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Bicompleteness in the Category of Partial Graphs with Total Homomorphisms

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Abstract

Category Theory is becoming an useful tool to formalize abstract concepts making easy to construct proofs and investigate properties while graphs are commonly used to model systems. Partiality is a important mathematical concept used in Mathematics and Computer Science. In this paper we define a category where objects are partial graphs whose arcs may have source and/or target nodes undefined and morphisms are total homomorphisms of partial graphs and prove that this category, named $\mathcal{G}r_p$ is bicomplete.

Keywords: *partial graphs, category theory, bicompleteness*

1 Introduction

Category Theory is becoming an useful tool to formalize abstract concepts making easy to construct proofs and investigate properties in many areas, speccially in Semmantics and Type Theory. The constructions about universal mappings like limits and adjunctions are getting usefull interpretations in terms of compositionality of systems.

Graphs are commonly used to model systems, either by simple graphs or by graph-based structures like automata [7, 1] and Petri nets [11, 10, 9].

The notion of partiality appears naturally in the main concepts from Computer Science and also in many other mathematical formalisms. In Computer Science, it can be used to express computations that don't terminate and to define partial recursive functions (due to partiality the class of partial recursive functions becomes equivalent to Turing Machines).

We defined a category where objects are partial graphs whose arcs may have source and/or target nodes undefined and morphisms are total homomorphisms of partial graphs. Our goal is to prove that the category of partial graphs with total morphisms, named $\mathcal{G}r_p$, is bicomplete. In graphs, limits and colimits can be used to compose systems like a syncronous composition, but interpretations of these constructions and examples are out of the scope of this paper.

In literature, partial graphs are not very common (compared to total graphs). Moreover, as far as we know, the proposed category is new. In [13] partial graphs (but with different morphisms) are used to modelling flow control of programs and in [5] partial functions in automata (in a different categorical approach) are used to allow the possibility of computations that do not terminate.

In section 2, we give the formal definitions of the category of partial graphs and in section 3 we give the proof of bicompletness, also given an overview of how to construct limits and colimits in this category. Finnaly, we give the conclusions and some future works.

2 Partial Graphs

A partial graph is a graph whose arcs can have source and/or target nodes undefined. Traditionally, a graph has a set of vertices, a set of arcs and two total functions named source and target that takes an arc and go to the respective source or target vertice. In partial graphs, the source and target functions are *partial functions*. We can see an arc without source but with a target defined as an entry-point of the graph (or the systems that this graph represents), an arc without target but with source defined is an exit-point, and an arc without source and target can be seen as an transaction of the system. This last interpretation can be confirmed when we give semmantics to the computations of a graph through span composition of graphs [6].

To define the category of partial graphs is needed to define first the Partial Comma Category, that is used to partial graphs. The resulting category has objects with partiality in its internal structure. We used comma category in the sense of [2].

Definition 2.1 (*$p\mathcal{C}omma$*)
Consider the finitely complete category $\mathcal{C}$ and the functors $\mathbf{inc_p} : \mathcal{C} \to p\mathcal{C}$ (the canonical inclusion functor), $\mathbf{f} : \mathcal{F} \to \mathcal{C}$ and $\mathbf{g} : \mathcal{G} \to \mathcal{C}$.

Therefore, $pComma(\mathbf{f}, \mathbf{g})$ is such that:

- the objects are triples $S = \langle F, s, G \rangle$, where F is a $\mathcal{F}$-object, G is a $\mathcal{G}$-object and $s : \mathbf{inc_p} \circ \mathbf{f}F \to \mathbf{inc_p} \circ \mathbf{g}G$ is a $p\mathcal{C}$-morphism;

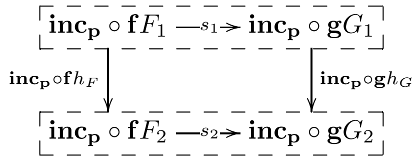

Figure 1: Diagram of Partial Comma Category

- a morphism $h : S_1 \to S_2$ where $S_1 = \langle F, s_1, G\rangle$, $S_2 = \langle F, s_2, G\rangle$ is a pair $h = \langle h_F : F_1 \to F_2, h_G : G_1 \to G_2\rangle$ where h_F and h_G are morphisms in $\mathcal{F}$ and $\mathcal{G}$ respectively, and are such that in $p\mathcal{C}$ (see figure 1) $(\mathbf{inc_p} \circ \mathbf{g}h_G) \circ s_1 = s_2 \circ (\mathbf{inc_p} \circ \mathbf{f}h_F)$
- the identity morphism of an object $S = \langle F, s, G\rangle$ is $\iota_S = \langle \iota_F : F \to F, \iota_G : G \to G\rangle$;
- the composition of $u = \langle u_F, u_G\rangle : S_1 \to S_2$, $v = \langle v_F, v_G\rangle : S_2 \to S_3$ is $v \circ u = \langle v_F \circ u_F, v_G \circ u_G\rangle : S_1 \to S_3$

Definition 2.2 (Category of Partial Graphs)
The category of partial graphs with total homomorphisms, named $\mathcal{G}r_p$, is the partial comma category $p\mathcal{C}omma(\mathbf{\Delta}, \mathbf{\Delta})$ (beeing $\mathbf{\Delta} : \mathcal{S}et \to \mathcal{S}et^2$ the diagonal functor).

Note that the objects are partial graphs, i.e., graphs where source and target functions are *partial* functions, and morphisms are *total* homomorphims of partial graphs.

A homomorphism $h : G \to H$ (where G and H are two partial graphs) in $\mathcal{G}r_p$ is a triple $\langle h_V, h_D, h_T\rangle$ where $h_V : V_G \to V_H$, $h_D : D_G \to D_H$ and $h_T : T_G \to T_H$ are total functions mapping vertices (h_V), arcs (h_T) and the domain such that the source and target functions are defined (h_D), such that source and target functions are preserved. By simplicity, we usually omit the function h_D, that is obvious.

3 Bicompletness of $\mathcal{G}r_p$

In this section, we describe the constructions about limits and colimits of $\mathcal{G}r_p$ and prove the bicompletness of this category.

3.1 Colimits

Theorem 3.1
$\mathcal{G}r_p$ is cocomplete.

Proof: The proof that $\mathcal{G}r_p$ is cocomplete is given by the inheritance of colimits in comma categories [3]. In this case, the categories involved to the definition of $\mathcal{G}r_p$ must be cocomplete and the functor $\mathbf{inc_p} \circ \mathbf{f}$ must preserve colimits. Both $\mathcal{S}et$ and $p\mathcal{S}et$ are cocomplete, and both Δ and $\mathbf{inc_p}$ has right-adjoint [8, 4]. If a functor has right-adjoint, then the functor preserves colimits [12] and the composition of left-adjoints is a left-adjoint, thus, the proof is done. ■

Due to the inheritance, the colimits in $p\mathcal{C}omma(\Delta, \Delta)$ are calculated in $\mathcal{S}et$, as illustrated in figure 2 for coproduct.

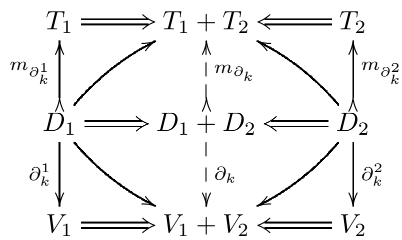

Figure 2: Coprodut in $\mathcal{G}r_p$

The cocompleteness is an important property to define, p.g., pushouts. With this construction we can use graph grammars (with double-pushout approach, since morphisms are total) and define systems that evolves during their computation (see [6]).

3.2 Limits

The proof of the completeness of $\mathcal{G}r_p$ is not by inheritance since $\mathbf{inc_p}$ does not preserve limits, as shown before. But there are constructions for products and for equalizers in $\mathcal{G}r_p$, therefore the category is complete. In this paper we give the construction and the proof for binary product, terminal object and equalizers.

Calculate the binary product of partial graphs is not trivial, so is necessary follow some steps. Consider the partial graphs $G = \langle V_G, T_G, \partial_{0_G}, \partial_{1_G} \rangle$ and $H = \langle V_H, T_H, \partial_{0_H}, \partial_{1_H} \rangle$, the steps are:

1. Calculate $V_G \times V_H$, the product of the vertices in $\mathcal{S}et$;
2. Separate the arcs of each graph in four distinct classes: arcs with both source and target, arcs without source and without target, arcs that have only source and arcs that have only target;
3. Calculate the product in $\mathcal{S}et$ of each class of arcs.

The resulting partial graph will be the graph with vertices being the set $V_G \times V_H$, arcs being the union of the four products of the classes of arcs, and the source and target functions are given unically by the calculated projections of the products in $\mathcal{S}et$.

Definition 3.1 (Division of T)
Let $G = \langle V, T, \partial_0, \partial_1 \rangle$ a partial graph, $\varnothing : T \to \{*\}$ the empty partial function, $tot_T : T \to \{*\}, tot_V : V \to \{*\}$ both total functions and $\partial_0^* = tot_V \circ \partial_0$, $\partial_1^* = tot_V \circ \partial_1$. The following subobjects are given by the equalizers in $p\mathcal{S}et$ like in figure 3:

- $\langle K_0, \neg\partial_0 \rangle$ equalizer of ∂_0^* and $\varnothing$. Arcs of G with source undefined;
- $\langle K_1, \neg\partial_1 \rangle$ equalizer of ∂_1^* and $\varnothing$. Arcs of G with target undefined;
- $\langle E_0, `\partial_0' \rangle$ equalizer of ∂_0^* and tot. Arcs of G with source defined;
- $\langle E_1, `\partial_1' \rangle$ equalizer of ∂_1^* and tot. Arcs of G with target defined.

The pullbacks of figure 4 give the division of T in four classes, where:

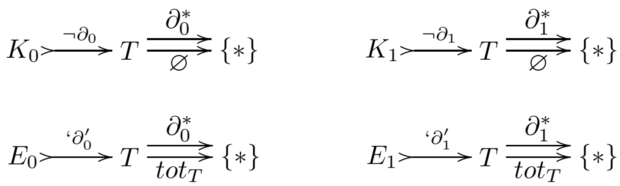

Figure 3: Equalizers in $p\mathcal{S}et$

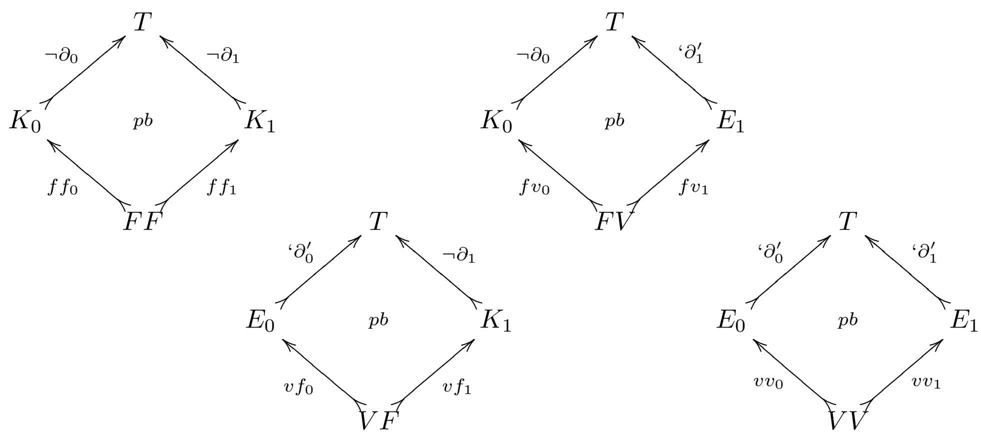

Figure 4: Division of Arcs

- $\langle VV, vv\rangle$, being $vv = {}`\partial_0' \circ vv_0 = {}`\partial_1' \circ vv_1$, arcs with ∂_0 and ∂_1 defined;
- $\langle VF, vf\rangle$, being $vf = {}`\partial_0' \circ vf_0 = \neg\partial_1 \circ vf_1$, arcs with ∂_0 defined only;
- $\langle FV, fv\rangle$, being $fv = \neg\partial_0 \circ fv_0 = {}`\partial_1' \circ fv_1$, arcs with ∂_1 defined only;
- $\langle FF, ff\rangle$, being $ff = \neg\partial_0 \circ ff_0 = \neg\partial_1 \circ ff_1$, arcs with ∂_0 and ∂_1 undefined;

Theorem 3.2
Let $\langle VV, vv\rangle$, $\langle VF, vf\rangle$, $\langle FV, fv\rangle$, $\langle FF, ff\rangle$ a division of T. Then, VV, VF, FV and FF are pairwise disjoint and $VV \cup VF \cup FV \cup FF = T$.

Proof: To prove that VV, VF, FV and FF are pairwise disjoints we prove that $VV \cap VF = \varnothing$. The proof of the other pairs are similar.

To prove $VV \cap VF = \varnothing$ we need to show that the pullback of vv and vf is the empty set. Let $\langle P, p_1, p_2\rangle$ the pullback of vv and vf illustraded in figure 5.

Let $\langle Q, q_1, q_2\rangle$ the pullback of ‘∂_1' and $\neg\partial_1$, illustraded in figure 6. Taking $\langle E_1, {}`\partial_1'\rangle$ the equalizer in $\mathcal{P}fn$ of $\langle tot, \partial_1\rangle$ and $\langle K_1, \neg\partial_1\rangle$ the equalizer in $\mathcal{P}fn$ of $\langle \partial_1, \varnothing\rangle$, we have that

$$\langle Q, q\rangle \text{ is a cone of } \langle tot, \partial_1\rangle \tag{1}$$

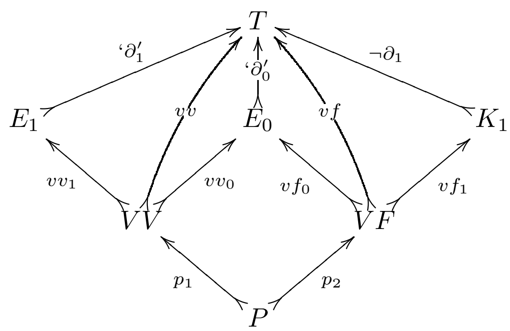

Figure 5: Pullback of $\langle vv, vf\rangle$

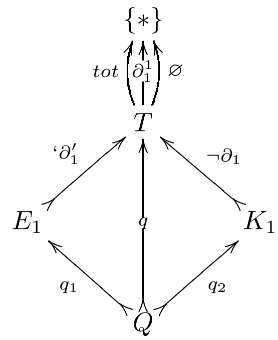

Figure 6: Pullback of $\langle `\partial'_1, \neg\partial_1\rangle$

$$\langle Q, q\rangle \text{ is a cone of } \langle \partial_1, \varnothing\rangle \tag{2}$$

Therefore, by (1), $tot \circ q = \partial_1 \circ q$ and by (2), $\partial_1 \circ q = \varnothing \circ q$, thus $tot \circ q = \varnothing \circ q$. Since $\varnothing \circ q = \varnothing$, $tot \circ q = \varnothing$. As tot is a total funtion, $q = \varnothing$. It's q also a monomorphism, so $Q = \varnothing$, i.e., $E_1 \cap K_1 = \varnothing$.

With this we know that $\langle \varnothing, \varnothing, \varnothing\rangle$ is the pullback of $\langle `\partial'_1, \neg\partial_1\rangle$. The pair $\langle P, vv_2 \circ p_1, vf_2 \circ p_2\rangle$ is a cone of $\langle `\partial'_1, \neg\partial_1\rangle$ as illustrated in figure 7.

Thus, there is a unique morphism from P to $\varnothing$. In $\mathcal{S}et$ there is only one morphisms with the empty set as target: the identity of the empty set. Therefore, $P = \varnothing$.

$VV \cup VF \cup FV \cup FF = T$ (by contradiction):

Suppose that $a \in T \wedge a \notin VV \cup VF \cup FV \cup FF$.

1. If $a \notin VV$, then $\partial_0(a)$ or $\partial_1(a)$ are undefined.
   If only $\partial_0(a)$ is undefined, then $a \in VF$;
   if only $\partial_1(a)$ is undefined, then $a \in FV$;
   if $\partial_0(a)$ and $\partial_1(a)$ are undefined, then $a \in FF$.

2. If $a \notin VF$, ento $\partial_0(a)$ indefinido ou $\partial_1(a)$ definido.
   If only $\partial_0(a)$ is undefined, then $a \in FF$;
   if only $\partial_1(a)$ is defined, then $a \in VV$;
   if $\partial_0(a)$ is undefinedo and $\partial_1(a)$ is defined, then $a \in FV$.

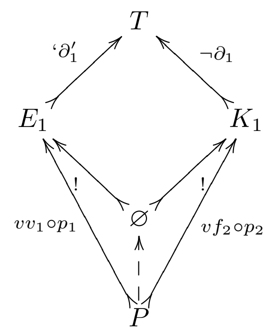

Figure 7: A Cone of $\langle `\partial_1', \neg\partial_1\rangle$ in $\mathcal{S}et$

3. If $a \notin FV$, then $\partial_0(a)$ is defined or $\partial_1(a)$ is undefined.
   If only $\partial_0(a)$ is defined, then $a \in VV$;
   if only $\partial_1(a)$ is undefined, then $a \in FF$;
   if $\partial_0(a)$ is defined and $\partial_1(a)$ is undefined, then $a \in VF$.

4. If $a \notin FF$, then $\partial_0(a)$ or $\partial_1(a)$ are defined.
   If only $\partial_0(a)$ is defined, then $a \in VF$;
   if only $\partial_1(a)$ is defined, then $a \in FV$;
   if $\partial_0(a)$ e $\partial_1(a)$ are defined, then $a \in VV$.

Therefore, $a \in T \Rightarrow a \in VV \cup VF \cup FV \cup FF$.
Now suppose that $a \in VV \cup VF \cup FV \cup FF \wedge a \notin T$.

1. if $a \in VV$. Since $VV \subseteq T$, $a \in T$;
2. if $a \in VF$. Since $VF \subseteq T$, $a \in T$;
3. if $a \in FV$. Since $FV \subseteq T$, $a \in T$;
4. if $a \in FF$. Since $FF \subseteq T$, $a \in T$.

So, $a \in VV \cup VF \cup FV \cup FF \Rightarrow a \in T$, and we conclude that $T = VV \cup VF \cup FV \cup FF$. ■

Definition 3.2 (Binary Product in $\mathcal{G}r_p$)
Let $G_1 = \langle V_1, T_1, \partial_0^1, \partial_1^1\rangle$ and $G_2 = \langle V_2, T_2, \partial_0^2, \partial_1^2\rangle$ partial graphs, a binary product is $G_1 \times G_2 = \langle V_1 \times V_2, \bigcup_T, \partial_0, \partial_1\rangle$ where (being $k \in \{0, 1\}$):

- $V_1 \times V_2$ is the product in $\mathcal{S}et$ of V_1 and V_2;
- $\bigcup_T$ is the object of the product in $\mathcal{S}et$ of $\langle \times_{VV}, \times_{VF}, \times_{FV}, \times_{FF}\rangle$;
- ∂_k is the partial morphism $\langle m_{\partial_k} : \bigcup_{D_{\partial_k}} \rightarrowtail \bigcup_T, \partial_k : \bigcup_{D_{\partial_k}} \to V_1 \times V_2\rangle$ where
  - $\bigcup_{D_{\partial_k}}$ is the pullback of $\langle inc_{\times_T}, m_{\times_{D_{\partial_k}}}\rangle$ as in figure 8;

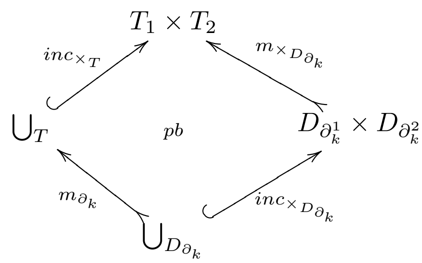

Figure 8: Midle Object of the Binary Product in $\mathcal{G}r_p$

- $m_{\partial_k} : \bigcup_{D_{\partial_k}} \rightarrowtail \bigcup_T$ is a projection of this pullback in figure 8;
- $\partial_k : \bigcup_{D_{\partial_k}} \to V_1 \times V_2$ is the induced morphism by the product $V_1 \times V_2$, i.e., (being $j \in \{1,2\}$) such that $\pi_{V_j} \circ \partial_k = \partial_{k_1} \circ \pi_{\bigcup_{D_k^j}}$ where $\pi_{\bigcup_{D_k^j}} = inc_{\times_{D_{\partial_k}}} \circ \pi_{D_j}$ being π_{D_j} the product projection of $D_1 \times D_2$;

with the morphisms $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ such that (being $j \in \{1,2\}$) $\pi_j = \langle \pi_{V_j}, \pi_{\bigcup_{T_j}} \rangle$ where:

- $\pi_{V_j} : V_1 \times V_2 \to V_j$ is the projection in $\mathcal{S}et$;
- $\pi_{\bigcup_{T_j}} : \bigcup_T \to T_j$ is the morphism such that $\pi_{\bigcup_{T_j}} \circ im_{\times_{VV}} = vv^j \circ \pi_{VV^j}$ and $\pi_{\bigcup_{T_j}} \circ im_{\times_{VF}} = vf^j \circ \pi_{VF^j}$ and $\pi_{\bigcup_{T_j}} \circ im_{\times_{FV}} = fv^j \circ \pi_{FV^j}$ and $\pi_{\bigcup_{T_j}} \circ im_{\times_{FF}} = ff^j \circ \pi_{FF^j}$

Proof: Let $G = \langle V_G, T_G, \partial_0^G, \partial_1^G \rangle$ a partial graph and the total homomorphisms $f : G \to G_1$ (where $f = \langle f_V : V_G \to V_1, f_T : T_G \to T_1 \rangle$) and $g : G \to G_2$ (where $g = \langle g_V : V_G \to V_2, g_T : T_G \to T_2 \rangle$), there is a morphism $h : G \to G_1 \times G_2$ where $h = \langle h_V : V_G \to V_1 \times V_2, h_T : T_G \to \bigcup_T \rangle$ such that

$$h_V(X) = \langle f_V(X), g_V(X) \rangle$$

$$h_T(x) = \langle f_T(x), g_T(x) \rangle$$

Thus,

$$\pi_{V_1}(h_V(X)) = \pi_{V_1}(\langle f_V(X), g_V(X) \rangle) = f_V(X)$$
$$\pi_{T_1}(h_T(x)) = \pi_{T_1}(\langle f_T(x), g_T(x) \rangle) = f_T(x)$$
$$\pi_{V_2}(h_V(X)) = \pi_{V_2}(\langle f_V(X), g_V(X) \rangle) = g_V(X)$$
$$\pi_{T_2}(h_T(x)) = \pi_{T_2}(\langle f_T(x), g_T(x) \rangle) = g_T(X)$$

Once we have the morphism h, we need to prove its uniqueness.
Suppose $m : G \to G_1 \times G_2$ such that (where $m = \langle m_V, m_T \rangle$)

$$\pi_{V_1} \circ m_V = f_V \quad (3)$$
$$\pi_{V_2} \circ m_V = g_V \quad (4)$$
$$\pi_{T_1} \circ m_T = f_T \quad (5)$$

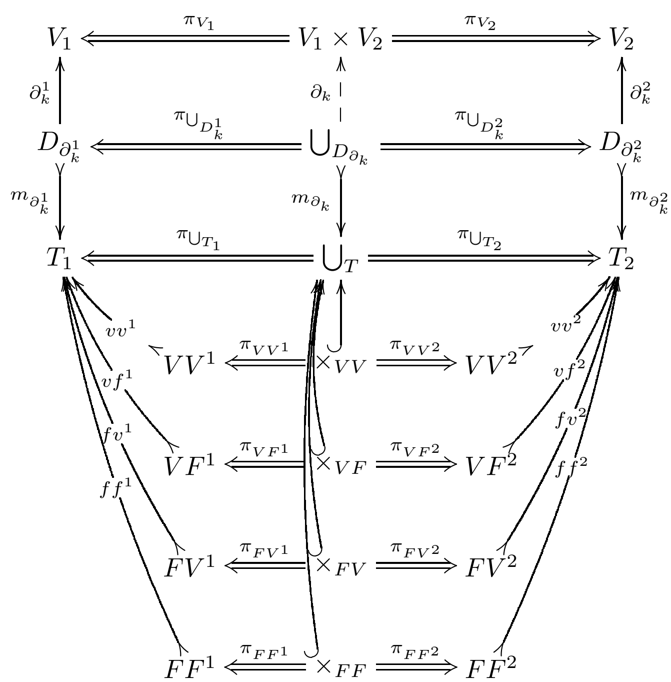

Figure 9: Binary Product in $\mathcal{G}r_p$

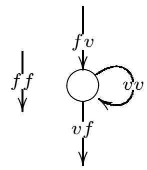

Figure 10: Terminal Object in $\mathcal{G}r_p$

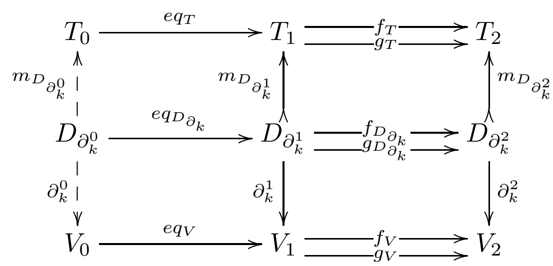

Figure 11: Equalizer in $\mathcal{G}r_p$

$$\pi_{T_2} \circ m_T = g_T \tag{6}$$

Let $m_V(X) = \langle Y, Z\rangle$ and $m_T(x) = \langle y, z\rangle$. So, $\pi_{V_1}(m_V(X)) = \pi_{V_1}(\langle Y, Z\rangle) = Y$ and by (3), $Y = f_V(X)$.

In the same way, by (4), $\pi_{V_2}(m_V(X)) = \pi_{V_2}(\langle Y, Z\rangle) = Z = g_V(X)$; by (5), $\pi_{T_1}(m_T(x)) = \pi_{T_1}(\langle y, z\rangle) = y = f_T(x)$ and by (6), $\pi_{T_2}(m_T(x)) = \pi_{T_2}(\langle y, z\rangle) = z = g_T(x)$.

Thus, $m = h$. ■

The product of two partial graphs can be seen as syncronous compositions of these systems. Note that an entry point in the resulting system occurrs only if it is possible to initiate both systems.

Definition 3.3 (Terminal Object in $\mathcal{G}r_p$)
The terminal object in $\mathcal{G}r_p$ is the graph $\mathbf{1} = \langle\{\bullet\}, \{vv, vf, fv, ff\}, \partial_0, \partial_1\rangle$ illustrated in figure 10.

Proof: Let $G = \langle V, T, \partial_0^G, \partial_1^G\rangle$ any partial graph, there is just one possible morphism $G \to \mathbf{1}$, and is such that all nodes of G are mapped in $\{\bullet\}$ and the arcs of G are divided in classes (using the division already given) and will be mapped in the arc vv of $\mathbf{1}$ if it belongs to the class VV, in the arc vf if it belongs to VF, in the arc fv if it belongs to FV and in the arc ff if it belongs to FF. ■

The construction of equalizers in $\mathcal{G}r_p$ is based on equalizers in $\mathcal{S}et$. Actually, the equalizer of two morphisms f and g are induced by the equalizers in $\mathcal{S}et$ of each componet function. In figure 11 we have the diagram of equalizer in $\mathcal{G}r_p$.

Definition 3.4 (Equalizer in $\mathcal{G}r_p$)
Let two partial graphs $G_1 = \langle V_1, T_1, \partial_0^1, \partial_1^1\rangle$ and $G_2 = \langle V_2, T_2, \partial_0^2, \partial_1^2\rangle$ and the total homomorphisms of partial graphs $f = \langle f_V, f_T\rangle, g = \langle g_V, g_T\rangle : G_1 \to G_2$, an equalizer of f and g is induced by the equalizer of the components functions of the homomorphisms in $\mathcal{S}et$, i.e., is the pair $\langle G_0, eq\rangle$ such that $G_0 = \langle V_0, T_0, \partial_0^0, \partial_1^0\rangle$ and $eq = \langle eq_V, eq_T\rangle$ where:

- $\langle V_0, eq_V \rangle$ is the equalizer in $\mathcal{S}et$ of $\langle f_V, g_V \rangle$;
- $\langle T_0, eq_T \rangle$ is the equalizer in $\mathcal{S}et$ of $\langle f_T, g_T \rangle$;
- ∂_k^0, where $k \in \{0,1\}$, is the partial morphism $\langle m_{D_{\partial_k^0}} : D_{\partial_k^0} \to T_0, \partial_k^0 : D_{\partial_k^0} \to V_0 \rangle$ where
  - $\langle D_{\partial_k^0}, eq_{D_{\partial_k}} \rangle$ is the equalizer in $\mathcal{S}et$ of $\langle f_{D_{\partial_k}}, g_{D_{\partial_k}} \rangle$;
  - $m_{D_{\partial_k^0}}$ is the morphism induced by the equalizer $\langle f_T, g_T \rangle$;
  - ∂_k^0 is the morphism induced by the equalizer $\langle f_V, g_V \rangle$.

Proof: It's interesting note that $m_{D_{\partial_k^0}}$ really is monic. Every equalizer is a monic and the composition of monics are monic. Therefore, $m_{D_{\partial_k^1}} \circ eq_{D_{\partial_k}}$ is monic. We also know that, if $g \circ f$ is monic, then f is monic. Since $eq_T \circ m_{D_{\partial_k^0}} = m_{D_{\partial_k^1}} \circ eq_{D_{\partial_k}}$, then $m_{D_{\partial_k^0}}$ is monic.

We need to prove that there is a unique morphism from a cone to the equalizer G_0.

Let X a node of V_1 and x an arc of T_1. Since eq is a monomorphism we can denote that $eq_V(X) = X$ and $eq_T(x) = x$ if $X \in V_0$ and $x \in T_0$.

Let $Q = \langle V_Q, T_Q, \partial_0^Q, \partial_1^Q \rangle$ a partial graph and $q : Q \to G_1$ a total homomorphism of partial graphs ($q = \langle q_V, q_T \rangle$) such that $f \circ q = g \circ q$. There is a homomorphism $h : Q \to G_0$, where $h = \langle h_V, h_T \rangle$, such that $h_T(x) = q_T(x)$ (with $x \in T_Q$) and $h_V(X) = q_V(X)$ (with $X \in V_Q$).

Suppose a morphism $j = \langle j_V, j_T \rangle : Q \to G_0$ such that $eq_V \circ j_V = q_V$ and $eq_T \circ j_T = q_T$. Therefore $eq_V(j_V(X)) = j_V(X) = q_V(X)$ and $eq_T(j_T(x)) = j_T(x) = q_T(x)$. Thus, $j = h$. ■

4 Conclusions

In this paper we prove the bicompleteness od a different category of graphs, named $\mathcal{G}r_p$, where objects are partial graphs and morphisms are total homomorphisms. Once $\mathcal{G}r_p$ is defined using an extension of Comma Categories we can inherit properties by construction, but in this case we just inherited colimits while limits must be constructed.

With this result is possible to use this category to construct, p.g., another categories of graph-based systems like automata and Petri Nets with partiality in the internal structure of the objects. Another future work is to investigate if $\mathcal{G}r_p$ is cartesian closed and if it is a topos.

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