A Roadmap to Decidability

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Abstract

It is well known that quantifier elimination plays a relevant role in proving decidability of theories. Herein the objective is to provide a toolbox that makes easier to establish quantifier elimination in a semantic way, capitalizing on the fact that a 1-model-complete theory with algebraically prime models has quantifier elimination. Iteration and adjunction are identified as important constructions that can be very helpful, by themselves or composed, in proving that a theory has algebraically prime models. Some guidelines are also discussed towards showing that a theory is 1-model-complete. Illustrations are provided for the theories of the natural numbers with successor, term algebras (having stacks as a particular case) and algebraically closed fields.

Keywords quantifier elimination, decidability of theories.

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1 Introduction

Quantifier elimination is a key property for proving decidability of a first-order theory. Decidable theories play an essential role in computer science applications. For example, the theory of real closed fields (firstly proved to be decidable by Alfred Tarski, see [14]) is of utmost interest when considering for instance probabilistic reasoning (see [4]) and hybrid system verification (see [12]), and the theory of algebraically closed fields has an important role in computer algebra systems (see [2]). On the other hand, decidable theories are very relevant in the areas of theorem proving and data abstraction [5, 13, 15, 1].

Proving decidability of new theories as well as finding new algorithms for proving quantifier elimination is still a very active research concern (see, for instance, [11, 10]). Moreover, it is foreseeable that future applications of computer science may require proving decidability of other relevant theories. Thus we need an effective toolbox of techniques for helping computer scientists in this task.

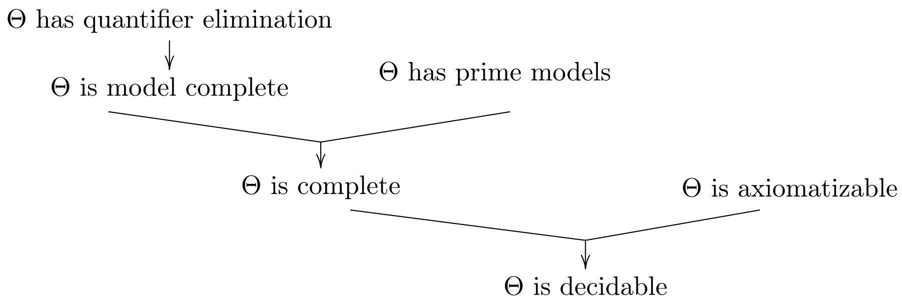

Figure 1: A roadmap to decidability

The role of quantifier elimination for proving decidability is depicted in Figure 1. From there we can conclude that if we manage to prove that a theory is axiomatizable, has quantifier elimination and has a prime model then we can conclude that it is decidable.

Quantifier elimination has been studied from a symbolic perspective (constructive in the sense that an algorithm is given to compute a quantifier free formula equivalent to a given formula) as well as from a semantic point of view. In applications, people want to investigate symbolic quantifier elimination techniques since they are more useful in algorithms. However, we believe that the semantic perspective is also very important since it may allow to prove in a simpler way that a theory enjoys quantifier elimination. If so, then the effort can be concentrated on the constructive algorithm but already with the knowledge that the theory enjoys quantifier elimination. And, of course, if this is not the case no investment on the constructive part is made. In this paper we concentrate our attention on general semantic techniques for quantifier elimination.

A well known sufficient condition for a theory Θ to have quantifier elimination (see, for instance Corollary 3.1.12 of [9]) requires that Θ has algebraically prime models and is 1-model-complete. However, looking at several well known theories we get the impression that more help could be provided, based on that sufficient condition, for proving quantifier elimination. This is even more important when there are more researchers who want to use (semantic) quantifier elimination.

The objective of this paper is to provide workable sufficient conditions for proving that a theory has algebraically prime models and is 1-model-complete.

We started by generalizing the notion of a theory having *algebraically prime models*, to a theory having *algebraically prime models with respect to* another theory. As we will see in Section 3, the extended notion coincides with the original one in some particular cases.

We identify two ways of establishing that a first-order theory Δ has algebraically prime models with respect to a theory Υ:

- (Iteration) Δ has algebraically prime models with respect to Υ whenever there is a map between the models of Δ satisfying particular conditions that guarantee that the image, by this map, of a model of Δ is "closer" to be a model of Υ than the original model and that, the successive application of this map converges, eventually in the limit, to a model of Υ which is an algebraic prime extension of the original model;

- (Adjunction) Δ has algebraically prime models with respect to Υ whenever there is a particular adjunction between their categories of models and embeddings.

The first condition is developed in Section 3 and the second in Section 4. By composing these two techniques, see the ending of Section 4, we have a more practical way to prove that a theory has algebraically prime models. Furthermore, we propose in Section 5 a sufficient condition, named adequacy for $\exists$, for a theory to be 1-model-complete.

We illustrate these conditions and techniques throughout the paper on the theory of naturals with successor, the theory of term algebras (having a theory of stacks as a particular case) and on the theory of algebraically closed fields.

Finally, in Section 6, we briefly draw some concluding remarks.

2 Preliminaries

We start this section by briefly reviewing some relevant notions. Given a first-order signature, by a *(first-order) theory* over that signature we mean a set of sentences over the signature, closed under semantic entailment.[1] In the sequel, in order to simplify the presentation, we may omit the reference to the signature when there is no ambiguity. A theory Γ is *decidable* if there is an algorithm that when receiving a formula returns 1 if the formula is in Γ and 0 otherwise. Usually the decidability of a theory is not proved directly. It is common to prove that the theory is complete and axiomatizable, since these conditions imply the decidability of the theory. A theory Γ over a signature Σ is *complete* if, for every closed formula φ, either

$$\Gamma \vDash_\Sigma \varphi \text{ or } \Gamma \vDash_\Sigma \neg\varphi$$

and is *axiomatizable* if there is a decidable set of sentences Θ such that $\Theta^{\vDash_\Sigma} = \Gamma$. For instance, Th($\mathbb{N}$), the theory of natural numbers with function symbols 0, successor, $+$ and $\times$ and predicate symbol $<$ is not axiomatizable as shown by Gödel in [6]. Herein we only consider axiomatizable theories, and, so, we identify a theory with a decidable set of sentences (the *axioms* of the theory) whose closure under semantic entailment coincides with the theory.

A theory Θ over a signature Σ has *quantifier elimination* if, for each formula φ there is a quantifier free formula φ^* such that $\Theta \vDash_\Sigma \varphi \Leftrightarrow \varphi^*$, and φ and φ^* have the same free variables. As examples of theories enjoying quantifier elimination note that the first-order theory of the:

- natural numbers with successor;

[1] In this work we consider first-order logic with equality $\cong$.

- natural numbers with successor, +, -, <;
- divisible torsion-free Abelian groups;
- term algebras;
- atomless Boolean algebras;
- algebraically closed fields;
- ordered real closed fields;
- differentially closed fields;
- dense orders without limits;

enjoy quantifier elimination. Amongst the results establishing sufficient conditions for quantifier elimination, there is one that is widely used when proving this property by symbolic and constructive techniques. It says that a theory Θ has quantifier elimination providing that for each formula

$$\exists x\, \varphi$$

where φ is a quantifier free formula, there is a formula φ^* such that: (i) φ is a quantifier free formula; (ii) φ has the same free variables of φ^* with the exception of x; and (iii) $\Theta \vdash_\Sigma (\exists x\, \varphi) \Leftrightarrow \varphi^*$.

3 Iteration

We start by defining when a model $\bar{I}$ of a theory Υ is an *algebraic prime extension of a model* I of a theory Δ over the same signature as Υ and such that $\text{Mod}(\Upsilon) \subseteq \text{Mod}(\Delta)$: this means that there is an embedding $\bar{\eta}_I : I \to \bar{I}$ such that for every embedding $h : I \to I'$ with I' in $\text{Mod}(\Upsilon)$ there is an embedding $h' : \bar{I} \to I'$ with $h = h' \circ \bar{\eta}_I$ (see Figure 2). In this case $\bar{I}$ is also said to be *algebraically prime with respect to* I *via* $\bar{\eta}_I$.

So, given first-order theories Δ and Υ over the same signature, we say that Δ has *algebraically prime models with respect* to Υ whenever

- $\text{Mod}(\Upsilon) \subseteq \text{Mod}(\Delta)$; and
- there is a map F that associates to each model I of Δ an *algebraically prime model* $\bar{I} = F(I)$ of Υ;

When Δ is $\Upsilon^\forall$, this notion coincides with the notion of Υ having *algebraically prime models.*[2]

[2]Recall that, for any first-order theory Θ, $\Theta^\forall$ is the set of all sentences entailed by Θ of the form $\forall x_1 \ldots \forall x_n \varphi$ where φ is a quantifier free formula.

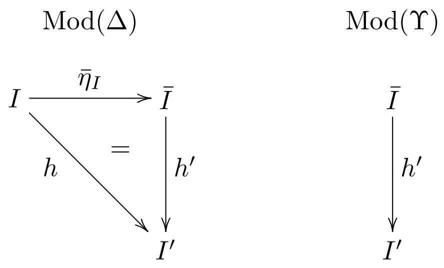

Figure 2: Δ has algebraically prime models with respect to Υ.

In general, the proof that a theory Δ has algebraically prime models with respect to another theory Υ, can be split in the following three steps: (a) definition of map $F : \mathrm{Mod}(\Delta) \to \mathrm{Mod}(\Upsilon)$; (b) for each model I of Δ, the definition of the embedding $\bar{\eta}_I$; (c) verification of the universal property in Figure 2.

Looking at several examples we arrived at the conclusion that the map F from $\mathrm{Mod}(\Delta)$ to $\mathrm{Mod}(\Upsilon)$ can be obtained by using an iterative construction over a one-step map E from $\mathrm{Mod}(\Delta)$ to $\mathrm{Mod}(\Delta)$ where $E(I)$ is closer to being algebraically prime with respect to I. By iteratively applying E to I we obtain $F(I)$ which is an algebraically prime model with respect to I.

The idea is that for showing that a theory has quantifier elimination using this method, it is only necessary to worry about the one step construction. A general result will state that if the one step construction fulfills certain properties then Δ will have algebraically prime models with respect to Υ. With this purpose in mind, assume that we start with:

- first-order theories Δ and Υ contained in $\forall_2$ over a signature Σ and with $\mathrm{Mod}(\Upsilon) \subseteq \mathrm{Mod}(\Delta)$;[3]
- a map $E : \mathrm{Mod}(\Delta) \to \mathrm{Mod}(\Delta)$;
- a family of embeddings $\eta = \{\eta_I : I \to E(I)\}_{I \in \mathrm{Mod}(\Delta)}$.

We say that E *extends in one step* Δ *towards* Υ *via* η whenever:

- E is *quasi-adjoint for* Υ *via* η, i.e.:
  - for every I in $\mathrm{Mod}(\Delta)$, I' in $\mathrm{Mod}(\Upsilon)$ and embedding $h : I \to I'$ there is an embedding h' from $E(I)$ to I' such that $h' \circ \eta_I = h$;
- E *increments local satisfaction for* Υ *via* η, i.e.,

[3]Recall that $\forall_2$ is the smallest class of formulas containing $\exists_1$ and closed under $\wedge$, $\vee$ and adding universal quantifiers at the front, where $\exists_1$ is the smallest class of formulas containing the quantifier free formulas and closed under $\wedge$, $\vee$ and adding existential quantifiers at the front. Observe that every $\forall_2$ formula is equivalent to a $\forall_2$ formula $\forall x_1 \ldots \forall x_n \psi$ with ψ in $\exists_1$ (for more details see for instance Section 2.4 of [7]). From now on we assume without loss of generality that the $\forall_2$ formulas are of this form, i.e., of the form $\forall x_1 \ldots \forall x_n \psi$ with ψ in $\exists_1$.

– for every $\forall x_1 \ldots \forall x_n \varphi$ in $\Upsilon \setminus \Delta$ where φ does not have universal quantifiers, I in $\mathrm{Mod}(\Delta)$ and assignment ρ over I, if $I\rho \not\Vdash_\Sigma \varphi$ then $E(I)\eta_I \circ \rho \Vdash_\Sigma \varphi$.[4]

The objective now is to prove, using these assumptions, that Δ has algebraically prime models with respect to Υ. We start by introducing E^n and $\eta^{p,q}$. Let

$$E^n : \mathrm{Mod}(\Delta) \to \mathrm{Mod}(\Delta) \text{ for } n \in \mathbb{N}$$

be the family of maps inductively defined as follows: E^0 is $\mathrm{id}_{\mathrm{Mod}(\Delta)}$ and E^{n+1} is $E \circ E^n$. Moreover, given natural numbers p and q with $p \leq q$, let $\eta^{p,q}$ be the family

$$\{\eta_I^{p,q} : E^p(I) \to E^q(I)\}_{I \in \mathrm{Mod}(\Delta)}$$

of embeddings inductively defined on $q-p$ as follows: $\eta_I^{p,p}$ is $\mathrm{id}_{E^p(I)}$ and $\eta_I^{p,q} = \eta_{E^{q-1}(I)} \circ \eta_I^{p,q-1}$.

Finally, we define the map E^ω as well as the family of embeddings η^ω. Given $I \in \mathrm{Mod}(\Delta)$, let

$$E^\omega(I) = (D_\omega, \cdot^{\mathsf{F}_\omega}, \cdot^{\mathsf{P}_\omega})$$

be as follows:

- D_ω is the quotient set of $\uplus_{n\in\mathbb{N}} D_n$ by the binary relation $\sim$ where

  $$d \sim e \quad \text{iff} \quad \eta_I^{k,m}(d) = e$$

  for every $d \in D_k, e \in D_m$ with $k, m \in \mathbb{N}$ and $k \leq m$;
- $f^{\mathsf{F}_\omega}([d_1], \ldots, [d_n]) = [f^{\mathsf{F}_k}(\eta_I^{k_1,k}(d_1), \ldots, \eta_I^{k_n,k}(d_n))]$, whenever $d_i \in D_{k_i}$ for $i = 1, \ldots, n$ and $k = \max\{k_1, \ldots, k_n\}$;
- $p^{\mathsf{P}_\omega}([d_1], \ldots, [d_n]) = p^{\mathsf{P}_k}(\eta_I^{k_1,k}(d_1), \ldots, \eta_I^{k_n,k}(d_n))$, whenever $d_i \in D_{k_i}$ for $i = 1, \ldots, n$ and $k = \max\{k_1, \ldots, k_n\}$.

and let η_I^ω be the family $\{\eta_I^{n,\omega} : E^n(I) \to E^\omega(I)\}_{n\in\mathbb{N}}$ of embeddings such that $\eta_I^{n,\omega}(d) = [d]$. We denote by η^ω the map $\eta^{0,\omega}$. Observe that $(E^\omega(I), \eta_I^\omega)$ is a direct limit of the directed diagram

$$(\{E^n(I)\}_{n\in\mathbb{N}}, \{\eta_I^{p,q} : E^p(I) \to E^q(I)\}_{p,q\in\mathbb{N}, p\leq q}).$$

[4]Given a signature Σ, an interpretation structure $I = (D, \cdot^F, \cdot^P)$ over Σ, a variable assignment $\rho : X \to D$, and a first-order formula φ, we denote by $I\rho \Vdash_\Sigma \varphi$ the satisfaction of φ by I and ρ. Recall that this relation is inductively defined as follows:

* $I\rho \Vdash_\Sigma p(t_1, \ldots, t_n)$ whenever $([\![t_1]\!]^{I\rho}, \ldots, [\![t_n]\!]^{I\rho}) \in p^P$ for every terms $t_1, \ldots, t_n$ and n-ary predicate symbol p, where $[\![t]\!]^{I\rho}$ is the interpretation of term t over I and ρ, inductively defined as follows: (a) $[\![x]\!]^{I\rho} = \rho(x)$ for every variable x; and (b) $[\![f(t_1, \ldots, t_n)]\!]^{I\rho} = f^F([\![t_1]\!]^{I\rho}, \ldots, [\![t_n]\!]^{I\rho})$ for every n-ary function symbol f and terms $t_1, \ldots, t_n$;
* $I\rho \Vdash_\Sigma \neg\varphi_1$ whenever $I\rho \not\Vdash_\Sigma \varphi_1$;
* $I\rho \Vdash_\Sigma (\varphi_1 \Rightarrow \varphi_2)$ whenever $I\rho \Vdash_\Sigma \varphi_1$ implies $I\rho \Vdash_\Sigma \varphi_2$;
* $I\rho \Vdash_\Sigma \forall x \varphi_1$ whenever for every ρ' over I with $\rho'(y) = \rho(y)$ for every $y \neq x$, $I\rho' \Vdash_\Sigma \varphi_1$.

The idea is that F is E^ω and $\bar{\eta}_I$ is η_I^ω for each model I of Δ. We start by proving that $E^\omega(I)$ is a model of Υ whenever I is a model of Δ, provided that E satisfies the conditions above.

Proposition 3.1 *Assuming that E extends in one step Δ towards Υ via η, then, E^ω is a map from $Mod(\Delta)$ to $Mod(\Upsilon)$.*

Proof: We must show that for each model I of Δ, $E^\omega(I)$ is a model of Υ. Let $\forall x_1 \dots \forall x_n \varphi$ be a sentence in Υ where φ does not have universal quantifiers, and ρ^ω an assignment over $E^\omega(I)$. Denote by $\{y_1, \dots, y_m\}$ the set of variables that occur free in φ and for each $j = 1, \dots, m$, let k_j be a natural number such that $\rho^\omega(y_j) \in \eta_I^{k_j,\omega}(E^{k_j}(I))$. Moreover, denote by k the maximum of $\{k_1, \dots, k_m\}$ and let ρ^k be an assignment over $E^k(I)$ such that

$$\eta_I^{k,\omega}(\rho^k(y_j)) = \rho^\omega(y_j)$$

for $j = 1, \dots, m$. Observe that such an assignment exists since

$$\eta_I^{k,\omega} \circ \eta_I^{k_j,k} = \eta_I^{k_j,\omega}.$$

One of the following two cases hold:

(a) $\forall x_1 \dots \forall x_n \varphi \in \Delta$. Then $I \Vdash_\Sigma \forall x_1 \dots \forall x_n \varphi$ and so $E^\omega(I) \Vdash_\Sigma \forall x_1 \dots \forall x_n \varphi$ since satisfaction of $\forall_2$ sentences is preserved by directed limits, see Theorem 2.4.4 and Theorem 2.4.6 in [7].

(b) $\forall x_1 \dots \forall x_n \varphi \notin \Delta$. Then, one of the following two cases hold:

(i) $E^k(I), \rho^k \Vdash_\Sigma \varphi$. Then $E^\omega(I), \eta_I^{k,\omega} \circ \rho^k \Vdash_\Sigma \varphi$ since φ is in the closure for $\wedge$ and $\vee$ of the class of formulas $\exists_1$, $\eta_I^{k,\omega}$ is an embedding from $E^k(I)$ to $E^\omega(I)$, and embeddings preserve satisfaction of such formulas. Therefore $E^\omega(I), \rho^\omega \Vdash_\Sigma \varphi$.

(ii) $E^k(I), \rho^k \nVdash_\Sigma \varphi$. Then, since E via η increments local satisfaction for Υ,

$$E^{k+1}(I), \eta_{E^k(I)} \circ \rho^k \Vdash_\Sigma \varphi$$

and so, since φ is in the closure for $\wedge$ and $\vee$ of the class of formulas $\exists_1$, $\eta_I^{k+1,\omega}$ is an embedding from $E^{k+1}(I)$ to $E^\omega(I)$, and embeddings preserve satisfaction of such formulas,

$$E^\omega(I), \eta_I^{k+1,\omega} \circ \eta_{E^k(I)} \circ \rho^k \Vdash_\Sigma \varphi.$$

Therefore, $E^\omega(I), \rho^\omega \Vdash_\Sigma \varphi$ since $\eta_I^{k+1,\omega} \circ \eta_{E^k(I)} = \eta_I^{k,\omega}$. QED

We now show that Δ has algebraically prime models with respect to Υ, under the conditions above.

Theorem 3.2 *Assuming that E extends in one step Δ towards Υ via η, then for each model I of Δ, $E^\omega(I)$ is algebraically prime with respect to I via η_I^ω. Hence, Δ has algebraically prime models with respect to Υ.*

Proof: Let I be a model of Δ, I' a model of Υ and h an embedding of I into I', Consider a family $\{g^k : E^k(I) \to I'\}_{k \in \mathbb{N}}$ of embeddings where:

- g^0 is h;
- g^k is such that $g^k \circ \eta_{E^{k-1}(I)} = g^{k-1}$ (there is such an embedding since E via η is quasi-adjoint for Υ).

Observe that

$$(\dagger) \quad g^q \circ \eta_I^{p,q} = g^p$$

for any natural numbers p and q with $p \leq q$, as can be shown by induction on $q - p$.

Let $h' : E^\omega(I) \to I'$ be such that

$$h'(b) = g^k(a)$$

where $k \in \mathbb{N}$ and $a \in E^k(I)$ are such that $b = \eta_I^{k,\omega}(a)$. Then:

(1) h' is well defined.

Let n be a natural number and d in $E^n(I)$ such that $b = \eta_I^{n,\omega}(d)$. Suppose without loss of generality that $k \leq n$. Then, $\eta_I^{k,n}(a) = d$, and, so, by (†), $g^k(a) = g^n(\eta_I^{k,n}(a)) = g^n(d)$;

(2) h' is injective.

Assume that $h'(b_1) = h'(b_2)$. Let k_1 and k_2 be natural numbers, and a_1 and a_2 elements of $E^{k_1}(I)$ and $E^{k_2}(I)$ respectively, such that $\eta_I^{k_1,\omega}(a_1) = b_1$ and $\eta_I^{k_2,\omega}(a_2) = b_2$. Then $g^{k_1}(a_1) = h'(b_1) = h'(b_2) = g^{k_2}(a_2)$ taking into account the definition of h'. Without loss of generality assume that $k_1 \leq k_2$. Then $g^{k_2}(\eta_I^{k_1,k_2}(a_1)) = g^{k_1}(a_1)$ by (†). Hence $\eta_I^{k_1,k_2}(a_1) = a_2$ since g^{k_2} is injective. Therefore $\eta_I^{k_1,\omega}(a_1) = \eta_I^{k_2,\omega}(a_2)$, and, so, $b_1 = b_2$;

(3) h' is an homomorphism. Straightforward;

(4) $h' \circ \eta_I^{0,\omega} = h$. Indeed: let d be an arbitrary element of I. Then $h'(\eta_I^{0,\omega}(d)) = g^0(d)$ by definition of h'. The thesis follows since $g^0(d)$ is $h(d)$. QED

We now illustrate these notions and results with several examples involving the theory of natural numbers with successor, the theory of term algebras and the theory of algebraically closed fields.

Natural Numbers with Successor

Let Σ_{S} be the signature with $F_0 = \{0\}$ and $F_1 = \{\mathbf{S}\}$, for the theory Θ_{S} containing the sentences:

S1 $\forall x(\neg(\mathbf{S}\, x \cong 0))$;

S2 $\forall x \forall y((\mathbf{S}\, x \cong \mathbf{S}\, y) \Rightarrow (x \cong y))$;

S3 $\forall y((\neg(y \cong 0)) \Rightarrow (\exists x(y \cong \mathbf{S}\, x)))$;

S4 $\forall x(\neg(\mathbf{S}^n\, x \cong x))$ for each $n \in \mathbb{N}^+$;

for the natural numbers with successor, see Section 3.1 of [3]. Taking into account Theorem 3.2, in order to show that $\Theta_{\mathrm{S}}^{\forall}$ has algebraically prime models with respect to Θ_{S}, it is enough to show the following conditions:

1. the sentences of $\Theta_{\mathrm{S}}^{\forall}$ and Θ_{S} are in $\forall_2$;
2. $\mathrm{Mod}(\Theta_{\mathrm{S}}) \subseteq \mathrm{Mod}(\Theta_{\mathrm{S}}^{\forall})$;
3. there is a map $E : \mathrm{Mod}(\Theta_{\mathrm{S}}^{\forall}) \to \mathrm{Mod}(\Theta_{\mathrm{S}}^{\forall})$ and a family of embeddings $\eta = \{\eta_I : I \to E(I)\}_{I \in \mathrm{Mod}(\Theta_{\mathrm{S}}^{\forall})}$ such that
   (a) E via η is quasi-adjoint for Θ_{S};
   (b) E via η increments local satisfaction for Θ_{S}.

It is immediate to see that Conditions 1 and 2 hold. We now show that Condition 3 also holds. Let $E : \mathrm{Mod}(\Theta_{\mathrm{S}}^{\forall}) \to \mathrm{Mod}(\Theta_{\mathrm{S}}^{\forall})$ be such that

$$E((D, \cdot^F, \cdot^P)) = (D^\bullet, \cdot^{F^\bullet}, \cdot^{P^\bullet})$$

where:

- $D^\bullet$ is $D \cup \{d^\bullet : d^\bullet \notin D, d \in D \setminus \{0^F\},$ there is no e in D with $\mathbf{S}^F(e) = d\}$;
- $0^{F^\bullet} = 0^F$;
- $\mathbf{S}^{F^\bullet}(e) = \begin{cases} \mathbf{S}^F(e) & \text{if } e \text{ is in } D \\ d & \text{if } e \text{ is } d^\bullet; \end{cases}$

and η be the family $\{\eta_I : I \to E(I)\}_{I \in \mathrm{Mod}(\Theta_{\mathrm{S}}^{\forall})}$ where each η_I is the inclusion of I into $E(I)$.

Clearly, for each model I of $\Theta_{\mathrm{S}}^{\forall}$, $E(I)$ is also a model of $\Theta_{\mathrm{S}}^{\forall}$. In fact, observe that I is a substructure of a model of Θ_{S}, see Exercise 2.5.10 of [9]. Hence, $E(I)$ is also a substructure of that model, and, so, it is a model of $\Theta_{\mathrm{S}}^{\forall}$.

Proposition 3.3 *The map E via η is quasi-adjoint for Θ_S.*

Proof: Let $I = (D, \cdot^F, \cdot^P)$ be a model of $\Theta_{\mathrm{S}}^{\forall}$, $I' = (D', \cdot^{F'}, \cdot^{P'})$ a model of Θ_{S}, h an embedding from I to I', and h' a map from $E(I)$ to I' defined as follows:

$$h'(e) = \begin{cases} h(e) & \text{if } e \text{ is in } D \\ (\mathbf{S}^{F'})^{-1}(h(d)) & \text{if } e \text{ is } d^\bullet. \end{cases}$$

Then:

(1) h' is well defined. It is enough to see that if e is $d^\bullet$ then there is one and only one e' in D' with $\mathbf{S}^{F'}(e') = h(d)$. Indeed: by S3 there is one e' in D' with $\mathbf{S}^{F'}(e') = h(d)$ since $h(d)$ is not $0^{F'}$ because d is not 0^F and h is an embedding. There is at most one e' in D' with $\mathbf{S}^{F'}(e') = h(d)$ since $\mathbf{S}^{F'}$ is one to one by S2.

(2) h' is one to one. Since h is an embedding, it is enough to show that:

(i) for each e_1 of the form $d_1^\bullet$, $h'(e_1) \notin h(D)$. Suppose by contradiction that $h'(e_1) \in h(D)$ and let $d \in D$ be such that $h'(e_1) = h(d)$. Then $(\mathbf{S}^{F'})^{-1}(h(d_1)) = h(d)$. Hence $h(d_1) = \mathbf{S}^{F'}((\mathbf{S}^{F'})^{-1}(h(d_1))) = \mathbf{S}^{F'}(h(d)) = h(\mathbf{S}^F(d))$. Since h is one to one then $d_1 = \mathbf{S}^F(d)$ which contradicts the fact that there is no e in D with $\mathbf{S}^F(e) = d_1$;

(ii) for each e_1 and e_2 of the forms $d_1^\bullet$ and $d_2^\bullet$ respectively, if $h'(e_1) = h'(e_2)$ then $e_1 = e_2$. Suppose that $h'(e_1) = h'(e_2)$. Then $(\mathbf{S}^{F'})^{-1}(h(d_1)) = (\mathbf{S}^{F'})^{-1}(h(d_2))$ and so $h(d_1) = \mathbf{S}^{F'}((\mathbf{S}^{F'})^{-1}(h(d_1))) = \mathbf{S}^{F'}((\mathbf{S}^{F'})^{-1}(h(d_2))) = h(d_2)$. Since h is an embedding then $d_1 = d_2$, and so $e_1 = e_1$.

(3) $h'(0^{F^\bullet}) = 0^{F'}$. Indeed: $h'(0^{F^\bullet}) = h'(0^F) = h(0^F) = 0^{F'}$ since h is an embedding;

(4) $h'(\mathbf{S}^{F^\bullet}(e)) = \mathbf{S}^{F'}(h'(e))$. In order to show this, consider two cases:

(i) $e \notin D$. Suppose e is $d^\bullet$. Then $h'(\mathbf{S}^{F^\bullet}(e)) = h'(d) = h(d) = \mathbf{S}^{F'}((\mathbf{S}^{F'})^{-1}(h(d))) = \mathbf{S}^{F'}(h'(e))$;

(ii) e is in D. Then $h'(\mathbf{S}^{F^\bullet}(e)) = h'(\mathbf{S}^F(e)) = h(\mathbf{S}^F(e)) = \mathbf{S}^{F'}(h(e)) = \mathbf{S}^{F'}(h'(e))$.

(5) $h' \circ \eta_I = h$. Indeed, let $d \in D$. Then $h'(\eta_I(d)) = h'(d) = h(d)$. QED

Proposition 3.4 *The map E via η increments local satisfaction for Θ_S.*

Proof: Let $I = (D, \cdot^F, \cdot^P)$ be a model of $\Theta_{\mathrm{S}}^\forall$ and ρ an assignment over I. Observe that $\Theta_{\mathrm{S}} \setminus \Theta_{\mathrm{S}}^\forall = \{\mathrm{S3}\}$. Suppose $I\rho \not\Vdash_{\Sigma_{\mathrm{S}}} (\neg(y \cong 0)) \Rightarrow (\exists x(y \cong \mathbf{S}\,x))$. Then $I\rho \Vdash_{\Sigma_{\mathrm{S}}} \neg(y \cong 0)$ and $I\rho \not\Vdash_{\Sigma_{\mathrm{S}}} \exists x(y \cong \mathbf{S}\,x)$. Hence $\rho(y) \neq 0^F$ and there is no e in D with $\rho(y) = \mathbf{S}^F(e)$. Observe that $\rho(y)^\bullet \in E(I)$, $\rho(y) = \mathbf{S}^{F^\bullet}(\rho(y)^\bullet)$ and $\eta_I(\rho(y)) = \rho(y)$. Therefore $E(I)\eta_I \circ \rho \Vdash_{\Sigma_{\mathrm{S}}} \exists x(y \cong \mathbf{S}\,x)$ and so $E(I)\eta_I \circ \rho \Vdash_{\Sigma_{\mathrm{S}}} (\neg(y \cong 0)) \Rightarrow (\exists x(y \cong \mathbf{S}\,x))$. QED

So, taking into account that E via η is quasi-adjoint for Θ_{S} and increments local satisfaction for Θ_{S}, see Proposition 3.3 and Proposition 3.4 respectively, we can use Theorem 3.2 to conclude that the following theorem holds.

Theorem 3.5 *Theory Θ_S has algebraically prime models.*

Term Algebras

Consider the first-order theory for term algebras induced by a given first-order signature with no predicate symbols, see [7]. We show that it has algebraically prime models using our iteration criteria.

Given a signature Σ with no predicate symbols, let Σ_{ta} be the signature induced by Σ with $F_{\mathrm{ta}_1} = \{f_i : n \in \mathbb{N}, f \in F_n \text{ and } i \in \{1, \ldots, n\}\}$ and $P_{\mathrm{ta}_1} = \{\mathbf{Is}_c : c \in F_0\} \cup \{\mathbf{Is}_f : n \in \mathbb{N} \text{ and } f \in F_n\}$ for the theory Θ_{ta} containing the sentences:

T1 $\exists^1 x\, \mathbf{Is}_c(x)$
for each constant symbol c in F_0;

T2 $\forall x_1 \ldots \forall x_n \exists^1 x(\mathbf{Is}_f(x) \wedge f_1(x) \cong x_1 \wedge \ldots \wedge f_n(x) \cong x_n)$
for each function symbol f in F_n and $n \in \mathbb{N}$;

T3 $\forall x\, \neg(\mathbf{Is}_{e_1}(x) \wedge \mathbf{Is}_{e_2}(x))$
where e_1 and e_2 are distinct constant or function symbols;

T4 $\forall x((\neg\, \mathbf{Is}_f(x)) \Rightarrow f_i(x) \cong x)$
for each function symbol f in F_n, $n \in \mathbb{N}$ and $i = 1, \ldots, n$;

T5 $\forall x((t(f_i(x)) \cong x) \Rightarrow \neg\, \mathbf{Is}_f(x))$
for each function symbol f in F_n, $n \in \mathbb{N}$, and sequence t of function symbols over Σ_{ta}, and $i = 1, \ldots, n$.

The idea is that, given a model of Θ_{ta}, the values in its carrier set that correspond to terms of Σ are the ones for which one of the predicates $\mathbf{Is}\_$ hold. For example, if a value of the domain satisfies predicate $\mathbf{Is}_f$ for a function symbol f in Σ, then that value is the denotation of a term of Σ whose main constructor is f. With this in mind, observe that T1 guarantees that each constant of Σ has a unique value corresponding to it, and T2 does the same for each function symbol of Σ and domain values (corresponding or not to terms of Σ) as arguments. Axiom T3 ensures that each value in the domain of a model of Θ_{ta} corresponds to at most one term of Σ, and axiom T4 requires that if a domain value is not a representative of a term with main constructor f then its projection along f is itself. Finally, axiom T5 requires that any representative of a non-constant term is different from each of its arguments, as well as from each of the arguments of its arguments, and so on.

Note that when Σ is the signature Σ^{stc} with $F_0^{\mathrm{stc}} = \mathbb{N}$, $F_2^{\mathrm{stc}} = \{\mathsf{push}\}$ and the other sets are empty, then $\Theta_{\mathrm{ta}}^{\mathrm{stc}}$ is a first-order theory for stacks. Observe that, in this case, push_1 and push_2 correspond to the usual stack operations of pop and top.

Taking into account Theorem 3.2, in order to show that $\Theta_{\mathrm{ta}}^{\forall}$ has algebraically prime models with respect to Θ_{ta}, it is enough to show the following conditions:

1. the sentences of $\Theta_{\mathrm{ta}}^{\forall}$ and Θ_{ta} are in $\forall_2$;
2. $\mathrm{Mod}(\Theta_{\mathrm{ta}}) \subseteq \mathrm{Mod}(\Theta_{\mathrm{ta}}^{\forall})$;
3. there are a map $E : \mathrm{Mod}(\Theta_{\mathrm{ta}}^{\forall}) \to \mathrm{Mod}(\Theta_{\mathrm{ta}}^{\forall})$ and a family of embeddings $\eta = \{\eta_I : I \to E(I)\}_{I \in \mathrm{Mod}(\Theta_{\mathrm{ta}}^{\forall})}$ such that
   (a) E via η is quasi-adjoint for Θ_{ta};
   (b) E via η increments local satisfaction for Θ_{ta}.

It is immediate to see that Conditions 1 and 2 hold. We now show that Condition 3 also holds.

Let $E : \mathrm{Mod}(\Theta_{\mathrm{ta}}^{\forall}) \to \mathrm{Mod}(\Theta_{\mathrm{ta}}^{\forall})$ be such that

$$E((D, \cdot^F, \cdot^P)) = (D^*, \cdot^{F^*}, \cdot^{P^*})$$

where:

- D^* is the union of D with
  - $\{d_c : c \text{ in } F_0, d_c \notin D \text{ and there is no } e \text{ in } D \text{ with } \mathbf{Is}_c^F(e) = 1\}$;
  - $\{\langle d_1, \ldots, d_n\rangle_f : d_1, \ldots, d_n \in D, n \in \mathbb{N}, f \in F_n, \langle d_1, \ldots, d_n\rangle_f \notin D$ and there is no e in D with $\mathbf{Is}_f^F(e) = 1$ and $f_1^F(e) = d_1, \ldots, f_n^F(e) = d_n\}$;
- for every $c \in F_0$,

$$\mathbf{Is}_c^{F^*}(d) = \begin{cases} \mathbf{Is}_c^F(d) & \text{if } d \in D \\ 1 & \text{if } d \text{ is } d_c \\ 0 & \text{otherwise;} \end{cases}$$

- for every $f \in F_n$,

$$\mathbf{Is}_f^{F^*}(d) = \begin{cases} \mathbf{Is}_f^F(d) & \text{if } d \in D \\ 1 & \text{if } d \text{ is } \langle d_1, \ldots, d_n\rangle_f \\ 0 & \text{otherwise;} \end{cases}$$

- for every $f \in F_n$ and $i \in \{1, \ldots, n\}$,

$$f_i^{F^*}(d) = \begin{cases} f_i^F(d) & \text{if } d \in D \\ d_i & \text{if } d \text{ is } \langle d_1, \ldots, d_n\rangle_f \\ d & \text{otherwise;} \end{cases}$$

and η be the family $\{\eta_I : I \to E(I)\}_{I \in \mathrm{Mod}(\Theta_{\mathrm{ta}}^{\forall})}$ where each η_I is the inclusion of I into $E(I)$.

Clearly, for each model I of $\Theta_{\mathrm{ta}}^{\forall}$, $E(I)$ is also a model of $\Theta_{\mathrm{ta}}^{\forall}$, since I is a substructure of a model of Θ_{ta}, by Exercise 2.5.10 of [9], and, so, by definition, $E(I)$ is also a substructure of a model of Θ_{ta}. Hence, see Exercise 2.5.10 of [9], $E(I)$ is a model of $\Theta_{\mathrm{ta}}^{\forall}$.

Proposition 3.6 *The map E via η is quasi-adjoint for Θ_{ta}.*

Proof: Let $I = (D, \cdot^F, \cdot^P)$ be a model of $\Theta_{\mathrm{ta}}^{\forall}$, $I' = (D', \cdot^{F'}, \cdot^{P'})$ a model of Θ_{ta}, h an embedding from I to I', and h' a map from $E(I)$ to I' defined as follows:

$$h'(e) = \begin{cases} h(e) & \text{if } e \text{ is in } D \\ d' & \text{if } e \text{ is } d_c \text{ and } d' \text{ is such that } \mathbf{Is}_c^{F'}(d') = 1 \\ d' & \text{if } e \text{ is } \langle d_1, \ldots, d_n\rangle_f \text{ and } d' \text{ is such that } \mathbf{Is}_f^{F'}(d') = 1 \text{ and} \\ & f_i^{F'}(d') = h(d_i) \text{ for } i = 1, \ldots, n. \end{cases}$$

Then:

(1) h' is well defined. It is enough to see that if e is d_c then by axiom T1 there is a unique d' in D' with $\mathbf{Is}_c^{F'}(d') = 1$, and if e is $\langle d_1, \ldots, d_n \rangle_f$ then by axiom T2 there is a unique d' in D' with $\mathbf{Is}_f^{F'}(d') = 1$ and $f_i^{F'}(d') = h(d_i)$ for $i = 1, \ldots, n$.

(2) h' is one to one. Since h is an embedding, it is enough to show that:

(i) for each e in D^* of the form d_c, $h'(e) \notin h(D)$. Suppose by contradiction that $h'(e) \in h(D)$ and let $d \in D$ be such that $h'(e) = h(d)$. Observe that $\mathbf{Is}_c^{F'}(h'(e)) = 1$ and thus $\mathbf{Is}_c^{F'}(h(d)) = 1$. Hence $\mathbf{Is}_c^{F}(d) = 1$ since h is an embedding which contradicts the existence of d_c in D^*;

(ii) for each e in D^* of the form $\langle d_1, \ldots, d_n \rangle_f$, $h'(e) \notin h(D)$. The proof of this case is similar to the proof of case (i) and so we omit it;

(iii) for each e_1 and e_2 in D^* of the forms d_{c_1} and d_{c_2} respectively, if $h'(e_1) = h'(e_2)$ then $e_1 = e_2$. Suppose that $h'(e_1) = h'(e_2)$. Then by definition of h' the constants c_1 and c_2 are the same and so e_1 and e_2 are the same.

(iv) for each e_1 and e_2 in D^* of the forms $\langle d_1^1, \ldots, d_n^1 \rangle_{f_1}$ and $\langle d_1^2, \ldots, d_n^2 \rangle_{f_2}$ respectively, if $h'(e_1) = h'(e_2)$ then $e_1 = e_2$. The proof proceeds as in (iii) and so we omit it;

(v) for each e_1 and e_2 of the forms $\langle d_1, \ldots, d_n \rangle_f$ and d_c respectively, $h'(e_1) \neq h'(e_2)$. Immediate by definition of h' taking also into account axiom T3.

(3) $h'(f_i^{F^*}(e)) = f_i^{F'}(h'(e))$ for every $e \in D^*$. Consider the following three cases:

(i) e is in D. Then $h'(f_i^{F^*}(e)) = h'(f_i^{F}(e)) = h(f_i^{F}(e)) = f_i^{F'}(h(e)) = f_i^{F'}(h'(e))$;

(ii) e is not in D and is not of the form $\langle d_1, \ldots, d_n \rangle_f$ for some $d_1, \ldots, d_n$ in D. Then $h'(f_i^{F^*}(e)) = h'(e) = f_i^{F'}(h'(e))$ by axiom T4;

(iii) e is of the form $\langle d_1, \ldots, d_n \rangle_f$. Then $h'(f_i^{F^*}(e)) = h'(d_i) = h(d_i) = f_i^{F'}(h'(e))$ by definition of h';

(4) $\mathbf{Is}_f^{F^*}(e) = 1$ iff $\mathbf{Is}_f^{F'}(h'(e)) = 1$. Consider the following three cases:

(i) $e \in D$. Then $\mathbf{Is}_f^{F^*}(e) = 1$ iff $\mathbf{Is}_f^{F}(e) = 1$ iff $\mathbf{Is}_f^{F'}(h(e)) = 1$ iff $\mathbf{Is}_f^{F'}(h'(e)) = 1$;

(ii) $e \notin D$ and is not of the form $\langle d_1, \ldots, d_n \rangle_f$ for some $d_1, \ldots, d_n$ in D. Then $\mathbf{Is}_f^{F^*}(e) = 0 = \mathbf{Is}_f^{F'}(h'(e))$ by definition of h';

(iii) e is of the form $\langle d_1, \ldots, d_n \rangle_f$. Then $\mathbf{Is}_f^{F^*}(e) = 1 = \mathbf{Is}_f^{F'}(h'(e))$ by definition of h';

(5) $\mathbf{Is}_c^{F^*}(e) = 1$ iff $\mathbf{Is}_c^{F'}(h'(e)) = 1$. The proof of this case is omitted since it is similar to the proof of case (4).

(6) $h' \circ \eta_I = h$. Indeed: $h'(\eta_I(d)) = h'(d) = h(d)$ for every $d \in D$. QED

Proposition 3.7 *The map E via η increments local satisfaction for Θ_{ta}.*

Proof: Let $I = (D, \cdot^F, \cdot^P)$ be a model of $\Theta_{\mathrm{ta}}^{\forall}$ and ρ an assignment over I. Observe that, in order to simplify the presentation of Θ_{ta}, the axioms $T1$ and $T2$

use the quantifier $\exists^1$. This quantifier is used as an abbreviation of two sentences, one having an existential quantifier and the other with only universal quantifiers. In the context of this proof it is important that we not use these abbreviations and so we are seeing axiom T_1 as two axioms, and the same for axiom T_2. Then, $\Theta_{\rm ta} \setminus \Theta_{\rm ta}^{\forall}$ is the set $\{\forall x_1 \ldots \forall x_n \exists x(\mathbf{Is}_f(x) \wedge f_1(x) \cong x_1 \wedge \ldots \wedge f_n(x) \cong x_n), \exists x_1\, \mathbf{Is}_c(x_1)\}$.

(1) Suppose $I\rho \not\Vdash_{\Sigma_{\rm ta}} \exists x_1\, \mathbf{Is}_c(x_1)$. Then $\mathbf{Is}_c^F(d) = 0$ for every d in D. Hence $d_c \in D^*$ and moreover $\mathbf{Is}_c^{F^*}(d_c) = 1$. Thus $E(I)\eta_I \circ \rho \Vdash_{\Sigma_{\rm ta}} \exists x_1\, \mathbf{Is}_c(x_1)$.

(2) Suppose $I\rho \not\Vdash_{\Sigma_{\rm ta}} \exists x(\mathbf{Is}_f(x) \wedge f_1(x) \cong x_1 \wedge \ldots \wedge f_n(x) \cong x_n)$. Then there is no e in D with $\mathbf{Is}_f^F(e) = 1$ and $f_1^F(e) = \rho(x_1), \ldots, f_n^F(e) = \rho(x_n)$. Hence $\langle \rho(x_1), \ldots, \rho(x_n)\rangle_f \in D^*$ with $f_i^{F^*}(\langle \rho(x_1), \ldots, \rho(x_n)\rangle_f) = \rho(x_i)$ for every $i = 1, \ldots, n$ and $\mathbf{Is}_f^{F^*}(\langle \rho(x_1), \ldots, \rho(x_n)\rangle_f) = 1$. Thus $E(I)\eta_I \circ \rho \Vdash_{\Sigma_{\rm ta}} \exists x(\mathbf{Is}_f(x) \wedge f_1(x) \cong x_1 \wedge \ldots \wedge f_n(x) \cong x_n)$. QED

So, taking into account that E via η is quasi-adjoint for $\Theta_{\rm ta}$ and increments local satisfaction for $\Theta_{\rm ta}$, see Proposition 3.6 and Proposition 3.7 respectively, we can use Theorem 3.2 to conclude that the following theorem holds.

Theorem 3.8 *Theory Θ_{ta} has algebraically prime models.*

Algebraically Closed Fields – from $\Theta_{\rm f}$ to $\Theta_{\rm acf}$

Let $\Sigma_{\rm f}$ be the signature for fields, that is: $F_0 = \{0, 1\}$, $F_1 = \{-\}$, and $F_2 = \{+, \times\}$, $\Theta_{\rm f}$ the theory containing the field axioms, and $\Theta_{\rm acf}$ the theory over $\Sigma_{\rm f}$ containing the axioms for algebraically closed fields, see [9, 7, 8]. We start by stating a well known theorem, Theorem 2.5 of [8], used extensively in this example.

Theorem 3.9 *Let k be a field. Then there exists an algebraically closed field containing k as a subfield.*

Proof: This is Theorem 2.5 in Chapter 5 of [8]. QED

Moreover, let $E = (\mathrm{QF} \circ \mathrm{PR})$ where PR is a map that associates to each field I a polynomial ring $I[V_I]$ where V_I is a set of names v_q in bijection with the set of all polynomials q in $I[x]$ with degree greater than 0, and QF associates to each polynomial ring $I[V_I]$ a field generated by the quotient with a maximal ideal containing the ideal generated by all polynomials $q(v_q)$ in $I[V_I]$ (see Figure 3 – for details consult the proof of Theorem 3.9 – Theorem 2.5 in Chapter 5 of [8]). Observe that E associates with each field an algebraic extension of it where every polynomial in one variable of degree at least one with coefficients in the field has a root.

Take η to be the family

$$\{\eta_I : I \to \mathrm{QF}(\mathrm{PR}(I))\}_{I \in \mathrm{Mod}(\Theta_{\rm f})}$$

of embeddings such that $\eta_I(d) = [p_d^I]$ where p_d^I is the constant polynomial d in $I[V_I]$.

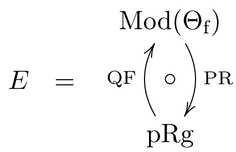

Figure 3: Definition of E as $\mathrm{QF} \circ \mathrm{PR}$.

Proposition 3.10 *The map $QF \circ PR$ via η is quasi-adjoint for Θ_{acf}.*

Proof: Let I' be a model of Θ_{acf} and $h : I \to I'$ an embedding. Since $\mathrm{QF} \circ \mathrm{PR}(I)$ is an algebraic extension of $\eta_I(D)$, there is an embedding $\bar{h}$ from $\mathrm{QF} \circ \mathrm{PR}(I)$ into I' such that $\bar{h} \circ \eta_I = h$ by Theorem 2.8 in Chapter 5 of [8]. QED

Proposition 3.11 *The map $QF \circ PR$ via η increments local satisfaction for Θ_{acf}.*

Proof: Let I be a model of Θ_{f} and ρ an assignment over I. Let n in $\mathbb{N}^+$ and γ be the formula $\exists y\,(y^n + x_1 y^{n-1} + \cdots + x_n \cong 0)$. Assume that $I\rho \nVdash_{\Sigma_{\mathrm{f}}} \exists y\,(y^n + x_1 y^{n-1} + \cdots + x_n \cong 0)$. Let q be the polynomial in $I[x]$ of the form $x^n + \rho(x_1)x^{n-1} + \ldots + \rho(x_{n-1})x^1 + \rho(x_n)$. Then, as explained in the proof of Theorem 3.9 (Theorem 2.5 of [8]) in Chapter 5 of [8], $\eta_I \circ q$ has a root in $\mathrm{PR}(\mathrm{QF}(I))$. Consider an assignment ρ' over $\mathrm{PR}(\mathrm{QF}(I))$ such that $\rho'(z) = \eta_I \circ \rho(z)$ for every variable $z \neq y$, such that $\rho'(y)$ is that root. Then $\mathrm{PR}(\mathrm{QF}(I))\rho' \Vdash_{\Sigma_{\mathrm{f}}} y^n + x_1 y^{n-1} + \cdots + x_n \cong 0$ and so $\mathrm{PR}(\mathrm{QF}(I))\eta_I \circ \rho \Vdash_{\Sigma_{\mathrm{f}}} \exists y(y^n + x_1 y^{n-1} + \cdots + x_n \cong 0)$. QED

Since $\mathrm{QF} \circ \mathrm{PR}$ via η is quasi-adjoint for Θ_{acf} and increments local satisfaction for Θ_{acf} by Proposition 3.10 and Proposition 3.11 respectively, we can use Theorem 3.2 to conclude that the following corollary holds.

Theorem 3.12 *Theory Θ_f has algebraically prime models with respect to Θ_{acf}.*

Observe that, although ACF admits quantifier elimination, it is not decidable. To obtain decidability, we must add an axiom to ACF specifying a fixed integral characteristic. Then, we can see that ACF_0 has QE and that the algebraic numbers (the algebraic closure of the rationals $\mathbb{Q}$) are a prime model for ACF_0.

4 Adjunction

An adjunction between two categories establishes a deep relationship between their objects and morphisms. Given first-order theories Υ and Δ with

$$\mathrm{Mod}(\Upsilon) \subseteq \mathrm{Mod}(\Delta),$$

we show in this section that every model of Δ has an algebraically prime model in Υ if the inclusion functor from the category of models of Υ and their embeddings into the category of models of Δ and their embeddings has a left adjoint.

We start by briefly recalling what is a natural transformation and a left adjoint. Given functors $F, H : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\alpha : F \to H$ is a family

$$\alpha = \{\alpha_c : F(c) \to H(c)\}_{c \in |\mathbf{C}|}$$

of morphisms in $\mathbf{D}$ such that

$$H(f) \circ \alpha_{c_1} = \alpha_{c_2} \circ F(f)$$

for every morphism $f : c_1 \to c_2$ in $\mathbf{C}$. Moreover, F is said to be *left adjoint* of functor H, denoted by

$$F \dashv H$$

if there is a natural transformation

$$\eta : \mathrm{id}_{\mathbf{C}} \to H \circ F,$$

called the *unit* of the adjunction satisfying the following universal property: Given any morphism $h : c \to H(d)$ in $\mathbf{C}$, there is a unique morphism $\bar{h} : F(c) \to d$ in $\mathbf{D}$ such that

$$H(\bar{h}) \circ \eta_c = h.$$

Theorem 4.1 *Let Υ and Δ be first-order theories with Mod(Υ) $\subseteq$ Mod(Δ). If the inclusion functor from Mod(Υ) to Mod(Δ) has a left adjoint*

$$\bar{E}$$

with unit η, then, $\bar{E}(I)$ is algebraically prime with respect to I via η_I.

Proof: Denote the inclusion functor from $\mathrm{Mod}(\Upsilon)$ to $\mathrm{Mod}(\Delta)$ by $J_{\Upsilon\Delta}$. Let I be a model of Δ, I' a model of Υ, and $h : I \to I'$ an embedding. Let h' be the unique embedding from $\bar{E}(I)$ to I' such that $h = J_{\Upsilon\Delta}(h') \circ \eta_I$, which exists since E is a left adjoint of $J_{\Upsilon\Delta}$ with unit η. The thesis follows immediately since $J_{\Upsilon\Delta}(h') = h'$. QED

Algebraically Closed Fields – from $\Theta^{\forall}_{\mathbf{acf}}$ to $\Theta_{\mathbf{f}}$

Observe that the models of $\Theta^{\forall}_{\mathrm{acf}}$ are the integral domains since

$$\Theta_{\mathrm{acf}} \vDash_{\Sigma_{\mathrm{f}}} \forall x_1 \forall x_2\, (((\neg(x_1 \cong 0)) \wedge (\neg(x_2 \cong 0))) \Rightarrow (\neg(x_1 \times x_2) \cong 0)),$$

and that every field is an integral domain, i.e., $\mathrm{Mod}(\Theta_{\mathrm{f}}) \subseteq \mathrm{Mod}(\Theta^{\forall}_{\mathrm{acf}})$.

Denote by J the inclusion functor from the category of models of Θ_{f} and their embeddings into the category of models of $\Theta^{\forall}_{\mathrm{acf}}$ and their embeddings, and by FF the functor that associates to each integral domain in $\mathrm{Mod}(\Theta^{\forall}_{\mathrm{acf}})$ its field of fractions (in $\mathrm{Mod}(\Theta_{\mathrm{f}})$) (see [8]). Then, it is not difficult to show that FF is a left adjoint of J, as is stated in the next proposition.

Proposition 4.2 *Functor FF is left adjoint of the inclusion functor J.*

So, we can use Theorem 4.1 to conclude the following corollary.

Theorem 4.3 *Theory $\Theta_{acf}^{\forall}$ has algebraically prime models with respect to Θ_f.*

It remains to show that the two forms of proving that a theory has algebraically prime models with respect to another theory can be composed.

Proposition 4.4 *Let Δ, Ω and Υ be first-order theories such that:*

- *Δ has algebraically prime models with respect to Ω;*
- *Ω has algebraically prime models with respect to Υ.*

Then Δ has algebraically prime models with respect to Υ.

Proof: Assume that every model of Δ has an algebraically prime model in Ω and that every model of Ω has an algebraically prime model in Υ. Let I be a model of Δ. Denote by I° a model of Ω and by $\eta_I^\circ : I \to I^\circ$ an embedding such that I° is algebraically prime with respect to I via η_I°. Since Ω has algebraically prime models with respect to Υ, I° has an algebraically prime model in Υ. Denote by $\bar{I}$ a model of Υ and by $\bar{\eta}_{I^\circ} : I^\circ \to \bar{I}$ an embedding such that $\bar{I}$ is algebraically prime with respect to I° via $\bar{\eta}_{I^\circ}$. We now show that $\bar{I}$ is algebraically prime with respect to I via $\bar{\eta}_{I^\circ} \circ \eta_I^\circ$. Indeed:
Let I' be a model of Υ and $g : I \to I'$ an embedding. Since $\mathrm{Mod}(\Upsilon) \subseteq \mathrm{Mod}(\Omega)$ we have that I' is a model of Ω and so, taking into account that I° is algebraically prime with respect to I via η_I°, let $g^\circ : I^\circ \to I'$ be an embedding such that $g = g^\circ \circ \eta_I^\circ$. Similarly, since $\bar{I}$ is algebraically prime with respect to I° via $\bar{\eta}_{I^\circ}$, let $\bar{g}^\circ : \bar{I} \to I'$ be an embedding such that $g^\circ = \bar{g}^\circ \circ \bar{\eta}_{I^\circ}$. Hence $g = (\bar{g}^\circ \circ \bar{\eta}_{I^\circ}) \circ \eta_I^\circ$, that is, $g = \bar{g}^\circ \circ (\bar{\eta}_{I^\circ} \circ \eta_I^\circ)$. Therefore, there is an embedding from $\bar{I}$ to I' whose composition with $\bar{\eta}_{I^\circ} \circ \eta_I^\circ$ is g. QED

Algebraically Closed Fields – from $\Theta_{\mathbf{acf}}^{\forall}$ to $\Theta_{\mathbf{acf}}$

$$\mathrm{Mod}(\Theta_{\mathrm{acf}}^{\forall}) \xrightarrow{\mathrm{FF}} \mathrm{Mod}(\Theta_{\mathrm{f}}) \xrightarrow{(\mathrm{QF}\circ\mathrm{PR})^{\omega}} \mathrm{Mod}(\Theta_{\mathrm{acf}})$$

Figure 4: $\Theta_{\mathrm{acf}}^{\forall}$ has algebraically prime models with respect to Θ_{acf}.

We omit the proof of the following Theorem since it follows immediately by Theorem 4.3, Theorem 3.12, and by Proposition 4.4.

Theorem 4.5 *The first-order theory Θ_{acf} has algebraically prime models.*

5 Adequacy for $\exists$

Local satisfaction of $\exists x\varphi$ formulas, where φ is a quantifier free formula, is preserved by embeddings. However, reflection of local satisfaction of those formulas by embeddings does not hold in general. A theory is *1-model-complete* whenever this property holds, that is,

$$I'h \circ \rho \Vdash_\Sigma \exists x\varphi \text{ implies } I\rho \Vdash_\Sigma \exists x\varphi$$

given an embedding $h : I \to I'$ and an assignment $\rho : X \to |I|$ over I.[5] We now provide a condition ("adequacy for $\exists$") sufficient for that reflection to hold. This condition tries to provide an answer to the following problem: What can be done when one wants to prove that a theory is 1-model-complete? The idea was to abstract the common aspects of proofs of reflection for several theories. As we detail below, the proposal consists on investigating the literals that really matter for the theory and variables at hand and try to show that their satisfaction is reflected. As we will see below, those literals are the ones that, when satisfied by a model of the theory, are not equivalent to literals without those variables.

Care must be taken with the variables that may appear existentially quantified when proving the reflection of satisfaction. So, we consider not a set of literals, but a family Ω^e of sets of literals indexed by the finite sets of variables that can be existentially quantified (those are the sets in the family X^e below).

In the sequel we denote by X the set of all variables and by L the set of all finite non empty sets of literals. Given a theory Θ and a variable x, a pair of families

- $X^e = \{X^{\bar\Lambda}\}_{\Lambda \in L}$ where $X^{\bar\Lambda}$ is finite and $\{x\} \subseteq X^{\bar\Lambda} \subseteq (X \setminus \mathrm{Vars}(\Lambda)) \cup \{x\}$;
- $\{\Omega(X^{\bar\Lambda}) : \Omega(X^{\bar\Lambda}) \text{ is a set of literals}\}_{X^{\bar\Lambda} \text{ in } X^e}$;

is said to be Θ *exhaustive* for $\exists$ and x, whenever for every finite set Λ of literals there are finite sets $\Lambda_1, \dots, \Lambda_n$ with literals in $\Omega(X^{\bar\Lambda})$ such that

$$\Theta \vDash_\Sigma \left(\exists x \bigwedge \Lambda\right) \Leftrightarrow \left(\bigvee_{i=1}^{n} \exists x_1 \dots \exists x_m \bigwedge \Lambda_i\right).$$

where $\{x_1, \dots, x_m\} = X^{\bar\Lambda}$. Moreover, we say that a set of variables X_1 is Θ *essential in* a literal ν *with respect to* a set of literals Ω_1 whenever for every μ in Ω_1 if $\Theta \vDash_\Sigma \mu \Leftrightarrow \nu$ then a variable of X_1 occurs in μ.

A theory Θ is *adequate for* $\exists$ whenever there are a variable x, a Θ exhaustive pair (X^e, Ω^e) for $\exists$ and x, and a family $A^e = \{A(X^{\bar\Lambda}) : A(X^{\bar\Lambda}) \subseteq \Omega(X^{\bar\Lambda})\}_{X^{\bar\Lambda} \text{ in } X^e}$, such that, given

- an embedding $h : I \to I'$ in $\mathrm{Mod}(\Theta)$;

[5]Given an embedding $h : I \to I'$ and an assignment $\rho : X \to |I|$ over I, observe that $h \circ \rho : X \to |I'|$ is an assignment of values of I' to the variables of X. So $I'h \circ \rho \Vdash \exists x\varphi$ means that formula $\exists x\varphi$ is satisfied by model I' and assignment $h \circ \rho$ over I'.

- finite and non-empty sets Λ and C of literals with $C \subseteq \Omega(X^{\bar{\Lambda}})$;
- assignment ρ' over I' such that $\rho'(z)$ is in $h(D)$ for every variable z not in $X^{\bar{\Lambda}}$, and $\rho'(z)$ is in $D' \setminus h(D)$ for some variable z in $X^{\bar{\Lambda}}$;

the following holds:

- for every ν in $C \setminus A(X^{\bar{\Lambda}})$, if $I'\rho' \Vdash \nu$ then $X^{\bar{\Lambda}}$ is not Θ essential in ν with respect to $\Omega(X^{\bar{\Lambda}})$;
- there is an assignment σ over I with $h \circ \sigma \equiv_{X^{\bar{\Lambda}}} \rho'$ such that: for every ν in $C \cap A(X^{\bar{\Lambda}})$, if $I'\rho' \Vdash_{\Sigma} \nu$ then $I\sigma \Vdash_{\Sigma} \nu$.

In this case Θ is said to be *adequate for* $\exists$ *and* x *with respect to* (X^e, Ω^e) *and* A^e.

Observe that the family A^e identifies for each finite set $X^{\overline{\Lambda}}$ of variables in X^e, a subset of the exhaustive set $\Omega(X^{\overline{\Lambda}})$ of literals associated with those variables containing the literals in which there is one such variable that is really essential. So, A^e contains the literals whose satisfaction really need to be preserved.

Proposition 5.1 *Every theory adequate for* $\exists$ *is 1-model-complete.*

Proof:
Let Θ be a theory over a signature Σ adequate for $\exists$, x a variable and X^e, Ω^e and A^e families such that Θ is adequate for $\exists$ and x with respect to (X^e, Ω^e) and A^e. Moreover, let I and I' be models of Θ, $h : I \to I'$ an embedding, ρ an assignment over I, and Λ a finite set of literals. Suppose that $I'\, h \circ \rho \Vdash_{\Sigma} \exists x \bigwedge \Lambda$.

Since (X^e, Ω^e) is Θ exhaustive for x, let $\Lambda_1, \ldots, \Lambda_n$ be finite sets of literals in $\Omega(X_{\bar{\Lambda}})$ where a variable of $X_{\bar{\Lambda}}$ only occurs if $X_{\bar{\Lambda}}$ is Θ essential in the literal in question for $\Omega(X_{\bar{\Lambda}})$, such that

$$I'\, h \circ \rho \Vdash_{\Sigma} \left(\bigvee_{i=1}^{n} \exists x_1 \ldots \exists x_m \bigwedge \Lambda_i \right).$$

where $\{x_1, \ldots, x_m\} = X_{\bar{\Lambda}}$. Let i be such that

$$I'\, h \circ \rho \Vdash_{\Sigma} \exists x_1 \ldots \exists x_m \bigwedge \Lambda_i.$$

We want to show that $I\rho \Vdash_{\Sigma} \exists x_1 \ldots \exists x_m \bigwedge \Lambda_i$. Let ρ' be an assignment over I' such that $\rho' \equiv_{X_{\bar{\Lambda}}} h \circ \rho$ and $I'\rho' \Vdash_{\Sigma} \bigwedge \Lambda_i$. Consider two cases:

(1) $\rho'(x_j) \in h(D)$ for every $j = 1, \ldots, m$. Let σ be such that $\sigma \equiv_{X_{\bar{\Lambda}}} \rho$ and $h \circ \sigma = \rho'$. Hence $I'h \circ \sigma \Vdash_{\Sigma} \bigwedge \Lambda_i$. Thus $I\sigma \Vdash_{\Sigma} \bigwedge \Lambda_i$. Therefore $I\rho \Vdash_{\Sigma} \exists x_1 \ldots \exists x_m \bigwedge \Lambda_i$ as we wanted to show.

(2) $\rho'(x_j)$ is in $D' \setminus h(D)$ for some j in $\{1, \ldots, m\}$. Observe that the image by ρ' of every variable in Λ_i not in $X^{\bar{\Lambda}}$ is in $h(D)$ since $\rho' \equiv_{X_{\bar{\Lambda}}} h \circ \rho$. Hence, since Θ is adequate for $\exists$ and x with respect to (X^e, Ω^e) and A^e, let σ be an

assignment over I such that $h \circ \sigma \equiv_{X_{\bar{\Lambda}}} \rho'$ and $I\sigma \Vdash_\Sigma \nu$ whenever $I'\rho' \Vdash_\Sigma \nu$ for every $\nu \in \Lambda_i \cap A(X^{\bar{\Lambda}})$. We now show that $I\sigma \Vdash_\Sigma \bigwedge \Lambda_i$. So let ν be a literal of Λ_i. Recall that $I'\rho' \Vdash_\Sigma \bigwedge \Lambda_i$. Consider two cases:

(a) $\nu \in A(X^{\bar{\Lambda}})$. Then $\nu \in \Lambda_i \cap A(X^{\bar{\Lambda}})$ and so $I\sigma \Vdash_\Sigma \nu$ by hypothesis since $I'\rho' \Vdash_\Sigma \bigwedge \Lambda_i$ and ν is in Λ_i;

(b) $\nu \notin A(X^{\bar{\Lambda}})$. Then, $X^{\bar{\Lambda}}$ is not Θ essential in ν for $\Omega(X^{\bar{\Lambda}})$ since $I'\rho' \Vdash_\Sigma \nu$. Hence no variable of $X^{\bar{\Lambda}}$ occurs in ν since by hypothesis a variable of $X^{\bar{\Lambda}}$ only occurs in literals of Λ_i when $X^{\bar{\Lambda}}$ is essential. So $I'h \circ \sigma \Vdash_\Sigma \nu$ since $I'\rho' \Vdash_\Sigma \nu$ and $(h \circ \sigma)(z) = \rho'(z)$ for every variable z not in $X^{\bar{\Lambda}}$ and no variable of $X^{\bar{\Lambda}}$ occurs in ν. Hence $I\sigma \Vdash_\Sigma \nu$ taking into account that ν is a literal.

So $I\sigma \Vdash_\Sigma \bigwedge \Lambda_i$. Observe that $h \circ \sigma \equiv_{X^{\bar{\Lambda}}} h \circ \rho$ and so $\sigma \equiv_{X^{\bar{\Lambda}}} \rho$. Therefore

$$I\rho \Vdash_\Sigma \exists x_1 \ldots \exists x_m \bigwedge \Lambda_i.$$

Thus

$$I\rho \Vdash_\Sigma \left(\bigvee_{i=1}^{n} \exists x_1 \ldots \exists x_m \bigwedge \Lambda_i \right)$$

and so

$$I\rho \Vdash_\Sigma \exists x \varphi$$

as we wanted to show. QED

Natural Numbers with Successor

The theory of natural numbers with successor is adequate for $\exists$, as we show below, and, so, by Proposition 5.1, also 1-model-complete. Recall the first-order theory $\Theta_{\rm S}$ for term algebras described in Section 3. Given a variable x and a finite and non-empty set Λ of literals over $\Sigma_{\rm S}$, let

- $X_{\rm S}^{\bar{\Lambda}}$ be the set $\{x\}$;
- $\Omega_{\rm S}(\{x\})$ be the set of all literals over the atomic formulas of the form $(\mathbf{S}^m\, t) \cong t'$ where t and t' are either 0 or a variable, and m is a natural number;

and denote by $X_{\rm S}$ the family $\{X_{\rm S}^{\bar{\Lambda}}\}_{\Lambda \in L_{\rm S}}$ and by $\Omega_{\rm S}$ the family $\{\Omega_{\rm S}(X_{\rm S}^{\bar{\Lambda}})\}_{X_{\rm S}^{\bar{\Lambda}} \in X_{\rm S}}$. Then the following proposition immediately follows:

Proposition 5.2 *The pair (X_S, Ω_S) is Θ_S exhaustive for $\exists$ and x.*

Denote by $A_{\rm S}$ the family

$$\{A_{\rm S}(X_{\rm S}^{\bar{\Lambda}})\}_{X_{\rm S}^{\bar{\Lambda}} \text{ in } X_{\rm S}}$$

where $A_{\rm S}(X_{\rm S}^{\bar{\Lambda}})$ is the set of literals of $\Omega_{\rm S}(X_{\rm S}^{\bar{\Lambda}})$ of the form $\neg((\mathbf{S}^m\, t) \cong t')$ where m is a natural number greater than zero, t and t' are either 0 or a variable, and either t or t' is x but not both.

Proposition 5.3 *The theory Θ_S is adequate for $\exists$ and x with respect to (X_S, Ω_S) and A_S.*

Proof: Let I and I' be models of Θ_{S}, $h : I \to I'$ an embedding, Λ a finite non-empty set of literals, C a finite non-empty set of literals in $\Omega_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$ and ρ' an assignment over I' such that $\rho'(z)$ is in $h(D)$ for every variable z not in $X_{\mathrm{S}}^{\bar{\Lambda}} = \{x\}$, and $\rho'(x)$ is in $D' \setminus h(D)$.

(1) Let ν be a literal in $C \setminus A_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$ such that $I'\rho' \Vdash \nu$. Then ν is either of the form $(\mathbf{S}^m\, t) \cong t'$ or $\neg(t \cong t')$ where t and t' are either 0 or a variable and m is a natural number, or of the form $\neg((\mathbf{S}^m\, t) \cong t')$ where m is a natural number greater than 0, t and t' are either 0 or a variable, and either both t and t' are x or both are not x. The proof proceeds by case analysis:

(a) m is 0 or ν is $\neg(t \cong t')$. Assume that ν is of the form $t \cong t'$. We have two cases: (i) t is t'. Then ν is equivalent to $0 \cong 0$ and so $X_{\mathrm{S}}^{\bar{\Lambda}}$ is not Θ_{S} essential in ν for $\Omega_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$; (ii) t is not t'. Then both t and t' are not x since $\rho'(x) \in D' \setminus h(D)$ and $0^{F'}$ and $\rho'(z) \in h(D)$ for every z different from x. Then $X_{\mathrm{S}}^{\bar{\Lambda}}$ is not Θ_{S} essential in ν for $\Omega_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$. We omit the proof when ν is $\neg(t \cong t')$ since it is similar to the case just proved;

(b) m is not 0 and t is t'. Assume that ν is of the form $(\mathbf{S}^m\, t) \cong t$. This case is not possible due to axiom S4. Assume now that ν is of the form $\neg((\mathbf{S}^m\, t) \cong t)$. Then ν is equivalent to $0 \cong 0$ and so $X_{\mathrm{S}}^{\bar{\Lambda}}$ is not Θ_{S} essential in ν for $\Omega_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$.

(c) m is not 0, t and t' are distinct and both are not x. Then $X_{\mathrm{S}}^{\bar{\Lambda}}$ is not Θ_{S} essential in ν for $\Omega_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$;

(d) ν is of the form $(\mathbf{S}^m\, t) \cong t'$, m is not 0, t and t' are distinct and either t or t' is x. This case is not possible since it implies that $\rho'(x) \in h(D)$ and we are assuming that $\rho'(x) \in D' \setminus h(D)$. We now detail the proof when $m = 1$. Assume, by contradiction that $\mathbf{S}^{F'}(\rho'(x)) = h(d)$ for some $d \in D$. Consider two cases: (i) $d = 0^F$. This case contradicts axiom S1. (ii) d is not 0^F. Let d_1 be such that $\mathbf{S}^F(d_1) = d$. Then $\mathbf{S}^{F'}(h(d_1)) = h(\mathbf{S}^F(d_1)) = h(d) = \mathbf{S}^{F'}(\rho'(x))$. Thus, by axiom S2, $h(d_1) = \rho'(x)$ which contradicts the hypothesis.

2) Without loss of generality denote the elements of $C \cap A_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$ by $\neg((\mathbf{S}^{m_1}\, t_1) \cong t_1'), \ldots, \neg((\mathbf{S}^{m_n}\, t_n) \cong t_n')$. Assume that $I'\rho' \Vdash_{\Sigma_{\mathrm{S}}} \nu$ for every $\nu \in C \cap A_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$. Let σ be an assignment over I such that $h \circ \sigma \equiv_x \rho'$ and such that $\sigma(x)$ satisfies the constraints imposed by those inequations. There is such a value in D since D is infinite and the set of values that $\sigma(x)$ should not take according to those inequations is finite. Then $I\sigma \Vdash_{\Sigma_{\mathrm{S}}} \nu$ for every $\nu \in C \cap A_{\mathrm{S}}(X_{\mathrm{S}}^{\bar{\Lambda}})$ by definition of σ. QED

So, capitalizing on the previous proposition and on Proposition 5.1 we can now establish the following theorem.

Theorem 5.4 *Theory Θ_S is 1-model-complete.*

Term Algebras

Recall the first-order theory Θ_{ta} for term algebras described in Section 3 over the signature Σ_{ta} induced by a signature Σ with no predicate symbols besides equality. Given a variable x and a finite and non-empty set Λ of literals over Σ_{ta},

- let $X^{\bar{\Lambda}}_{\mathrm{ta}}$ be a set $\{x, x_1, \ldots, x_m\}$ where $x_1, \ldots, x_m$ are variables not occurring in Λ and m in $\mathbb{N}$ is greater than the product of the maximum depth of a term in Λ containing x with the greatest arity of a function in Σ;
- let $\Omega_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ be the set of all literals of Σ_{ta} with no variables in $X^{\bar{\Lambda}}_{\mathrm{ta}}$ together with the literals of the following form:
  - $\neg(z \cong t)$ where z is in $X^{\bar{\Lambda}}_{\mathrm{ta}}$ and t is either a variable of $X^{\bar{\Lambda}}_{\mathrm{ta}}$ distinct of z or has no variables of $X^{\bar{\Lambda}}_{\mathrm{ta}}$;
  - $\mathbf{Is}_c(z)$, $\neg\,\mathbf{Is}_c(z)$ for every z in $X^{\bar{\Lambda}}_{\mathrm{ta}}$ and constant c in Σ;
  - $\mathbf{Is}_f(z)$, $\neg\,\mathbf{Is}_f(z)$ for every z in $X^{\bar{\Lambda}}_{\mathrm{ta}}$ and function symbol f in Σ.

Denote by X_{ta} the family $\{X^{\bar{\Lambda}}_{\mathrm{ta}}\}_{\Lambda \in L_{\mathrm{ta}}}$ and by Ω_{ta} the family $\{\Omega(X^{\bar{\Lambda}}_{\mathrm{ta}})\}_{X^{\bar{\Lambda}}_{\mathrm{ta}} \text{ in } X_{\mathrm{ta}}}$. Then the following proposition, Proposition 5.5, follows immediately from the first steps of the proof of Theorem 2.7.5 of [7].

Proposition 5.5 *The pair (X_{ta}, Ω_{ta}) is Θ_{ta} exhaustive for $\exists$ and x.*

Denote by A_{ta} the family $\{A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})\}_{X^{\bar{\Lambda}}_{\mathrm{ta}} \text{ in } X_{\mathrm{ta}}}$ where

$$A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$$

is the set of literals of $\Omega_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ with a variable of $X^{\bar{\Lambda}}_{\mathrm{ta}}$.

Proposition 5.6 *The theory Θ_{ta} is adequate for $\exists$ and x with respect to (X_{ta}, Ω_{ta}) and A_{ta}, provided that Σ has an infinite number of constants.*

Proof: Let I and I' be models of Θ_{ta}, $h : I \to I'$ an embedding, Λ a finite non-empty set of literals, C a finite non-empty set of literals in $\Omega_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ and ρ' an assignment over I' such that $\rho'(z)$ is in $h(D)$ for every variable z not in $X^{\bar{\Lambda}}$, and $\rho'(z)$ is in $D' \setminus h(D)$ for some variable z in $X^{\bar{\Lambda}}$.

(1) Let ν be a literal in $C \setminus A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ such that $I'\rho' \Vdash \nu$. Then, since $C \subseteq \Omega_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$, by definition of $A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ no variable of $X^{\bar{\Lambda}}_{\mathrm{ta}}$ occurs in ν. So $X^{\bar{\Lambda}}_{\mathrm{ta}}$ is not Θ_{ta} essential in ν for $\Omega_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$.

2) Assume that $I'\rho' \Vdash_{\Sigma_{\mathrm{ta}}} \nu$ for every $\nu \in C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$. Let σ be an assignment over I such that $(h \circ \sigma)(z) = \rho'(z)$ for every variable z with $\rho'(z) \in h(D)$ and such that if $\rho'(z) \notin h(D)$ then

- if $\mathbf{Is}_f(z)$ is in $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ for some function symbol f in Σ then $\sigma(z)$ is an element in D satisfying the predicate $\mathbf{Is}_f$ and different of $[\![t]\!]^{I\sigma}$ for every $\neg(z \cong t)$ in $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$. Observe that there is such an element in D since Σ has an infinite number of constants by assumption and so by axioms T2, T3 and T4 there is an infinite number of elements of D satisfying the predicate $\mathbf{Is}_f$, and since the number of elements of the domain to which $\sigma(z)$ must be different is finite because $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ has a finite number of literals;
- if $\mathbf{Is}_f(z)$ is not in $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ for every function symbol f in Σ then let $\sigma(z)$ be $[\![c']\!]^I$ for some constant c' in Σ, different of $[\![c]\!]^I$ for every $\neg\,\mathbf{Is}_c(z)$ in $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$, and different of $[\![t]\!]^{I\sigma}$ for every $\neg(z \cong t)$ in $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$. Observe that there is such an element in D since Σ has an infinite number of constants by assumption, which are interpreted as different elements of the domain by axiom T4, and since $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$ has a finite number of literals.

Hence $I\sigma \Vdash_{\Sigma_{\mathrm{ta}}} \nu$ for every literal ν of $C \cap A_{\mathrm{ta}}(X^{\bar{\Lambda}}_{\mathrm{ta}})$, by definition of σ. QED

So, capitalizing on the previous proposition and on Proposition 5.1 we can now establish the following theorem.

Theorem 5.7 *Theory Θ_{ta} is 1-model-complete.*

Algebraically Closed Fields

We now prove that Θ_{acf} is adequate for $\exists$, and, so, by Proposition 5.1, that Θ_{acf} is 1-model-complete. Given a variable x and a finite and non-empty set Λ of literals over Σ_{f}, let

- $X^{\bar{\Lambda}}_{\mathrm{acf}}$ be the set $\{x\}$;
- $\Omega_{\mathrm{acf}}(\{x\})$ be the set of all literals over the atomic formulas of the form

$$q(x_1, \ldots, x_n, x) \cong 0$$

  for some natural number n and distinct variables $x_1, \ldots, x_n, x$.

Denote by X_{acf} and Ω_{acf} the families of $\{X^{\bar{\Lambda}}_{\mathrm{acf}}\}_{\Lambda \in L_{\mathrm{acf}}}$ and $\{\Omega_{\mathrm{acf}}(X^{\bar{\Lambda}}_{\mathrm{acf}})\}_{X^{\bar{\Lambda}}_{\mathrm{acf}} \text{ in } X_{\mathrm{acf}}}$, respectively. Then the following proposition, Proposition 5.8, follows immediately.

Proposition 5.8 *The pair (X_{acf}, Ω_{acf}) is Θ_{acf} exhaustive for $\exists$ and x.*

Denote by A_{acf} the family $\{A_{\mathrm{acf}}(X^{\bar{\Lambda}}_{\mathrm{acf}})\}_{X^{\bar{\Lambda}}_{\mathrm{acf}} \text{ in } X_{\mathrm{acf}}}$ where $A_{\mathrm{acf}}(X^{\bar{\Lambda}}_{\mathrm{acf}})$ is the set of literals of $\Omega_{\mathrm{acf}}(X^{\bar{\Lambda}}_{\mathrm{acf}})$ whose main connective is negation.

Proposition 5.9 *The theory Θ_{acf} is adequate for $\exists$ and x with respect to (X_{acf}, Ω_{acf}) and A_{acf}.*

Proof: Let I and I' be models of $\Theta_{\rm acf}$, $h : I \to I'$ an embedding, Λ a finite non-empty set of literals, C a finite non-empty set of literals in $\Omega_{\rm acf}(X^{\bar{\Lambda}}_{\rm acf})$ and ρ' an assignment over I' such that $\rho'(z)$ is in $h(D)$ for every variable z not in $X^{\bar{\Lambda}}_{\rm acf} = \{x\}$, and $\rho'(x)$ is in $D' \setminus h(D)$. Then:

(1) Let ν be a literal in $C \setminus A_{\rm acf}(X^{\bar{\Lambda}}_{\rm acf})$. Assume that $I'\rho' \Vdash \nu$. Then, for some natural number n and distinct variables $x_1, \ldots, x_n, x$, ν is of the form $q(x_1, \ldots, x_n, x) \cong 0$. Assume by contradiction that $X^{\bar{\Lambda}}_{\rm acf}$ is $\Theta_{\rm acf}$ essential in ν for $\Omega_{\rm acf}(X^{\bar{\Lambda}}_{\rm acf})$. Observe that $\rho'(x)$ is a solution of the polynomial equation

$$q^{F'}(\rho'(x_1), \ldots, \rho'(x_n), x) = 0^{F'}.$$

Let ρ be an assignment over I such that $\rho(x_i) = h^{-1}(\rho'(x_i))$ for $i = 1, \ldots, n$ and let m be the number of roots of the equation $q^F(\rho(x_1), \ldots, \rho(x_n), x) = 0^F$ in D and $d_1, \ldots, d_m$ those roots. Note that $h(d_1), \ldots, h(d_m)$ are also the m roots in D' of the equation $q^{F'}(h(\rho(x_1)), \ldots, h(\rho(x_n)), x) = 0^{F'}$ that is of the equation $q^{F'}(\rho'(x_1), \ldots, \rho'(x_n), x) = 0^{F'}$. So $\rho'(x) = h(d_j)$ for some j in $\{1, \ldots, m\}$ which contradicts the fact that $\rho'(x) \in D' \setminus h(D)$.

2) Assume with no loss of generality that $C \cap A_{\rm acf}(X^{\bar{\Lambda}}_{\rm acf})$ contains the literals $\neg(q_1(x_{11}, \ldots, x_{1n_1}, x) \cong 0), \ldots, \neg(q_k(x_{k1}, \ldots, x_{kn_k}, x) \cong 0)$. Let σ be an assignment over I such that $h \circ \sigma \equiv_x \rho'$ and $\sigma(x)$ is not a root of the polynomial equation $q_j^F(h^{-1}(\rho'(x_{j1})), \ldots, h^{-1}(\rho'(x_{jn_j})), x) = 0^F$ for all $j = 1, \ldots, k$. There is such a value in D since D is infinite and the number of such roots is finite. Then $I\sigma \Vdash_{\Sigma_{\rm f}} \nu$ for every $\nu \in C \cap A_{\rm acf}(X^{\bar{\Lambda}}_{\rm acf})$ by definition of σ. QED

So, capitalizing on the previous proposition and on Proposition 5.1, we can establish the following theorem.

Theorem 5.10 *Theory Θ_{acf} is 1-model-complete.*

6 Concluding Remarks

The importance of decidability of first-order theories is well recognized in computer science applications. In many cases proving decidability involves proving that the theory at hand has quantifier elimination. Quantifier elimination can be proved constructively using symbolic methods or non constructively using semantic methods. The objective of this paper is to contribute to a toolbox making easier to prove, in a semantic way, that a theory has quantifier elimination. We believe that this step is useful when we still do not know if a theory has quantifier elimination. The work capitalises on a well known result stating that a theory with algebraically prime models and 1-complete has quantifier elimination.

Proving that a theory Θ has algebraically prime models with respect to another theory Υ involves providing a map F from the models of Θ to the models of Υ, an embedding from each model I of Θ to $F(I)$ and involves verifying that a certain universal property holds. In this paper we state that such a proof can be obtained either by iteration or by adjunction or by a combination of both. We also provide a sufficient condition for a theory to be 1-model-complete. This sufficient condition is targeted to identify the key literals for the theory at hand and to prove the reflection of satisfaction of these literals through embeddings. We illustrate the techniques on the theories of natural numbers with successor, term algebras and algebraically closed fields.

When applying our results to show that a theory has algebraically prime models with respect to another theory, one may wonder which method to use: the iterative or the category theoretic. In order to help to answer this question observe that the iterative approach is more general than the category theoretic approach in the sense that the relationship considered between the categories of models of the theories and their embeddings does not need to be a functor, as it should be in the category theoretic approach. For example the relationship used in the proof that Θ_{f} has algebraically prime models with respect to Θ_{acf}, defined by the map $\mathrm{QF} \circ \mathrm{PR} : \mathrm{Mod}(\Theta_{\mathrm{f}}) \to \mathrm{Mod}(\Theta_{\mathrm{f}})$, is not a functor, which means that in this case the category theoretic approach could not be used.

There are several ways to go on with this work. The most interesting and perhaps important aspect is to obtain results of preservation of quantifier elimination when adding axioms to a given theory.

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References

[1] A. R. Bradley and Z. Manna. *The Calculus of Computation*. Springer, 2007.

[2] D. Delahaye and M. Mayero. Quantifier elimination over algebraically closed fields in a proof assistant using a computer algebra system. *Electronic Notes in Theoretical Computer Science*, 151(1), 2006.

[3] H. B. Enderton. *A Mathematical Introduction to Logic*. Harcourt/Academic Press, second edition, 2001.

[4] R. Fagin, J. Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation*, 87(1-2):78–128, 1990.

[5] D. M. Gabbay and A. Szałas. Second-order quantifier elimination in higher-order contexts with applications to the semantical analysis of conditionals. *Studia Logica*, 87(1):37–50, 2007.

[6] K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38(1):173–198, 1931.

[7] W. Hodges. *Model Theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[8] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, third edition, 2002.

[9] D. Marker. *Model Theory: An Introduction*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, 2002.

[10] D. Monniaux. Quantifier elimination by lazy model enumeration. In *Computer Aided Verification*, volume 6174 of *Lecture Notes in Computer Science*, pages 585–599. Springer, Berlin, 2010.

[11] T. Nipkow. Linear quantifier elimination. *Journal of Automated Reasoning*, 45(2):189–212, 2010.

[12] A. Platzer. Differential-algebraic dynamic logic for differential-algebraic programs. *Journal of Logic and Computation*, 20(1):309–352, 2010.

[13] T. Sturm and A. Tiwari. Verification and synthesis using real quantifier elimination. In *ISSAC 2011—Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation*, pages 329–336. ACM, 2011.

[14] A. Tarski. *A Decision Method for Elementary Algebra and Geometry*. University of California Press, 1951. 2nd ed.

[15] V. Weispfenning. Simulation and optimization by quantifier elimination. *Journal of Symbolic Computation*, 24(2):189–208, 1997.