Labelled deduction over algebras of truth-values*

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Abstract. We introduce a framework for presenting non-classical logics in a modular and uniform way as labelled natural deduction systems. The use of algebras of truth-values as the labelling algebras of our systems allows us to give generalized systems for multiple-valued logics. More specifically, our framework generalizes previous work where labels represent worlds in the underlying Kripke structure: since we can take multiple-valued logics as meaning not only finitely or infinitely many-valued logics but also power-set logics, our framework allows us to present also logics such as modal, intuitionistic and relevance logics, thus providing a first step towards fibring these logics with many-valued ones.

1 Introduction

Context *Labelled Deduction* is an approach to presenting different logics in a uniform and natural way as Gentzen-style deduction systems, such as natural deduction, sequent or tableaux systems; see, for instance, [2, 3, 8, 9, 11, 16, 21]. It has been applied, for example, to formalize and reason about dynamic "state-oriented" properties, such as knowledge, belief, time, space, and resources, and thereby formalize deduction systems for a wide range of non-classical logics, such as modal, temporal, intuitionistic, relevance and other substructural logics. The intuition behind Labelled Deduction is that the *labelling* (sometimes also called prefixing, annotating or subscripting) allows one to explicitly encode additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the logic one wants to capture. To illustrate this, take the simple, standard example of modal logics, where the additional information encoded into the syntax usually comes from the underlying Kripke semantics: instead of considering a modal formula φ, we can consider the *labelled formula* $x : \varphi$, which intuitively means that φ holds at the world denoted by x within the underlying Kripke structure (i.e. model). We can also use labels to specify the way in which the different worlds are related in the Kripke structures; for example, we can use the formula xRy to specify that the world denoted by y is accessible from that denoted by x. A modal labelled natural deduction system over this extended

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language is then obtained by giving inference rules for deriving labelled formulae, introducing or eliminating formula constructors such as implication $\supset$ and modal necessity $\Box$, and by defining a suitable *labelling algebra*, which governs the inferences of formulae *about* labels, such as xRy.

Labelled deduction systems are *modular* for families of logics, such as the family of normal modal logics, in that to capture logics in the family we only need to vary appropriately the labelling algebra, while leaving the language and the rules for the formula constructors unchanged. Labelled deduction systems are also *uniform*, in that the same philosophy and technique can be applied for different, unrelated logic families. More specifically, changes in the labelling, i.e. in how formulae are labelled and with what labels (as we might need labels that are structurally more complex than the simple equivalents of Kripke worlds), together with changes in the language and rules, allow for the formalization of systems for non-classical logics other than the modal ones. For instance, labels can also be employed to give Gentzen-style systems for many-valued logics; in this case labels are used to represent the set of truth-values of the particular logic, which can be either the unit interval $[0,1]$ on the rational numbers or a finite set of rational numbers of the form $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, e.g. the set $\{0, 0.5, 1\}$ that is used in 3-valued Gödel logic (see Example 3 and note that in this paper we employ falsum $\bot$ and verum $\top$ instead of 0 and 1). For examples of many-valued labelled deduction systems, mostly tableaux systems, see [1, 6, 7, 13, 14]. In these systems, the labelled formula $x : \varphi$ intuitively means that the formula φ has truth-value x; or, when x is a set of truth-values as in the "sets-as-signs" approach, it means that φ has one of the values in x. The inference rules and the labelling algebra of a system then essentially mirror the truth-tables of the formula constructors of the corresponding logic.

Contributions We here introduce a framework for presenting non-classical logics in a modular and uniform way as labelled natural deduction systems. The main idea underlying our approach is the use of *algebras of truth-values* as the labelling algebras of our systems, which allows us to give generalized systems for *multiple-valued logics*. More specifically, our framework generalizes previous work, including our own [3, 16, 21], on labelled deduction systems where labels represent worlds in the underlying Kripke structures, and this generalization is illustrated by the following observation: since we can take multiple-valued logics as meaning not only finitely or infinitely many-valued logics but also *power-set logics*, i.e. logics for which the denotation of a formula can be seen as a set of worlds, our framework allows us to give systems also for logics such as modal, intuitionistic and relevance ones. In a nutshell, the novelty of our approach with respect to previous approaches based on labelling is that we can capture all these different logics *within the same formalism*. (It is interesting to note that this also provides a first, large, step towards the *fibring* [4, 12] of these logics with many-valued ones; we have begun [15] investigating in this direction as part of our research program on fibring of logics [16, 18, 19, 22].)

The fact that the labels constitute an algebra of truth-values means that we can have operations on truth-values, and that formula constructors can be associated with these operations. To this end, the syntax of our systems defines operators that build labels as terms of truth-values, and uses these "complex" labels to build two kinds of formulae: (i) *labelled formulae*, which are built by prefixing "standard", unlabelled, formulae with a label (and with an infix operator : or ::), and (ii) *truth-value formulae*, which are equalities and inequalities between labels. For labels β, β_1, β_2 and a formula φ, the semantic intuition behind these formulae is as follows: $\beta : \varphi$ means that the value of φ is equal to that of the truth-value term β, $\beta :: \varphi$ means that the value of φ is greater than or equal to that of β, $\beta_1 = \beta_2$ means that the value of β_2 is equal to that of β_1, and $\beta_1 \leq \beta_2$ means that the value of β_2 is greater than or equal to that of β_1.

A system for a particular logic comprises then inference rules that define how these formulae can be derived, e.g. "basic" rules expressing the properties of :, ::, $=$ and $\leq$, rules defining how the formula constructors are introduced and eliminated and how this is reflected in the associated label operators, and rules defining properties of these operators. For example, the modal constructor $\Box$ is associated with the label operator $\check{\Box}$, and different modal logics are obtained modularly, by varying the properties of (i.e. adding or deleting rules for) $\check{\Box}$.

We give here introduction and elimination rules for general formula constructors, which subsume, as special cases, non-classical constructors like necessity or intuitionistic, relevant, or many-valued implication, as well as classical (i.e. material) implication and classical conjunction and disjunction. We then consider three particular kinds of labelled deduction systems that can be expressed within our framework: (i) *exhaustive systems* where each formula constructor is associated with a truth-value operator of the same arity, (ii) *well-behaved systems* where the arity of the operator is less than or equal to the arity of the associated constructor, and for each constructor there are both introduction and elimination rules, and (iii) *finitely-valued systems* where the rules capture the truth-tables of formula constructors, as is common in systems for finitely-valued logics. As examples, we give systems for modal and relevance logics (which provide the basis for systems for other non-classical logics such as intuitionistic logic), and systems that capture many-valued logics, e.g. the 3-valued Gödel logic.

As exemplified in the intuitive explanations above, the semantics of our systems is given by structures interpreting both formulae and labels as truth-values, and checking if the relationship between them complies with the labelling, e.g. for the labelled formula $\beta : \varphi$ if the value of φ is indeed that of β. Rather than simply proving soundness and completeness for a particular system, we analyze soundness and completeness in the general context of our framework. That is, we establish the conditions under which our systems are sound and complete with respect to the algebraic semantics for the corresponding logics.

Organization In §2 we introduce our general labelled deduction systems on algebras of truth-values, and in §3 we consider some special systems. In §4 we focus on semantics and define interpretation systems, and in §5 we establish conditions for soundness and completeness. In §6 we draw conclusions and discuss related and future work. Note that discussions and proofs have been shortened or omitted altogether; a detailed account is given in [15].

2 Labelled deduction systems

We define the language of our labelled deduction systems (LDSs), specifying how to build labels, formulae and judgements, and then define inference rules and derivations. We then consider the case of exhaustive LDSs, which is important for establishing completeness (as each LDS can be transformed into an exhaustive one).

2.1 Truth-value labels and formulae

The use of terms of truth-values as labels allows us to build two kinds of formulae, namely (i) labelled formulae, obtained by prefixing "standard", unlabelled, formulae with a label (and with an infix operator : or ::), and (ii) truth-value formulae, which are equalities and inequalities between labels. To define these formulae, we introduce the general concept of signature, and then define a truth-value signature as the composition of a signature for formulae with one for labels.

A *signature* is a pair $\langle F, S\rangle$ where $F = \{F_k\}_{k\in\mathbb{N}}$ is a family of sets of *constructors* (F_k is the set of constructors of arity k) and S is a set of *meta-variables.* The set of propositions over a signature is the free algebra $L(\langle F, S\rangle)$ where the 0-ary operations are the elements of $F_0 \cup S$ and, for $k > 0$, the k-ary operations are the elements of F_k.

Definition 1. *A* truth-value signature TVS *is a tuple* $\Sigma = \langle C^f, \Xi^f, C^v, \Xi^v, D\rangle$ *where* $\langle C^f, \Xi^f\rangle$ *and* $\langle C^v, \Xi^v\rangle$ *are signatures with* $\mathbf{t} \in C_0^f$*,* $\top \in D$ *and* $D \subseteq C_0^v$*.*

The elements of the sets C_k^f and C_k^v are the *formula constructors of arity* k and the *truth-value operators of arity* k, respectively, where the *true* $\mathbf{t}$ is a 0-ary constructor. The elements of Ξ^f and Ξ^v are *schema formula variables* and *schema truth-value variables*, respectively. The elements of D are the *distinguished elements*, i.e. the designated elements of the set of truth-values; D is not empty and contains at least the top element (verum) $\top$. We call the elements of $L(\langle C^f, \Xi^f\rangle)$ *formulae* and the elements of $L(\langle C^v, \Xi^v\rangle)$ *truth-value terms.*

Definition 2. *Given a TVS* Σ*, truth-value terms* $\beta, \beta_1, \beta_2 \in L(\langle C^v, \Xi^v\rangle)$*, and a formula* $\varphi \in L(\langle C^f, \Xi^f\rangle)$*, the equalities* $\beta_1 = \beta_2$ *and the inequalities* $\beta_1 \leq \beta_2$ *are* truth-value formulae*, and* $\beta : \varphi$ *and* $\beta :: \varphi$ *are* labelled formulae*. We denote by* $L(\Sigma)$ *the set of truth-value and labelled formulae, and call its elements* composed formulae*.*

We employ the following notation: φ and γ are formulae in $L(\langle C^f, \Xi^f\rangle)$, β is a truth-value term in $L(\langle C^v, \Xi^v\rangle)$, ξ is a schema formula variable in Ξ^f, δ is a schema truth-value variable in Ξ^v, and ψ and η are composed formulae in $L(\Sigma)$. All these variables may be annotated with subscripts.

2.2 Deduction systems and derivations

A LDS allows us to infer a composed formula in $L(\Sigma)$ from a set of composed formulae in $L(\Sigma)$ or, in other words, to infer judgements.

Definition 3. *A* judgement J *over a TVS* Σ *is a triple* $\langle \Theta, \eta, \Upsilon \rangle$*, written* $\Theta \,/\, \eta \rhd \Upsilon$ *(or* $\Theta \,/\, \eta$ *if* $\Upsilon = \emptyset$*), where* $\Theta \subseteq L(\Sigma)$*,* $\eta \in L(\Sigma)$ *and* $\Upsilon \subseteq \Xi^v$*.*

The set of composed formulae Θ and the composed formula η are the *antecedent* and the *consequent* of the judgement, respectively, and the schema truth-value variables in the set Υ are the *fresh* variables of the judgement. As will become clear when we will give rules for formula constructors, e.g. in Definition 10 and in Example 1, the fresh variables allow us to impose constraints on substitutions and thereby express universal quantification over truth-values.

In order to prove assertions using hypotheses, we need to say when a judgement follows (i.e. can be derived) from a set of judgements. Hence, we now introduce inference rules and then define generically a "basic" LDS that can be extended to systems for particular non-classical logics.

Definition 4. *A* rule r *over a TVS* Σ *is a pair* $\langle \{J_1, \ldots, J_k\}, J \rangle$*, graphically*

$$\frac{J_1 \cdots J_k}{J}\; r\,,$$

where $J_1, \ldots, J_k, J$ *are judgements, and* J *is such that* $\Theta = \emptyset$ *and* $\Upsilon = \emptyset$*.*

Definition 5. *A* labelled deduction system LDS *is a pair* $\langle \Sigma, R \rangle$*, where* Σ *is a TVS and* R *is a set of rules including at least the following rules:*

$$\frac{\delta_1 \leq \delta_2 \quad \delta_2 :: \xi}{\delta_1 :: \xi}\; ::_I\,, \qquad \frac{\delta_1 :: \xi \quad \delta_2 : \xi}{\delta_1 \leq \delta_2}\; ::_E\,, \qquad \frac{\delta_1 = \delta_2 \quad \delta_2 : \xi}{\delta_1 : \xi}\; :_I\,, \qquad \frac{\delta : \xi}{\delta :: \xi}\; :_E\,,$$

$$\frac{}{\delta \leq \delta}\; \leq_r\,, \qquad \frac{\delta_1 \leq \delta_2 \quad \delta_2 \leq \delta_3}{\delta_1 \leq \delta_3}\; \leq_t\,, \qquad \frac{\delta_1 \leq \delta_2 \quad \delta_2 \leq \delta_1}{\delta_1 = \delta_2}\; =_I\,, \qquad \frac{\delta_1 = \delta_2}{\delta_1 \leq \delta_2}\; =_E\,,$$

$$\frac{\delta_2 = \delta_1}{\delta_1 = \delta_2}\; =_s\,, \qquad \frac{\delta_1 = \delta_1' \quad \cdots \quad \delta_k = \delta_k'}{o(\delta_1, \ldots, \delta_k) = o(\delta_1', \ldots, \delta_k')}\; =_{cong}\,, \qquad \frac{}{\delta \leq \top}\; \top\,, \qquad \frac{}{\top : \mathbf{t}}\; \mathbf{t}\,,$$

where $k \in \mathbb{N}$*,* $o \in C_k^v$*,* $\delta, \delta_1, \ldots, \delta_k, \delta_1', \ldots, \delta_k' \in \Xi^v$ *and* $\xi \in \Xi^f$*.*

LDSs for particular logics are obtained by fixing a particular TVS Σ and adding rules to the ones in Definition 5, which establish minimal properties on labelling, common to all our LDSs. So, $\leq_r$ and $\leq_t$ establish that $\leq$ is reflexive and transitive, $=_s$ establishes that $=$ is symmetric, $=_I$ and $=_E$ introduce and eliminate $=$, and together with $=_{cong}$ they say that $=$ constitutes a congruence relation. The rules $::_I$, $::_E$ and $:_I$ define conditions that $::$ and $:$ should obey, relating labelled and truth-value formulae. Similarly to $=_E$, rule $:_E$ relates $:$ to $::$ (intuitively because if the value of ξ is equal to δ, i.e. $\delta : \xi$ holds, then a fortiori it is greater than or equal to it, i.e. $\delta :: \xi$ holds). The rule $\top$ says that "top" is the greatest element, and the rule $\mathbf{t}$ says that the true $\mathbf{t}$ is a formula corresponding to $\top$. Note also that, as we indicate below, other rules about $:$, $::$, $=$ and $\leq$ can be derived, such as the rules for the reflexivity and transitivity of $=$.

To define derivations in our LDSs, we introduce additional terminology and notation. A *substitution* is a pair $\sigma = \langle \sigma^f, \sigma^v \rangle$, where $\sigma^f : \Xi^f \to L(\langle C^f, \Xi^f \rangle)$ and

$\sigma^v : \Xi^v \to L(\langle C^v, \Xi^v \rangle)$ are maps from schema variables to formulae and truth-value terms, respectively. Given a composed formula $\psi \in L(\Sigma)$, we denote by $\psi\sigma$ the composed formula that results from ψ by the simultaneous substitution of each ξ by $\sigma(\xi) = \sigma^f(\xi)$ and of each δ by $\sigma(\delta) = \sigma^v(\delta)$. By extension, $\Psi\sigma$ denotes the set consisting of $\psi\sigma$ for all $\psi \in \Psi$. Given a set of schema truth-value variables $\{\delta_1, \ldots, \delta_k\} \subseteq \Xi^v$, we say that the substitutions σ_1 and σ_2 are $\{\delta_1, \ldots, \delta_k\}$-*co-equivalent*, in symbols $\sigma_1 \equiv_{\{\delta_1,\ldots,\delta_k\}} \sigma_2$, iff $\sigma_1^f = \sigma_2^f$ and $\sigma_1^v(\delta) = \sigma_2^v(\delta)$ for each $\delta \in \Xi^v$ such that $\delta \notin \{\delta_1, \ldots, \delta_k\}$. We use *label*$(\Theta)$ to denote the set of all schema truth-value variables in a set of composed formulae Θ.

Definition 6. *A judgement J is* derivable *from a set Ω of judgements in a LDS $\langle \Sigma, R \rangle$, in symbols $\Omega \vdash_{\langle \Sigma, R \rangle} J$, iff there is a finite sequence of judgements $J_1, \ldots, J_n$ such that J_n is J and, for $l = 1, \ldots, n$, either*

1. *J_l is an axiom Ax, that is of the form $\psi_1, \ldots, \psi_m, \eta \,/\, \eta \ \triangleright \Upsilon$; or*
2. *there are a rule $r' = \langle \{\Theta'_1 \,/\, \eta'_1 \ \triangleright \Upsilon'_1, \ldots, \Theta'_k \,/\, \eta'_k \ \triangleright \Upsilon'_k\}, \eta' \rangle$ in R and substitutions $\sigma, \sigma_1, \ldots, \sigma_k$ such that $\eta_l = \eta'\sigma$ and for each $i = 1, \ldots, k$ there is $\Theta_i \,/\, \eta_i \ \triangleright \Upsilon_i$ in $J_1, \ldots, J_{l-1}$ such that $\Theta_i = \Theta_l \cup \Theta'_i\sigma_i$, $\eta_i = \eta'_i\sigma_i$, $\Upsilon'_i\sigma_i \subseteq \Upsilon_i$ and $\Upsilon_i \setminus \Upsilon'_i\sigma_i = \Upsilon_l$, label$(\Theta_l) \cap \Upsilon'_i\sigma_i = \emptyset$, $\sigma_i \equiv_{\Upsilon'_i} \sigma$ and $\sigma_i(\delta) \notin (\Xi^v \setminus \{\delta\})\sigma_i$ for each $\delta \in \Upsilon'_i$; or*
3. *there is $\Theta \,/\, \eta \ \triangleright \Upsilon \in \Omega$ and a substitution σ with $\sigma \equiv_\Upsilon id$ such that $\Theta\sigma \subseteq \Theta_l$, label$(\Theta \cup \{\eta\})\sigma \cap \Upsilon_l \subseteq \Upsilon\sigma$ and η_l is $\eta\sigma$ (where id is the pair of identity substitutions on Ξ^f and Ξ^v).*

The judgements $\Theta_l \,/\, \eta_l \ \triangleright \Upsilon_l$ and $\Theta_i \,/\, \eta_i \ \triangleright \Upsilon_i$ for $i = 1, \ldots, k$ in condition 2 constitute a rule, referred to as an instance *of r' by the substitutions $\sigma, \sigma_1, \ldots, \sigma_k$.*

Definition 7. *We say that a composed formula $\eta \in L(\Sigma)$ is* inferred *from a set of composed formulae $\Psi \subseteq L(\Sigma)$ in a LDS $\langle \Sigma, R \rangle$, in symbols $\Psi \vdash_{\langle \Sigma, R \rangle} \eta$, iff there are $\psi_1, \ldots, \psi_n \in \Psi$ such that $\vdash_{\langle \Sigma, R \rangle} \psi_1, \ldots, \psi_n \,/\, \eta$. When there is no risk of confusion we will simply write $\Psi \vdash \eta$.*

We say that a formula φ is a theorem *of a LDS $\langle \Sigma, R \rangle$ with set D of designated truth-values whenever $\vdash_{\langle \Sigma, R \rangle} \delta : \varphi$ or $\vdash_{\langle \Sigma, R \rangle} \delta :: \varphi$ for some $\delta \in D$.*

A rule $\langle \{J_1, \ldots, J_k\}, J \rangle$ is a *derived rule* in a LDS $\langle \Sigma, R \rangle$ iff $J_1, \ldots, J_k \vdash_{\langle \Sigma, R \rangle} J$. We will use derived rules like primitive ones. For example, among others, from the rules in Definition 5 we can derive in any LDS the rules for the reflexivity and transitivity of equality for labels ($=_r$ follows straightforwardly by the reflexivity $\leq_r$ of $\leq$ and $=_t$ follows by the transitivity $\leq_t$ of $\leq$ and the symmetry $=_s$ of $=$):

$$\frac{}{\delta = \delta} =_r \qquad \text{and} \qquad \frac{\delta_1 = \delta_2 \quad \delta_2 = \delta_3}{\delta_1 = \delta_3} =_t \ .$$

We give example derivations in the next section; we conclude this one by defining exhaustive systems, in which each formula constructor c is associated with a truth-value operation symbol $\overline{c}$ of the same arity.

Definition 8. *An* exhaustive LDS *is a triple $\langle \Sigma, R, \bar{\cdot} \rangle$ such that: (i) $\langle \Sigma, R \rangle$ is a LDS, (ii) $\bar{\cdot}$ is a family $\{\bar{\cdot}_k\}_{k \in \mathbb{N}}$ of maps where $\bar{\cdot}_k : C_k^f \to C_k^v$ for $k \geq 1$, and $\bar{\cdot}_0 : C_0^f \cup \Xi^f \to C_0^v \cup \Xi^v$ with $\overline{c}_0 \in C_0^v$ and $\overline{\xi}_0 \in \Xi^v$, and (iii) R includes $\dfrac{}{\overline{\xi} : \xi}\ \bar{\cdot}$.*

It is then easy to derive the following rules in any exhaustive LDS:

$$\frac{\delta = \overline{\xi}}{\delta : \xi} :_{I_2}\,, \qquad \frac{\delta : \xi}{\delta = \overline{\xi}} :_{E_2}\,, \qquad \frac{\delta \leq \overline{\xi}}{\delta :: \xi} ::_{I_2}\,, \qquad \frac{\delta :: \xi}{\delta \leq \overline{\xi}} ::_{E_2}\,.$$

Given a formula φ, $\overline{\varphi}$ denotes the term in $L(\langle C^v, \Xi^v\rangle)$ inductively induced by $\bar{\cdot}$, and for a labelled formula $\beta : \varphi$, $\overline{\beta : \varphi}$ denotes the truth-value formula $\beta = \overline{\varphi}$. That is, $\bar{:}$ corresponds to $=$. Similarly for $\beta :: \varphi$, where $\overline{::}$ corresponds to $\leq$.

By induction on the length of derivations we can then show that if $\Theta \vdash_{\langle \Sigma, R\rangle} \eta$ then $\overline{\Theta} \vdash_{\langle \Sigma, R, \bar{\cdot}\rangle} \overline{\eta}$ where $\langle \Sigma, R, \bar{\cdot}\rangle$ is the exhaustive LDS induced by $\langle \Sigma, R\rangle$.

3 Special deduction systems

In our LDSs we can formalize formula constructors that are common to several multiple-valued logics. One such constructor is conjunction, which, interpreted as a binary meet, is common to classical, intuitionistic, relevance, Gödel and many other logics. Another example is implication, in all its variants ranging from material to intuitionistic to relevance or to other non-classical, substructural implications. In fact, our framework allows us to define general introduction and elimination rules that encompass formula constructors in several logics like modal, intuitionistic and relevance logics thus obtaining the counterpart of the rules in [21], where the labels represent worlds in the underlying Kripke structures. To formalize this, we now introduce well-behaved systems, which are based on a stronger definition of signature than the simple one given above.

3.1 Well-behaved deduction systems

A well-behaved signature associates, by means of the operation $\check{\cdot}$, each formula constructor of arity k with a truth-value operator of arity n with $n \leq k$. For instance, the unary constructor $\Box$ is associated with the unary operator $\check{\Box}$, while the binary "local" constructor $\wedge$ for conjunction is associated with the unary operator $\check{\wedge}$ (see the examples below).

Definition 9. *A* well-behaved signature *is a tuple $\langle \Sigma, \dot{C}^f, \check{\cdot}\rangle$, where Σ is a TVS $\dot{C}^f = \{\dot{C}^f_k\}_{k \in \mathbb{N}^+}$ with $\dot{C}^f_k \subseteq C^f_k$, and $\check{\cdot} : \cup_{k \in \mathbb{N}^+} C^f_k \to \cup_{k \in \mathbb{N}^+} C^v_k \cup \{\top\}$ such that if $c \in C^f_k$ then $\check{c} \in C^v_n$ with $n \leq k$.*

To illustrate the intuition behind $\check{\cdot}$ and $\dot{C}^f$, let us first define a well-behaved LDS as a system that is based on a well-behaved signature, and that has an introduction and an elimination rule for each constructor in C^f (as is usual for *natural* deduction systems; see, e.g., [20]).

Definition 10. *A* well-behaved labelled deduction system *is a LDS $\langle \Sigma, R\rangle$, where Σ is a well-behaved signature and R includes the rules:*

- *for each $c \in \dot{C}^f_k$ where $\check{c} \in C^v_n$, $\boldsymbol{\delta} = \delta_1, \ldots, \delta_{n-1}$, $\boldsymbol{\xi} = \xi_1, \ldots, \xi_{n-1}$, and $i = n, \ldots, k$*

$$\frac{\boldsymbol{\delta} :: \boldsymbol{\xi} \,/\, \check{c}(\delta, \boldsymbol{\delta}) :: \xi_j \;\triangleright \boldsymbol{\delta},\; j = n, \ldots, k}{\delta :: c(\xi_1, \ldots, \xi_k)}\; c_I\,, \qquad \frac{\delta :: c(\xi_1, \ldots, \xi_k) \quad \boldsymbol{\delta} :: \boldsymbol{\xi}}{\check{c}(\delta, \boldsymbol{\delta}) :: \xi_i}\; c_{E^i}\,,$$

– *for each* $c \in C_k^f \setminus \dot{C}_k^f$ *where* $\check{c} \in C_n^v$, $\boldsymbol{\delta} = \delta_1, \ldots, \delta_n$, $\boldsymbol{\xi} = \xi_1, \ldots, \xi_n$ *and* $i = n+1, \ldots, k$

$$\frac{\delta \leq \check{c}(\boldsymbol{\delta}) \quad \boldsymbol{\delta} :: \boldsymbol{\xi} \quad \delta :: \xi_i}{\delta :: c(\xi_1, \ldots, \xi_k)}\; c_{I^i}\,,$$

$$\frac{\delta :: c(\xi_1, \ldots, \xi_k) \quad [\delta \leq \check{c}(\boldsymbol{\delta}), \boldsymbol{\delta} :: \boldsymbol{\xi} \,/\, \delta' :: \xi',\; if\, n \neq 0] \quad \delta :: \xi_j \,/\, \delta' :: \xi',\, j = n+1, \ldots, k}{\delta' :: \xi'}\; c_E\,,$$

where, for $\boldsymbol{\delta} = \delta_1, \ldots, \delta_l$ *and* $\boldsymbol{\xi} = \xi_1, \ldots, \xi_l$, *the vector* $\boldsymbol{\delta} :: \boldsymbol{\xi}$ *stands for the* l *hypothesis formulae* $\delta_1 :: \xi_1 \cdots \delta_l :: \xi_l$, *and* $\delta \leq \check{c}(\boldsymbol{\delta})$ *stands for* $\delta \leq \check{c}(\delta_1, \ldots, \delta_l)$. *Observe also that* $[\delta \leq \check{c}(\boldsymbol{\delta}), \boldsymbol{\delta} :: \boldsymbol{\xi} \,/\, \delta' :: \xi',\ if\, n \neq 0]$ *indicates that this judgement does not appear in the rule when* $n = 0$ *(e.g. as in the case of disjunction).*

The introduction and elimination rules of Definition 10 are general and cover a large number of constructors and logics. The constructors in $\dot{C}_k^f$ are those that have a "universal" nature, such as necessity $\Box$ and classical, intuitionistic and relevant implications $\sqsupset$, as well as conjunction $\wedge$. For instance, recall that in the Kripke semantics a modal formula $\Box\varphi$ holds at a world x iff for *all* worlds y, xRy implies that φ holds at y. Similarly, intuitionistic implication is universal since to be satisfied in a Kripke world it has to be satisfied in *all* the accessible worlds, and it is locally implicative for each world. Analogously, the constructors in $C_k^f \setminus \dot{C}_k^f$ are those that have an "existential" nature, such as modal possibility $\Diamond$, relevant fusion $\circ$, and disjunction $\vee$.

To illustrate this further, we give examples of well-behaved LDSs for powerset logics, i.e. logics for which the denotation of a formula can be seen as a set of worlds. In particular, we give a LDS for the normal modal logic K [5] and one for the basic positive relevance logic B^+ [10, 17]. We first need one additional definition, namely that of a local constructor, which allows us to identify the constructors whose evaluation in the corresponding Kripke-style semantics only depends on one world (and is thus "local" to that world), such as classical implication, conjunction and disjunction.

Definition 11. *A constructor* $c \in \dot{C}_k^f$ *in a well-behaved LDS* $\langle \Sigma, R \rangle$ *is* local *iff* $\check{c} \in C_n^v$ *and* R *includes the following rules (where* $\boldsymbol{\delta} = \delta_1, \ldots, \delta_{n-1}$*)*

$$\frac{}{\delta \leq \check{c}(\delta, \boldsymbol{\delta})}\; U_1 \qquad \text{and} \qquad \frac{}{\check{c}(\delta, \boldsymbol{\delta}) \leq \delta}\; U_2\,.$$

Example 1. The *modal system* $\mathcal{L}(K)$ is a well-behaved LDS $\langle \Sigma, R \rangle$ where Σ is a well-behaved signature such that

- $C_1^f = \dot{C}_1^f = \{\Box\}$, $C_2^f = \dot{C}_2^f = \{\sqsupset\}$, $\mathbf{f}, \mathbf{t} \in C_0^f$ and $C_k^f = \emptyset$ for $k \geq 3$,
- $\bot, \top \in C_0^v$, $C_1^v = \{\check{\Box}\}$, $C_2^v = \{\check{\sqsupset}\}$ and $C_k^v = \emptyset$ for $k > 2$,
- $\sqsupset$ is a local constructor (so that the rules U_1 and U_2 hold for $\check{\sqsupset}$),
- $D = \{\top\}$,

and R includes the rules $\dfrac{}{\bot :: \xi}\,\bot^f$, $\dfrac{}{\bot \leq \delta}\,\bot^v$, and $\dfrac{}{\bot : \mathbf{f}}\,\mathbf{f}_I$.

The rules for the operators $\Box$ and $\sqsupset$ of the system $\mathcal{L}(K)$ are thus the following instances of the rules in the Definitions 10 and 11:

$$\dfrac{\check{\Box}\delta :: \xi}{\delta :: \Box\xi}\,\Box_I\,, \qquad \dfrac{\delta :: \Box\xi}{\check{\Box}\delta :: \xi}\,\Box_E\,, \qquad \dfrac{\delta :: \xi_1 \,/\, \delta :: \xi_2}{\delta :: \xi_1 \sqsupset \xi_2}\,\sqsupset_I\,, \qquad \dfrac{\delta :: \xi_1 \sqsupset \xi_2 \quad \delta :: \xi_1}{\delta :: \xi_2}\,\sqsupset_E\,.$$

The intuition behind the additional rules of $\mathcal{L}(K)$ is the following: $\bot^f$ and $\bot^v$ define $\bot$ as the least element of the set of truth-values in $\mathcal{L}(K)$, and $\mathbf{f}_I$ says that the *false* $\mathbf{f}$ is indeed false. As usual for modal logics, negation is a derived constructor, i.e. $\neg\varphi$ abbreviates $\varphi \sqsupset \mathbf{f}$, and so is $\Diamond$, which abbreviates $\neg\Box\neg$.

That our system $\mathcal{L}(K)$ indeed represents the logic K will become clear below when defining the algebraic semantics (which allows us to show that the system is sound and complete with respect to it), but we can already observe that our system is justified, albeit informally, by the way in which one can obtain a modal algebra from a Kripke structure [5]. As another informal justification for $\mathcal{L}(K)$, observe that it is straightforward to derive the axioms and rules of the standard Hilbert system $H(K)$ [5]. For instance, we can show that the normality axiom $\Box(\xi_1 \sqsupset \xi_2) \sqsupset ((\Box\xi_1) \sqsupset (\Box\xi_2))$ is a theorem of $\mathcal{L}(K)$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\top :: \Box(\xi_1 \sqsupset \xi_2), \top :: \Box\xi_1 \,/\, \top :: \Box(\xi_1 \sqsupset \xi_2)}\,Ax}{\top :: \Box(\xi_1 \sqsupset \xi_2), \top :: \Box\xi_1 \,/\, \check{\Box}\top :: \xi_1 \sqsupset \xi_2}\,\Box_E \quad \dfrac{\dfrac{}{\top :: \Box(\xi_1 \sqsupset \xi_2), \top :: \Box\xi_1 \,/\, \top :: \Box\xi_1}\,Ax}{\top :: \Box(\xi_1 \sqsupset \xi_2), \top :: \Box\xi_1 \,/\, \check{\Box}\top :: \xi_1}\,\Box_E}{\top :: \Box(\xi_1 \sqsupset \xi_2), \top :: \Box\xi_1 \,/\, \check{\Box}\top :: \xi_2}\,\sqsupset_E}{\top :: \Box(\xi_1 \sqsupset \xi_2), \top :: \Box\xi_1 \,/\, \top :: \Box\xi_2}\,\Box_I}{\top :: \Box(\xi_1 \sqsupset \xi_2) \,/\, \top :: (\Box\xi_1) \sqsupset (\Box\xi_2)}\,\sqsupset_I}{\top :: \Box(\xi_1 \sqsupset \xi_2) \sqsupset ((\Box\xi_1) \sqsupset (\Box\xi_2))}\,\sqsupset_I\,.$$

Labelled equivalents of the necessitation rule and of modus ponens can be derived similarly, e.g.

$$\dfrac{\dfrac{\dfrac{}{\top :: \xi \,/\, \check{\Box}\top \leq \top}\,\top \quad \dfrac{}{\top :: \xi \,/\, \top :: \xi}\,Ax}{\top :: \xi \,/\, \check{\Box}\top :: \xi}\,::_I}{\top :: \xi \,/\, \top :: \Box\xi}\,\Box_I\,.$$

Compare our system here with the labelled deduction systems where labels are worlds in a Kripke structure, e.g as in [3, 16, 21] and [2, 11]: the unary operator $\check{\Box}$ corresponds at the algebraic level to the accessibility relation of the Kripke semantics. In fact, our system $\mathcal{L}(K)$ is modular, as it directly mirrors the modularity of modal logics: all we have to do is add rules expressing properties of $\check{\Box}$, and thereby corresponding to properties of the accessibility relation of the underlying Kripke semantics (and thus also to axioms of the corresponding Hilbert axiomatization). For instance, to get a LDS for the modal logic $S4$ we only have to add the rules

$$\dfrac{}{\delta \leq \check{\Box}\delta}\,T \qquad \text{and} \qquad \dfrac{\delta \leq \check{\Box}\delta' \quad \delta' \leq \check{\Box}\delta''}{\delta \leq \check{\Box}\delta''}\,4\,.$$

We can then show that the Hilbert axiom $\Box\xi \sqsupset \xi$, corresponding to the reflexivity of the Kripke accessibility relation [5], can be derived in our LDS for T, i.e. in the system extending $\mathcal{L}(K)$ with the rule T, and that $\Box\xi \sqsupset \Box\Box\xi$ is a theorem of $\mathcal{L}(K)$ extended with the rule 4.

Example 2. The *basic positive relevance system* $\mathcal{L}(B^+)$ is a well-behaved LDS $\langle \Sigma, R\rangle$ where Σ is a well-behaved signature such that

- $\dot{C}_2^f = \{\wedge, \sqsupset\}$, $C_2^f = \{\wedge, \sqsupset, \vee\}$, $\mathbf{t} \in C_0^f$ and $C_k^f = \emptyset$ for $k \geq 3$ and $k = 1$,
- $\wedge$ is a local constructor, and $\check{\vee} = \top$,
- $\top \in C_0^v$, $C_1^v = \{\check{\wedge}\}$, $C_2^v = \{\check{\sqsupset}\}$, and $C_k^v = \emptyset$ for $k \geq 3$,
- $D = \{\top\}$,

and R includes the rules $\dfrac{\delta :: \xi}{\check{\sqsupset}(\top, \delta) :: \xi}\ her$ and $\dfrac{}{\delta \leq \check{\sqsupset}(\top, \delta)}\ iden$.

The rules for the operators $\sqsupset$, $\wedge$, $\vee$ of $\mathcal{L}(B^+)$ are thus the following instances of the rules in the Definition 10 (for example, $\wedge_I$ is an instance of the rule c_I since $\wedge$ is local and $\check{\wedge}$ is unary):

$$\frac{\delta_1 :: \xi_1 \,/\, \check{\sqsupset}(\delta, \delta_1) :: \xi_2 \ \triangleright \delta_1}{\delta :: \xi_1 \sqsupset \xi_2}\ \sqsupset_I\,, \quad \frac{\delta :: \xi_1 \sqsupset \xi_2 \quad \delta_1 :: \xi_1}{\check{\sqsupset}(\delta, \delta_1) :: \xi_2}\ \sqsupset_E\,, \quad \frac{\delta :: \xi_1 \quad \delta :: \xi_2}{\delta :: \xi_1 \wedge \xi_2}\ \wedge_I\,,$$

$$\frac{\delta :: \xi_1 \wedge \xi_2}{\delta :: \xi_i}\ \wedge_{E^i}\,, \quad \frac{\delta :: \xi_i}{\delta :: \xi_1 \vee \xi_2}\ \vee_{I^i}\,, \quad \frac{\delta :: \xi_1 \vee \xi_2 \quad \delta :: \xi_1 \,/\, \delta' :: \xi' \quad \delta :: \xi_2 \,/\, \delta' :: \xi'}{\delta' :: \xi'}\ \vee_E\,.$$

Note that the rules *her* and *iden* capture the hereditary and identity properties of the ternary compossibility relation of the Kripke semantics of relevance logics [10, 17]. The use of fresh variables in the rule $\sqsupset_I$ for the introduction of relevant implication $\sqsupset$ also mirrors the Kripke semantics, namely the evaluation clause of $\sqsupset$ in terms of the compossibility relation: the rule says that $\delta :: \xi_1 \sqsupset \xi_2$ holds whenever for an arbitrary δ_1 the antecedent $\delta_1 :: \xi_1$ yields the consequent $\check{\sqsupset}(\delta, \delta_1) :: \xi_2$, which in turn says that the value of ξ_2 is greater than or equal to the value of the arbitrary (due to the arbitrariness of δ_1) term $\check{\sqsupset}(\delta, \delta_1)$.

As for modal logics, an intuitive justification for the LDS $\mathcal{L}(B^+)$ can be obtained by showing that we can derive the axioms and rules of the corresponding Hilbert system $H(B^+)$ for basic positive relevance logic [10, 17, 21], e.g.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\delta :: \xi_1 \,/\, \delta :: \xi_1}\ Ax}{\delta :: \xi_1 \,/\, \check{\sqsupset}(\top, \delta) :: \xi_1}\ her}{\delta :: \xi_1 \,/\, \check{\sqsupset}(\top, \delta) :: \xi_1 \vee \xi_2}\ \vee_{I^1}}{\top :: \xi_1 \sqsupset (\xi_1 \vee \xi_2)}\ \sqsupset_I \quad .$$

Similarly, we can prove both $\top :: (\xi_1 \wedge (\xi_2 \vee \xi_3)) \sqsupset ((\xi_1 \wedge \xi_2) \vee (\xi_1 \wedge \xi_3))$ and $\top :: \xi_1 \sqsupset \xi_2, \top :: \xi_3 \sqsupset \xi_4 \,/\, \top :: (\xi_2 \sqsupset \xi_3) \sqsupset (\xi_1 \sqsupset \xi_4)$.

Similar to the case of modal logics, other logics in the relevance family (up to the notable examples of the relevance logic R and, perhaps even more importantly, of intuitionistic logic) are obtained by extending $\mathcal{L}(B^+)$ with rules formalizing other properties of the truth-value operator $\check{\sqsupset}$, e.g. its transitivity.

3.2 Finitely-valued deductive systems

We now consider systems for finitely many-valued logics; in particular, we give a system for the 3-valued Gödel logic with one distinguished element [13]. In fact, our framework allows us to impose that a LDS has a finite number of values in a very simple way: we just need to require the set C_0^v of operation symbols of arity zero to have as many elements as we want.

Definition 12. *Let $Var(\eta)$ denote the set of elements of Ξ^v that occur in the composed formula η, and let $Var(\Theta)$ denote the set $\cup_{\eta \in \Theta} Var(\eta)$. A* finitely-valued *LDS is an exhaustive LDS $\langle \Sigma, R, \bar{\cdot} \rangle$ such that C_0^v is finite and R includes*

$$\frac{\eta_1, \ldots, \eta_n / \psi}{{\eta_1}^{\delta_1 \ldots \delta_k}_{\beta_1 \ldots \beta_k}, \ldots, {\eta_n}^{\delta_1 \ldots \delta_k}_{\beta_1 \ldots \beta_k} / \psi^{\delta_1 \ldots \delta_k}_{\beta_1 \ldots \beta_k}} \; val_I \; (\textit{with } \beta_1 \ldots \beta_k \in C_0^v)\,,$$

$$\frac{{\eta_1}^{\delta_1 \ldots \delta_k}_{\beta_1 \ldots \beta_k}, \ldots, {\eta_n}^{\delta_1 \ldots \delta_k}_{\beta_1 \ldots \beta_k} / \psi^{\delta_1 \ldots \delta_k}_{\beta_1 \ldots \beta_k} \; \textit{for every } \beta_1 \ldots \beta_k \textit{ in } C_0^v}{\eta_1, \ldots, \eta_n / \psi} \; val_E\,,$$

where $\eta_1, \ldots, \eta_n, \psi$ are composed formulae and $Var(\{\eta_1, \ldots, \eta_n, \psi\}) = \{\delta_1, \ldots, \delta_k\}$. A k-LDS is a finitely-valued LDS such that the cardinality of C_0^v is k.

The rule val_E states that to derive a formula we have to derive it by considering all the possible instances of schema truth-value variables with elements in C_0^v. The rule val_I states the inverse.

Example 3. The 3-*valued Gödel LDS* $\mathcal{L}(\mathrm{G})$ is a 3-LDS $\langle \Sigma, R \rangle$ where Σ is such that

- $C_1^f = \{\neg\}$, $C_2^f = \{\sqsupset\}$, and $C_k^f = \emptyset$ for $k \geq 3$ and $k = 0$,
- $C_0^v = \{\bot, 0.5, \top\}$, $C_1^v = \{\bar{\neg}\}$, $C_2^v = \{\overline{\sqsupset}\}$, and $C_k^v = \emptyset$ for $k \geq 3$,
- $D = \{\top\}$,

and R includes the rules $\dfrac{}{\bar{\neg}\delta = \delta \overline{\sqsupset} \bot}\,\bar{\neg}_I\,,\quad \dfrac{}{\bot :: \xi}\,\bot^f,\quad \dfrac{}{\bot \leq \delta}\,\bot^v,$ and

$$\frac{\delta_1 \leq \delta_2}{\delta_1 \overline{\sqsupset} \delta_2 = \top}\,\overline{\sqsupset}_{I^1}\,,\qquad \frac{\delta_1 = 0.5 \quad \delta_2 = \bot}{\delta_1 \overline{\sqsupset} \delta_2 = \bot}\,\overline{\sqsupset}_{I^2}\,,\qquad \frac{\delta_1 = \top}{\delta_1 \overline{\sqsupset} \delta_2 = \delta_2}\,\overline{\sqsupset}_{I^3}\,,$$

$$\frac{\delta_1 \overline{\sqsupset} \delta_2 = \top}{\delta_1 \leq \delta_2}\,\overline{\sqsupset}_{E^1}\,,\qquad \frac{\delta_1 \overline{\sqsupset} \delta_2 = \bot \quad \delta_1 = 0.5}{\delta_2 = \bot}\,\overline{\sqsupset}_{E^2}\,,\qquad \frac{\delta_1 \overline{\sqsupset} \delta_2 = \delta \quad \delta_1 = \top}{\delta = \delta_2}\,\overline{\sqsupset}_{E^3}\,.$$

Observe that the rules $\overline{\sqsupset}_{I^i}$ and $\overline{\sqsupset}_{E^i}$ actually capture the entries of the truth-table for 3-valued Gödel implication

$\sqsupset$	$\bot$	0.5	$\top$
$\bot$	$\top$	$\top$	$\top$
0.5	$\bot$	$\top$	$\top$
$\top$	$\bot$	0.5	$\top$

.

For instance, that $\delta = \top \overline{\sqsupset} 0.5$ yields $\delta = 0.5$ can then be shown as follows:

$$\frac{\dfrac{}{\delta = \top \overline{\sqsupset} 0.5 \,/\, \delta = \top \overline{\sqsupset} 0.5}\,Ax \qquad \dfrac{\dfrac{}{\delta = \top \overline{\sqsupset} 0.5 \,/\, \top = \top}\,=_r}{\delta = \top \overline{\sqsupset} 0.5 \,/\, \top \overline{\sqsupset} 0.5 = 0.5}\,\overline{\sqsupset}_{I^3}}{\delta = \top \overline{\sqsupset} 0.5 \,/\, \delta = 0.5}\,=_t\,.$$

To illustrate the rule val_E at work, we show that the Heyting chain axiom $((\xi_1 \sqsupset \xi_2) \sqsupset \bot) \sqsupset (\xi_2 \sqsupset \xi_1)$, i.e. $(\xi_1 \sqsupset \xi_2) \vee (\xi_2 \sqsupset \xi_1)$, is a theorem of $\mathcal{L}(\mathrm{G})$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\bot \overline{\sqsupset} 0.5) \leq \top}\,\top \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\bot \leq 0.5}\,\bot^v}{(\bot \overline{\sqsupset} 0.5) = \top}\,\overline{\sqsupset}_{I^1}}{\top = (\bot \overline{\sqsupset} 0.5)}\,=_s}{\top \leq (\bot \overline{\sqsupset} 0.5)}\,=_E}{(\bot \overline{\sqsupset} 0.5) = \top}\,=_I}{((\bot \overline{\sqsupset} 0.5) \sqsupset \bot) = \bot}\,\overline{\sqsupset}_{I^3}}{((\bot \overline{\sqsupset} 0.5) \sqsupset \bot) \leq \bot}\,=_E \qquad \dfrac{}{\bot \leq (0.5 \overline{\sqsupset} \bot)}\,\bot^v}{((\bot \overline{\sqsupset} 0.5) \sqsupset \bot) \leq (0.5 \overline{\sqsupset} \bot)}\,\leq_t
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((\bot \overline{\sqsupset} 0.5) \sqsupset \bot) \leq (0.5 \overline{\sqsupset} \bot)}{((\bot \overline{\sqsupset} 0.5) \sqsupset \bot) \overline{\sqsupset} (0.5 \overline{\sqsupset} \bot) = \top}\,\overline{\sqsupset}_{I^1}}{\top = ((\bot \overline{\sqsupset} 0.5) \sqsupset \bot) \overline{\sqsupset} (0.5 \overline{\sqsupset} \bot)}\,=_s}{\top \leq ((\bot \overline{\sqsupset} 0.5) \sqsupset \bot) \overline{\sqsupset} (0.5 \overline{\sqsupset} \bot)}\,=_E \qquad \cdots}{\dfrac{\top \leq ((\overline{\xi}_1 \overline{\sqsupset} \overline{\xi}_2) \sqsupset \bot) \overline{\sqsupset} (\overline{\xi}_2 \overline{\sqsupset} \overline{\xi}_1)}{\top :: ((\xi_1 \sqsupset \xi_2) \sqsupset \bot) \sqsupset (\xi_2 \sqsupset \xi_1)}\,::_2}\,val_E
$$

where the elided cases are similar to the one displayed (with $\overline{\xi}_1 = \bot$ and $\overline{\xi}_2 = 0.5$).

4 Interpretation systems

The structure for a signature is an algebra providing the interpretation of formula constructors and truth-value operators.

Definition 13. *A* structure *for a signature Σ (or Σ-*structure *for short) is a tuple $\mathcal{B} = \langle B, B_0, \leq, [\cdot]^f, [\cdot]^v \rangle$, where $\langle B, \leq, [\cdot]^f \rangle$ is a pre-ordered algebra for the formula constructors C^f, $\langle B, \leq, [\cdot]^v \rangle$ is a pre-ordered algebra for the truth-value operators C^v, $B_0 \subseteq B$ and B_0 has a top element $\top$. A structure for an exhaustive signature Σ is a Σ-structure such that $[c]^f = [\overline{c}]^v$.*

The elements of the set B are the truth-values and those of B_0 are the distinguished truth-values; in most cases, B_0 is a singleton set, but there are many-valued logics that have more than one distinguished element (see [7]). The other components of $\mathcal{B}$ provide the denotations of the constructors and operators, e.g. $[\cdot]^f$ is a family of maps $[c]^f_k : B^k \to B$ for each $c \in C^f_k$, and similarly for $[\cdot]^v$.

For the modal signatures we now introduce structures that are based on generalized Kripke frames $\langle W, G, R \rangle$, where W is a set of worlds, $G \subseteq \wp W$ is a set of sets of worlds and R is a binary accessibility relation on worlds (see [5]).

Example 4. Each generalized Kripke frame $\langle W, G, R \rangle$ induces a structure for a modal signature, as defined in Example 1, as follows:

- B is G with W as $\top$ and $\emptyset$ as $\bot$, and $B_0 = \{\top\}$;
- $[\check{\Box}]^v(b) = \{u \in W \mid wRu, \text{ for some } w \in b\}$;
- $[\mathbf{t}]^f = \top$ and $[\mathbf{f}]^f = \bot$;

- $[\sqsupset]^f(b_1, b_2) = (W \setminus b_1) \sqcup b_2$;
- $[\Box]^f(b) = \{w \in W \mid w \in [\check{\Box}]^v(b)\}$.

Each such structure is of course a modal algebra.

Similarly, the structures for the basic positive relevance signature are based on the algebras associated with the relational structures $\langle W, R\rangle$ for relevance logic, where R is a ternary compossibility relation on worlds in W (see [10, 17]).

Example 5. Each relational structure $\langle W, R\rangle$ induces a structure for the basic positive relevance signature of Example 2 as follows:

- B is $\wp W$ with W as $\top$, and $B_0 = \{\top\}$;
- $[\check{\sqsupset}]^v(b, b_1) = \{w_2 \in W \mid Rww_1w_2$ for some $w \in b$ and $w_1 \in b_1\}$;
- $[\mathbf{t}]^f = \top$;
- $[\wedge]^f(b_1, b_2) = b_1 \cap b_2$;
- $[\sqsupset]^f(b_1, b_2)$ is the greatest set b such that for every $b' \subseteq W$ if $b' \subseteq b_1$ then $[\check{\sqsupset}]^v(b, b') \subseteq b_2$;
- $[\vee]^f(b_1, b_2) = b_1 \cup b_2$.

Example 6. A structure $\mathcal{B}$ for the 3-valued Gödel signature Σ of Example 3 is defined as follows:

- $\langle B, \leq\rangle$ is a total order with a top $\top$ and a bottom $\perp$ and $B = \{\perp, b, \top\}$, and $B_0 = \{\top\}$;
- $\langle B, [\cdot]^f\rangle$ and $\langle B, [\cdot]^v\rangle$ are 3-valued Gödel algebras such that $[c]^f = [\overline{c}]^v$ for each $c \in \{\neg, \sqsupset\}$, $[\perp]^v = \perp$, $[\top]^v = \top$ and $[0.5]^v = b$.

For more details about Gödel algebras see [13].

In the sequel, we will sometimes omit the reference to the arity of the constructors and operators in order to make the notation lighter.

Definition 14. *An* interpretation system *is a pair $\mathcal{I} = \langle \Sigma, M\rangle$ where Σ is a signature and M is a class of structures for Σ.*

Definition 15. *Let $\mathcal{B} = \langle B, B_0, \leq, [\cdot]^f, [\cdot]^v\rangle$ be a Σ-structure. An* assignment α *over $\mathcal{B}$ is a pair $\langle \alpha^f, \alpha^v\rangle$ such that $\alpha^f : \Xi^f \to B$ and $\alpha^v = \Xi^v \to B$.*

The interpretation of formulae *over $\mathcal{B}$ and α is a map $[\![.]\!]^{\mathcal{B}}_{\alpha} : L(\langle C^f, \Xi^f\rangle) \to B$ inductively defined as follows:*

- $[\![c]\!]^{\mathcal{B}}_{\alpha} = [c]^f$, *whenever* $c \in C^f_0$;
- $[\![\xi]\!]^{\mathcal{B}}_{\alpha} = \alpha^f(\xi)$, *whenever* $\xi \in \Xi^f$;
- $[\![c(\varphi_1, \dots, \varphi_k)]\!]^{\mathcal{B}}_{\alpha} = [c]^f([\![\varphi_1]\!]^{\mathcal{B}}_{\alpha}, \dots, [\![\varphi_k]\!]^{\mathcal{B}}_{\alpha})$, *whenever* $c \in C^f_k$ *and* $\varphi_1, \dots, \varphi_k \in L(\langle C^f, \Xi^f\rangle)$ *for* $k \in \mathbb{N}$.

The interpretation of truth-value terms *over $\mathcal{B}$ and α is a map $[\![.]\!]^{\mathcal{B}}_{\alpha} : L(\langle C^v, \Xi^v\rangle) \to B$ inductively defined as follows:*

- $[\![o]\!]^{\mathcal{B}}_{\alpha} = [o]^v$, *whenever* $o \in C^v_0$;

- $[\![\delta]\!]^{\mathcal{B}}_{\alpha} = \alpha^v(\delta)$, *whenever* $\delta \in \Xi^v$*;*
- $[\![o(\beta_1, \ldots, \beta_k)]\!]^{\mathcal{B}}_{\alpha} = [o]^v([\![\beta_1]\!]^{\mathcal{B}}_{\alpha}, \ldots, [\![\beta_k]\!]^{\mathcal{B}}_{\alpha})$, *whenever* $o \in C^v_k$ *and* $\beta_1, \ldots, \beta_k \in L(\langle C^v, \Xi^v \rangle)$ *for* $k \in \mathbb{N}$.

We say that $\mathcal{B}$ *and* α satisfy *a composed formula* ψ*, in symbols* $\mathcal{B}, \alpha \Vdash \psi$*, iff*

- $[\![\beta]\!]^{\mathcal{B}}_{\alpha} = [\![\varphi]\!]^{\mathcal{B}}_{\alpha}$ *whenever* ψ *is* $\beta : \varphi$*;*
- $[\![\beta]\!]^{\mathcal{B}}_{\alpha} \leq [\![\varphi]\!]^{\mathcal{B}}_{\alpha}$ *whenever* ψ *is* $\beta :: \varphi$*;*
- $[\![\beta_1]\!]^{\mathcal{B}}_{\alpha} \bullet [\![\beta_2]\!]^{\mathcal{B}}_{\alpha}$ *whenever* ψ *is* $\beta_1 \bullet \beta_2$ *with* $\bullet \in \{=, \leq\}$.

To introduce satisfaction of judgements, we define *co-equivalent assignments* with respect to a set of schema truth-value variables as follows: $\alpha_1 \equiv_{\{\delta_1, \ldots, \delta_k\}} \alpha_2$ iff $\alpha^v_1(\delta) = \alpha^v_2(\delta)$ for every $\delta \in \Xi^v \setminus \{\delta_1, \ldots, \delta_k\}$ and $\alpha^f_1 = \alpha^f_2$.

Definition 16. *A structure* $\mathcal{B}$ *and an assignment* α satisfy $\Theta \,/\, \eta \rhd \delta_1, \ldots, \delta_k$*, in symbols* $\mathcal{B}, \alpha \Vdash \Theta \,/\, \eta \rhd \delta_1, \ldots, \delta_k$*, iff for every assignment* α' *with* $\alpha' \equiv_{\{\delta_1, \ldots, \delta_k\}} \alpha$*,* $\mathcal{B}, \alpha' \Vdash \eta$ *whenever* $\mathcal{B}, \alpha' \Vdash \psi$ *for every* $\psi \in \Theta$.

A structure $\mathcal{B}$ validates *a rule* $\langle \{J_1, \ldots, J_k\}, J \rangle$ *iff for every assignment* α *over* $\mathcal{B}$ *we have* $\mathcal{B}, \alpha \Vdash J$ *whenever* $\mathcal{B}, \alpha \Vdash J_i$ *for each* $i = 1, \ldots, k$.

A composed formula $\eta \in L(\Sigma)$ *is* entailed *by* $\Theta \subseteq L(\Sigma)$ *in a LDS* $\langle \Sigma, R \rangle$*, in symbols* $\Theta \vDash_{\langle \Sigma, R \rangle} \eta$*, iff* $\mathcal{B}, \alpha \Vdash \Theta \,/\, \eta$ *for every* Σ*-structure* $\mathcal{B}$ *and assignment* α*. When there is no risk of confusion we will simply write* $\Theta \vDash \eta$.

5 Soundness and completeness

Rather than focussing on particular systems and logics, we will now analyze soundness and completeness in the general context of our framework. We first introduce the concept of logic system and then identify the conditions under which a logic system is sound and complete.

Definition 17. *A* logic system *is a tuple* $\langle \Sigma, R, M \rangle$ *where* $\langle \Sigma, R \rangle$ *is a LDS and* $\langle \Sigma, M \rangle$ *is an interpretation system. An* exhaustive logic system $\langle \Sigma, R, \ddot{\cdot}, M \rangle$ *is a logic system based on an exhaustive LDS.*

Definition 18. *Let* $\Theta \subseteq L(\Sigma)$ *and* $\eta \in L(\Sigma)$*. A logic system is* sound *iff* $\Theta \vDash \eta$ *whenever* $\Theta \vdash \eta$*, and it is* complete *iff* $\Theta \vdash \eta$ *whenever* $\Theta \vDash \eta$.

We can prove soundness by showing that every Σ-structure in M satisfies all the instances of the inference rules in $\langle \Sigma, R \rangle$. By a simple case analysis we have:

Proposition 1. *Let* $\langle \Sigma, R, M \rangle$ *(or* $\langle \Sigma, R, \ddot{\cdot}, M \rangle$*) be a logic system. If a* Σ*-structure validates a rule in* R*, then it validates all instances of that rule.*

We can then show by induction that:

Theorem 1. *A logic system* $\langle \Sigma, R, M \rangle$ *(or* $\langle \Sigma, R, \ddot{\cdot}, M \rangle$*) is sound whenever every* Σ*-structure in* M *validates all rules of* R.

For example, the Σ-structures for modal signatures, as introduced in Example 4, validate the rules of $\mathcal{L}(K)$, the Σ-structures of Example 5 for basic positive relevance signatures validate the rules of $\mathcal{L}(B^+)$, and the Σ-structures of Example 6 for 3-valued Gödel signatures validate the rules of $\mathcal{L}(\mathrm{G})$.

For completeness, we consider exhaustive LDSs; this is by no means a restriction since, as we argued in Section 2, it is always possible to get an exhaustive system out of a LDS. The proof of completeness follows a standard approach [5] by defining the induced Lindenbaum-Tarski algebra for maximally consistent sets and showing that this algebra validates the rules. The next definition adapts the standard definition to introduce the notion of maximality of a set of composed formulae with respect to a composed formula.

Definition 19. *Let $\langle \Sigma, R, \bar{\cdot} \rangle$ be an exhaustive LDS. A set Θ of composed formulae is* maximal *with respect to a composed formula $\eta \in L(\Sigma)$ iff (i) $\Theta \not\vdash \eta$ and (ii) for any Θ' such that $\Theta \subset \Theta'$ we have $\Theta' \vdash \eta$. We say Θ is* maximally consistent *iff Θ is maximal with respect to η for some $\eta \in L(\Sigma)$.*

In the proof of completeness (cf. Theorem 2 below), it is necessary to extend a set Θ_0 of composed formulae, with $\Theta_0 \not\vdash \eta$ for some $\eta \in L(\Sigma)$, to a set Θ maximal with respect to η. The details of this construction, of course, depend on the particular logic we are considering, which might be a common modal logic or some not so common non-classical logic. (For examples of similar constructions for labelled deduction systems based on Kripke-style semantics see [3, 16, 21].) So, here we only illustrate the main ideas underlying the construction.

Definition 20. *An exhaustive LDS $\langle \Sigma, R, \bar{\cdot} \rangle$ induces, for every (maximally) consistent set $\Theta \subseteq L(\Sigma)$ of composed formulae, the* Lindenbaum-Tarski *algebra $\lambda\tau_\Theta = \langle B, B_0, \leq_\Theta, [.]^f, [.]^v \rangle$ where*

- $B = L(\langle C^v, \Xi^v \rangle)$, *and* $B_0 = D$;
- $\leq_\Theta$ *is such that* $b_1 \leq_\Theta b_2$ *iff* $b_1 \leq b_2 \in \Theta$;
- $[c]^f(b_1, \ldots, b_n) = \overline{c}(b_1, \ldots b_n)$;
- $[o]^v(b_1, \ldots, b_n) = o(b_1, \ldots, b_n)$.

Note that the construction of the Lindenbaum-Tarski algebra is as expected, and that exhaustiveness is required to define the denotations of the constructors over tuples of truth-values. The following two propositions are useful in establishing the completeness of logic systems under some conditions (namely, exhaustiveness and fullness, which we will formalize below).

Proposition 2. *Let $\langle \Sigma, R, \bar{\cdot} \rangle$ be an exhaustive LDS and $\Theta \subseteq L(\Sigma)$ a maximally consistent set of composed formulae . Then, for any assignment α and substitution σ_α such that $\overline{\sigma^f_\alpha(\delta)} = \alpha^f(\delta)$ and $\overline{\sigma^v_\alpha(\xi)} = \alpha^v(\xi)$ we have:*

1. $b : \varphi \in \Theta$ *iff* $b \leq_\Theta \overline{\varphi}$ *and* $\overline{\varphi} \leq_\Theta b$, *and* $b :: \varphi \in \Theta$ *iff* $b \leq_\Theta \overline{\varphi}$;
2. $[\![\varphi]\!]^{\lambda\tau_\Theta}_\alpha = \overline{\varphi\sigma_\alpha}$ *and* $[\![b]\!]^{\lambda\tau_\Theta}_\alpha = b\sigma_\alpha$;
3. $\lambda\tau_\Theta\alpha \Vdash \eta$ *iff* $\eta\sigma_\alpha \in \Theta$;
4. $\lambda\tau_\Theta\alpha \Vdash \Psi \,/\, \eta \;\triangleright \Upsilon$ *iff* $\Theta \cup \Psi\sigma_{\alpha'} \vdash \eta\sigma_{\alpha'}$, *for any assignment α' with $\alpha' \equiv_\Upsilon \alpha$.*

Next we prove that the Lindenbaum-Tarski algebra validates the rules.

Proposition 3. *Let $\langle \Sigma, R, \bar{\cdot} \rangle$ be an exhaustive LDS and Θ a maximally consistent set of composed formulae. Then the induced algebra $\lambda\tau_\Theta$ validates all rules of R.*

Proof. Let α be an assignment over $\lambda\tau_\Theta$, σ_α a substitution related to α according to Proposition 2, and $r = \langle \{\Theta_1 \,/\, \eta_1 \;\triangleright \Upsilon_1, \ldots, \Theta_k \,/\, \eta_k \;\triangleright \Upsilon_k\}, \eta \rangle \in R$. Assume $\lambda\tau_\Theta \alpha \Vdash \Theta_i \,/\, \eta_i \;\triangleright \Upsilon_i$ for every $i \in \{1, \ldots, k\}$. Then, by Proposition 2, for any assignment α' with $\alpha' \equiv_{\Upsilon_i} \alpha$ we have $\Theta \cup \Theta_i \sigma_{\alpha'} \vdash \eta_i \sigma_{\alpha'}$. So $\vdash \Theta \cup \Theta_i \sigma_{\alpha'} \,/\, \eta_i \sigma_{\alpha'}$. Hence, using rule r we have that $\vdash \Theta \,/\, \eta\sigma_\alpha$. Thus $\Theta \vdash \eta\sigma_\alpha$, and since Θ is maximal then $\eta\sigma_\alpha \in \Theta$. So, again by Proposition 2, we have $\lambda\tau_\Theta \alpha \Vdash \eta$. QED

To prove completeness we need to require that our exhaustive systems are full, in the sense that the Lindenbaum-Tarski algebras are among the structures that we consider.[1]

Definition 21. *An exhaustive logic system $\langle \Sigma, R, \bar{\cdot}, M \rangle$ is* full *iff $\lambda\tau_\Theta \in M$ for every maximally consistent set $\Theta \subseteq L(\Sigma)$.*

Theorem 2. *Every full exhaustive logic system is complete.*

Proof. Let $\langle \Sigma, R, \bar{\cdot}, M \rangle$ be a full exhaustive logic system. Assume $\Theta_0 \not\vdash \eta$. Then $\eta \notin \Theta$ where Θ is an extension of Θ_0 maximal with respect to η. So, by Proposition 2, we have that $\lambda\tau_\Theta\, id \not\Vdash \eta$. Since $\lambda\tau_\Theta \in M$, then $\Theta_0 \not\vDash \eta$. QED

With these general results at hand, it is not difficult to prove the soundness and completeness of particular systems, such as the ones we considered above.

6 Concluding remarks

We have given a framework for presenting non-classical logics in a modular and uniform way as labelled natural deduction systems, where the use of algebras of truth-values as the labelling algebras of our systems, as opposed to the more customary labelling based on Kripke semantics, allows us to give generalized systems for multiple-valued logics. In the tradition of Labelled Deduction for many-valued logics [1, 6, 7, 13, 14] and for modal, relevance and other power-set logics [2, 3, 8, 9, 11, 16, 21], our systems make use of labels to give natural deduction rules for a large number of formula constructors. The novelty of our approach is that these constructors, and thus the logics they appear in, are all captured within the same formalism. This also opens up the possibility of investigating many-valued variants of power-set logics, and, more generally, their fibring [4, 12], as we have already begun [15] to do as part of our research program on fibring of logics and of their deduction systems [16, 18, 19, 22]. As future work, we plan to investigate also extensions to the first-order case of the propositional multiple-valued logics we considered here.

[1] Note that it is always possible to make full an exhaustive logic system by considering all Σ-structures that validate the rules, and that soundness is preserved by the closure for fullness.

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