Combining Logics: Parchments Revisited

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Abstract. In the general context of the theory of institutions, several notions of parchment and parchment morphism have been proposed as the adequate setting for combining logics. However, so far, they seem to lack one of the main advantages of the combination mechanism known as fibring: general results of transference of important logical properties from the logics being combined to the resulting fibred logic. Herein, in order to bring fibring to the institutional setting, we propose to work with the novel notion of **c**-*parchment*. We show how both free and constrained fibring can be characterized as colimits of **c**-*parchments*, and illustrate both the construction and its preservation capabilities by exploring the idea of obtaining partial equational logic by fibring equational logic with a suitable logic of partiality. Last but not least, in the restricted context of propositional based, we state and prove a collection of meaningful soundness and completeness preservation results for fibring, with respect to Hilbert-like proof-calculi.

1 Introduction

Recently, the problem of combining logics has been deserving much attention. The practical impact of a theory of logic combination is clear for anyone working in knowledge representation or in formal specification and verification. In the fields of artificial intelligence and software engineering, the need for working with several formalisms at the same time is widely recognized. Besides, combinations of logics are also of great theoretical interest [4]. Among the different combination techniques, both *fibring* [12, 13, 22] and combinations of parchments [20, 21] deserve close attention. In fact, although the work on parchments has found its way into practice, see for instance [19], it lacks a feature that we consider essential: transference results for relevant properties of logics, such as soundness and completeness. For *fibring*, however, recent significant preservation results have been obtained [28, 7]. Our goal in this paper is to bring both *fibring* and these transference results to the setting of institutions.

This leads us, first of all, to a revised notion of parchment. It shall be made clear that the detail provided by early definitions [14, 20, 21] is not enough to capture the finer structure of models. In particular, for a smooth characterization of *fibring*, we need a notion that promotes logical consequence as a whole, rather than just validity. In previous work [22, 28, 7], a validity based consequence has also been considered and related to this more "internal" notion [7]. Herein, however, we shall not make explicit use of it. Still, the distinction is crucial to the full understanding of many logics, including first-order logic and modal logic, and plays an essential role in the process. So, we propose to work with **c**-*parchments*, that essentially extend the model-theoretic parchments of [21] by endowing the algebras of truth-values with more than just a set of designated values. Namely, we require the set of truth-values to be structured according to a Tarskian closure operation as in [7], thus recovering an early proposal of Smiley [25].

Besides showing how **c**-*parchments* can be seen as presentations of institutions, a suitable notion of *morphism* is also proposed and shown to present institution op-morphisms. The reason for this relationship to the dual of the category of institutions and institution morphisms is precisely our intention to follow the "old slogan" in its strict sense, and use colimits for combination. Therefore, building on the fact that **c**-*parchments* are essentially functors over a suitable category of **c**-*rooms*, we manage to characterize both *free* and *constrained fibring* as colimits of **c**-*parchments*. We illustrate *fibring* by providing a detailed construction of an equational logic dealing with partiality, by combining equational logic with a suitable logic of partiality. This example, when compared with the way partiality is dealt with using previous notions of parchment [20, 21, 19], is in fact paradigmatic of the modular power of *fibring*. Along with the fibred semantics of partial equational logic, we also show that by simultaneously combining Hilbert-like proof-calculi for the given logics, a sound and complete calculus for partial equational logic can also be obtained.

In fact, given that the right amount of *structurality* [6] is embodied in the deduction rules of proof-calculi, their *fibring* is well understood [7] and meaningful. As in previous treatments of this issue, we shall achieve this by using *schema variables* to write *schema rules* that can then be instantiated with arbitrary formulae while building deductions. In this context, although just for the particular case of propositionally based logics, we then state and prove a collection of soundness and completeness transference results for *fibring*. Preservation of soundness is easily just a consequence of the construction underlying *fibring*, as shown in [7]. On the contrary, as should be expected, completeness preservation results are in general not so easy to obtain. The completeness transference results that we shall present are based on the fundamental notion of *fullness*, as a means of guaranteeing that we always have enough models, extending original ideas from [28], further worked out in [7]. We provide completeness proofs for several classes of interpretation structures, including partially-ordered ones, using standard techniques in logic and algebra, such as congruence and Lindenbaum-Tarski algebras. Rephrasing the main Theorem of [28], we also mention the case of powerset structures inspired by general models for modal logic (see for instance [16]) whose completeness proof uses a Henkin-style technique.

The rest of the paper is organized as follows. In Section 2 we introduce the novel notion of **c**-*parchment* and show how it relates to institutions. For the sake of illustration, we show how to represent two well known logics as **c**-parchments. Section 3 is devoted to developing the categorial setting of **c**-parchments as indexed categories of **c**-*rooms* and to establishing the cocompleteness of the corresponding categories. In section 4, we define *fibring* of **c**-rooms (and **c**-parchments) and provide its characterization as a categorial colimit. In order to bring the necessary insight on the construction, we then dwell on the interesting example of obtaining partial equational logic by fibring equational logic with a logic of partiality. Moreover, paving the way to the subsequent sections, we also show how a sound and complete calculus for the resulting logic can be obtained by putting together (*fibring*) the proof-calculus of equational logic with a suited calculus for the logic of partiality. Section 5 introduces the details of the appropriate notion of Hilbert-like *proof-calculi*, using schematic rules, and shows also that their *fibring* corresponds to a colimit in the appropriate category. The whole treatment is restricted to the particular case of propositional based logics, since that is sufficient for the forthcoming preservation results. Finally, in Section 6, we state and prove our soundness and completeness preservation results for fibring of propositional based logics. We conclude by discussing the limitations of the work presented, and how we expect to overcome them in future work.

2 The notion of c-parchment

We start by introducing some notation. In the sequel, $\mathbf{AlgSig}_\phi$ denotes the category of algebraic many-sorted signatures $\langle S, O\rangle$, where S is a set (of *sorts*) and $O = \{O_u\}_{u\in S^+}$ is a family of sets (of *operators*) indexed by their type, with a distinguished sort $\phi \in S$ (for formulae) and morphisms preserving it. Given such a signature $\langle S, O\rangle$, we denote by $\mathbf{Alg}(\langle S, O\rangle)$ the category of $\langle S, O\rangle$-algebras and homomorphisms, and by $\mathrm{cAlg}(\langle S, O\rangle)$ the class of all pairs $\langle \mathcal{A}, \mathbf{c}\rangle$ with $\mathcal{A} \in |\mathbf{Alg}(\langle S, O\rangle)|$ and $\mathbf{c}$ a closure operation on $|\mathcal{A}|_\phi$ (the carrier of sort ϕ, that we can see as the set of truth-values). Recall that $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is required to be extensive - $B \subseteq B^{\mathbf{c}}$, monotonous - $B^{\mathbf{c}} \subseteq (B \cup B')^{\mathbf{c}}$, and idempotent - $(B^{\mathbf{c}})^{\mathbf{c}} = B^{\mathbf{c}}$, for all $B, B' \subseteq |\mathcal{A}|_\phi$. We shall use $\mathcal{W}_{\langle S,O\rangle}$ to denote the free $\langle S, O\rangle$-algebra (the *word* algebra), and $[\![\_]\!]^{\mathcal{A}}$ (for word *interpretation*) to denote the unique $\mathbf{Alg}(\langle S, O\rangle)$-homomorphism from $\mathcal{W}_{\langle S,O\rangle}$ to any given $\langle S, O\rangle$-algebra $\mathcal{A}$. Also recall that every $\mathbf{AlgSig}_\phi$-morphism $h : \langle S_1, O_1\rangle \to \langle S_2, O_2\rangle$ has an associated reduct functor $\_|_h : \mathbf{Alg}(\langle S_2, O_2\rangle) \to \mathbf{Alg}(\langle S_1, O_1\rangle)$. As usual, we shall preferably write $\widehat{h}$ (for word *translation*) instead of $[\![\_]\!]^{\mathcal{W}_{\langle S_2,O_2\rangle}|_h}$ to denote the unique $\mathbf{Alg}(\langle S_1, O_1\rangle)$-homomorphism from $\mathcal{W}_{\langle S_1,O_1\rangle}$ to $\mathcal{W}_{\langle S_2,O_2\rangle}|_h$.

Definition 1. A $\mathbf{c}$-*parchment* is a tuple $P = \langle \mathbf{Sig}, L, \mathcal{M}\rangle$ where:

- $\mathbf{Sig}$ is a category (of *abstract signatures*);
- $L : \mathbf{Sig} \to \mathbf{AlgSig}_\phi$ is a functor (of *concrete syntax*);
- $\mathcal{M} = \{M_\Sigma\}_{\Sigma\in|\mathbf{Sig}|}$, with $M_\Sigma \subseteq \mathrm{cAlg}(L(\Sigma))$ (a class of *structured algebras*),

such that the following condition holds for every $\mathbf{Sig}$-morphism $\sigma : \Sigma_1 \to \Sigma_2$:

- $\langle \mathcal{A}|_{L(\sigma)}, \mathbf{c}\rangle \in M_{\Sigma_1}$ for every $\langle \mathcal{A}, \mathbf{c}\rangle \in M_{\Sigma_2}$.

Note that although this coherence condition is stricter than the one considered in [21], all the examples presented there indeed correspond to this particular case.

It is easy to see that a model-theoretic parchment can be extracted from a $\mathbf{c}$-*parchment*. However, the idea here is to take advantage of the closure operation in each structured algebra $\langle \mathcal{A}, \mathbf{c}\rangle$ and to go beyond the obvious choice of the set $D = \emptyset^{\mathbf{c}} \subseteq |\mathcal{A}|_\phi$ of designated values. Instead, we shall allow the set of distinguished values to vary freely among all possible $\mathbf{c}$-closed sets $T \subseteq |\mathcal{A}|_\phi$. Of course, given $\Sigma \in |\mathbf{Sig}|$, we can recognize $|\mathcal{W}_{L(\Sigma)}|_\phi$ (the carrier of sort ϕ in the word algebra) as the set $Sen(\Sigma)$ of formulae. We shall use φ, ψ with or without primes and subscripts to denote formulae, and Φ, Ψ to denote sets of formulae. Furthermore, we can set $Mod(\Sigma) = \{\langle\langle \mathcal{A}, \mathbf{c}\rangle, T\rangle : \langle \mathcal{A}, \mathbf{c}\rangle \in M_\Sigma, T^{\mathbf{c}} = T \subseteq |\mathcal{A}|_\phi\}$ to be the class of models, and define the *satisfaction* relation $\Vdash_\Sigma$ by:

- $\langle\langle \mathcal{A}, \mathbf{c}\rangle, T\rangle \Vdash_\Sigma \varphi$ if $[\![\varphi]\!]^{\mathcal{A}} \in T$.

As usual, we extend satisfaction to sets of sentences by letting $\langle\langle \mathcal{A}, \mathbf{c}\rangle, T\rangle \Vdash_\Sigma \Phi$ if and only if $\langle\langle \mathcal{A}, \mathbf{c}\rangle, T\rangle \Vdash_\Sigma \varphi$ for every $\varphi \in \Phi$. We denote the induced *semantic entailment* relation by $\vDash_\Sigma$. It is defined as usual, from satisfaction, by $\Phi \vDash_\Sigma \psi$ if $\langle\langle \mathcal{A}, \mathbf{c}\rangle, T\rangle \Vdash_\Sigma \psi$ whenever $\langle\langle \mathcal{A}, \mathbf{c}\rangle, T\rangle \Vdash_\Sigma \Phi$, for every $\langle\langle \mathcal{A}, \mathbf{c}\rangle, T\rangle \in Mod(\Sigma)$. However, easily, this definition can be seen to correspond precisely to:

- $\Phi \vDash_\Sigma \psi$ if $[\![\psi]\!]^{\mathcal{A}} \in \{[\![\varphi]\!]^{\mathcal{A}} : \varphi \in \Phi\}^{\mathbf{c}}$, for every $\langle \mathcal{A}, \mathbf{c}\rangle \in M_\Sigma$.

Note that validity is still represented by $\emptyset \vDash_\Sigma \varphi$. For the sake of illustration, let us develop two well known examples: classical first-order logic and propositional normal modal logic.

Example 1. Classical first-order logic.
Let X be a fixed denumerable set of variables.

- Abstract signatures are given by $\mathbf{Set}^{\mathbb{N}} \times \mathbf{Set}^{\mathbb{N}}$, the category of pairs of $\mathbb{N}$-ranked alphabets $\langle F, P \rangle$ (of function and predicate symbols) and rank preserving functions.
- The concrete syntax functor $L : \mathbf{Set}^{\mathbb{N}} \times \mathbf{Set}^{\mathbb{N}} \to \mathbf{AlgSig}_{\phi}$ is defined by:
  - $L(\langle F, P \rangle) = \langle \{\tau, \phi\}, O \rangle$ with $O_{\tau} = X \cup F_0$, $O_{\tau^n \tau} = F_n$ for $n > 0$, $O_{\tau^n \phi} = P_n$ for $n \in \mathbb{N}$, $O_{\phi\phi} = \{\neg\} \cup \{\forall x : x \in X\}$, $O_{\phi^2 \phi} = \{\Rightarrow\}$ and $O_w = \emptyset$ otherwise;
  - $L(h : \langle F, P \rangle \to \langle F', P' \rangle)$ is the identity on the sorts τ, ϕ, the variables in X, the quantifiers $\forall x$ and the logical connectives $\neg, \Rightarrow$, and maps each n-ary function symbol $f \in F_n$ to $h(f) \in F'_n$ and each n-ary predicate symbol $p \in P_n$ to $h(p) \in P'_n$.
- Each $M_{\langle F,P \rangle}$ is the class of all structured algebras $\langle \mathcal{A}, \mathbf{c} \rangle$ obtained from $\langle F, P \rangle$-interpretations $I = \langle D, \_{}_I \rangle$ with $D \neq \emptyset$ a set, $f_I : D^n \to D$ for $f \in F_n$, and $p_I \subseteq D^n$ for $p \in P_n$, as follows:
  - $|\mathcal{A}|_{\tau} = D^{\mathbf{Asg}(X,D)}$ and $|\mathcal{A}|_{\phi} = \wp(\mathbf{Asg}(X, D))$, where $\mathbf{Asg}(X, D) = D^X$ is the set of all assignments ρ to variables in D;
  - $x_{\mathcal{A}}(\rho) = \rho(x)$ for $x \in X$, $f_{\mathcal{A}}(e_1, \ldots, e_n)(\rho) = f_I(e_1(\rho), \ldots, e_n(\rho))$ for $f \in F_n$, $p_{\mathcal{A}}(e_1, \ldots, e_n) = \{\rho \in \mathbf{Asg}(X, D) : \langle e_1(\rho), \ldots, e_n(\rho) \rangle \in p_I\}$ for $p \in P_n$, $\neg_{\mathcal{A}}(r) = \mathbf{Asg}(X, D) \backslash r$, $\forall x_{\mathcal{A}}(r) = \{\rho \in \mathbf{Asg}(X, D) : \rho[x/d] \in r$ for every $d \in D\}$, and $\Rightarrow_{\mathcal{A}}(r_1, r_2) = (\mathbf{Asg}(X, D) \setminus r_1) \cup r_2$;
  - $\mathbf{c} : \wp(|\mathcal{A}|_{\phi}) \to \wp(|\mathcal{A}|_{\phi})$ is the cut closure operation induced by set inclusion, that is, for every $R \subseteq \wp(|\mathcal{A}|_{\phi})$, $R^{\mathbf{c}} = \{r \subseteq |\mathcal{A}|_{\phi} : (\bigcap R) \subseteq r\}$ is the principal ideal determined by $(\bigcap R)$ on the complete lattice $\langle \wp(|\mathcal{A}|_{\phi}), \supseteq \rangle$.

$Sen(\langle F, P \rangle)$ is the set of all first-order formulae build using the predicate symbols in P from the terms build using the function symbols in F. Easily, in such interpretation structures, the denotation of a formula corresponds to the set of all assignments for which it holds, and $\vDash_{\langle F,P \rangle}$ reflects reasoning with a fixed assignment. Moreover, $\emptyset^{\mathbf{c}} = \{\mathbf{Asg}(X, D)\}$ and thus a formula holds in a model if it is true for all possible assignments, as usual. Of course, expectedly, $\vDash_{\langle F,P \rangle}$ does not comply with the rule of generalization. Namely, in general $\{\varphi\} \not\vDash_{\langle F,P \rangle} (\forall x\, \varphi)$. Instead, however, as a rule for generating theorems out of theorems, generalization is such that $\langle \langle \mathcal{A}, \mathbf{c} \rangle, \emptyset^{\mathbf{c}} \rangle \Vdash_{\langle F,P \rangle} (\forall x\, \varphi)$ whenever $\langle \langle \mathcal{A}, \mathbf{c} \rangle, \emptyset^{\mathbf{c}} \rangle \Vdash_{\langle F,P \rangle} \varphi$.

Example 2. Propositional normal modal logic.

- Abstract signatures are given by $\mathbf{Set}$.
- The concrete syntax functor $L : \mathbf{Set} \to \mathbf{AlgSig}_{\phi}$ is defined by:
  - $L(\mathrm{PS}) = \langle \{\phi\}, O \rangle$ with $O_{\phi} = \mathrm{PS}$, $O_{\phi\phi} = \{\Box, \neg\}$, $O_{\phi^2 \phi} = \{\Rightarrow\}$ and $O_w = \emptyset$ otherwise;
  - $L(h : \mathrm{PS} \to \mathrm{PS}')$ is the identity on the sort ϕ, the modality $\Box$ and the connectives $\neg, \Rightarrow$, and maps each $p \in \mathrm{PS}$ to $h(p) \in \mathrm{PS}'$;
- Each M_{PS} is the class of all structured algebras $\langle \mathcal{A}, \mathbf{c} \rangle$ obtained from a Kripke frame $\langle W, R \rangle$ and a valuation $\vartheta : \mathrm{PS} \to \wp(W)$, where $W \neq \emptyset$ is a set and $R \subseteq W^2$, as follows:
  - $|\mathcal{A}|_{\phi} = \wp(W)$;
  - $p_{\mathcal{A}} = \vartheta(p)$ for $p \in \mathrm{PS}$, $\Box_{\mathcal{A}}(U) = \{w \in W : \{u \in W : w R u\} \subseteq U\}$, $\neg_{\mathcal{A}}(U) = W \setminus U$ and $\Rightarrow_{\mathcal{A}}(U_1, U_2) = (W \setminus U_1) \cup U_2$;
  - $\mathbf{c} : \wp(|\mathcal{A}|_{\phi}) \to \wp(|\mathcal{A}|_{\phi})$ is, as before, the cut closure operation induced by set inclusion.

Easily, in such a structure, $\emptyset^{\mathbf{c}} = \{W\}$ and a modal formula holds in a model if it is true for all worlds in that model. In fact, the denotation of a formula corresponds precisely to the set of all worlds where it holds. Therefore, $\vDash_{\mathrm{PS}}$ reflects reasoning over a given fixed world. Again, it does not comply with the rule of necessitation. In general, $\{\varphi\} \not\vDash_{\mathrm{PS}} (\Box\varphi)$. Instead, as a rule for theorem generation, necessitation is such that $\langle \langle \mathcal{A}, \mathbf{c} \rangle, \emptyset^{\mathbf{c}} \rangle \Vdash_{\mathrm{PS}} (\Box\varphi)$ whenever $\langle \langle \mathcal{A}, \mathbf{c} \rangle, \emptyset^{\mathbf{c}} \rangle \Vdash_{\mathrm{PS}} \varphi$.

3 The category of c-parchments

As for *morphisms of* **c**-*parchments* we propose a version specially tailored for *fibring*. It is essentially dual to the notion of morphism of model-theoretic parchments used in [21], although in our case the relation between algebras is a little stricter (in exactly the same sense of the previous coherence condition for **c**-parchments).

Definition 2. A *morphism of* **c**-*parchments* from $P_1 = \langle \mathbf{Sig}_1, L_1, \mathcal{M}_1 \rangle$ to $P_2 = \langle \mathbf{Sig}_2, L_2, \mathcal{M}_2 \rangle$ is a pair $\langle \Phi, \eta \rangle : P_1 \to P_2$ where:

- $\Phi : \mathbf{Sig}_2 \to \mathbf{Sig}_1$ is a functor;
- $\eta : L_1 \circ \Phi \to L_2$ is a natural transformation,

such that, for every $\Sigma \in |\mathbf{Sig}_2|$, the following condition holds:

- $\langle \mathcal{A}|_{\eta_\Sigma}, \mathbf{c} \rangle \in M_{1,\Phi(\Sigma)}$ for every $\langle \mathcal{A}, \mathbf{c} \rangle \in M_{2,\Sigma}$.

Clearly, **c**-parchments and their morphisms constitute a category **CPar**. Moreover, the construction of an institution out of a **c**-parchment easily extends to a functor from **CPar** to the dual of the category **Ins** of institutions and institution morphisms [15]. In fact, a **c**-parchment is no more than a functor from a signature category to the following category **CPRoom** of *rooms for* **c**-*parchments*, or just **c**-*rooms*.

Definition 3. A **c**-*room* is a pair $R = \langle \langle S, O \rangle, M \rangle$ with $\langle S, O \rangle \in |\mathbf{AlgSig}_\phi|$ and $M \subseteq \mathrm{cAlg}(\langle S, O \rangle)$. A *morphism of* **c**-*rooms* from $R_1 = \langle \langle S_1, O_1 \rangle, M_1 \rangle$ to $R_2 = \langle \langle S_2, O_2 \rangle, M_2 \rangle$ is an $\mathbf{AlgSig}_\phi$-morphism $h : \langle S_1, O_1 \rangle \to \langle S_2, O_2 \rangle$ such that $\langle \mathcal{A}|_h, \mathbf{c} \rangle \in M_1$ for every $\langle \mathcal{A}, \mathbf{c} \rangle \in M_2$.

A **c**-parchment $P = \langle \mathbf{Sig}, L, \mathcal{M} \rangle$ essentially corresponds to the functor $P : \mathbf{Sig} \to \mathbf{CPRoom}$ such that $P(\Sigma) = \langle L(\Sigma), M_\Sigma \rangle$ and $P(\sigma) = L(\sigma)$. It is straightforward to show that this correspondence is not only a bijection, but that it indeed extends to an isomorphism of categories. Namely, a morphism $\langle \Phi, \eta \rangle : P_1 \to P_2$ of **c**-parchments corresponds precisely to a functor $\Phi : \mathbf{Sig}_2 \to \mathbf{Sig}_1$ and a natural transformation $\eta : P_1 \circ \Phi \to P_2$. Thus, in exactly the same way that the category **Ins** of institutions and institution morphisms corresponds to a Grothendieck construction on categories of functors to the category **Room(2)** of [14], and the category **MPar** of model-theoretic parchments and morphisms corresponds to a Grothendieck construction on categories of functors to the category **MPRoom** of [21], the dual of our category **CPar** corresponds to a Grothendieck construction on categories of functors to the dual of **CPRoom**. As in the other cases, the cocompleteness of **CPar** follows immediately from the cocompleteness of **CPRoom**.

Proposition 1. CPRoom *is cocomplete.*

Proof. We just show what coproducts and coequalizers look like in **CPRoom**. For the purpose, let I be a set and $\{R_i = \langle \langle S_i, O_i \rangle, M_i \rangle\}_{i \in I}$ a family of **c**-rooms. Of course, we capitalize on the well known fact that $\mathbf{AlgSig}_\phi$ is cocomplete.
(Coproducts) Let $\{h_j : \langle S_j, O_j \rangle \to \coprod_{i \in I} \langle S_i, O_i \rangle\}_{j \in I}$ be a coproduct in $\mathbf{AlgSig}_\phi$. A coproduct $\{h_j : R_j \to \langle \coprod_{i \in I} \langle S_i, O_i \rangle, M \rangle\}_{j \in I}$ in **CPRoom** can be obtained by taking $M \subseteq \mathrm{cAlg}(\coprod_{i \in I} \langle S_i, O_i \rangle)$ to be the class of all pairs $\langle \mathcal{A}, \mathbf{c} \rangle$ such that $\langle \mathcal{A}|_{h_j}, \mathbf{c} \rangle \in M_j$ for every $j \in I$.
(Coequalizers) Let $I = \{1, 2\}$, $h', h'' : R_1 \to R_2$ be **CPRoom**-morphisms and $h : \langle S_2, O_2 \rangle \to \langle S, O \rangle$ a coequalizer of $h', h'' : \langle S_1, O_1 \rangle \to \langle S_2, O_2 \rangle$ in $\mathbf{AlgSig}_\phi$. A coequalizer $h : R_2 \to \langle \langle S, O \rangle, M \rangle$ of $h', h'' : R_1 \to R_2$ in **CPRoom** can be obtained by taking $M \subseteq \mathrm{cAlg}(\langle S, O \rangle)$ to be the class of all pairs $\langle \mathcal{A}, \mathbf{c} \rangle$ such that $\langle \mathcal{A}|_h, \mathbf{c} \rangle \in M_2$. □

The following result is a simple corollary of the corresponding Grothendieck construction, similar to the one in [26].

Proposition 2. **CPar** *is cocomplete.*

Proof. Again, we just show what coproducts and coequalizers look like in **CPar**. For the purpose, let I be a set and $\{P_i : \mathbf{Sig}_i \to \mathbf{CPRoom}\}_{i\in I}$ a family of **c**-parchments.
(Coproducts) A coproduct $\{\langle \Pi_i, \iota_i\rangle : P_i \to P\}_{i\in I}$ in **CPar** can be obtained by taking the functor $P = (\coprod_I \circ (\prod_{i\in I} P_i)) : \prod_{i\in I} \mathbf{Sig}_i \to \mathbf{CPRoom}$, where: each $\Pi_j : \prod_{i\in I} \mathbf{Sig}_i \to \mathbf{Sig}_j$ is the corresponding projection functor; $\coprod_I$ is the coproduct functor left adjoint to the diagonal functor $\Delta_I : \mathbf{CPRoom} \to \mathbf{CPRoom}^I$; $\prod_{i\in I} P_i : \prod_{i\in I} \mathbf{Sig}_i \to \mathbf{CPRoom}^I$ is the unique functor such that $(\prod_{i\in I} P_i)_j = P_j \circ \Pi_j$ for every $j \in I$, resulting from the universality of the product $\{\__i : \mathbf{CPRoom}^I \to \mathbf{CPRoom}\}_{i\in I}$ in **Cat**; and each $\iota_{j,\Sigma} : P_j(\Sigma_j) \to \coprod_{i\in I} P_i(\Sigma_i)$ is the corresponding injection on the coproduct, for every $\Sigma = \langle \Sigma_i\rangle_{i\in I} \in |\prod_{i\in I} \mathbf{Sig}_i|$.
(Coequalizers) Let $I = \{1, 2\}$. A coequalizer $\langle \Phi, \eta\rangle : P_2 \to P$ of a pair of morphisms $\langle \Phi', \eta'\rangle, \langle \Phi'', \eta''\rangle : P_1 \to P_2$ in **CPar** can be obtained by taking the functor $P = \circ\langle \overline{\eta}'_\Phi, \overline{\eta}''_\Phi\rangle : \mathbf{Sig} \to \mathbf{CPRoom}$, where: $\Phi : \mathbf{Sig} \to \mathbf{Sig}_2$ is an equalizer of the functors $\Phi', \Phi'' : \mathbf{Sig}_2 \to \mathbf{Sig}_1$ in **Cat**; is the coequalizer functor left adjoint to the diagonal functor $\Delta_{\downarrow\downarrow} : \mathbf{CPRoom} \to \mathbf{CPRoom}^{\downarrow\downarrow}$; $\langle \overline{\eta}'_\Phi, \overline{\eta}''_\Phi\rangle : \mathbf{Sig} \to \mathbf{CPRoom}^{\downarrow\downarrow}$ is the unique functor such that $\langle \overline{\eta}'_\Phi, \overline{\eta}''_\Phi\rangle_1 = \overline{\eta}' \circ \Phi$ and $\langle \overline{\eta}'_\Phi, \overline{\eta}''_\Phi\rangle_2 = \overline{\eta}'' \circ \Phi$, with $\overline{\eta}', \overline{\eta}'' : \mathbf{Sig}_2 \to \mathbf{CPRoom}^{\downarrow}$ the functors corresponding to the natural transformations η' and η'', resulting from the universality of the pullback $\__1, \__2 : \mathbf{CPRoom}^{\downarrow\downarrow} \to \mathbf{CPRoom}^{\downarrow}$ of $\langle \mathrm{dom}, \mathrm{cod}\rangle, \langle \mathrm{dom}, \mathrm{cod}\rangle : \mathbf{CPRoom}^{\downarrow} \to A \times A$ in **Cat**; and $\eta_\Sigma : P_2(\Phi(\Sigma)) \to (\langle \eta'_{\Phi(\Sigma)}, \eta''_{\Phi(\Sigma)}\rangle)$ is the corresponding coequalizer, for $\Sigma \in |\mathbf{Sig}|$. □

4 Fibred semantics

As we have said before, morphisms of **c**-parchments (as well as morphisms of **c**-rooms) have been set up having in mind the characterization of *fibring* via colimits. We now extend our previous characterizations of the construction [22, 28, 7] to the level of **c**-rooms and **c**-parchments, and concentrate on the particular cases of colimit defining *fibring constrained by sharing of symbols*. In the remainder of the paper, in fact, we restrict our attention to just **c**-rooms. As we have seen, colimits can be smoothly lifted to the level of **c**-parchments.

Let us consider fixed two arbitrary **c**-rooms $R_1 = \langle\langle S_1, O_1\rangle, M_1\rangle$ and $R_2 = \langle\langle S_2, O_2\rangle, M_2\rangle$. For simplicity, we shall assume that when fibring R_1 and R_2 the required sharing of syntax is specified by means of the largest common subsignature of $\langle S_1, O_1\rangle$ and $\langle S_2, O_2\rangle$. That is, by default, we shall assume to be sharing the signature $\langle S_0, O_0\rangle$ with $S_0 = S_1 \cap S_2$ (it always includes at least the sort ϕ) and $O_{0,u} = O_{1,u} \cap O_{2,u}$ for $u \in S_0^+$ is shared according to the corresponding signature inclusion morphisms $h_1 : \langle S_0, O_0\rangle \to \langle S_1, O_1\rangle$ and $h_2 : \langle S_0, O_0\rangle \to \langle S_2, O_2\rangle$. We denote by R_0 the canonical **c**-room $\langle\langle S_0, O_0\rangle, M_0\rangle$ where $M_0 = \mathrm{cAlg}(\langle S_0, O_0\rangle)$. In the simplest possible case when $S_0 = \{\phi\}$ and $O_0 = \emptyset$ we say that the *fibring* is *free* or *unconstrained*.

Definition 4. The *fibring* of R_1 and R_2 (constrained by sharing $\langle S_0, O_0\rangle$) is the **c**-room $R_1 \circledast R_2 = \langle\langle S, O\rangle, M\rangle$ such that:

- $S = S_1 \cup S_2$, with inclusions $f_i : S_i \to S$;
- $O_u = O_{1,u} \cup O_{2,u}$ if $u \in S_0^+$, $O_u = O_{i,u}$ if $u \in S_i^+ \setminus S_0^+$ and $O_u = \emptyset$ otherwise, with inclusions $g_i : O_i \to O$;

– $M \subseteq \text{cAlg}(\langle S, O \rangle)$ is the class of all pairs $\langle \mathcal{A}, \mathbf{c} \rangle$ such that $\langle \mathcal{A}|_{\langle f_1, g_1 \rangle}, \mathbf{c} \rangle \in M_1$ and $\langle \mathcal{A}|_{\langle f_2, g_2 \rangle}, \mathbf{c} \rangle \in M_2$.

Clearly, M consists precisely of all those $\langle \mathcal{A}, \mathbf{c} \rangle$ that can be obtained by joining together any two $\langle \mathcal{A}_1, \mathbf{c}_1 \rangle \in M_1$ and $\langle \mathcal{A}_2, \mathbf{c}_2 \rangle \in M_2$ such that $|\mathcal{A}_1|_s = |\mathcal{A}_2|_s = |\mathcal{A}|_s$ for every $s \in S_0$, $o_{\mathcal{A}_1} = o_{\mathcal{A}_2} = o_{\mathcal{A}}$ for every $o \in O_{0,u}$ with $u \in S_0^+$, and $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}$.

Proposition 3. *The fibring of* $\mathbf{c}$*-rooms* R_1 *and* R_2 *(constrained by sharing* $\langle S_0, O_0 \rangle$*) is a pushout of* $\{h_i : R_0 \to R_i\}_{i \in \{1,2\}}$ *in* **CPRoom**.

As a simple corollary, when the fibring is free, $R_1 \circledast R_2$ is a coproduct of R_1 and R_2 in **CPRoom**.

Let us now analyze in some detail a new application of fibring, made possible in this setting. The example concerns partial equational logic and the way it can be obtained by fibring equational logic with a logic of partiality. A similar idea had already been proposed in [19–21], but for a different notion of parchment and without any concerns for preservation results. For this reason, we do think that the approach followed here is much more ellegant, direct and modular. Moreover, as we show, a nice proof-calculus for the fibred partial equational logic is obtained by putting together the proof-calculi for equational logic together with a calculus suited for the logic of partiality adopted.

Example 3. Partial equational logic.
We start by representing equational logic as a $\mathbf{c}$-room, for a given $\mathbb{N}$-ranked alphabet F of function symbols. In order to keep the focus on partiality, we shall just consider an unsorted version of equational logic. Let X be a fixed denumerable set of variables and Eq an equational specification (set of equations) over F and X.

- The concrete syntax signature $\langle S, O \rangle$ is such that:
  - $S = \{\tau, \phi\}$;
  - $O_\tau = X \cup F_0$, $O_{\tau^n \tau} = F_n$ for $n > 0$, $O_{\tau^2 \phi} = \{=\}$ and $O_w = \emptyset$ otherwise.
- M is the class of all structured algebras $\langle \mathcal{A}, \mathbf{c} \rangle$ obtained from an F-algebra $\mathcal{F}$ that is a model[1] of Eq as follows:
  - $|\mathcal{A}|_\tau = |\mathcal{F}|^{\mathbf{Asg}(X, |\mathcal{F}|)}$ and $|\mathcal{A}|_\phi = \wp(\mathbf{Asg}(X, |\mathcal{F}|))$, where $\mathbf{Asg}(X, |\mathcal{F}|) = |\mathcal{F}|^X$ is the set of all assignments ρ to variables in $|\mathcal{F}|$;
  - $x_{\mathcal{A}}(\rho) = \rho(x)$ for $x \in X$, $f_{\mathcal{A}}(e_1, \ldots, e_n)(\rho) = f_{\mathcal{F}}(e_1(\rho), \ldots, e_n(\rho))$ for $f \in F_n$, and $=_{\mathcal{A}} (e_1, e_2) = \{\rho \in \mathbf{Asg}(X, |\mathcal{F}|) : e_1(\rho) = e_2(\rho)\}$;
  - $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is defined as in Example 1 for first-order logic.

Clearly, as formulae, we have precisely the equations between terms build using the function symbols in F and the variables in X. Moreover, in all such interpretation structures, the denotation of an equation is precisely the set of all assignments where it holds. Moreover, $\emptyset^{\mathbf{c}} = \{\mathbf{Asg}(X, |\mathcal{F}|)\}$ and an equation holds in a model if the values of the two terms coincide for all possible assignments. Again, $\vDash$ reflects reasoning with a fixed assignment and does not comply with the rule of substitution. In general, $\{t_1 = t_2\} \nvDash t_1\sigma = t_2\sigma$, where σ is some substitution of variables by terms. Instead, we have that if $\langle \langle \mathcal{A}, \mathbf{c} \rangle, \emptyset^{\mathbf{c}} \rangle \Vdash t_1 = t_2$ then $\langle \langle \mathcal{A}, \mathbf{c} \rangle, \emptyset^{\mathbf{c}} \rangle \Vdash t_1\sigma = t_2\sigma$.

We aim at obtaining a $\mathbf{c}$-room for unsorted partial equational logic by fibring the $\mathbf{c}$-room above with a suitable room $\langle \langle S, O \rangle, M \rangle$ for partiality. For the purpose, let $G = \{G_n\}_{n \in \mathbb{N}}$ be a ranked subalphabet of F. Operations in G shall be considered total, whereas all other operations in F can be partial. Here is a very simple possibility:

[1] This means that $[\![t_1]\!]_\rho^{\mathcal{F}} = [\![t_2]\!]_\rho^{\mathcal{F}}$ for every assignment $\alpha : X \to |\mathcal{F}|$ and every equation $t_1 = t_2 \in \text{Eq}$.

- The concrete syntax signature $\langle S, O\rangle$ is such that:
  - $S = \{\tau, \phi\}$;
  - $O_\tau = X \cup F_0$, $O_{\tau^n\tau} = F_n$ for $n > 0$, $O_{\tau\phi} = \{D\}$, $O_{\tau^2\phi} = \{=\}$ and $O_w = \emptyset$ otherwise;
- M is the class of all structured algebras $\langle \mathcal{A}, \mathbf{c}\rangle$ obtained from an F-algebra $\mathcal{F}$ with a distinguished element $* \in |\mathcal{F}|$ and satisfying:
  - $f_\mathcal{F}(a_1, \ldots, a_n) = *$ whenever some $a_i = *$, for every $f \in F_n$;
  - $g_\mathcal{F}(a_1, \ldots, a_n) \neq *$ whenever all $a_i \neq *$, for every $g \in G_n$,

  plus a given binary relation $R \subseteq |\mathcal{F}| \times |\mathcal{F}|$ satisfying:
  - $a\,R\,*$ or $*\,R\,a$ imply $a = *$,

  as follows:
  - $|\mathcal{A}|_\tau = |\mathcal{F}|^{\mathbf{Asg}(X,|\mathcal{F}|)}$ and $|\mathcal{A}|_\phi = \wp(\mathbf{Asg}(X, |\mathcal{F}|))$;
  - $x_\mathcal{A}(\rho) = \rho(x)$ for $x \in X$, $f_\mathcal{A}(e_1, \ldots, e_n)(\rho) = f_\mathcal{F}(e_1(\rho), \ldots, e_n(\rho))$ for $f \in F_n$, $D_\mathcal{A}(e) = \{\rho \in \mathbf{Asg}(X, |\mathcal{F}|) : e(\rho) \neq *\}$, and $=_\mathcal{A} (e_1, e_2) = \{\rho \in \mathbf{Asg}(X, |\mathcal{F}|) : e_1(\rho)\, R\, e_2(\rho)\}$;
  - $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is again the cut closure operation induced by set inclusion.

In this case we are only concerned with partiality. Hence, we impose the least possible constraints to the interpretation R of equality.

The desired $\mathbf{c}$-room for unsorted partial equational logic, obtained by constrained fibring and sharing both sorts τ and ϕ, the variables X, the equality symbol $=$, and the operations in F is as follows:

- The concrete syntax signature $\langle S, O\rangle$ is such that:
  - $S = \{\tau, \phi\}$;
  - $O_\tau = X \cup F_0$, $O_{\tau^n\tau} = F_n$ for $n > 0$, $O_{\tau\phi} = \{D\}$, $O_{\tau^2\phi} = \{=\}$ and $O_w = \emptyset$ otherwise;
- M is the class of all structured algebras $\langle \mathcal{A}, \mathbf{c}\rangle$ obtained from an F-algebra $\mathcal{F}$ that is a model of Eq with a distinguished element $* \in |\mathcal{F}|$ and satisfying:
  - $f_\mathcal{F}(a_1, \ldots, a_n) = *$ whenever some $a_i = *$, for every $f \in F_n$;
  - $g_\mathcal{F}(a_1, \ldots, a_n) \neq *$ whenever all $a_i \neq *$, for every $g \in G_n$,

  as follows:
  - $|\mathcal{A}|_\tau = |\mathcal{F}|^{\mathbf{Asg}(X,|\mathcal{F}|)}$ and $|\mathcal{A}|_\phi = \wp(\mathbf{Asg}(X, |\mathcal{F}|))$;
  - $x_\mathcal{A}(\rho) = \rho(x)$ for $x \in X$, $f_\mathcal{A}(e_1, \ldots, e_n)(\rho) = f_\mathcal{F}(e_1(\rho), \ldots, e_n(\rho))$ for $f \in F_n$, $D_\mathcal{A}(e) = \{\rho \in \mathbf{Asg}(X, |\mathcal{F}|) : e(\rho) \neq *\}$, and $=_\mathcal{A} (e_1, e_2) = \{\rho \in \mathbf{Asg}(X, |\mathcal{F}|) : e_1(\rho) = e_2(\rho)\}$;
  - $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is again the cut closure operation induced by set inclusion.

Note that, in the resulting $\mathbf{c}$-room, equality is necessarily interpreted as *strong equality*. *Existential equality* can also be introduced, by adding the symbol $\stackrel{e}{=}$ to the partial logic room, where it should be interpreted as follows:

$$\stackrel{e}{=}_\mathcal{A} (e_1, e_2) = \{\rho \in \mathbf{Asg}(X, |\mathcal{F}|) : e_1(\rho) \neq * \text{ and } e_1(\rho)\, R\, e_2(\rho)\}.$$

One may ask what can be achieved at the proof-theoretic level. Can we expect to obtain a calculus for partial equational logic, by putting together a calculus for equational logic and a calculus for the logic of partiality? The answer is yes. Just consider the following deduction rules for partiality:

$$\frac{D(f(t_1, \ldots, t_n))}{D(t_i)}\; f \in F_n, i = 1, \ldots, n \qquad \frac{D(t_1)\ \ldots\ D(t_n)}{D(g(t_1, \ldots, t_n))}\; g \in G_n$$

$$\frac{D(t_1) \quad t_1 = t_2}{D(t_2)} \qquad \frac{D(t_2) \quad t_1 = t_2}{D(t_1)}$$

and for the partial logic with existential equality ($\overset{e}{=}$) we may consider the additional rules:

$$\frac{t_1 \overset{e}{=} t_2}{D(t_1)} \qquad \frac{t_1 \overset{e}{=} t_2}{D(t_2)} \qquad \frac{t_1 \overset{e}{=} t_2}{t_1 = t_2} \qquad \frac{D(t_1) \quad t_1 = t_2}{t_1 \overset{e}{=} t_2}.$$

It is easy to check that not only this calculus is sound and complete for the simple logic of partiality being adopted, but furthermore the calculus for partial equational logic obtained by adding these rules to the usual calculus for equational logic with specific axioms for each equation in Eq is also sound and complete, even with existential equality. The proof of soundness is straightforward. In what concerns completeness, we can easily adapt the usual techniques for partiality (see, for instance, [27, 2]). We should also note that similar results could have been obtained by taking from the beginning a conditional equational specification instead of Eq.

Note that other approaches to partiality, namely using three-valued or versions of equality, could also be considered by just performing a closure operation on the interpretation structures of equational logic, similarly to what is done in [28, 7]. This just means that, for instance, all three-valued interpretation structures whose two-valued fragment is now an interpretation structure should be added to the **c**-room. This operation clearly has no effect on the entailment consequence [28, 7] and can be used to obtain the desired fibred logic by suitably defining the desired logic of partiality, necessarily different from the one we have used above.

5 Fibred deduction

The most promising advantage of fibring concerns preservation results [12, 13]. In the next section, we present a collection of soundness and completeness preservation results for fibring, within the context of a suitable notion of Hilbert-like proof-calculus that explicitly distinguishes theorem generating rules. We dedicate this section to presenting the details of such calculi and their fibring. However, we shall restrict ourselves to logics with a propositional base, namely, whose concrete syntax is given by a one-sorted signature whose only sort is the sort of formulae. Thus, from now on, we shall refer to ϕ-sorted signatures as those having ϕ as the unique sort. The forthcoming proof-theoretic definitions could of course be extended in order to encompass also logics without a propositional base, such as first-order logic or equational logic as in the preceding examples. However, we refrain from doing so, for simplicity, since the preservation results that we shall present are only applicable to the propositional base case.

The way the proof-calculi of equational logic and of the logic of partiality were put together in Example 3 to obtain the proof-calculus of partial equational logic, can in fact be systematized as a corresponding proof-theoretic form of fibring. The idea, originally used in [22], is to use schema variables in rules and schema variable substitutions in rule application. For the purpose, we consider fixed a denumerable set $\Xi = \{\xi_n : n \in \mathbb{N}\}$ of *schema formula variables*, to be used in writing *schema rules*. Letting $\langle\{\phi\}, O\rangle$ be a signature, we shall use $\mathcal{W}_{\langle\{\phi\},O\rangle}(\Xi)$ to denote the free $\langle\{\phi\}, O\rangle$-algebra on the set Ξ of generators (the *schema word* algebra), and $[\![\_]\!]^{\mathcal{A}}_{\alpha}$ (for schema word *interpretation under* α) to denote the unique $\mathbf{Alg}(\langle\{\phi\}, O\rangle)$-homomorphism from $\mathcal{W}_{\langle\{\phi\},O\rangle}(\Xi)$ to any given $\langle\{\phi\}, O\rangle$-algebra $\mathcal{A}$ that extends a given *assignment* $\alpha : \Xi \to |\mathcal{A}|_\phi$. As before, given $\mathbf{AlgSig}_\phi$-morphism $h : \langle\{\phi\}, O_1\rangle \to \langle\{\phi\}, O_2\rangle$, we shall preferably write $\widehat{h}$ (for schema word *translation*) instead of $[\![\_]\!]^{\mathcal{W}_{\langle\{\phi\},O_2\rangle|_\sigma}}_{id_\Xi}$ to denote the unique $\mathbf{Alg}(\langle\{\phi\}, O_1\rangle)$-homomorphism from $\mathcal{W}_{\langle\{\phi\},O_1\rangle}(\Xi)$ to $\mathcal{W}_{\langle\{\phi\},O_2\rangle}|_\sigma(\Xi)$ that identifies schema variables.

We can recognize $|\mathcal{W}_{\langle\{\phi\},O\rangle}(\Xi)|_\phi$ (the carrier of sort ϕ in the schema word algebra) as the set of schema formulae over $\langle\{\phi\},O\rangle$. We shall use γ, δ with or without primes and subscripts to denote schema formulae, and Γ, Δ to denote sets of schema formulae.

Definition 5. An *inference rule* over a signature $\langle\{\phi\},O\rangle$ is a pair $r = \langle\Gamma, \delta\rangle$ where $\Gamma \cup \{\delta\} \subseteq |\mathcal{W}_{\langle\{\phi\},O\rangle}(\Xi)|_\phi$ is a finite set.

Given such an inference rule r, we shall often use $\mathrm{Prem}(r) = \Gamma$ and $\mathrm{Conc}(r) = \delta$ to denote its premises and conclusion. As usual, we shall sometimes represent r simply by $\frac{\mathrm{Prem}(r)}{\mathrm{Conc}(r)}$, $\frac{\Gamma}{\delta}$, or even by $\frac{\gamma_1 \ldots \gamma_n}{\delta}$ if $\Gamma = \{\gamma_1 \ldots \gamma_n\}$. If the set of premises is empty the rule is sometimes also identified with its conclusion and referred to as an *axiom*. The following definition is similar to those in [22, 28, 7].

Definition 6. A *proof-calculus* is a triple $C = \langle\langle\{\phi\},O\rangle, dR, gR\rangle$ where $\langle\{\phi\},O\rangle$ is a signature and $dR \cup gR$ is a set of inference rules over $\langle\{\phi\},O\rangle$ such that $\mathrm{Prem}(r) \neq \emptyset$ for every $r \in gR$. A *morphism* between proof-calculi $C_1 = \langle\langle\{\phi\},O_1\rangle, dR_1, gR_1\rangle$ and $C_2 = \langle\langle\{\phi\},O_2\rangle, dR_2, gR_2\rangle$ is an $\mathbf{AlgSig}_\phi$-morphism $h : \langle\{\phi\},O_1\rangle \to \langle\{\phi\},O_2\rangle$ such that $\widehat{h}(r) \in dR_2$ for every $r \in dR_1$, and $\widehat{h}(r) \in gR_2$ for every $r \in gR_1$.

In the definition above, dR denotes the set of rules allowed for building deductions in the calculus, while gR stands for the set of rules that are to be used just for building theorems out of theorems. Thus, the requirement that gR rules must have premises is quite natural. In the context of a signature morphism $h : \langle\{\phi\},O_1\rangle \to \langle\{\phi\},O_2\rangle$, given a rule $r = \frac{\gamma_1 \ldots \gamma_n}{\delta}$ over $\langle\{\phi\},O_1\rangle$, we are of course writing $\widehat{h}(r)$ to denote the translated rule $\frac{\widehat{h}(\gamma_1) \ldots \widehat{h}(\gamma_n)}{\widehat{h}(\delta)}$ over $\langle\{\phi\},O_2\rangle$. It is straightforward that proof-calculi consitute a category $\mathbf{PCalc}$. Moreover, exactly as the category of $\mathbf{c}$-parchments can be build over the category of $\mathbf{c}$-rooms, we can use a similar Grothendieck construction to set up our proof-theoretic notion of logic as a category $\mathbf{Prf}$ of proof-systems built over $\mathbf{PCalc}$.

A proof-calculus $C = \langle\langle\{\phi\},O\rangle, dR, gR\rangle$ presents, as expected, a *deducibility* relation. However, this relation is build on top of a notion of *theoremhood*. For convenience, we define both notions over schema formulae. We say that a schema formula δ is a *theorem schema generated* from the set of schema formulae Γ if there exists a finite sequence $\gamma_1 \ldots \gamma_n$ of schema formulae such that:

- γ_n is δ;
- for each $i = 1, \ldots, n$ either $\gamma_i \in \Gamma$, or there exist a rule $r \in dR \cup gR$ and a schema variable substitution $\sigma : \Xi \to |\mathcal{W}_{\langle\{\phi\},O\rangle}(\Xi)|_\phi$ such that $\mathrm{Prem}(r)\sigma \subseteq \{\gamma_j : j < i\}$ and $\mathrm{Conc}(r)\sigma$ is γ_i.

In the sequel, we shall refer to theorem schemata generated from $\emptyset$ simply by *theorem schemata*. In deductions now, only instances of rules in dR are allowed, together with theorem schemata. We say that δ is *deducible* from Γ, $\Gamma \vdash \delta$ if there exists a finite sequence $\gamma_1 \ldots \gamma_n$ such that:

- γ_n is δ;
- for each $i = 1, \ldots, n$ either $\gamma_i \in \Gamma$, or γ_i is a theorem schema, or there exist a rule $r \in dR$ and a schema variable substitution σ such that $\mathrm{Prem}(r)\sigma \subseteq \{\gamma_j : j < i\}$ and $\mathrm{Conc}(r)\sigma$ is γ_i.

Easily, $\emptyset \vdash \gamma$ if and only if γ is a theorem schema. Of course, since formulae are special cases of schema formulae, the notions of theorem and deduction also apply. The convenience for having introduced schema formulae is that both theoremhood and deducibility are *structural* [22] in the following sense: for every schema variable substitution σ, if δ is a theorem schema generated from Γ then $\delta\sigma$ is a theorem schema generated from $\Gamma\sigma$, and if $\Gamma \vdash \delta$ then $\Gamma\sigma \vdash \delta\sigma$.

Often, we shall work with a **c**-room $R = \langle\langle\{\phi\}, O\rangle, M\rangle$ equipped with a proof-calculus $C = \langle\langle\{\phi\}, O\rangle, dR, gR\rangle$. In this context, since R defines a semantic entailment relation $\vDash$ and C defines a deducibility relation $\vdash$, we shall use the following usual definitions:

- C is *sound* for R if $\Phi \vdash \psi$ implies $\Phi \vDash \psi$;
- C is *weak complete* for R if $\emptyset \vDash \varphi$ implies $\emptyset \vdash \varphi$;
- C is *finite complete* for R if Φ finite and $\Phi \vDash \psi$ imply $\Phi \vdash \psi$;
- C is *complete* for R if $\Phi \vDash \psi$ implies $\Phi \vdash \psi$,

for every set of formulae Φ and formula ψ.

The usual way of proving the soundness of a proof-calculus consists in establishing the *soundness* of each of its rules. An inference rule r is said to be *d-sound* for R if for every $\langle\mathcal{A}, \mathbf{c}\rangle \in M$ and every assignment $\alpha : \Xi \to |\mathcal{A}|_\phi$ we have that $[\![\mathrm{Conc}(r)]\!]^{\mathcal{A}}_\alpha \in \{[\![\gamma]\!]^{\mathcal{A}}_\alpha : \gamma \in \mathrm{Prem}(r)\}^{\mathbf{c}}$. On the other hand, r is said to be *g-sound* for R if for every $\langle\mathcal{A}, \mathbf{c}\rangle \in M$ and every assignment $\alpha : \Xi \to |\mathcal{A}|_\phi$, we have that $[\![\mathrm{Conc}(r)]\!]^{\mathcal{A}}_\alpha \in \emptyset^{\mathbf{c}}$ whenever $\{[\![\gamma]\!]^{\mathcal{A}}_\alpha : \gamma \in \mathrm{Prem}(r)\} \subseteq \emptyset^{\mathbf{c}}$. Clearly, if all the rules of C are sound for R, that is the rules in dR are d-sound and the rules in gR are g-sound, then C is sound for R [22, 28, 7].

For the sake of illustration, let us now present the well known calculus for propositional normal modal logic.

Example 4. Propositional normal modal logic.

- Consider any concrete syntax signature defined as in Example 2;
- The set dR of deduction rules is composed of the schema axioms

$$(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$$

$$((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))$$

$$(((\neg\, \xi_1) \Rightarrow (\neg\, \xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))$$

$$((\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2)))$$

  and the schema rule

$$\frac{\xi_1 \quad (\xi_1 \Rightarrow \xi_2)}{\xi_2}$$

- The set gR of theorem generating rules contains just the schema rule

$$\frac{\xi_1}{(\Box\xi_1)}$$

It is well known (see, for instance, [16]) that this proof-calculus is both sound and complete for the **c**-room of propositional modal logic in Example 2.

As in the case of rooms, fibring of proof-calculi is to be characterized as a colimit. And in fact the corresponding category **PCalc** is cocomplete.

Proposition 4. PCalc *is cocomplete.*

Proof. We just show what coproducts and coequalizers look like in **PCalc**. For the purpose, let I be a set and $\{C_i = \langle\langle\{\phi\}, O_i\rangle, dR_i, gR_i\rangle\}_{i\in I}$ a family of proof-calculi. Again, we capitalize on the fact that $\mathbf{AlgSig}_\phi$ is cocomplete.
(Coproducts) Let $\{h_j : \langle\{\phi\}, O_j\rangle \to \coprod_{i\in I}\langle\{\phi\}, O_i\rangle\}_{j\in I}$ be a coproduct in $\mathbf{AlgSig}_\phi$. A coproduct $\{h_j : C_j \to \langle\coprod_{i\in I}\langle\{\phi\}, O_i\rangle, dR, gR\rangle\}_{j\in I}$ in **PCalc** can be obtained by simply taking $dR = \bigcup_{i\in I}\widehat{h}_i(dR_i)$ and $gR = \bigcup_{i\in I}\widehat{h}_i(gR_i)$.
(Coequalizers) Let $I = \{1,2\}$, $h', h'' : C_1 \to C_2$ be **PCalc**-morphisms and $h : \langle\{\phi\}, O_2\rangle \to \langle\{\phi\}, O\rangle$ a coequalizer of $h', h'' : \langle\{\phi\}, O_1\rangle \to \langle\{\phi\}, O_2\rangle$ in $\mathbf{AlgSig}_\phi$. A coequalizer $h : C_2 \to \langle\langle\{\phi\}, O\rangle, dR, gR\rangle$ of $h', h'' : C_1 \to C_2$ in **PCalc** can be obtained by simply taking $dR = \widehat{h}(dR_2)$ and $gR = \widehat{h}(gR_2)$. □

As a consequence of the Grothendieck construction, **Prf** is also cocomplete. The same applies to the following definition of fibring of proof-calculi. Let us consider fixed two arbitrary proof-calculi $C_1 = \langle\langle\{\phi\}, O_1\rangle, dR_1, gR_1\rangle$ and $C_2 = \langle\langle\{\phi\}, O_2\rangle, dR_2, gR_2\rangle$, and recall from the previous section the definition of the sharing common subsignature $\langle\{\phi\}, O_0\rangle$ and corresponding signature inclusion morphisms $h_1 : \langle\{\phi\}, O_0\rangle \to \langle\{\phi\}, O_1\rangle$ and $h_2 : \langle\{\phi\}, O_0\rangle \to \langle\{\phi\}, O_2\rangle$. We shall denote by C_0 the canonical proof-calculus $\langle\langle\{\phi\}, O_0\rangle, dR_0, gR_0\rangle$ where $dR_0 = gR_0 = \emptyset$.

Definition 7. The *fibring* of C_1 and C_2 (constrained by sharing $\langle\{\phi\}, O_0\rangle$) is the proof-calculus $C_1 \circledast C_2 = \langle\langle\{\phi\}, O\rangle, dR, gR\rangle$ such that:

- $O_u = O_{1,u} \cup O_{2,u}$ if $u \in S_0^+$, $O_u = O_{i,u}$ if $u \in S_i^+ \setminus S_0^+$ and $O_u = \emptyset$ otherwise, with inclusions $g_i : O_i \to O$;
- $dR = \widehat{\langle id_{\{\phi\}}, g_1\rangle}(dR_1) \cup \widehat{\langle id_{\{\phi\}}, g_2\rangle}(dR_2)$;
- $gR = \widehat{\langle id_{\{\phi\}}, g_1\rangle}(gR_1) \cup \widehat{\langle id_{\{\phi\}}, g_2\rangle}(gR_2)$.

Thus, the schema inference rules in the fibring are just the translation of the rules of each given proof-calculus translated to the fibred language.

Proposition 5. *The fibring of proof-calculi C_1 and C_2 (constrained by sharing $\langle S_0, O_0\rangle$) is a pushout of $\{h_i : C_0 \to C_i\}_{i\in\{1,2\}}$ in* **PCalc**.

As a simple corollary, if the fibring is free, $C_1 \circledast C_2$ also corresponds to a coproduct of C_1 and C_2 in **PCalc**.

6 Preservation results

We are now in the position of stating and proving our soundness and completeness preservation results. Excluding the very last completeness transference result, that is an instance of the one in [28], all the other results are suitably adapted from [7]. Recall that these are, for the moment, only concerned with propositional based logics. We shall discuss in the conclusions how these results can be extended to the general case. In the remainder of this section we consider fixed two **c**-rooms $R_1 = \langle\langle\{\phi\}, O_1\rangle, M_1\rangle$ and $R_2 = \langle\langle\{\phi\}, O_2\rangle, M_2\rangle$, and two corresponding proof-calculi $C_1 = \langle\langle\{\phi\}, O_1\rangle, dR_1, gR_1\rangle$ and $C_2 = \langle\langle\{\phi\}, O_2\rangle, dR_2, gR_2\rangle$.

To prove preservation of soundness we shall rely on guaranteeing the soundness of all inference rules.

Theorem 1. Soundness Preservation.
Assume that all the rules of C_1 are sound for R_1 and all the rules of C_2 are sound for R_2. Then, the fibred proof-calculus $C_1 \circledast C_2$ is sound for the fibred **c**-*room* $R_1 \circledast R_2$.

Proof. It is immediate, by definition of fibring, that in the conditions of the theorem all the rules of $C_1 \circledast C_2$ are sound for $R_1 \circledast R_2$. □

Completeness preservation is, as expected, much harder to obtain. Our completeness preservation results are based on the following notion of *fullness*.

Definition 8. Let $\mathcal{P}$ be a class of closure operations. A $\mathbf{c}$-room $\langle\langle\{\phi\},O\rangle,M\rangle$ equipped with a proof-calculus C is said to be *full with respect to* $\mathcal{P}$ if M contains every structure $\langle\mathcal{A},\mathbf{c}\rangle\in \mathrm{cAlg}(\langle\{\phi\},O\rangle)$ with $\langle|\mathcal{A}|_\phi,\mathbf{c}\rangle\in\mathcal{P}$ that makes all the rules in C sound.

Although fullness may seem to be a fairly strong requirement, note that the operation of turning a $\mathbf{c}$-room full with respect to some given class $\mathcal{P}$ in the context of a proof-calculus does not change its semantic entailment [28, 7]. Moreover, the essential bit is that fullness is always preserved by fibring.

Proposition 6. *Let the* $\mathbf{c}$*-rooms* R_1 *and* R_2 *equipped with the proof-calculi* C_1 *and* C_2 *be full with respect to a class* $\mathcal{P}$ *of closure operations. Then, the fibred* $\mathbf{c}$*-room* $R_1\circledast R_2$ *equipped with the fibred proof-calculus* $C_1\circledast C_2$ *is full with respect to* $\mathcal{P}$*.*

Proof. Suppose that $\langle\mathcal{A},\mathbf{c}\rangle$ makes all the rules in $C_1\circledast C_2$ sound and $\langle|\mathcal{A}|_\phi,\mathbf{c}\rangle\in\mathcal{P}$. Easily, then, being each $\langle id_{\{\phi\}},g_i\rangle:\langle\{\phi\},O_i\rangle\to\langle\{\phi\},O\rangle$ the signature morphism underlying both the morphisms from R_i to $R_1\circledast R_2$ and from C_i to $C_1\circledast C_2$ as defined in the fibring, and given that the rules of $C_1\circledast C_2$ are corresponding translations of rules of C_1 or C_2, it easily follows that $\langle\mathcal{A}|_{\langle id_{\{\phi\}},g_i\rangle},\mathbf{c}\rangle$ makes all the rules of C_i sound. Moreover, $|(\mathcal{A}|_{\langle id_{\{\phi\}},g_i\rangle})|_\phi=|\mathcal{A}|_\phi$ and therefore, by the fullness of each R_i equipped with C_i, it follows that $\langle\mathcal{A}|_{\langle id_{\{\phi\}},g_i\rangle},\mathbf{c}\rangle\in M_i$. Thus, by definition of fibring, $\langle\mathcal{A},\mathbf{c}\rangle$ is a structure of the fibred room $R_1\circledast R_2$. □

We shall now present completeness preservation results for several choices of $\mathcal{P}$.

Theorem 2. Completeness Preservation - All Structures.
Let the $\mathbf{c}$*-rooms* R_1 *and* R_2 *equipped with the proof-calculi* C_1 *and* C_2 *be full with respect to the class of all closure operations. Then, the fibred proof-calculus* $C_1\circledast C_2$ *is complete for the fibred* $\mathbf{c}$*-room* $R_1\circledast R_2$*.*

Proof. We know that $R_1\circledast R_2$ equipped with $C_1\circledast C_2$ is full with respect to the class of all closure operations. It is very easy to see that the structure $\langle\mathcal{W}_{\langle\{\phi\},O\rangle},\mathbf{c}\rangle$ such that $\mathbf{c}=\vdash$, where $\langle\{\phi\},O\rangle$ stands for the fibred signature, makes all the rules in $C_1\circledast C_2$ sound. Therefore, the structure belongs to the fibred room $R_1\circledast R_2$ as a consequence of fullness. Suppose now that, in the fibring, $\Phi\not\vdash\varphi$. To show that $\Phi\not\vDash\varphi$ it is enough to note that $[\![\_]\!]^{\mathcal{W}_{\langle\{\phi\},O\rangle}}$ is the identity on formulae. □

Due to the plain use of the free word algebra, this result is, so to say, a little too much syntactic. Let us try and avoid such structures. A closure operation $\langle A,\mathbf{c}\rangle$ is said to be *elementary* if for every $a_1,a_2\in A$, $a_1\in\{a_2\}^{\mathbf{c}}$ and $a_2\in\{a_1\}^{\mathbf{c}}$ together imply $a_1=a_2$. In most of the interesting cases, the structure used in the previous proof clearly fails to be elementary.

In this case, however, we need to make a few further assumptions about the systems being fibred. A proof-calculus $C=\langle\langle\{\phi\},O\rangle,dR,gR\rangle$ is said to be *congruent* if for every operator $o\in O_{\phi^n\phi}$, every set Γ of schema formulae closed for theorem generation, and all schema formulae $\gamma_1\ldots,\gamma_n,\delta_1,\ldots,\delta_n$, it is the case that

$$\Gamma\cup\{o(\gamma_1\ldots,\gamma_n)\}\vdash o(\delta_1,\ldots,\delta_n)$$

whenever

$$\Gamma\cup\{\gamma_i\}\vdash\delta_i \text{ and } \Gamma\cup\{\delta_i\}\vdash\gamma_i \text{ for } i=1,\ldots,n.$$

Many proof-calculi, including the one for modal logic presented in Example 4, are indeed congruent. However, there are exceptions, as for instance the paraconsistent systems of [11].

Moreover, C is said to have an *implication* connective if there exists $\Rightarrow\, \in O_{\phi^2\phi}$ such that, for every set Γ of schema formulae and all schema formulae γ, δ it is the case that

$$\Gamma \vdash (\gamma \Rightarrow \delta) \text{ if and only if } \Gamma \cup \{\gamma\} \vdash \delta.$$

This condition can be easily shown [28] to be equivalent to requiring

$$\emptyset \vdash (\xi_1 \Rightarrow \xi_1),$$

$$\{\xi_1, (\xi_1 \Rightarrow \xi_2)\} \vdash \xi_2,$$

$$\{\xi_2\} \vdash (\xi_1 \Rightarrow \xi_2)$$

and

$$\{(\xi_r \Rightarrow \gamma) : \gamma \in \text{Prem}(r)\} \vdash (\xi_r \Rightarrow \text{Conc}(r))$$

for each $r \in dR$ and some ξ_r not occurring in r.

Furthermore, in the presence of an implication, it is known [28] that congruence is equivalent to the condition that, for each operator $o \in O_{\phi^n\phi}$, the schema formula

$$(o(\xi_1 \ldots, \xi_n) \Rightarrow o(\xi_1', \ldots, \xi_n'))$$

is a theorem schema generated from

$$\{(\xi_1 \Rightarrow \xi_1'), (\xi_1' \Rightarrow \xi_1), \ldots, (\xi_n \Rightarrow \xi_n'), (\xi_n' \Rightarrow \xi_n)\}.$$

Given these characterizations of implication and congruence in the presence of an implication, it is very easy to see that the fibring of two congruent proof-calculi sharing an implication connective is also a congruent proof-calculus with implication. The proof of this property can be found in [28, 7] and just uses the obvious fact that both theorem generation and deducibility are preserved by morphisms of proof-calculi.

Theorem 3. Completeness Preservation - Elementary Structures.
Let the $\mathbf{c}$*-rooms* R_1 *and* R_2 *equipped with the proof-calculi* C_1 *and* C_2 *be full with respect to the class of all elementary closure operations. If both* C_1 *and* C_2 *are congruent and there is a shared implication connective, then the fibred proof-calculus* $C_1 \circledast C_2$ *is complete for the fibred* $\mathbf{c}$*-room* $R_1 \circledast R_2$.

Proof. We know that $R_1 \circledast R_2$ equipped with $C_1 \circledast C_2$ is full with respect to the class of all elementary closure operations. Moreover, we also know that $C_1 \circledast C_2$ is congruent and has an implication connective $\Rightarrow$. It is easy to see that the binary relation $\equiv$ defined on formulae by $\varphi_1 \equiv \varphi_2$ if both $\{\varphi_1\} \vdash \varphi_2$ and $\{\varphi_2\} \vdash \varphi_1$ is an equivalence relation. Moreover, if $\langle\{\phi\}, O\rangle$ is the fibred signature, the congruence of $C_1 \circledast C_2$ immediately implies that $\equiv$ is a congruence relation on the word algebra $\mathcal{W}_{\langle\{\phi\},O\rangle}$. Thus, let us consider the structure $\langle \mathcal{W}_{\langle\{\phi\},O\rangle}/\equiv, \mathbf{c}\rangle$, corresponding to the Lindenbaum-Tarski quotient algebra together with the closure defined by $\{[\psi] : \psi \in \Psi\}^{\mathbf{c}} = \{[\psi'] : \Psi \vdash^l \psi'\}$, where $[\_]$ applied to a formula denotes its equivalence class under $\equiv$. Once again, it is straightforward to show that this structure makes all the rules in $C_1 \circledast C_2$ sound. In fact, it is clear that $[\![\_]\!]^{\mathcal{W}_{\langle\{\phi\},O\rangle}/\equiv} = [\_]$. Therefore, the structure belongs to the fibred room $R_1 \circledast R_2$ as a consequence of fullness. Suppose that, in the fibring, $\Phi \nvdash \varphi$. The structure just built clearly shows that $\Phi \not\models \psi$. □

Let us try and improve even more on this result, by requiring that the algebra of truth-values has the usual property of being ordered. Every partial-order $\langle A, \leq\rangle$ easily induces two polarities $\mathrm{Upp}(B) = \{a \in A : b \leq a \text{ for every } b \in B\}$ and $\mathrm{Low}(B) = \{a \in A : a \leq b \text{ for every } b \in B\}$, and a cut closure $\mathbf{c}$ on A defined by $B^{\mathbf{c}} = \mathrm{Upp}(\mathrm{Low}(B))$, where $B \subseteq A$, as in [3]. In such case, $\langle A, \mathbf{c}\rangle$ is said to be a *partially ordered* closure operation.

Theorem 4. Completeness Preservation - Partially Ordered Structures I.
Let the $\mathbf{c}$*-rooms* R_1 *and* R_2 *equipped with the proof-calculi* C_1 *and* C_2 *be full with respect to the class of all partial-order closure operations. If both* C_1 *and* C_2 *are congruent and there is a shared implication connective, then the fibred proof-calculus* $C_1 \circledast C_2$ *is weak complete for the fibred* $\mathbf{c}$*-room* $R_1 \circledast R_2$.

Proof. The proof of this result is similar to the previous one, but now considering a different closure operation on the Lindenbaum-Tarski algebra. Just note that the binary relation $\leq$ defined by $[\varphi_1] \leq [\varphi_2]$ if $\emptyset \vdash (\varphi_1 \Rightarrow \varphi_2)$ is a partial order on the quotient of the set of formulae. Therefore, let us consider the structure $\langle \mathcal{W}_{\langle\{\phi\},O\rangle}/{\equiv}, \mathbf{c}\rangle$ where $\mathbf{c}$ is the cut closure induced by $\leq$ as explained above. Let us check, in this less trivial case, that this structure indeed makes all the rules of $C_1 \circledast C_2$ sound. Just note that an assignment α in the quotient algebra sends each schema variable to an equivalence class of formulae. Therefore, if we set a schema variable substitution σ such that $[\sigma(\xi)] = \alpha(\xi)$, it immediately follows that $[\![\_]\!]_{\alpha}^{\mathcal{W}_{\langle\{\phi\},O\rangle}/\equiv} = [\_\sigma]$. Consider a d-rule $r = \frac{\gamma_1 \ldots \gamma_n}{\delta}$ and fix an assignment α. We need to show that $[\delta\sigma] \in \{[\gamma_1\sigma], \ldots, \gamma_n\sigma]\}^{\mathbf{c}}$. Therefore, let φ be a sentence such that $[\varphi] \leq [\gamma_i\sigma]$ for $i = 1, \ldots, n$. By definition, this means that $\emptyset \vdash (\varphi \Rightarrow \gamma_i\sigma)$ for each i. Therefore, by using the last requirement (concerning d-rules) in the characterization of implication together with the structurality of deducibility, we can chose σ as above such that $\sigma(\xi_r) = \varphi$ and therefore conclude that also $\emptyset \vdash (\varphi \Rightarrow \delta\sigma)$. Equivalently, this means that $[\varphi] \leq [\delta\sigma]$ and the d-rule is sound. Assume now that r is a g-rule. We need to show that if $\{[\gamma_1\sigma], \ldots, [\gamma_n\sigma]\} \in \emptyset^{\mathbf{c}}$ then also $[\delta\sigma] \in \emptyset^{\mathbf{c}}$. In this case, it is easy to see that $\emptyset^{\mathbf{c}}$ has precisely one element, corresponding to the equivalence class of all theorems of $C_1 \circledast C_2$. Thus, if all $\gamma_i\sigma$ are theorems, then by using the dR rule r we can conclude that also $\delta\sigma$ is a theorem, and the g-rule is sound. Therefore, as a consequence of fullness, the structure belongs to the fibred room $R_1 \circledast R_2$. Now, if $\emptyset \not\vdash \psi$ this structure clearly shows that $\emptyset \not\models \psi$. □

One may wonder why the construction fails if we want to go beyond weak completeness. If $\Phi \not\vdash \psi$, in the structure built above, we would need to find a sentence that would imply all the formulae in Φ but not ψ. This, in general, can only be solved in the presence of some form of infinitary conjunction, since Φ may very well be an infinite set. However, we can still improve the result a little bit by also dealing with the finite case.

A proof-calculus $C = \langle\langle\{\phi\}, O\rangle, dR, gR\rangle$ is said to have a *conjunction* connective if there exists $\wedge \in O_{\phi^2\phi}$ such that, for every set of schema formulae Γ and all schema formulae $\delta_1, \delta_2, \gamma$ it is the case that

$$\Gamma \cup \{\delta_1, \delta_2\} \vdash \gamma \text{ if and only if } \Gamma \cup \{(\delta_1 \wedge \delta_2)\} \vdash \gamma.$$

This condition can be easily shown to be equivalent to requiring

$$\{(\xi_1 \wedge \xi_2)\} \vdash \xi_1$$

$$\{(\xi_1 \wedge \xi_2)\} \vdash \xi_2$$

and

$$\{\xi_1, \xi_2\} \vdash (\xi_1 \wedge \xi_2).$$

Theorem 5. Completeness Preservation - Partially Ordered Structures II.
Let the $\mathbf{c}$*-rooms* R_1 *and* R_2 *equipped with the proof-calculi* C_1 *and* C_2 *be full with respect to the class of all partial-order closure operations. If both* C_1 *and* C_2 *are congruent, there is a shared implication connective and at least one of them has a conjunction connective, then the fibred proof-calculus* $C_1 \circledast C_2$ *is finite complete for the fibred* $\mathbf{c}$*-room* $R_1 \circledast R_2$.

Proof. Consider exactly the same structure as in the previous proof, and suppose that $\{\varphi_1, \ldots, \varphi_n\} \nvdash \psi$. In that case, just consider the sentence $\varphi = (\varphi_1 \wedge \ldots \wedge \varphi_n)$. It is trivial to check that $\emptyset \vdash (\varphi \Rightarrow \varphi_i)$ for each $i = 1, \ldots, n$. However, as easily, it is also the case that $\emptyset \nvdash (\varphi \Rightarrow \psi)$ and the structure shows that $\{\varphi_1, \ldots, \varphi_n\} \not\models \psi$. □

Being unfeasible to require an infinitary conjunction connective and therefore obtain completeness, we now just point out to an alternative. The following result, that we just state, is an instance of the one proved in [28] and concerns algebras of sets in the style of general frames for modal logic (see [16]) and uses a Henkin-style construction. For a powerset lattice $\langle \wp(U), \subseteq \rangle$, the cut closure $\mathbf{c}$ induced by the polarities as above is such that $B^{\mathbf{c}} = \{b \subseteq \wp(U) : (\bigcap B) \subseteq b\}$. We have already used such closures in the examples. A closure operation $\langle A, \not{c} \rangle$ is said to be a *general powerset* closure operation if $A \subseteq \wp(U)$, $U \in A$ and $\not{c}$ is the closure induced by $\mathbf{c}$ on A, i.e., $B^{\not{c}} = B^{\mathbf{c}} \cap A$ for each $B \subseteq A$.

Theorem 6. Completeness Preservation - General Powerset Structures.
Let the $\mathbf{c}$*-rooms* R_1 *and* R_2 *equipped with the proof-calculi* C_1 *and* C_2 *be full with respect to the class of all general powerset closure operations. If both* C_1 *and* C_2 *are congruent and there is a shared implication connective, then the fibred proof-calculus* $C_1 \circledast C_2$ *is complete for the fibred* $\mathbf{c}$*-room* $R_1 \circledast R_2$.

Other approaches to completeness preservation, namely via encoding in a suitable meta-logic have already been employed in the specific context of paraconsistent non-truth-functional logics [8] but should of course be also workable in this setting. Some forms of conditional equational logic, namely rewriting logic [17], seem to be good candidate meta-logics for this process of algebraization.

7 Conclusion

We have introduced the novel notion of $\mathbf{c}$-parchment, by enriching the structure of interpretation algebras, as a means of presenting an institution, and we have shown how it can be used to bring the combination mechanism of fibring to the institutional setting. Moreover, $\mathbf{c}$-parchments were shown to correspond to indexed categories of $\mathbf{c}$-rooms and their fibring characterized as a colimit. Furthermore, we have put the construction into practice by exploring partial equational logic as a fibred logic. The example is even more interesting since we also obtain a proof-calculus for partial equational logic by fibring calculi for equational logic and the logic of partiality adopted. In fact, although in the simplest case of propositionally based logics, we have established a collection of soundness and completeness preservation results for fibring, thus showing that the successfulness of the example is not just a mere coincidence.

Of course, this line work is far from over. The most important topic to be followed is precisely the extension of these preservation results to a more general context. Namely, the work reported in [24] seems to bring ideas that can also be explored in this context. This comment applies not only to preservation results but, first of all, also with the way we should deal with more complex notions of proof-calculus than the one used here. Work in this direction has already been done in [23], including ways of representing constrained schematic rules such as those used in many logics with terms and quantifiers. Two well known such examples appear in classical first-order logic's axioms $((\forall x(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow (\forall x\, \psi)))$ and $((\forall x\, \varphi) \Rightarrow \varphi^x_t)$. However, in both cases there are well known constraints. Namely, in the first case, there is the requirement that x must not occur free in φ and, in the second, the requirement that t must be free for x in φ.

Another interesting possibility to be pursued is to explore a process of algebraization using a suitable meta-logic, as mentioned above. This alternative does not only bring us closer to the realm of algebraic logic [10, 5, 1], but it also has the advantage of being able to deal, at an adequate level, with non-truth-functional logics. In fact, several interesting paraconsistent and paracomplete logics, with meaningful applications in computer science, fail to be truth-functional in the sense that some of its operators are not congruent. Some preliminary work also in the setting up of a meaningful notion of non-truth-functional parchment can be found in [9]. However, in this context, there is still a lot of research work to be done in order to extend the usual algebraic techniques of logic to cope with the possible absence of congruence. We are also interested in studying the representation of fibring in logical frameworks. Namely, capitalizing on Meseguer's theory of general logics [18], we aim at characterizing the mechanism of fibring of logics within rewriting logic [17]. In particular, general representation preservation results are envisaged, that may determine the exact extent to which representations of fibred logics can be obtained out of representations of the logics being fibred. Last but not least, future work should also cover the characterization of fibring of logics presented by other means, either model or proof-theoretic, as well as the search for transference results for other interesting properties, such as decidability, the finite model property, interpolation and amalgamation.

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