$. If there is no v such that $\rho\, w\vec{w}_{k-1}v$ then $w \in g_{m,\vec{b}_{b-1}}(d)$. Otherwise, $\{v \in W \mid \rho\, w\vec{w}_{k-1}v\} \leq f_{m,\vec{w}_{k-1}}(b)$, hence $\{v \in W \mid \rho\, w\vec{w}_{k-1}v\} \leq d$, and so $w \in g_{m,\vec{w}_{k-1}}(d)$. Finally, we show that

$$b \leq g_{m,\vec{b}_{k-1}}(d) \text{ iff } f_{m,\vec{b}_{k-1}}(b) \leq d.$$

This follows because $b \leq g_{m,\vec{b}_{k-1}}(d)$ iff, using 1, $b \leq g_{m,\vec{w}_{k-1}}(d)$ for some $\vec{w}_{k-1} \in \vec{b}_{k-1}$, iff $f_{m,\vec{w}_{k-1}}(b) \leq d$ for some $\vec{w}_{k-1} \in \vec{b}_{k-1}$, iff $f_{m,\vec{b}_{k-1}}(b) \leq d$, using 2. QED

In particular, c has the universal property when M is the class of all power-set algebras over $\mathcal{W}$. Observe that when b is W then the universal property states that

$$W \leq c^v_m(\vec{b}_{k-1}, d) \text{ iff } \{v \in W \mid \rho\, w\vec{w}_{k-1}v\} \leq d.$$

Observe also that $\{w\} \leq g_{m,\vec{w}_{k-1}}(d)$ iff $\{v \in B_m \mid \rho\, w\vec{w}_{k-1}v\} \leq d$.

Example 4.16 (Modal logic K) Necessitation has the 1-universal property in the context of the exhaustive interpretation system $\mathcal{I}(K) = \langle \Sigma, M\rangle$ where M is the class of all power-set algebras over the class of Kripke models for Σ (consequence of Proposition 4.12). Note that $\mathcal{I}(K)$ is with truth and has an intuitionistic implication. Then $\mathcal{D}(\mathcal{I})$ includes additionally the rule

$$\frac{\delta_1 \leq \delta_2}{\Box^\circ(\delta_1) \leq \Box^\circ(\delta_2)}\ mon_{\Box^\circ}.$$

We can show that the Hilbert system $H(K)$ for modal logic K is sound with respect to $\mathcal{D}(K)$. Observe that it is simple to deduce the axioms and rules of the Hilbert system $H(K)$ [7]. For instance, we can show that the normality axiom is a theorem of $\mathcal{D}(K)$ as follows (where some obvious steps were omitted):

$$
\begin{array}{lll}
1 & \delta \prec \Box(\xi_1 \sqsupset \xi_2) & hyp \\
2 & \sqsupset^\circ(\delta, \delta_1) \prec \Box(\xi_1 \sqsupset \xi_2) & \sqsupset^\circ_{I_2}\ 1 \\
3 & \Box^\circ(\sqsupset^\circ(\delta, \delta_1)) \prec \xi_1 \sqsupset \xi_2 & \Box_E 2 \\
4 & \delta_1 \prec \Box(\xi_1) & hyp \\
5 & \sqsupset^\circ(\delta, \delta_1) \prec \Box(\xi_1) & \sqsupset^\circ_{I_2}\ 4 \\
6 & \Box^\circ(\sqsupset^\circ(\delta, \delta_1)) \prec \xi_1 & \Box_E 5 \\
7 & \sqsupset^\circ\ (\Box^\circ(\sqsupset^\circ(\delta, \delta_1)), \Box^\circ(\sqsupset^\circ(\delta, \delta_1))) \prec \xi_2 & \sqsupset_E\ 3, 6 \\
8 & \Box^\circ(\sqsupset^\circ(\delta, \delta_1)) = \Box^\circ(\sqsupset^\circ(\delta, \delta_1)) & =_r \\
9 & \Box^\circ(\sqsupset^\circ(\delta, \delta_1)) \leq \Box^\circ(\sqsupset^\circ(\delta, \delta_1)) & =_E\ 8 \\
10 & \Box^\circ(\sqsupset^\circ(\delta, \delta_1)) \prec \xi_2 & \sqsupset^\circ_E\ 7, 9, 9 \\
11 & \sqsupset^\circ(\delta, \delta_1) \prec \Box(\xi_2) & \Box_I 10 \\
12 & \delta \prec \Box(\xi_1) \sqsupset \Box(\xi_2) & \sqsupset_I\ 11 \\
13 & \sqsupset^\circ\ (\top, \delta) \prec \Box(\xi_1) \sqsupset \Box(\xi_2) & \sqsupset^\circ_{I_2}\ 12 \\
14 & \top \prec \Box(\xi_1 \sqsupset \xi_2) \sqsupset (\Box(\xi_1) \sqsupset \Box(\xi_2)) & \sqsupset_I\ 13
\end{array}
$$

with $\Psi_{13} = \{\delta \prec \Box(\xi_1 \sqsupset \xi_2)\}$ and $\Psi_{11} = \Psi_{13} \cup \{\delta_1 \prec \Box(\xi_1)\}$.

Labelled equivalents of the necessitation rule (cf. Example 2.4) and of modus ponens can be derived similarly. The deduction for the modus ponens rule is as follows:

$$
\begin{array}{lll}
1 & \xi_1^v \approx \xi_1 & v \\
2 & \xi_1^v \prec \xi_1 & \approx_E\ 1 \\
3 & \delta \prec \xi_1 & hyp \\
4 & \delta \leq \xi_1^v & \prec_E\ 1, 3 \\
5 & \delta \prec \xi_1 \sqsupset \xi_2 & hyp \\
6 & \delta = \delta & =_r \\
7 & \delta \leq \delta & =_E\ 6 \\
8 & \sqsupset^\circ(\delta, \xi_1^v) \prec \xi_2 & \sqsupset_E\ 2, 5 \\
9 & \delta \prec \xi_2 & \sqsupset^\circ_E\ 8, 4, 7
\end{array}
$$

with $\Psi_9 = \{\delta \prec \xi_1 \sqsupset \xi_2, \delta \prec \xi_1\}$. Observe that modus ponens is the unique derivation rule in $H(K)$. △

Example 4.17 (Modal logic $S4$) Let $\mathcal{I}(S4)$ be an exhaustive interpretation system for modal logic where M is the class of all power-set algebras over the class $\mathcal{W}$ of Kripke models whose accessibility relation is reflexive and transitive. Then $\mathcal{D}(S4)$, besides the induced additional rule for modal logic K presented in Example 4.16, includes the following additional rules:

$$
\frac{}{\delta \leq \Box^\circ(\delta)}\ T \qquad \text{and} \qquad \frac{\delta \leq \Box^\circ(\delta') \quad \delta' \leq \Box^\circ(\delta'')}{\delta \leq \Box^\circ(\delta'')}\ 4\,.
$$

The Hilbert axiom $\Box(\xi) \sqsupset \xi$, corresponding to reflexivity of the Kripke accessibility relation [7], can then be deduced as follows:

$$
\begin{array}{lll}
1 & \delta \prec \Box(\xi) & hyp \\
2 & \Box^\circ(\delta) \prec \xi & \Box_E\ 1 \\
3 & \delta \leq \Box^\circ(\delta) & T \\
4 & \delta \prec \xi & \prec_I\ 3, 2 \\
5 & \sqsupset^\circ\ (\top, \delta) \prec \xi & \sqsupset^\circ_{I_2} \\
6 & \top \prec \Box(\xi) \sqsupset \xi & \sqsupset_I\ 5
\end{array}
$$

where $\Psi_5 = \{\delta \prec \Box(\xi)\}$. The transitivity axiom $\Box(\xi) \sqsupset \Box(\Box(\xi))$ can also be deduced. Therefore, the Hilbert system $H(S4)$ for modal logic $S4$ is sound with respect to $\mathcal{D}(S4)$. $\triangle$

Example 4.18 (First-order universal quantification) The universal quantifier for each variable x has the 1-universal property over the exhaustive interpretation system $\mathcal{I} = \langle \Sigma, M \rangle$ where:

- Σ is an exhaustive signature with $C_1 = \{\neg\} \cup \{(\forall x) \mid x \in X\}$ and $C_2 = \{\sqsupset\}$;
- M is the class of all power-set algebras $\langle B_m, \leq_m, \cdot_m \rangle$ where B_m is the collection of all sets of assignments σ of variables to a fixed set S, where S is a non-empty set (the domain of individuals);
- $(\forall x)_m b = \{\sigma \mid \text{ for all } \sigma',\ \sigma\, x\text{-equivalent to } \sigma' \text{ implies } \sigma' \in b\}$ (where σ is x-equivalent to σ' iff $\sigma(y) = \sigma'(y)$ for $y \neq x$);
- the interpretations of the other constructors are straightforward. $\triangle$

Example 4.19 (Intuitionistic implication) Let $\mathcal{I}$ be an exhaustive interpretation system such that C^f is the intuitionistic signature and M is the class of general power-set algebras over $\mathcal{W}$ where each element $\langle W, B, \rho, \cdot \rangle$ of $\mathcal{W}$ has the following properties:

1. $\rho \subseteq W^3$;
2. $\rho w v_1 v_2$ implies $v_1 = v_2$ for all w, v_1 and v_2 in W;
3. $\rho v v v$ for all v in W;
4. $\rho v_1 v_3 v_3$, whenever $\rho v_1 v_2 v_2$ and $\rho v_2 v_3 v_3$ for all v_1, v_2 and v_3 in W;
5. each $b \in B$ is upwards closed, that is if $w \in b$ then $v \in b$ for all v in W such that $\rho w v v$.

Observe that, on the one hand, ${\sqsupset^\circ}_m(b, \{w\}) = \emptyset$ if $w \notin b$, and ${\sqsupset^\circ}_m(b, \{w_1\}) = \{w_1\}$ otherwise. Therefore ${\sqsupset^\circ}_m(b, b_1) = \bigcup_{w_1 \in b_1} {\sqsupset^\circ}_m(b, \{w_1\}) = b \sqcap b_1$. On the other hand, $b_1 \sqsupset^v_m b_2 = \{w \in W \mid \text{ for all } v, \text{ if } \rho w v v \text{ then } v \in b_1, v \in b_2\}$ and $b_1 \sqsupset^v_m b_2$ is in B; and so intuitionistic implication has the 2-universal property. We expected so because M is a subclass of the class of Heyting algebras and intuitionistic implication has the 2-universal property in this class. $\triangle$

Example 4.20 (Relevant implication) Let $\langle \Sigma, M \rangle$, denoted by $\mathcal{I}(R^+)$, be the exhaustive interpretation system where Σ is the signature for the positive relevance logic R^+ and M is the class of all power-set algebras induced by the relational relevance models [26]. Then, the Hilbert system $H(R^+)$ is sound with respect to $\mathcal{D}(R^+)$. For instance, we can show

$$\vdash_g (\xi_1 \sqsupset \xi_2) \sqsupset ((\xi_3 \sqsupset \xi_1) \sqsupset (\xi_3 \sqsupset \xi_2))$$

as follows:

1	$\delta_1 \prec \xi_3 \sqsupset \xi_1$	*hyp*
2	$\delta_2 \prec \xi_3$	*hyp*
3	$\sqsupset^\circ(\delta_1, \delta_2) \prec \xi_1$	$\sqsupset_E$ 1, 2
4	$\delta \prec \xi_1 \sqsupset \xi_2$	*hyp*
5	$\sqsupset^\circ(\delta, \sqsupset^\circ(\delta_1, \delta_2)) \prec \xi_2$	$\sqsupset_E$ 4, 3
6	$\sqsupset^\circ(\sqsupset^\circ(\delta, \delta_1), \delta_2) \prec \xi_2$	$\sqsupset^\circ_{a_2}$ 5
7	$\sqsupset^\circ(\delta, \delta_1) \prec \xi_3 \sqsupset \xi_2$	$\sqsupset_I$ 6
8	$\delta \prec (\xi_3 \sqsupset \xi_1) \sqsupset (\xi_3 \sqsupset \xi_2)$	$\sqsupset_I$ 7
9	$\sqsupset^\circ(1, \delta) \prec (\xi_3 \sqsupset \xi_1) \sqsupset (\xi_3 \sqsupset \xi_2)$	$\sqsupset^\circ_{1_2}$ 8
10	$1 \prec (\xi_1 \sqsupset \xi_2) \sqsupset ((\xi_3 \sqsupset \xi_1) \sqsupset (\xi_3 \sqsupset \xi_2))$	$\sqsupset_I$ 9

where $\Psi_9 = \{\delta \prec \xi_1 \sqsupset \xi_2\}$, $\Psi_7 = \Psi_9 \cup \{\delta_1 \prec \xi_3 \sqsupset \xi_1\}$ and $\Psi_6 = \Psi_7 \cup \{\delta_2 \prec \xi_3\}$. Similarly, we can show

$$\vdash_g \xi_1 \sqsupset ((\xi_1 \sqsupset \xi_2) \sqsupset \xi_2)$$

as follows:

1	$\delta_1 \prec \xi_1 \sqsupset \xi_2$	*hyp*
2	$\delta \prec \xi_1$	*hyp*
3	$\sqsupset^\circ(\delta_1, \delta) \prec \xi_2$	$\sqsupset_E$ 1, 2
4	$\sqsupset^\circ(\delta, \delta_1) \prec \xi_2$	$\sqsupset^\circ_c$ 3
5	$\delta \prec (\xi_1 \sqsupset \xi_2) \sqsupset \xi_2$	$\sqsupset_I$ 4
6	$\sqsupset^\circ(1, \delta) \prec (\xi_1 \sqsupset \xi_2) \sqsupset \xi_2$	$\sqsupset^\circ_{1_2}$ 5
7	$1 \prec \xi_1 \sqsupset ((\xi_1 \sqsupset \xi_2) \sqsupset \xi_2)$	$\sqsupset_I$ 6

where $\Psi_6 = \{\delta \prec \xi_1\}$ and $\Psi_4 = \Psi_6 \cup \{\delta_1 \prec \xi_1 \sqsupset \xi_2\}$. △

4.1.4 Completeness issues

We end this subsection with a result on completeness for the universal property:

Theorem 4.21 Let $\mathcal{D} = \mathcal{D}(\mathcal{I})$ be the exhaustive labelled deduction system induced by $\mathcal{I} = \langle \Sigma, M \rangle$ where $c \in C^f_k$ is a constructor with the n-universal property. Then c has the universal property in $LS^*(\mathcal{D})$, i.e. in the logic system of all structures that validate the rules c_I and c_{E^i} for $i = n, \ldots, k$.

Proof Assume that m validates c_I and $c^\circ{}_m(b, \vec{b}_{n-1}) \leq b_j$ for $j = n, \ldots, k$. Consider the assignment α such that $\alpha(\delta_i) = b_i$ for $i = 1, \ldots, n-1$, $\alpha(\xi_i) = b_i$ for $i = 1, \ldots, k$ and $\alpha(\delta) = b$. Then $m, \alpha \Vdash \vec{\delta}_{n-1} \prec \vec{\xi}_{n-1} \,/\, c^\circ(\delta, \vec{\delta}_{n-1}) \prec \xi_j$ for $j = n, \ldots, k$, hence $m, \alpha \Vdash \delta \prec c(\xi_1, \ldots, \xi_k)$ since m validates c_I, and so $b \leq c^v_m(\vec{b}_{k-1}, b_k)$. Assume that m validates c_{E^i} where $i = n, \ldots, k$, and that $b \leq c^v_m(\vec{b}_{k-1}, b_k)$. Consider the assignment α such that $\alpha(\delta_j) = b_j$ for $j = 1, \ldots, n-1$, $\alpha(\xi_j) = b_j$ for $j = 1, \ldots, k$ and $\alpha(\delta) = b$. Then, $m, \alpha \Vdash c^\circ(\delta, \vec{\delta}_{n-1}) \prec \xi_i$ for $i = n, \ldots, k$ and so $c^\circ{}_m(b, \vec{b}_{n-1}) \leq b_i$ for $i = n, \ldots, k$. QED

Corollary 4.22 Let $\mathcal{D}(\mathcal{I})$ be the exhaustive labelled deduction system induced by $\mathcal{I} = \langle \Sigma, M \rangle$ where $c \in C^f_k$ is a constructor with the n-universal property. Then c has the universal property in the Lindenbaum-Tarski algebra for every consistent set of composed formulae. △

Proof The result follows from the Theorem 4.21 using Lemma 2.9, where it was proved that the Lindenbaum-Tarski algebra is a structure in $LS^*(\mathcal{D})$. QED

4.2 The unary co-universal property

The unary co-universal property is the dual of the universal property for constructors of arity one. Constructors of arity one that have the unary co-universal property are negations; herein, we do not explore constructors of other arities with this kind of property.

Definition 4.23 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system. The constructor $c \in C_1^f$ has the *unary co-universal property* iff there is a $c^\circ \in C_1^v$ such that

$$b \leq c_m^v(d) \text{ iff } d \leq c_m^\circ(b)$$

for every $m \in M$ and $b, d \in B_m$. △

Observe that the co-universal property is the minimum requirement for negation, and stronger forms of negation appear by adding new rules like we did, for instance, for implication and necessitation. (We return to this in the examples below.) A constructor c has the unary co-universal property iff it has a left coadjoint.

4.2.1 Deduction system

Deduction rules for constructors with the unary co-universal property are different from the ones that we introduced before because of the inequality $d \leq c^\circ(b)$.

Definition 4.24 Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system and $c \in C_1^f$ a constructor with the unary co-universal property. The labelled deduction system $\mathcal{D}(\mathcal{I})$ induced by $\mathcal{I}$ includes the following *introduction* and *elimination rules* for c:

$$\frac{\xi^v \leq c^\circ(\delta)}{\delta \prec c(\xi)}\; c_I \qquad \text{and} \qquad \frac{\beta \prec \xi \quad \delta \prec c(\xi)}{\beta \leq c^\circ(\delta)}\; c_E$$

where $\xi \in \Xi^f$ and $\delta \in \Xi^v$. △

Then, we can, for instance, show $\vdash_g \xi \sqsupset c(c(\xi))$ in any exhaustive labelled deduction system where c satisfies the unary co-universal property and $\sqsupset$ is an intuitionistic implication.

To illustrate how by stating additional properties we obtain different kinds of negation, we begin with the weakest form of negation as in [26].

Example 4.25 (Weak negation) Let $\mathcal{I}$ be an exhaustive interpretation system where $c \in C_1^f$ is a constructor with the co-universal property and M is a class of structures such that $c_m^v = c_m^\circ$ for each structure m. Then $\mathcal{D}(\mathcal{I})$ should include the following additional rule:

$$\frac{}{c^v(\delta) = c^\circ(\delta)}\; v\circ\ .$$

Moreover, we can "merge" the introduction and elimination rules in Definition 4.24 and rule $v\circ$ to obtain

$$\frac{\delta \prec \xi_1 \qquad \xi^v \prec c(\xi_1)}{\delta \prec c(\xi)}\; c_w.$$

Observe that negation in intuitionistic and modal logics is weak and in the context of an exhaustive labelled deduction system with truth and with an intuitionistic implication we can prove $\xi_1 \sqsupset c(\xi_2) \vdash_g \xi_2 \sqsupset c(\xi_1)$ as follows:

1	$\top \prec \xi_1 \sqsupset c(\xi_2)$	*hyp*
2	$\xi_1^v \approx \xi_1$	v
3	$\xi_1^v \prec \xi_1$	$\approx_E 2$
4	$\sqsupset^\circ(\top, \xi_1^v) \prec c(\xi_2)$	$\sqsupset_E 1,3$
5	$\xi_1^v \leq \top$	$\top$
6	$\xi_1^v = \xi_1^v$	$=_r$
7	$\xi_1^v \leq \xi_1^v$	$=_E 6$
8	$\xi_1^v \prec c(\xi_2)$	$\sqsupset^\circ{}_E\, 4,5,7$
9	$\xi_2^v \approx \xi_2$	v
10	$\xi_2^v \prec \xi_2$	$\approx_E 9$
11	$\xi_2^v \prec c(\xi_1)$	$c_w\, 10,8$
12	$\sqsupset^\circ(\top, \xi_2^v) \prec c(\xi_1)$	$\sqsupset^\circ{}_{I_2}\, 11$
13	$\top \prec \xi_2 \sqsupset c(\xi_1)$	$\sqsupset_I 12$

where $\Psi_{12} = \{\top \prec \xi_1 \sqsupset c(\xi_2)\}$. △

Example 4.26 (Intuitionistic negation) Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system with a weak negation $\neg \in C_1^f$. Then $\neg$ is an *intuitionistic* negation iff $\mathcal{D}(\mathcal{I})$ includes the additional rule

$$\frac{\delta \prec \xi \qquad \delta \prec \neg\xi}{\delta \prec \xi'}\ \neg^i\,.$$

Then, in the context of an exhaustive labelled deduction system with a intuitionistic negation, a intuitionistic implication, a conjunction, and with true, we can for instance show $(\xi_1 \sqsupset \xi_2) \wedge (\xi_1 \sqsupset \neg\xi_2) \vdash_g \neg\xi_1$ as follows:

1	$\xi_1^v \leq \top$	$\top$
2	$\xi_1^v = \xi_1^v$	$=_r$
3	$\xi_1^v \leq \xi_1^v$	$=_E 2$
4	$\xi_1^v \approx \xi_1$	v
5	$\xi_1^v \prec \xi_1$	$\approx_E 4$
6	$\top \prec (\xi_1 \sqsupset \xi_2) \wedge (\xi_1 \sqsupset \neg\xi_2)$	*hyp*
7	$\top \prec \xi_1 \sqsupset \xi_2$	$\wedge_{E^1} 6$
8	$\sqsupset^\circ(\top, \xi_1^v) \prec \xi_2$	$\sqsupset_E 7,5$
9	$\xi_1^v \prec \xi_2$	$\sqsupset_E^\circ 8,1,3$
10	$\top \prec \xi_1 \sqsupset \neg\xi_2$	$\wedge_{E^2} 6$
11	$\sqsupset^\circ(\top, \xi_1^v) \prec \neg\xi_2$	$\sqsupset_E 10,5$
12	$\xi_1^v \prec \neg\xi_2$	$\sqsupset_E^\circ 11,1,3$
13	$\xi_1^v \prec \neg\mathbf{t}$	$\neg^i 9,12$
14	$\top \prec \mathbf{t}$	$\mathbf{t}$
15	$\top \prec \neg\xi_1$	$\neg_I 14,13$

where $\Psi_{15} = \{\top \prec (\xi_1 \sqsupset \xi_2) \wedge (\xi_1 \sqsupset \neg\xi_2)\}$. △

Example 4.27 (Classical negation) Let $\mathcal{I} = \langle \Sigma, M \rangle$ be an exhaustive interpretation system with an intuitionistic negation $\neg \in C_1^f$. Then $\neg$ is a *classical* negation iff $\mathcal{D}(\mathcal{I})$ includes the additional rule

$$\frac{\delta \prec \neg\neg\xi}{\delta \prec \xi}\ \neg^c\,,$$

which guarantees that $\vdash_g^{\mathcal{D}(\mathcal{I})} (\neg\,\neg\,\xi) \sqsupset \xi$. △

Example 4.28 (Irreflexivity) Recall the labelled deduction system for the modal logic K given in Example 4.16. We can express all the properties in [1] which are not definable in pure modal logic. For instance, irreflexivity can be expressed by $\Box^\circ(\delta) \leq \neg^v \delta$, and with this axiom we can then prove Gabbay's irreflexivity rule $((\Box\varphi) \sqsupset \varphi) \vee \varphi \vdash_g \varphi$. $\triangle$

4.3 Example

We now present an exhaustive labelled logic system that puts together and illustrates some of the properties above introduced related to intuitionistic constructors.

Example 4.29 Consider the exhaustive labelled logic system $\langle \Sigma, R, M \rangle$, where Σ is the truth-value signature where $\mathbf{t}$ is in C_0^f, $C_1^f = \{\neg\}$, $C_2^f = \{\sqsupset, \wedge, \vee\}$, $C_0^v = \{\top\}$, $C_1^v = \{\neg^v\}$, $C_2^v = \{\sqsupset^v, \sqsupset^\circ, \wedge^\circ, \vee^v\}$, all the other sets are empty, s is $\top$, $\wedge^v$ is $\sqsupset^\circ$, M is the class of Heyting algebras, and R, besides the rules mentioned in Definition 2.2 common to all deduction systems, includes the following deduction rules:

$$\frac{}{\xi^v \approx \xi}\ v$$

$$\frac{\wedge^\circ(\delta) \prec \xi_1 \quad \wedge^\circ(\delta) \prec \xi_2}{\delta \prec \xi_1 \wedge \xi_2}\ \wedge_I \qquad \frac{\delta \prec \xi_1 \wedge \xi_2}{\wedge^\circ(\delta) \prec \xi_1}\ \wedge_{E^1} \qquad \frac{\delta \prec \delta_1 \wedge \xi_2}{\wedge^\circ(\delta) \prec \xi_2}\ \wedge_{E^2}$$

$$\frac{}{\delta = \wedge^\circ(\delta)}\ \wedge^\circ$$

$$\frac{\delta \prec \xi_1}{\delta \prec \xi_1 \vee \xi_2}\ \vee_I^1 \qquad \frac{\delta \prec \xi_2}{\delta \prec \xi_1 \vee \xi_2}\ \vee_I^2 \qquad \frac{\delta \prec \xi_1 \vee \xi_2 \quad \delta \prec \xi_1 \,/\, \eta \quad \delta \prec \xi_2 \,/\, \eta}{\eta}\ \vee_E$$

$$\frac{}{\delta \leq \top}\ \top \qquad \frac{}{\top \prec \mathbf{t}}\ \mathbf{t}$$

$$\frac{\delta_1 \prec \xi_1 \,/\, \sqsupset^\circ(\delta, \delta_1) \prec \xi_2}{\delta \prec \xi_1 \sqsupset \xi_2}\ \sqsupset_I \qquad \frac{\delta \prec \xi_1 \sqsupset \xi_2 \quad \delta_1 \prec \xi_1}{\sqsupset^\circ(\delta, \delta_1) \prec \xi_2}\ \sqsupset_E$$

$$\frac{\delta_1 \prec \xi}{\sqsupset^\circ(\delta_1, \delta_2) \prec \xi}\ \sqsupset^\circ{}_{I_1} \qquad \frac{\delta_2 \prec \xi}{\sqsupset^\circ(\delta_1, \delta_2) \prec \xi}\ \sqsupset^\circ{}_{I_2} \qquad \frac{\sqsupset^\circ(\delta_1, \delta_2) \prec \xi \quad \delta \leq \delta_1 \quad \delta \leq \delta_2}{\delta \prec \xi}\ \sqsupset^\circ{}_E$$

$$\frac{\delta \prec \xi_1 \qquad \xi^v \prec \neg\xi_1}{\delta \prec \neg\xi}\ \neg_w \qquad \frac{\delta \prec \xi \qquad \delta \prec \neg\xi}{\delta \prec \xi'}\ \neg^i\,.$$

$\triangle$

Note that most of the deductions presented in the above sub-sections can be done in the context of the exhaustive labelled logic system just introduced.

5 Concluding remarks

We have given a general recipe for presenting non-classical logics in a modular and uniform way as labelled deduction systems with algebras of truth-values whose operators reflect the semantics we have in mind. Previous approaches based on labelling appear as particular cases of our "truth-values as labels" approach. Moreover, more logics can be considered, i.e. we can present a larger number of non-classical logics as labelled deduction systems possessing all the good uniformity and modularity features of labelled deduction. Indeed, our approach allows us to give, within the same formalism, generalized systems for multiple-valued logics, i.e. for many-valued logics and for power-set logics, such as modal, intuitionistic and relevance logics.

A uniform way of choosing the labels seems to be of great impact in particular for the problem of combining logics. Interest in this problem has been growing in the last decade, driven by foundational issues and applications (see, for instance, [6]). The theoretical interest in the problem is clear. Every logician will have looked at some logic as a combination of two simpler logics. For instance, one is tempted to look at predicate modal logic as resulting from the combination of first-order logic and propositional modal logic. But the approach will be significant if and only if general transference (or preservation) results are available about the mechanism used for combining the logics. For example, if it has been established that completeness is preserved by the logic-combination mechanism $\bullet$ and it is known that logic $\mathcal{L}$ is given by $\mathcal{L}' \bullet \mathcal{L}''$, then the completeness of the combination $\mathcal{L}$ follows from the completeness of $\mathcal{L}'$ and $\mathcal{L}''$. No wonder that much effort has been dedicated to establishing such transference results and/or counterexamples for mechanisms for combining logics.

The practical interest in the problem is also clear, at least from the point of view of those working in recent application areas of mathematical logic, like knowledge representation (within artificial intelligence) and formal specification and verification (within software engineering). Indeed, in these application areas, the need for working with several formalisms (read logics) at the same time is the rule rather than the exception. For instance, in a knowledge representation problem it may be necessary to work with both temporal and deontic knowledge, and in a software specification problem it may be necessary to work with both equational and temporal specifications. In such applications, theoretical results on the problem of combining logics are of immediate relevance.

Among the combination mechanisms, fibring has deserved special attention because it is quite general (see, for instance, [14, 16]). Since, as mentioned above, we have begun to apply our algebraic labelled deduction systems within our research program on the fibring of logics [25, 27, 28, 32], we will conclude with a brief description of our current and future work.

For those not familiar with the concept, a (very) rough description of fibring is possible in a few sentences. Assume that we have two logics $\mathcal{L}'$ and $\mathcal{L}''$. The main idea of fibring, at the signature level, is to define a new logic $\mathcal{L}$ whose signature includes all the symbols of the signatures in $\mathcal{L}'$ and $\mathcal{L}''$. Therefore, in logic $\mathcal{L}$ we can have mixed formulae like for instance $c'_1(c''_1, c'_2, c''_2)$ where c'_1 and c'_2 are constructors in the signature of $\mathcal{L}'$ and c''_1 and c''_2 are constructors in the signature of $\mathcal{L}''$. From a semantic point of view fibring is more complex [27, 32], and thus herein we briefly point out the problem of labelled deduction.

That is, assume that we have a labelled deduction system $\mathcal{D}'$ for logic $\mathcal{L}'$ and a labelled deduction system $\mathcal{D}''$ for logic $\mathcal{L}''$. How can we get a labelled deduction system $\mathcal{D}$ for logic $\mathcal{L}$? Provided that the two given logics are endowed with deduction systems of the same type (meaning with the same kind of labels), the deduction system of the fibring will be obtained by the free use of the deduction rules from both. As a first step, in [25] we assumed that although working with general logics (not necessarily modal-like), the deduction systems were endowed with the same kind of labels and the same kind of labelled formulae, where labels were worlds related via predicates. Of course, in this setting we were not able to cope with the fibring of two labelled deduction systems for, let us say, modal and 3-valued Gödel logics. In [24], a preliminary attempt is made towards having more general labels, and here we have significantly generalized and extended this attempt to demonstrate how the use of truth-values as labels allows us to give a general recipe for giving labelled deduction systems for a large number of different non-classical logics.

As future work, we also plan to investigate the proof-theoretical properties of our systems, e.g. proof normalization, proof search and decidability in order to support automated deduction with our systems. Our work [21] on labelled sequent systems, which are better suited than natural deduction systems for these kinds of metatheoretic analysis, provides a first step in this direction.

Besides for the theoretical interest in using our systems for fibring, it is worth pointing out that applications of fibring to cryptography and spatial reasoning are under way. In [9], for example, Caires and Cardelli give the outline of a logic for concurrency endowed with spatial features for coping with security aspects. The aim is to define a "computationally meaningful" deduction system, and thus choosing labelled deduction seems quite natural. The problem is that labels for temporal aspects can be worlds but spatial features are described via a process algebra not endowed with worlds. We thus believe that our new results will have an immediate impact on this work and plan to report on this soon. An investigation of the relationships between our work and the logic of bunched implications of O'Hearn and Pym [18, 22], which can be seen as a merging of intuitionistic logic and multiplicative intuitionistic linear logic, and which has strong links with spatial logics, will provide a good starting point for this.

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