Equivalence of Shiny and Strongly Polite Theories

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Abstract

In this paper we show that a many-sorted shiny theory with respect to a set of sorts is strongly polite with respect to that set, and vice-versa, assuming that the theory has a decidable quantifier-free satisfiability problem. Moreover, we provide sufficient conditions for a many-sorted polite theory with respect to the set of all sorts to be shiny/strongly polite with respect to that set. Relying on these theoretical results, algorithms for crucial parts of these reductions are presented and their time complexities analysed. Besides providing alternative ways of proving that a theory satisfies the conditions of some of the Nelson-Oppen style combination theorems that have been proposed, these results allowed us to generalize a Nelson-Oppen style combination theorem. An application to graph colouring is presented.

Keywords: Nelson-Oppen method, combination of satisfiability procedures, shiny theories, polite theories, strongly polite theories, first-order logic, many-sorted logic.

1 Introduction

The Nelson-Oppen method is a well-known method for modularly combining satisfiability procedures of given theories. The method was proposed by Nelson and Oppen in 1979, [5], and provides a way of deciding the satisfiability of quantifier-free formulas in the union of two theories, as long as both of them have their own procedure for deciding the satisfiability problem of quantifier-free formulas. After a correction, see [6], the two main restrictions of the Nelson-Oppen[1] method are that:

[1] A correctness proof of the method was presented by Tinelli and Harandi in [8].

- the theories are *stably infinite*,
- their signatures are disjoint.

Concerned about the fact that many theories of interest, such as those admitting only finite models, are not stably infinite, Tinelli and Zarba, in [10], showed that the Nelson-Oppen combination procedure still applies when the stable infiniteness condition is replaced by the requirement that all but one of the theories is *shiny*. However, a shiny theory must be equipped with a particular function called mincard, which is inherently hard to compute.

In order to overcome the problem in computing the mincard function and in proving that a theory is shiny, Ranise, Ringeissen and Zarba proposed an alternative requirement, *politeness*, in [7], and analysed its relationship with shininess. A polite theory has to be equipped with a witness function, which was thought to be easier to compute than the mincard function. They showed that given a polite theory and an arbitrary one, the Nelson-Oppen combination procedure is still valid when the theories have disjoint signatures and both have their own procedure for deciding the satisfiability problem of quantifier-free formulas. Some time later, in [3], Jovanović and Barrett reported that the politeness notion provided in [7] allowed, after all, witness functions that are not sufficiently strong to prove the combination theorem. In order to overcome this issue they provided a seemingly stronger notion of politeness, in the sequel called *strong politeness*, equipped with a seemingly stronger witness function, s-witness, that allowed to prove the combination theorem. However, the relationship between the two notions of politeness and between the strong politeness notion and shininess was not studied. This motivated the work in [4], where the authors investigated the relationship between shiny and strongly polite theories in the one-sorted case. They showed that a shiny theory with a decidable quantifier-free satisfiability problem is strongly polite, and, for the other direction, they provided two different sets of conditions under which a polite theory is shiny.

Herein we show that shiny theories with respect to a set of sorts are strongly polite with respect to the same set, when the theory is equipped with a quantifier-free satisfiability solver. Furthermore, we show that when equipped with such a solver, strongly polite theories with respect to a set of sorts are shiny with respect to the same set. Observe that these results do not have the restriction imposed in [7] when relating polite theories with shiny theories, that the set of sorts has to be the full set of sorts in the signature. For that we had to propose a new way of computing the mincard function that does not rely on enumerating interpretations.

Moreover, we provide sufficient conditions for a polite theory with respect to the set of all sorts to be shiny with respect to that set, generalizing the relationship between these two classes to the many-sorted case.

Relying on these theoretical results, algorithms for crucial parts of these reductions are presented and their time complexities analysed. Furthermore, a colouring algorithm particularly suitable for applications where several graphs need to be coloured is presented. Finally, based on these results, we propose a Nelson-Oppen style combination theorem that generalizes to the many-sorted case a similar theorem in [4].

1.1 Organization of the Paper

The paper is organized as follows: in Section 2 we introduce the main notions and definitions used throughout the paper. In Section 3 we show that when equipped with a quantifier-free satisfiability solver, the classes of shiny and strongly polite theories coincide. Moreover, capitalizing on these results, we derive an algorithm with applications to graph colouring. In Section 4, we investigate the relation between shiny/strongly polite theories and polite theories. In Section 5 we generalize to the many-sorted case the Nelson-Oppen style combination result in [4] and illustrate its application. In Section 6 we conclude the paper and provide some directions for further research.

2 Preliminaries

The results in this paper concern many-sorted first-order logic with equality. For each sort, we assume given disjoint countably infinite sets of variables. We mainly follow the notation in [10] and [7].

2.1 Syntax

A *signature* is a tuple $\Sigma = \langle \Sigma^S, \Sigma^F, \Sigma^P, \alpha, \tau \rangle$ where Σ^S is the set of sorts, Σ^F is the set of function symbols, Σ^P is the set of predicate symbols, α is a map that for each function and predicate symbol returns its arity and τ is a map that for each function and predicate symbol returns its type. For each sort $\sigma \in \Sigma^S$, we use $\cong_\sigma$ to denote the equality logic symbol over pairs of terms of sort σ and assume the standard many-sorted definitions of Σ-*atom* and Σ-*term*. A Σ-*formula* is inductively defined as usual over Σ-atoms using the connectives $\wedge, \vee, \neg, \rightarrow$ or the quantifiers $\forall$ and $\exists$. We denote by $\mathsf{QF}(\Sigma)$ the set of Σ-formulas with no occurrences of quantifiers and, given a

Σ-formula φ, by $\mathsf{vars}(\varphi)$ the set of free variables of φ. Furthermore, denote by $\mathsf{vars}_\sigma(\varphi)$ the set of free variables of sort σ occuring in φ. Given a set of terms T and a sort σ, we denote by T_σ the set of terms in T of sort σ, and we say that a Σ-formula is a Σ-*sentence* if it has no free variables. In the sequel, when there is no ambiguity, we may omit the reference to the signature when referring to atoms, terms, formulas and sentences.

Given a finite set of variables Y over a set of sorts S and $E \subseteq Y^2$, we write

$$E \sqsubseteq Y^2$$

to denote that E is a family of sort-wise equivalence relations over Y, i.e.,

$$E = \bigcup_{\sigma \in S} E_\sigma,$$

and for each sort $\sigma \in S$, E_σ is an equivalence relation on Y_σ^2.

Definition 2.1 (Arrangement formula) *Given a finite set of variables Y over a set of sorts S and $E \sqsubseteq Y^2$, the* arrangement formula *induced by E over Y, denoted by*

$$\delta_E^Y$$

is the formula

$$\bigwedge_{\sigma \in S} \delta_{E_\sigma}^{Y_\sigma}$$

where $\delta_{E_\sigma}^{Y_\sigma}$ is

$$\bigwedge_{(x,y) \in E_\sigma} (x \cong y) \wedge \bigwedge_{(x,y) \in Y_\sigma^2 \setminus E_\sigma} \neg(x \cong y).$$

In the sequel, when there is no ambiguity, we may simply denote δ_E^Y by δ_E.

2.2 Semantics

Given a signature Σ, a Σ-*interpretation* $\mathcal{A}$ over a set of variables X is a map that interprets:

- each sort $\sigma \in \Sigma^S$ as a non-empty set A_σ;
- each variable $x \in X$ with sort σ as an element $x^{\mathcal{A}} \in A_\sigma$;
- each function symbol $f \in \Sigma^F$ of arity n and type $\sigma_1 \times \ldots \times \sigma_n \to \sigma$ as a map $f^{\mathcal{A}} : A_{\sigma_1} \times \ldots \times A_{\sigma_n} \to A_\sigma$, and

- each predicate symbol $p \in \Sigma^P$ of arity n and type $\sigma_1 \times \ldots \times \sigma_n$ as a subset $p^{\mathcal{A}}$ of $A_{\sigma_1} \times \ldots \times A_{\sigma_n}$.

We denote by $\mathsf{dom}(\mathcal{A})$ the domain of a Σ-interpretation $\mathcal{A}$, i.e., $\mathsf{dom}(\mathcal{A}) = \{A_\sigma : \sigma \in \Sigma^S\}$. In the sequel, when there is no ambiguity, we may omit the reference to the signature when referring to interpretations.

Given an interpretation $\mathcal{A}$ and a term t, we denote by $t^{\mathcal{A}}$ the interpretation of t under $\mathcal{A}$. Similarly, we denote by $\varphi^{\mathcal{A}}$ the truth value of the formula φ under the interpretation $\mathcal{A}$. Furthermore, given a set Γ of formulas, we denote by $[\![\Gamma]\!]^{\mathcal{A}}$ the set $\{\varphi^{\mathcal{A}} : \varphi \in \Gamma\}$, and similarly for a set of terms. Finally, we write $\mathcal{A} \Vdash \varphi$ when the formula φ is true under the interpretation $\mathcal{A}$, i.e., $\mathcal{A}$ satisfies φ.

A formula φ is *satisfiable* if it is true under some interpretation. It is *unsatisfiable* otherwise.

Given sets of variables Y and X, we say that two interpretations $\mathcal{A}$ and $\mathcal{B}$ over X are *equivalent modulo* Y whenever $\mathsf{dom}(\mathcal{A}) = \mathsf{dom}(\mathcal{B})$, $f^{\mathcal{A}} = f^{\mathcal{B}}$ for each function symbol f, $p^{\mathcal{A}} = p^{\mathcal{B}}$ for each predicate symbol p, and $x^{\mathcal{A}} = x^{\mathcal{B}}$ for each variable x in $X \setminus Y$.

We also say that an *interpretation* $\mathcal{A}$ *is finite (infinite)* with respect to a set S of sorts when, for each sort $\sigma \in S$, the set A_σ is finite (infinite).

2.3 Theories

Given a signature Σ, a Σ-*theory* is a set of Σ-sentences, and given a Σ-theory $\mathcal{T}$, a $\mathcal{T}$-*model* is a Σ-interpretation that satisfies all the sentences of $\mathcal{T}$. We say that a formula φ is $\mathcal{T}$-*satisfiable* when there is a $\mathcal{T}$-model that satisfies it, and, we say that two formulas are $\mathcal{T}$-*equivalent* if they are interpreted to the same truth value in every $\mathcal{T}$-model. Given a Σ_1-theory $\mathcal{T}_1$ and a Σ_2-theory $\mathcal{T}_2$, their union, $\mathcal{T}_1 \oplus \mathcal{T}_2$, is a $\Sigma_1 \cup \Sigma_2$-theory defined by the union of the sentences of $\mathcal{T}_1$ with the sentences of $\mathcal{T}_2$. In the sequel, when there is no ambiguity, we may omit the reference to the signature when referring to theories.

Definition 2.2 (Smoothness) *We say that a theory $\mathcal{T}$ over a signature Σ is* smooth *with respect to a set of sorts $S \subseteq \Sigma^S$ if for every $\mathcal{T}$-satisfiable quantifier-free formula φ, $\mathcal{T}$-model $\mathcal{A}$ satisfying φ and cardinals $\kappa_\sigma \geq |A_\sigma|$ for each $\sigma \in S$, there exists a $\mathcal{T}$-model $\mathcal{B}$ satisfying φ such that $|B_\sigma| = \kappa_\sigma$ for all $\sigma \in S$.*

Definition 2.3 (Stable finiteness) *We say that a theory $\mathcal{T}$ over a signature Σ is* stably finite *with respect to a set of sorts $S \subseteq \Sigma^S$ if for every*

$\mathcal{T}$-satisfiable quantifier-free formula φ there exists a $\mathcal{T}$-model $\mathcal{A}$ of φ finite with respect to S.

Definition 2.4 (Stable infiniteness) *We say that a theory $\mathcal{T}$ over a signature Σ is* stably infinite *with respect to a set of sorts $S \subseteq \Sigma^S$ if for every $\mathcal{T}$-satisfiable quantifier-free formula φ there exists a $\mathcal{T}$-model $\mathcal{A}$ of φ infinite with respect to S.*

Definition 2.5 (Finite witnessability, [7]) *We say that a theory $\mathcal{T}$ over a signature Σ is* finitely witnessable *with respect to a set of sorts $S \subseteq \Sigma^S$ if there exists a computable function* $\mathsf{witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$ *such that for every quantifier-free formula φ the following conditions hold:*

- *φ and $\exists \vec{w}\ \mathsf{witness}(\varphi)$ are $\mathcal{T}$-equivalent, where $\vec{w}$ are the variables in $\mathsf{witness}(\varphi)$ which do not occur in φ;*
- *if $\mathsf{witness}(\varphi)$ is satisfiable in $\mathcal{T}$ then there exists a $\mathcal{T}$-model $\mathcal{A}$ of $\mathsf{witness}(\varphi)$ such that $A_\sigma = [\![\mathsf{vars}_\sigma(\mathsf{witness}(\varphi))]\!]^{\mathcal{A}}$, for each $\sigma \in S$.*

A function satisfying the above properties is called a *witness function for $\mathcal{T}$* with respect to S. In [3], a stronger finite witnessability notion was introduced in order to clarify an issue found on [7].

Definition 2.6 (Strong finite witnessability, [3]) *We say that a theory $\mathcal{T}$ over a signature Σ is* strongly finitely witnessable *with respect to a set of sorts $S \subseteq \Sigma^S$ if there exists a computable function* $\mathsf{s\text{-}witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$ *such that for every quantifier-free formula φ the following conditions hold:*

- *φ and $\exists \vec{w}\ \mathsf{s\text{-}witness}(\varphi)$ are $\mathcal{T}$-equivalent, where $\vec{w}$ are the variables in the formula $\mathsf{s\text{-}witness}(\varphi)$ which do not occur in φ;*
- *for every finite set of variables Y and $E \sqsubseteq Y^2$, if $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is satisfiable in $\mathcal{T}$ then there exists a $\mathcal{T}$-model $\mathcal{A}$ of $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ such that $A_\sigma = [\![\mathsf{vars}_\sigma\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right)]\!]^{\mathcal{A}}$, for all $\sigma \in S$.*

A function satisfying the above properties is called a *strong witness function for $\mathcal{T}$* with respect to S.

The following notion was introduced by Tinelli and Zarba in [10] and extended to the many-sorted case by Ranise, Ringeissen and Zarba in [7].

Definition 2.7 ($\mathsf{mincard}$ **function, [10, 7]**) *Given a theory $\mathcal{T}$ over a signature Σ and a set of sorts $S \subseteq \Sigma^S$, let $\mathsf{mincard}_{\mathcal{T},S}$ be the function such that*

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \min\left\{\max_{\sigma\in S} |A_\sigma| : \mathcal{A} \Vdash \varphi\right\}$$

if φ is $\mathcal{T}$-satisfiable, otherwise $\mathsf{mincard}_{\mathcal{T},S}(\varphi)$ is undefined.

We may simply write $\mathsf{mincard}_S$ when there is no ambiguity to which theory this function refers to.

Definition 2.8 (Shininess, [10, 7]) *A theory over a signature Σ is* shiny *with respect to a set of sorts $S \subseteq \Sigma^S$ whenever it is smooth and stably finite with respect to S, and its $\mathsf{mincard}_S$ function is computable.*

Several one-sorted theories were proved to be shiny, such as the theory of equality, the theory of partial orders and the theory of total orders, in [10].

Definition 2.9 (Strong politeness, [3]) *A theory over a signature Σ is* strongly polite *with respect to a set of sorts $S \subseteq \Sigma^S$ whenever it is smooth and strongly finitely witnessable with respect to S.*

Definition 2.10 (Politeness, [7]) *A theory over a signature Σ is* polite *with respect to a set of sorts $S \subseteq \Sigma^S$ whenever it is smooth and finitely witnessable with respect to S.*

3 Shiny and Strongly Polite Theories

In this section we analyse the relationship between many-sorted shiny and strongly polite theories. We start by showing that a shiny theory with respect to a set of sorts is strongly polite with respect to the same set, assuming that the theory has a decidable quantifier-free satisfiability problem.

3.1 A Shiny Theory with a Decidable Quantifier-free Satisfiability Problem is a Strongly Polite Theory

Proposition 3.1 *A shiny theory with respect to a set of sorts S with a decidable quantifier-free satisfiability problem is strongly polite with respect to S.*

Proof: Let $\mathcal{T}$ be a shiny Σ-theory with respect to a set $S \subseteq \Sigma^S$ of sorts and Sat an algorithm that solves its quantifier-free satisfiability problem. Since a shiny theory is by definition smooth, in order to conclude that $\mathcal{T}$ is strongly polite with respect to S, we are left to prove that $\mathcal{T}$ is strongly finitely witnessable with respect to S. In the sequel, given a $\mathcal{T}$-satisfiable quantifier-free formula φ and a family of equivalence relations $E \sqsubseteq \mathsf{vars}(\varphi)^2$, if $\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$ is $\mathcal{T}$-satisfiable, we denote by $k_E^{\varphi,\sigma}$ the result of $\mathsf{mincard}_{\mathcal{T},\sigma}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)})$.

Let

$$\mathsf{s\text{-}witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$$

be such that $\mathsf{s\text{-}witness}(\varphi) = \varphi \wedge \Omega$, where Ω is

$$\bigwedge_{\substack{E \sqsubseteq \mathsf{vars}(\varphi)^2 \\ \mathsf{Sat}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)})=1}} \left(\delta_E^{\mathsf{vars}(\varphi)} \to \bigwedge_{\sigma \in S} \gamma_{k_E^{\varphi,\sigma}} \right)$$

and $\gamma_{k_E^{\varphi,\sigma}}$ is

$$\bigwedge_{\substack{i,j=1 \\ i \neq j}}^{k_E^{\varphi,\sigma}} \neg(w_{i,\sigma} \cong_\sigma w_{j,\sigma})$$

and $w_{1,\sigma}, \ldots, w_{k_E^{\varphi,\sigma},\sigma}$ are distinct σ-variables not occurring in φ and in $\gamma_{k_{E'}^{\varphi}}$ for all $E' \neq E$ with $E' \sqsubseteq \mathsf{vars}(\varphi)^2$ and $\mathsf{Sat}(\varphi \wedge \delta_{E'}^{\mathsf{vars}(\varphi)}) = 1$. It is immediate to conclude that $\mathsf{s\text{-}witness}$ is computable since:

- there is a finite number of sets E with $E \sqsubseteq \mathsf{vars}(\varphi)^2$ since $\mathsf{vars}(\varphi)$ is finite;

- formula $\delta_E^{\mathsf{vars}(\varphi)}$ can be computed in a finite number of steps since E and $\mathsf{vars}(\varphi)$ are finite;

- the value $k_E^{\varphi,\sigma}$ is computable since: (i) the $\mathsf{mincard}$ function is computable; (ii) we can decide the satisfiability of $\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$ with Sat; and (iii) $\mathcal{T}$ is stably finite with respect to S;

- the formula $\gamma_{k_E^{\varphi,\sigma}}$ is computable in a finite number of steps because $k_E^{\varphi,\sigma}$ is a natural number.

Let φ be a quantifier free formula. We now show that φ and $\exists \vec{w}\ \mathsf{s\text{-}witness}(\varphi)$ are $\mathcal{T}$-equivalent. Let $\mathcal{A}$ be a $\mathcal{T}$-model. Assume that $\mathcal{A} \Vdash \exists \vec{w}\ \mathsf{s\text{-}witness}(\varphi)$.

Then $\mathcal{A} \Vdash \varphi \wedge \exists \vec{w}\, \Omega$, and so $\mathcal{A} \Vdash \varphi$. For the other direction, assume $\mathcal{A} \Vdash \varphi$. We need to show that

$$\mathcal{A} \Vdash \exists \vec{w} \bigwedge_{\substack{E \sqsubseteq \mathsf{vars}(\varphi)^2 \\ \mathsf{Sat}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)})=1}} \left(\delta_E^{\mathsf{vars}(\varphi)} \to \bigwedge_{\sigma \in S} \gamma_{k_E^{\varphi,\sigma}} \right).$$

Let $\mathcal{A}'$ be an interpretation equivalent modulo $\vec{w}$ to $\mathcal{A}$ (and so with the same domain and the same interpretation of functions, predicates and of all variables except possibly those in $\vec{w}$) such that for each sort $\sigma \in S$:

- if A_σ is infinite then $w_{1,\sigma}^{\mathcal{A}'} \neq w_{2,\sigma}^{\mathcal{A}'}$ for every $w_{1,\sigma}, w_{2,\sigma} \in \vec{w}_\sigma$;
- if A_σ is finite then for each $E \sqsubseteq \mathsf{vars}(\varphi)^2$ with $\mathsf{Sat}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}) = 1$:
  - if $k_E^{\varphi,\sigma} \leq |A_\sigma|$ then $w_1^{\mathcal{A}'} \neq w_2^{\mathcal{A}'}$ for every $w_1, w_2 \in \mathsf{vars}(\gamma_{k_E^{\varphi,\sigma}})$;
  - otherwise, $w_1^{\mathcal{A}'} = w_2^{\mathcal{A}'}$ for every $w_1, w_2 \in \mathsf{vars}(\gamma_{k_E^{\varphi,\sigma}})$.

Then

$$\mathcal{A}' \Vdash \bigwedge_{\substack{E \sqsubseteq \mathsf{vars}(\varphi)^2 \\ \mathsf{Sat}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)})=1}} \left(\delta_E^{\mathsf{vars}(\varphi)} \to \bigwedge_{\sigma \in S} \gamma_{k_E^{\varphi,\sigma}} \right)$$

since for each $E \sqsubseteq \mathsf{vars}(\varphi)^2$ with $\mathsf{Sat}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}) = 1$ either

- $\mathcal{A}' \nVdash \delta_E^{\mathsf{vars}(\varphi)}$ and so $\mathcal{A}' \Vdash \delta_E^{\mathsf{vars}(\varphi)} \to \bigwedge_{\sigma \in S} \gamma_{k_E^{\varphi,\sigma}}$; or
- $\mathcal{A}' \Vdash \delta_E^{\mathsf{vars}(\varphi)}$ and so $\mathcal{A}' \Vdash \varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$ since $\mathcal{A} \Vdash \varphi$ and $\mathcal{A}$ and $\mathcal{A}'$ may only differ in the interpretation of the variables in $\vec{w}$ which do not occur in φ. Since $\mathcal{A}'$ is a model for $\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$, for each sort σ in S the cardinality of A_σ has to be greater or equal than $k_E^{\varphi,\sigma} = \mathsf{mincard}_{\mathcal{T},\sigma}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)$. Hence $\mathcal{A}' \Vdash \gamma_{k_E^{\varphi,\sigma}}$ for each $\sigma \in S$ and so $\mathcal{A}' \Vdash \delta_E^{\mathsf{vars}(\varphi)} \to \bigwedge_{\sigma \in S} \gamma_{k_E^{\varphi,\sigma}}$.

Let Y be a finite set of variables. We now show that given a family $E' \sqsubseteq Y$, if $\varphi \wedge \Omega \wedge \delta_{E'}^{Y}$ is $\mathcal{T}$-satisfiable, then there exists a $\mathcal{T}$-model $\mathcal{B}'$ that satisfies $\varphi \wedge \Omega \wedge \delta_{E'}^{Y}$ such that for each sort σ in S, $B'_\sigma = [\![\mathsf{vars}_\sigma(\varphi \wedge \Omega \wedge \delta_{E'}^{Y})]\!]^{\mathcal{B}'}$. So, let $E' \sqsubseteq Y$ be such that $\varphi \wedge \Omega \wedge \delta_{E'}^{Y}$ is $\mathcal{T}$-satisfiable. For each $\sigma \in S$, let p_σ be a natural number and $Y_1^\sigma, \ldots, Y_{p_\sigma}^\sigma$ pairwise disjoint non-empty sets of variables of sort σ such that

- $Y_\sigma = \bigcup_{i=1}^{p_\sigma} Y_i^\sigma$; and
- for each $i = 1, \ldots, p_\sigma$, and $y \in Y_i^\sigma$,
  - $(y \cong_\sigma x)$ and $(x \cong_\sigma y)$ are in $\delta_{E'_\sigma}^{Y_\sigma}$ for each $x \in Y_i^\sigma$;
  - $\neg(y \cong_\sigma x)$ and $\neg(x \cong_\sigma y)$ are in $\delta_{E'_\sigma}^{Y_\sigma}$ for each $x \in Y_\sigma \setminus Y_i^\sigma$;

and observe that each variable in Y can be either in $\mathsf{vars}(\varphi)$ or in $\mathsf{vars}\left(\wedge_{\sigma \in S} \gamma_{k_E^{\varphi,\sigma}}\right)$ for some E or not in $\mathsf{vars}(\varphi \wedge \Omega)$. Furthermore, let $p = \max\{p_\sigma : \sigma \in S\}$ and $\mathcal{A}$ be a $\mathcal{T}$-model that satisfies

$$\varphi \wedge \Omega \wedge \delta_{E'}^{Y}$$

and let $\delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ be the arrangement formula induced by

$$E_\mathcal{A} = \bigcup_{\sigma \in S} \left\{(x, y) : x, y \in \mathsf{vars}_\sigma(\varphi) \text{ and } x^\mathcal{A} = y^\mathcal{A}\right\}.$$

Then, obviously, $\delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ is satisfied by $\mathcal{A}$. Hence, $\varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ is satisfiable and

$$\mathsf{mincard}_{\mathcal{T},S}\left(\varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}\right) = \min\left\{\max_{\sigma \in S} |A_\sigma| : \mathcal{A} \Vdash \varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}\right\}$$

is defined. Observe that no other formula in

$$\left\{\delta_E^{\mathsf{vars}(\varphi)} : E \sqsubseteq \mathsf{vars}(\varphi)^2 \text{ and } \mathsf{Sat}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}) = 1\right\}$$

is satisfied by $\mathcal{A}$. Let $K = \max\left\{\mathsf{mincard}_{\mathcal{T},S}(\varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}), p\right\}$. By the smoothness of $\mathcal{T}$ and since $\varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ is $\mathcal{T}$-satisfiable, let $\mathcal{B}$ be a $\mathcal{T}$-model such that

$$\mathcal{B} \Vdash \varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)} \quad \text{and} \quad |B_\sigma| = K \text{ for each sort } \sigma \in S,$$

and, for each sort σ in S, let $d_1^\sigma, \ldots, d_{p_\sigma}^\sigma$ be distinct elements of B_σ such that for $i = 1, \ldots, p_\sigma$

$$d_i^\sigma = y^\mathcal{B} \text{ for each } y \in Y_i^\sigma \cap \mathsf{vars}(\varphi),$$

and assuming that the variables of $\gamma_{k^{\varphi,\sigma}_{E_\mathcal{A}}}$ are $w_1, \ldots, w_{k^{\varphi,\sigma}_{E_\mathcal{A}}}$ let $e^\sigma_1, \ldots, e^\sigma_{k^{\varphi,\sigma}_{E_\mathcal{A}}}$ be distinct elements of B_σ such that

$$e^\sigma_j = d^\sigma_i \text{ if } w_j \in Y^\sigma_i$$

for $j = 1, \ldots, k^{\varphi,\sigma}_{E_\mathcal{A}}$ and $i = 1, \ldots, p_\sigma$. Observe that distinct variables in $w_1, \ldots, w_{k^{\varphi,\sigma}_{E_\mathcal{A}}}$ are in distinct sets in $Y^\sigma_1, \ldots, Y^\sigma_{p_\sigma}$ since $\mathcal{A} \Vdash \delta^Y_{E'}$ and $\mathcal{A} \Vdash \bigwedge_{\sigma \in S} \gamma_{k^{\varphi,\sigma}_{E_\mathcal{A}}}$ taking into account that $\mathcal{A} \Vdash \delta^{\mathsf{vars}(\varphi)}_{E_\mathcal{A}}$ and $\mathcal{A} \Vdash \Omega$. Let $\mathcal{B}'$ be the $\mathcal{T}$-model equivalent modulo $(\vec{w} \cup (Y \setminus \mathsf{vars}(\varphi)))$ to $\mathcal{B}$ such that

$$x^{\mathcal{B}'} = \begin{cases} d^\sigma_i & \text{if } x \in Y^\sigma_i \text{ for some } i \in \{1, \ldots, p\} \text{ and } \sigma \in S \\ e^\sigma_j & \text{if } x \notin Y \text{ and } x \text{ is } w_j \text{ with } w_j \in \mathsf{vars}\left(\gamma_{k^{\varphi,\sigma}_{E_\mathcal{A}}}\right) \text{ and } \sigma \in S \\ x^\mathcal{B} & \text{if } x \notin Y \text{ and } x \notin \mathsf{vars}\left(\gamma_{k^{\varphi,\sigma}_{E_\mathcal{A}}}\right) \text{ for all } \sigma \in S \end{cases}$$

for each $x \in \vec{w} \cup (Y \setminus \mathsf{vars}(\varphi))$. Let us now prove that $\mathcal{B}' \Vdash \varphi \wedge \Omega \wedge \delta^Y_{E'}$:

(a) $\mathcal{B}' \Vdash \varphi$. This follows immediately taking into account that $\mathcal{B} \Vdash \varphi$ and that $\mathcal{B}$ and $\mathcal{B}'$ may only differ in variables in $\vec{w} \cup (Y \setminus \mathsf{vars}(\varphi))$, hence not occurring in φ;

(b) $\mathcal{B}' \Vdash \Omega$. Observe that $\mathcal{B}' \Vdash \varphi \wedge \delta^{\mathsf{vars}(\varphi)}_{E_\mathcal{A}}$ since $\mathcal{B}$ and $\mathcal{B}'$ may only differ in variables in $\vec{w} \cup (Y \setminus \mathsf{vars}(\varphi))$, hence not occurring in $\varphi \wedge \delta^{\mathsf{vars}(\varphi)}_{E_\mathcal{A}}$. Moreover $\mathcal{B}' \Vdash \bigwedge_{\sigma \in S} \gamma_{k^{\varphi,\sigma}_{E_\mathcal{A}}}$ by definition, and so $\mathcal{B}' \Vdash \delta^{\mathsf{vars}(\varphi)}_{E_\mathcal{A}} \to \bigwedge_{\sigma \in S} \gamma_{k^{\varphi,\sigma}_{E_\mathcal{A}}}$. Since $\mathcal{B}' \Vdash \delta^{\mathsf{vars}(\varphi)}_{E_\mathcal{A}}$, we have that $\mathcal{B}' \nVdash \delta^{\mathsf{vars}(\varphi)}_{E}$ for all $E \neq E_\mathcal{A}$ with $E \sqsubseteq \mathsf{vars}(\varphi)^2$. Hence $\mathcal{B}' \Vdash \delta^{\mathsf{vars}(\varphi)}_{E} \to \bigwedge_{\sigma \in S} \gamma_{k^{\varphi,\sigma}_{E}}$ for all $E \sqsubseteq \mathsf{vars}(\varphi)^2$ with $\mathsf{Sat}(\varphi \wedge \delta^{\mathsf{vars}(\varphi)}_{E}) = 1$ and so $\mathcal{B}' \Vdash \Omega$;

(c) $\mathcal{B}' \Vdash \delta^Y_{E'}$. We only need to verify that $\mathcal{B}'$ satisfies the equalities and disequalities induced by E'. This holds since by construction, for each $\sigma \in S$, $\mathcal{B}'$ assigns the same value to variables in Y^σ_i, and assigns different values to variables in different sets Y^σ_i.

Finally it remains to show that $B'_\sigma = [\![\mathsf{vars}_\sigma(\varphi \wedge \Omega \wedge \delta^Y_{E'})]\!]^{\mathcal{B}'}$ for each sort $\sigma \in S$:

($\subseteq$): Let $d \in B'_\sigma$. Then d is either a d^σ_i for some $i = 1, \ldots, p$ or a e^σ_j for some $j = 1, \ldots, k^{\varphi,\sigma}_{E_\mathcal{A}}$. In the case that $d = d^\sigma_i$ then we have that $d = x^{\mathcal{B}'}$ for all $x \in Y^\sigma_i$. On the other hand, if $d = e^\sigma_j$ then $d = w^{\mathcal{B}'}_j$ for the w_j variable in $\mathsf{vars}(\gamma_{k^{\varphi,\sigma}_{E_\mathcal{A}}})$;

($\supseteq$): Obviously, $[\![\mathsf{vars}_\sigma(\varphi \wedge \Omega \wedge \delta^Y_{E'})]\!]^{\mathcal{B}'} \subseteq B'_\sigma$ by definition.

Algorithm 1 – Computes a strong witness function for a shiny theory $\mathcal{T}$ with respect to S

Input: φ, where φ is a quantifier-free satisfiable formula

Output: $\mathsf{s\text{-}witness}(\varphi)$

Requires: access to an algorithm Sat that decides satisfiability of quantifier-free formulas, and to the function $\mathsf{mincard}_{\mathcal{T},S}$ for $\mathcal{T}$

$\psi = \varphi$
for $E \sqsubseteq \mathsf{vars}(\varphi)^2$
  $\delta_E^{\mathsf{vars}(\varphi)} = \varepsilon$
  for $\sigma \in S$
    $\delta_{E_\sigma}^{\mathsf{vars}_\sigma(\varphi)} = \varepsilon$
    for all pairs $(x, y) \in \mathsf{vars}_\sigma(\varphi)^2$
      if $(x, y) \in E_\sigma$
        then $\delta_{E_\sigma}^{\mathsf{vars}_\sigma(\varphi)} = \delta_{E_\sigma}^{\mathsf{vars}_\sigma(\varphi)} \wedge (x \cong_\sigma y)$
        else $\delta_{E_\sigma}^{\mathsf{vars}_\sigma(\varphi)} = \delta_{E_\sigma}^{\mathsf{vars}_\sigma(\varphi)} \wedge \neg(x \cong_\sigma y)$
      end if
    end for
    $\delta_E^{\mathsf{vars}(\varphi)} = \delta_E^{\mathsf{vars}(\varphi)} \wedge \delta_{E_\sigma}^{\mathsf{vars}_\sigma(\varphi)}$
  end for
  if $\mathsf{Sat}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right) == 1$
    then
      for $\sigma \in S$
        $k_E^{\varphi,\sigma} = \mathsf{mincard}_{\mathcal{T},\sigma}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)$
        $\gamma_{k_E^{\varphi,\sigma}} = \varepsilon$
        for $i, j = 1,\ i \neq j$ **to** $k_E^{\varphi,\sigma}$
          $\gamma_{k_E^{\varphi,\sigma}} = \gamma_{k_E^{\varphi,\sigma}} \wedge \neg(w_{i,\sigma} \cong_\sigma w_{j,\sigma})$
        end for
      end for
      $\psi = \psi \wedge \left(\delta_E^{\mathsf{vars}(\varphi)} \rightarrow \bigwedge_{\sigma \in S} \gamma_{k_E^{\varphi,\sigma}}\right)$
  end if
end for
return ψ

Combining the previous items, we conclude that a shiny theory with respect to a set S is strongly finitely witnessable with respect to S, hence strongly polite with respect to S since it is smooth by definition. QED

Based on the proof of Proposition 3.1, we construct Algorithm 1 that builds a strong witness function for a shiny theory, provided that it has a decidable quantifier-free satisfiability problem.

3.2 A Strongly Polite Theory with a Decidable Quantifier-free Satisfiability Problem is a Shiny Theory

In this section we show that a strongly polite theory with respect to a set of sorts is shiny with respect to that set, assuming that the theory has a decidable quantifier-free satisfiability problem. The result holds for any set of sorts S since the computation of the $\mathsf{mincard}_S$ function will not rely on enumerating interpretations. This circumvents the restriction that $S = \Sigma^S$ imposed in [7]. We begin by proving some technical lemmas relating the mincard function with equivalence classes.

Lemma 3.2 *Let Y be a set of variables and $E \sqsubseteq Y^2$ a sort-wise family of equivalence relations over Y. Given the arrangement formula δ_E^Y and a sort $\sigma \in \tau(Y)$, the number of equivalence classes in the quotient set of Y_σ by E_σ is computable in at most $\mathcal{O}(|Y_\sigma|^2)$ steps.*

Proof:

Consider the following algorithm to compute the number of equivalence classes in the quotient set of Y_σ by E_σ, denoted in the sequel by ${}^{Y_\sigma}\!/_{E_\sigma}$, given the arrangement formula δ_E^Y and sort σ.

Algorithm 2 – Counting equivalence classes given an arrangement formula and a sort

---

Input: an arrangement formula δ_E^Y, and a sort σ
Output: cardinality of the quotient set ${}^{Y_\sigma}\!/_{E_\sigma}$

1: Construct a graph G_σ with vertices Y_σ and edge set $\left\{(x, y) : (x \cong_\sigma y) \in \delta_{E_\sigma}^{Y_\sigma}\right\}$
2: Compute connected components of G_σ
3: Return number of connected components of the graph

In step 1, the adjacency list for G_σ is built from $\delta_{E_\sigma}^{Y_\sigma}$. This can be made in $\mathcal{O}(|Y_\sigma|^2)$. To compute the number of connected components of the graph, we refer to the algorithm described by Hopcroft and Tarjan in [11] which has time complexity of the order of

$$\max\{|Ed|, |V|\} = \max\left\{\left|\left\{(x, y) : (x \cong_\sigma y) \in \delta_{E_\sigma}^{Y_\sigma}\right\}\right|, |Y_\sigma|\right\} \le |Y_\sigma|^2 .$$

So the time complexity of Algorithm 2 is $\mathcal{O}(|Y_\sigma|^2)$.

QED

Given a finite set of variables Y and a sort-wise family of equivalence relations $E \sqsubseteq Y^2$, denote

$$\max_{\sigma \in S} \left|{}^{Y_\sigma}\!/_{E_\sigma}\right| \quad \text{by} \quad \left|{}^{Y}\!/_{E}\right| .$$

Furthermore, denote by $\mathcal{E}$ the set $\left\{E : E \sqsubseteq Y^2\right\}$ of all sort-wise families of equivalence relations over Y, and define an ordering $\leq_{\mathsf{card}} \subseteq \mathcal{E}^2$ such that

$$E \leq_{\mathsf{card}} E' \text{ iff } \left|{}^{Y}\!/_{E}\right| \leq \left|{}^{Y}\!/_{E'}\right|.$$

Observe that the value $\left|{}^{Y}\!/_{E}\right|$ is computable since, for each $\sigma \in \tau(Y)$, the term $\left|{}^{Y_\sigma}\!/_{E_\sigma}\right|$ is computable by Lemma 3.2 and since $\tau(Y)$ is finite.

Lemma 3.3 *Consider a strongly polite theory $\mathcal{T}$ with respect to a set of sorts S and let φ be a quantifier free formula and $Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi))$. Then, for all $E \sqsubseteq Y^2$ such that $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is $\mathcal{T}$-satisfiable,*

$$\mathsf{mincard}_{\mathcal{T},S}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right) = \left|{}^{Y}\!/_{E}\right|.$$

Proof: Let $E \sqsubseteq Y^2$ be such that $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is $\mathcal{T}$-satisfiable. By the strong finite witnessability property, there is a $\mathcal{T}$-model $\mathcal{A}$ of $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ such that $A_\sigma = [\![\mathsf{vars}_\sigma(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y)]\!]^{\mathcal{A}}$, for each $\sigma \in S$. Hence,

$$\begin{aligned}
\mathsf{mincard}_{\mathcal{T},S}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right) &\leq \max_{\sigma \in S} |A_\sigma| \\
&= \max_{\sigma \in S} \left|[\![\mathsf{vars}_\sigma(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y)]\!]^{\mathcal{A}}\right| \\
&= \max_{\sigma \in S} \left|[\![Y_\sigma]\!]^{\mathcal{A}}\right| \\
&= \max_{\sigma \in S} \left|{}^{Y_\sigma}\!/_{E_\sigma}\right| \\
&= \left|{}^{Y}\!/_{E}\right|.
\end{aligned}$$

On the other hand, since any model that satisfies an arrangement formula must have at least a point in the domain for each equivalence class, we have that

$$\mathsf{mincard}_{\mathcal{T},S}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right) \geq \left|{}^{Y}\!/_{E}\right|.$$

Hence, we conclude that

$$\mathsf{mincard}_{\mathcal{T},S}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right) = \left|{}^{Y}\!/_{E}\right|.$$

QED

Lemma 3.4 *Let $\mathcal{T}$ be a strongly polite Σ-theory with respect to a set of sorts $S \subseteq \Sigma^S$ with a decidable quantifier-free satisfiability problem. Given a $\mathcal{T}$-satisfiable quantifier-free formula φ, there is a family of sort-wise equivalence relations E^* over $Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi))$ such that*

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \left|{}^{Y}\!/_{E^*}\right| \quad \textit{and} \quad \mathsf{Sat}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{E^*}^Y\right) = 1.$$

Proof: First, observe that the cardinality of the smallest $\mathcal{T}$-models of φ and $\mathsf{s\text{-}witness}(\varphi)$ is the same since these formulas are $\mathcal{T}$-equivalent. So

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \mathsf{mincard}_{\mathcal{T},S}(\mathsf{s\text{-}witness}(\varphi)).$$

Let $\mathcal{M}$ be a smallest $\mathcal{T}$-model of $\mathsf{s\text{-}witness}(\varphi)$ and consider the equivalence relationship on $Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi))$ induced by $\mathcal{M}$, i.e.,

$$E_{\mathcal{M}} = \bigcup_{\sigma \in S} \left\{(x,y) : x^{\mathcal{M}} = y^{\mathcal{M}} \text{ with } x, y \text{ in } Y_\sigma\right\}.$$

Hence, $\mathcal{M}$ is a smallest model of $\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_{E_{\mathcal{M}}}$, and so

$$\mathsf{mincard}_{\mathcal{T},S}(\mathsf{s\text{-}witness}(\varphi)) = \mathsf{mincard}_{\mathcal{T},S}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_{E_{\mathcal{M}}}).$$

By Lemma 3.3,

$$\mathsf{mincard}_{\mathcal{T},S}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_{E_{\mathcal{M}}}) = \left|{}^{Y}/_{E_{\mathcal{M}}}\right|.$$

Therefore,

$$\begin{aligned}\mathsf{mincard}_{\mathcal{T},S}(\varphi) &= \mathsf{mincard}_{\mathcal{T},S}(\mathsf{s\text{-}witness}(\varphi)) \\ &= \mathsf{mincard}_{\mathcal{T},S}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_{E_{\mathcal{M}}}) \\ &= \left|{}^{Y}/_{E_{\mathcal{M}}}\right|.\end{aligned}$$

QED

Lemma 3.5 *Let $\mathcal{T}$ be a strongly polite theory with respect to a set of sorts S with a decidable quantifier-free satisfiability problem. For all $\mathcal{T}$-satisfiable quantifier-free formulas φ,*

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \min\left\{\left|{}^{Y}/_{E}\right| : E \sqsubseteq Y^2,\ \mathsf{Sat}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_E\right) = 1\right\}$$

where $Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi))$.

Proof: Given a $\mathcal{T}$-satisfiable quantifier-free formula φ and $E \sqsubseteq Y^2$ such that $\mathsf{Sat}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_E\right) = 1$, since

$$\{\mathcal{A} : \mathcal{A} \Vdash \mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_E\} \subseteq \{\mathcal{A} : \mathcal{A} \Vdash \varphi\},$$

then

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) \leq \mathsf{mincard}_{\mathcal{T},S}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_E\right).$$

So, by Lemma 3.3, for each $E \sqsubseteq Y^2$ with $\mathsf{Sat}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right) = 1$,

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) \leq \left|{}^{Y}/_{E}\right|. \tag{1}$$

On the other hand, by Lemma 3.4, there is a family of sort-wise equivalence relations E^* over Y with

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \left|{}^{Y}/_{E^*}\right| \quad \text{and} \quad \mathsf{Sat}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{E^*}^Y\right) = 1. \tag{2}$$

Hence, combining (1) and (2), we conclude that

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \min\left\{\left|{}^{Y}/_{E}\right| : E \sqsubseteq Y^2,\ \mathsf{Sat}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right) = 1\right\}.$$

QED

We recall that a strongly polite theory with respect to a set of sorts S is stably finite with respect to S, as mentioned in Remark 10 of [7].

Lemma 3.6 ([7]) *A strongly polite theory with respect to a set of sorts is stably finite with respect to that set.*

Proposition 3.7 *A strongly polite theory with respect to a set of sorts S with a decidable quantifier-free satisfiability problem is shiny with respect to S. Moreover, Algorithm 3 computes its $\mathsf{mincard}_S$ function.*

Proof: Let $\mathcal{T}$ be a strongly polite theory with respect to a set of sorts S. By definition of strong politeness, $\mathcal{T}$ is smooth with respect to S. Moreover, by Lemma 3.6, $\mathcal{T}$ is stably finite with respect to S and so, in order to show that $\mathcal{T}$ is shiny with respect to S, it remains to prove that $\mathsf{mincard}_{\mathcal{T},S}$ is computable. Consider the $\mathsf{mincard}_{\mathcal{T},S}$ function described in Algorithm 3.

Concerning the termination of Algorithm 3, it is enough to see that Y is a finite set, hence the number of possible equivalence relations over Y is also finite, and so $\mathcal{E}$ is finite. Moreover, sorting $\mathcal{E}$ terminates since $\leq_{\mathsf{card}}$ is computable taking into account Lemma 3.2.

Concerning the partial correctness of Algorithm 3 observe that if the algorithm exits at step 5 in the i-th iteration, then this means that $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{\mathcal{E}_k}^Y$ is not satisfiable for all $k < i$, i.e., that $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is not satisfiable for all $E \sqsubseteq Y^2$ with $\left|{}^{Y}/_{E}\right| < \left|{}^{Y}/_{\mathcal{E}_i}\right|$. Moreover, $\mathsf{Sat}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{\mathcal{E}_i}^Y) = 1$ and since we have that

$$\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \min\left\{\left|{}^{Y}/_{E}\right| : E \sqsubseteq Y^2,\ \mathsf{Sat}\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right) = 1\right\}$$

by Lemma 3.5, this means that $\mathsf{mincard}_{\mathcal{T},S}(\varphi) = \left|{}^{Y}/_{\mathcal{E}_i}\right|$.

QED

Algorithm 3 – $\mathsf{mincard}_{\mathcal{T},S}$ algorithm for a strongly polite theory $\mathcal{T}$ with respect to a set of sorts S

Input: φ, a $\mathcal{T}$-satisfiable quantifier-free formula
Output: $\mathsf{mincard}_{\mathcal{T},S}(\varphi)$

$Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi))$
$\mathcal{E} = \mathsf{sort}(\{E : E \sqsubseteq Y^2\}, \leq_{\mathsf{card}})$
for $i = 1$ to $|\mathcal{E}|$ do
  if $\mathsf{Sat}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_{\mathcal{E}_i}) = 1$
    then return $\left|{}^{Y}/_{\mathcal{E}_i}\right|$
  $i = i + 1$
end for

Observe that the number of equivalence relations of a set with n elements is the n-th Bell number, B_n. Therefore, given a satisfiable quantifier-free formula φ and a set of sorts S, and letting $Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi))$, for each $\sigma \in \tau(Y)$ there are $B_{|Y_\sigma|}$ equivalence relations.

Proposition 3.8 *The time complexity of Algorithm 3 is*

$$\mathcal{O}\left(\left(\prod_{\sigma\in S} B_{|Y_\sigma|}\right)\cdot\left(|Y|^2 + C_{\mathsf{Sat}}(|Y|^2 + |\mathsf{s\text{-}witness}(\varphi)|)\right)\right),$$

where $C_{\mathsf{Sat}}(\cdot)$ *is the time complexity of the* Sat *solver.*

Proof: Let $\mathcal{T}$ be a strongly polite theory with respect to a set of sorts S and φ a $\mathcal{T}$-satisfiable quantifier-free formula. For each $\sigma \in S$ there are $B_{|Y_\sigma|}$ equivalence relations on Y_σ. Hence, there are $\prod_{\sigma\in S} B_{|Y_\sigma|}$ families of equivalence relations $E \sqsubseteq Y^2$, which is the cardinality of $\mathcal{E}$. To sort $\mathcal{E}$, construct $|Y|$ lists and for each $E \in \mathcal{E}$, insert the formula δ^Y_E in the $\left|{}^{Y}/_{E}\right|$-th list. Thus, for each $E \in \mathcal{E}$ it takes

- $\mathcal{O}\left(|Y|^2\right)$ to describe the formula δ^Y_E;
- $\mathcal{O}\left(\Sigma_{\sigma\in S}|Y_\sigma|^2\right) \subseteq \mathcal{O}\left(|Y|^2\right)$ to compute $\left|{}^{Y}/_{E}\right|$ using Algorithm 2 and the fact that if $|Y| = \Sigma_{\sigma\in S}|Y_\sigma|$ then $|Y|^2 \geq \Sigma_{\sigma\in S}|Y_\sigma|^2$;

which accounts to $\mathcal{O}\left(|Y|^2 \cdot \prod_{\sigma\in S} B_{|Y_\sigma|}\right)$ time steps to process the set $\mathcal{E}$.

The cycle runs at most $|\mathcal{E}|$ iterations each one with time complexity bounded by $C_{\mathsf{Sat}}(|Y|^2 + |\mathsf{s\text{-}witness}(\varphi)|)$. Observe that, if $\mathsf{Sat}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta^Y_{\mathcal{E}_i}) = 1$, computing $|{}^{Y}/_{\mathcal{E}_i}|$ takes at most $\mathcal{O}(|Y|^2)$ steps, by Lemma 3.2.

Hence, the time complexity of Algorithm 3 is of order

$$\mathcal{O}\left(|Y|^2 \cdot \prod_{\sigma\in S} B_{|Y_\sigma|} + |Y|^2 + \left(\prod_{\sigma\in\tau(Y)} B_{|Y_\sigma|}\right) \cdot C_{\mathsf{Sat}}(|Y|^2 + |\mathsf{s\text{-}witness}(\varphi)|)\right),$$

that is

$$\mathcal{O}\left(\left(\prod_{\sigma\in S} B_{|Y_\sigma|}\right) \cdot (|Y|^2 + C_{\mathsf{Sat}}(|Y|^2 + |\mathsf{s\text{-}witness}(\varphi)|))\right).$$

QED

Combining Propositions 3.1 and 3.7 we obtain the equivalence between shininess and strong politeness in the many-sorted case. This result shows that the class of shiny theories coincides with the class of strongly polite theories when the theories are equipped with a quantifier-free satisfiability solver. It should be made clear that the requirement that the theories have satisfiability solvers is not a real restriction since shiny, polite and strongly polite theories were proposed in view of a more general Nelson-Oppen result, which is about combination of satisfiability solvers.

Theorem 3.9 *Let $\mathcal{T}$ be a first-order Σ-theory with a decidable quantifier-free satisfiability problem and $S \subseteq \Sigma^S$ a set of sorts. Then, the following statements are equivalent:*

1. *$\mathcal{T}$ is shiny with respect to S;*
2. *$\mathcal{T}$ is strongly polite with respect to S.*

The proof that 1. implies 2. follows by Proposition 3.1 and for 2. implies 1., by Proposition 3.7.

3.3 Applications and Examples

Theory of equality. Consider the theory $\mathcal{T}$ of equality over the signature

$$\Sigma = \langle\{1,2,3\}, \emptyset, \emptyset, \alpha, \tau\rangle,$$

with the following extra axioms

- $\mathsf{Ax1}$: $\exists x \exists y\ \neg(x \cong_1 y)$,
- $\mathsf{Ax2}$: $\exists x \exists y \exists z\ \neg(x \cong_2 y) \wedge \neg(x \cong_2 z) \wedge \neg(y \cong_2 z)$,

- $\mathsf{Ax3} : \forall x \forall y\ (x \cong_3 y)$.

This theory is strongly polite with respect to $S = \{1, 2\}$ but not with respect to sort 3, since $\mathsf{Ax3}$ makes $\mathcal{T}$ not smooth with respect to sort 3. Consider the strong witness function for $\mathcal{T}$, defined as

$$\mathsf{s\text{-}witness} = \lambda\varphi.\varphi \wedge \neg(w_1 \cong_1 w_2) \wedge \neg(w_3 \cong_2 w_4) \wedge \neg(w_3 \cong_2 w_5) \wedge \neg(w_4 \cong_2 w_5).$$

For the sake of illustration, we now follow Algorithm 3 in order to calculate $\mathsf{mincard}_{\mathcal{T},S}(\varphi)$ where φ is the formula $(x \cong_1 x)$.

Let $Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi)) = Y_1 \cup Y_2$, where $Y_1 = \{x, w_1, w_2\}$ and $Y_2 = \{w_3, w_4, w_5\}$. Then, we need to consider the families of equivalence relations on Y. These relations are of the kind $E_1 \cup E_2$ where E_1 and E_2 are equivalence relations on Y_1 and Y_2 respectively. A smallest satisfiable equivalence relation on sort 1, E_1, is $\{(x, w_1)\}$ which has 2 equivalence classes. On sort 2, the variables w_3, w_4, w_5 must be different and so a smallest satisfiable relation on sort 2 has 3 equivalence classes. Hence, the smallest $\mathcal{T}$-model of φ has 3 elements, and so $\mathsf{mincard}_{\mathcal{T},S}(\varphi) = 3$ by Algorithm 3.

Graph colouring. As pointed out in [10], colourings of a graph and cardinality of models are intimately related. In fact, deciding the k-colorability of a graph reduces to deciding the k-cardinality of a set of formulas in the empty theory of equality with one sort.[2] Here, we adapt Algorithm 3 for computing the minimal cardinality of a set of formulas to obtain a k-colorability algorithm.

It is worth mentioning that the empty theory of equality is self strongly finitely witnessable, i.e., $\mathsf{s\text{-}witness}(\varphi) = \varphi$. This vastly simplifies Algorithm 3 for computing the $\mathsf{mincard}$ function and its complexity.

Let $\mathcal{G}$ be a graph with vertex set V and edges $E \subseteq V^2$. We say that γ is the *graph formula* of $\mathcal{G}$ if γ is of the form

$$\bigwedge_{(u,v) \in E} \neg(u \cong v) \wedge \bigwedge_{u \in V} (u \cong u).$$

We now propose a new algorithm to decide the k-colorability of $\mathcal{G}$.

There are $\left\{ {|V| \atop k} \right\}$ equivalence relations with k non-empty classes, where $\left\{ {\cdot \atop \cdot} \right\}$ is the Stirling number of the second kind. Since the complexity of the quantifier-free satisfiability algorithm for the theory of equality is of the

[2] Given a set of formulas, deciding its k-cardinality problem is deciding whether this set of formulas is satisfied by a model of cardinality k.

Algorithm 4 – k-colorability algorithm

Input: γ, the graph formula of $\mathcal{G}$
Output: k-colorability of $\mathcal{G}$

```
E = {E : E ⊆ V^2 with k equivalence classes};
for i = 1 to |E| do
    if Sat(γ ∧ δ_{E_i}) == 1
        then return Yes
    i = i + 1
end for
return No
```

order of n^2 for a formula that is a conjunction of n literals, we have that the complexity of Algorithm 4 is of order

$$\mathcal{O}\left(\left\{ {|V| \atop k} \right\} \times |V|^4\right).$$

The presented k-colorability algorithm (based on the previously presented mincard algorithm) differs from the one in [10] since it does not rely on enumerating interpretations and building their diagrams – here we enumerate equivalence relations with k classes which are, in a sense, structures of the interpretations.

Batch algorithm for k-colorability Suppose we are given a set of P undirected graphs of n vertices which we need to decide whether they are k-colorable or not. The batch algorithm first preprocesses the set of equivalence relations with k classes, $\mathcal{E}$, in the following way: since the graphs are undirected, we can assume that all the possible edges are representable by $\binom{n}{2} = \frac{1}{2}(n-1)n$ numbers. Then, given an equivalence relation in $\mathcal{E}$, represent it by the list of its edge-numbers and sort it.[3] Hence, we have a set L of $\left\{ {n \atop k} \right\}$ ordered lists, each a subset of $\{1, \ldots, \binom{n}{2}\}$. Now, we shall create a structure in which checking whether an arbitrary list is a subset of a list in L takes the size of the list operations to complete. Such structure A can be created using Ukkonen's Algorithm, see [13], for generalized suffix trees, viewing the lists in L as the following parsed strings:

Algorithm 5 basically writes the missing numbers from the set $\left\{1, \ldots, \binom{n}{2}\right\}$ as # symbols and trims the edges. For instance, consider the set of all equivalence relations on 4 elements with 2 classes represented by its edge-numbers

[3] Here we represent an equivalence relation E by numbers that represent the negated formulas in δ_E. These numbers are called *edge-numbers*.

Algorithm 5 – Parse: parses a list to a string

Input: Ordered list $\ell \subseteq \left\{1, \ldots, \binom{n}{2}\right\}$
Output: s, special string

$s[1] = \ell(1)$
$\nu = \ell(1) + 1$
$j = 2$
for $i = 2$ to $|\ell|$ do
    if $\ell(i) == \nu$
        then $s[j] = \ell(i)$
            $i = i + 1$
        else $s[j] = \#$
    $\nu = \nu + 1$; $j = j + 1$
end for
return s

1	$\mapsto$	(3,4)
2	$\mapsto$	(2,4)
3	$\mapsto$	(2,3)
4	$\mapsto$	(1,4)
5	$\mapsto$	(1,3)
6	$\mapsto$	(1,2)

Table 1: Edge-number to edges mapping

as well as its associated parsed strings (using Algorithm 5), where the edge-numbers are mapped to the possible $\binom{4}{2} = 6$ edges as in Table 1.

Given the strings in Table **??**, we use Ukkonen's Algorithm to build the generalized suffix tree, [13]. In the pseudo-code, we will refer to Ukkonen's Algorithm as GenSuffTree, that when receiving a list of strings will create its associated generalized suffix tree. Given that Algorithm 5 has time complexity of order $\mathcal{O}\left(n^2\right)$ and since we have to parse $\left\{ {n \atop k} \right\}$ strings, the total parsing time is of order $\mathcal{O}\left(\left\{ {n \atop k} \right\} n^2\right)$. Furthermore, building the generalized suffix tree takes linear time on the sum of the size of the strings, which is again of order $\mathcal{O}\left(\left\{ {n \atop k} \right\} n^2\right)$. This concludes the preprocessing part of the algorithm, detailed in Algorithm 6.

Now, in order to check whether a sorted list of edges of a graph is k-colorable, we need to determine if it is a subset of one of the lists in L. Equivalently, we need to parse it with Algorithm 5 and traverse the suffix tree as usual, except when it encounters a # in either the tree or the queried

$\{1, 2, 4\}$		12#4
$\{1, 2, 5, 6\}$		12##56
$\{1, 3, 4, 6\}$		1#34#6
$\{1, 3, 5\}$	maps to	1#3#5
$\{2, 3, 4, 5\}$		2345
$\{2, 3, 6\}$		23##6
$\{4, 5, 6\}$		456

Algorithm 6 – Preprocess: preprocessing for k-colorability algorithm

Input: Number of vertices in the graph, n, and k
Output: A, a generalized suffix tree

```
Store in L the {n k} equivalence relations represented by ordered subsets of edge-numbers
𝓛 = {}
for i = 1 to |L| do
    Append Parse(L_i) to 𝓛
    i = i + 1
end for
A = GenSuffTree(𝓛)
```

string. This is summarized in Table 2.

An accepted string will always be a substring of one of the strings in the tree since:

- the first symbol of the input string is never a # symbol and thus, we will continue iff there is a branch in the tree that matches it;
- the traversing algorithm will only continue iff the input symbol is in the tree or it is a # symbol.

Input symbol	**Tree symbol**	**Operation**
n	n	Continue
n	$p (p \neq n)$	Fail
#	#	Continue
#	n	Continue
n	#	Fail

Table 2: Query rules

This concludes the query part of the algorithm, detailed in Algorithm 7.

Algorithm 7 – Query: query procedure for k-colorability algorithm

Input: Generalized suffix tree A and a graph $\mathcal{G}$ to be queried
Output: k-colorability of $\mathcal{G}$

```
Represent the graph in an ordered list with its edge-numbers, ℓ.
Query A with input Parse(ℓ) and traverse as in Table 2.
```

In conclusion, to decide the k-colorability we obtain Algorithm 8 such that:

- the preprocessing time of the generation of the equivalence relations, parsing to special strings and building the generalized suffix tree is in $\mathcal{O}\left(\left\{ n \atop k \right\} n^2\right)$;
- the query time is of order $\mathcal{O}\left(n^2\right)$.

Algorithm 8 – k-colorability algorithm for a batch of graphs

Input: Set B of graphs with n vertices
Output: k-colorability of graphs in B

```
A = Preprocess(n, k)
for i = 1 to |B| do
    v[i] = Query(A, B_i) (*store answers in v*)
    i = i + 1
end for
return v
```

Hence, the complexity of solving the k-colorability for the batch of P graphs with n vertices is of order

$$\mathcal{O}\left(\left\{ n \atop k \right\} n^2 + P \cdot n^2\right).$$

Observe that for a fixed value of k,

$$\left\{ n \atop k \right\} \sim \frac{k^n}{k!},$$

whence

$$\mathcal{O}\left(\left\{ n \atop k \right\} n^2 + P \cdot n^2\right) \sim \mathcal{O}\left(\frac{k^n}{k!} n^2 + P n^2\right).$$

Furthermore, since $k! > \left(\frac{k}{3}\right)^k$, we get that

$$\mathcal{O}\left(\frac{k^n}{k!}n^2 + Pn^2\right) < \mathcal{O}\left(3^k k^{n-k} n^2 + Pn^2\right)$$
$$= \mathcal{O}\left(3^k \cdot 2^{(\log k)(n-k)} n^2 + Pn^2\right).$$

To the best of our knowledge, the fastest algorithm for deciding k-colorability has time complexity of order $\mathcal{O}(2^n nk \cdot \mathsf{polylog}(nk))$, [12]. Hence, deciding the k-colorability of a set of P graphs with n vertices, would take $\mathcal{O}(P \cdot 2^n nk \cdot \mathsf{polylog}(nk))$ using the algorithm in [12].

Despite both having exponential complexity orders, since our proposed algorithm only preprocesses once, we believe that in practical terms, our proposed Algorithm 8 may be advantageous when deciding colorability of a large number of graphs.

4 Shiny/Strongly Polite and Polite Theories

Although the Nelson-Oppen method proposed by [7] (Sat algorithm for quantifier-free formulas in the union of a polite theory with an arbitrary theory) does not hold in general, as seen in [3], there are conditions under which it does.

4.1 A Polite Theory is a Shiny/Strongly Polite Theory: Sufficient Conditions

Herein, we investigate conditions on polite theories that make them shiny in the many-sorted case. This will allow us to characterize shiny theories with a different set of conditions, generalizing results from [4].

The first result on this relationship appeared in [7], where the authors provide conditions under which a polite theory is shiny with respect to a set S of sorts. However, it is imposed that $S = \Sigma^S$. This requirement is necessary for computing the mincard function by enumerating models when the theory is polite (in fact, the rationale behind Proposition 3.7 that allowed us to compute the mincard function without enumerating models does not work without requiring strong politeness). For completion and self-containment, we recall that result.

Proposition 4.1 ([7]) *Let Σ be a finite signature, $S \subseteq \Sigma^S$ be a set of sorts and $\mathcal{T}$ a Σ-theory. Assume that:*

- *it is decidable to check if a finite Σ-interpretation is a $\mathcal{T}$-model;*
- $S = \Sigma^S$.

If $\mathcal{T}$ is polite with respect to S then $\mathcal{T}$ is shiny with respect to S and Algorithm 9 computes its $\mathsf{mincard}_{\mathcal{T},S}$ *function.*

Algorithm 9 – $\mathsf{mincard}_{\mathcal{T},S}$ algorithm given a $\mathsf{witness}$ function, [7]
Input: φ, where φ is a quantifier-free satisfiable formula
Output: k, where k is the cardinality of the smallest $\mathcal{T}$-model of φ
Requires: access to a witness function $\mathsf{witness}$ for $\mathcal{T}$

$n = \max_{\sigma \in S} |\mathsf{vars}_\sigma(\mathsf{witness}(\varphi))|$
for $k = 1$ **to** n
  for all non-isomorphic $\mathcal{T}$-models $\mathcal{A}$ s.t. $\max_{\sigma \in S} |A_\sigma| = k$ do
    if $\mathcal{A} \Vdash \varphi$ **then** return k
  end for
  $k = k + 1$
end for

Even for theories $\mathcal{T}$ where checking whether a finite interpretation is a model of the theory is not known to be decidable, it may be possible to construct the $\mathsf{mincard}$ function provided that $\mathcal{T}$ is universal, as is stated in the next proposition. This proposition uses a result of Tinelli and Zarba in [10] and generalizes the result in [4] to the many-sorted case. Similarly to the result of Ranise, Ringeissen and Zarba, the requirement that $S = \Sigma^S$ arises again in order to enumerate the interpretation structures.

Proposition 4.2 *Let Σ be a finite signature, $S \subseteq \Sigma^S$ be a set of sorts and $\mathcal{T}$ a Σ-theory. Assume that:*

- *$\mathcal{T}$ is universal;*
- *$\mathcal{T}$ has a decidable quantifier-free satisfiability problem;*
- $S = \Sigma^S$.

If $\mathcal{T}$ is polite with respect to S then $\mathcal{T}$ is shiny with respect to S and Algorithm 10 computes its $\mathsf{mincard}_{\mathcal{T},S}$ *function.*

Proof: Since $\mathcal{T}$ is polite with respect to S, we trivially obtain that $\mathcal{T}$ is stably finite with respect to S. The thesis follows immediately by a generalization to the many-sorted case of Proposition 23 in [10] that establishes that the mincard function of any theory is computable by Algorithm 10, if that theory is stably finite, universal, is over a finite signature, and has a decidable quantifier-free satisfiability problem. Given that the algorithm relies on enumerating interpretation structures this generalization is possible since $S = \Sigma^S$. QED

Observe that Algorithm 10 makes use of a *simple diagram* of an interpretation. We suggest [10] for this definition.

Algorithm 10 – $\mathsf{mincard}_{\mathcal{T},S}$ algorithm given a Sat procedure, [10]
Input: φ, where φ is a quantifier-free satisfiable formula
Output: k, where k is the cardinality of the smallest $\mathcal{T}$-model of φ
Requires: access to a procedure Sat that decides satisfiability of quantifier-free formulas and where $\Delta(\mathcal{A})$ denotes the simple diagram of $\mathcal{A}$

$k = 1$
while *true* **do**
  for all non-isomorphic interpretations $\mathcal{A}$ s.t. $\max_{\sigma \in S} |A_\sigma| = k$ do
    if $\mathsf{Sat}(\Delta(\mathcal{A}) \wedge \varphi) == 1$ **then** return k
  end for
  $k = k + 1$
end while

The next result summarizes the conditions on polite theories that make them *shiny/strongly polite*.

Corollary 4.3 *Let $\mathcal{T}$ be a theory over a finite signature Σ with a decidable quantifier-free satisfiability problem. If $\mathcal{T}$ is polite with respect to $S = \Sigma^S$ and either*

1. *$\mathcal{T}$ is universal; or*
2. *checking whether a finite interpretation is a $\mathcal{T}$-model is decidable;*

then $\mathcal{T}$ is shiny with respect to S.

4.2 A Shiny/Strongly Polite Theory is a Polite Theory

On the other hand, we obtain that all shiny/strongly polite theories with a decidable quantifier-free satisfiability procedure with respect to a set of sorts are indeed polite with respect to the same set, by the equivalence Theorem 3.9, taking into account that

Fact 1 *A strongly finitely witnessable theory with respect to a set of sorts is finitely witnessable with respect to that set.*[4]

5 Combining Theories

Capitalizing on the relationships between the classes of polite, shiny and strongly polite theories established in the previous sections, we can now generalize the Nelson-Oppen style theorem in [4].

Observe that, as previously motivated, the conditions described in Corollary 4.3 for a theory to be shiny/strongly polite may be more manageable than proving by the definition that the theory is strongly finitely witnessable, particularly because many theories of interest to SMT applications are either universal or finitely axiomatized.

In other words, these results show that in the many-sorted context, if a polite theory with respect to the set of all sorts in its signature is either universal or is such that checking whether a finite interpretation is a model is decidable, then we can construct both a strong witness function and the $\mathsf{mincard}_S$ function. These functions can then be used in the application of the Nelson-Oppen method for the combination of strongly polite theories or shiny theories with an arbitrary theory. The following result formalizes these statements in a Nelson-Oppen style combination theorem.

Given a natural k and a sort σ, define γ_k^σ as the formula

$$\bigwedge_{\substack{i,j=1 \\ i \neq j}}^{k} \neg(w_{i,\sigma} \cong_\sigma w_{j,\sigma})$$

where $w_{1,\sigma}, \ldots, w_{k,\sigma}$ are distinct fresh σ-variables.

Theorem 5.1 *Let Σ_2 be a finite signature and $\mathcal{T}_i$ a Σ_i-theory with a decidable quantifier-free satisfiability problem, for $i = 1, 2$, such that $\Sigma_1^P \cap \Sigma_2^P = \emptyset$ and $\Sigma_1^F \cap \Sigma_2^F = \emptyset$. Assume that $\mathcal{T}_2$ is polite with respect to $S = \Sigma_2^S$ and that either:*

- *$\mathcal{T}_2$ is universal; or*
- *checking whether a finite interpretation is a $\mathcal{T}_2$-model is decidable.*

[4]Observe that a strong witness function is also a witness function (specifically, with respect to the second condition of the finite witnessability, let E and Y to be the empty set and the result follows).

Then, the function $\mathsf{mincard}_{\mathcal{T}_2,S}$ *is computable and there is a computable strong witness function,* $\mathsf{s\text{-}witness}_{\mathcal{T}_2}$*, for* $\mathcal{T}_2$ *with respect to* $S = \Sigma_2^S$*, such that the following statements are equivalent:*

1. $\Gamma_1 \wedge \Gamma_2$ *is* $\mathcal{T}_1 \oplus \mathcal{T}_2$*-satisfiable;*

2. *there exists* $E \sqsubseteq Y^2$*, where* Y *is* $\mathsf{vars}_S(\Gamma_1) \cap \mathsf{vars}_S(\Gamma_2)$*, such that*

   - $\Gamma_1 \wedge \delta_E^Y \wedge \bigwedge_{\sigma \in S} \gamma_{k_E^\sigma}^\sigma$ *is* $\mathcal{T}_1$*-satisfiable, where* k_E^σ *is* $\mathsf{mincard}_{\mathcal{T}_2,\sigma}(\Gamma_2 \wedge \delta_E^Y)$*;*
   - $\Gamma_2 \wedge \delta_E^Y$ *is* $\mathcal{T}_2$*-satisfiable;*

3. *there exists* $E \sqsubseteq Y^2$*, where* Y *is* $\mathsf{vars}_S(\mathsf{s\text{-}witness}(\Gamma_2))$*, such that*

   - $\Gamma_1 \wedge \delta_E^Y$ *is* $\mathcal{T}_1$*-satisfiable;*
   - $\mathsf{s\text{-}witness}_{\mathcal{T}_2}(\Gamma_2) \wedge \delta_E^Y$ *is* $\mathcal{T}_2$*-satisfiable;*

for every conjunction Γ_1 *of* Σ_1*-literals and* Γ_2 *of* Σ_2*-literals.*

Proof: Observe that the theory $\mathcal{T}_2$ is polite with respect to $S = \Sigma_2^S$ and using Corollary 4.3 also shiny. Hence the function $\mathsf{mincard}_{\mathcal{T}_2,S}$ is computable. Moreover, $\mathcal{T}_2$ is also strongly polite with respect to S by Theorem 3.9 since it is shiny and has a decidable quantifier-free satisfiability procedure. Hence, since $\mathcal{T}_2$ is shiny with respect to S, the equivalence between (1) and (2) follows from the combination theorem, Theorem 18, in [10], and since $\mathcal{T}_2$ is strongly polite with respect to S, the equivalence between (1) and (3) follows from the combination proposition, Proposition 2, in [3]. QED

We now provide an example showing an application of the previous theorem.

Example 5.2 *Consider the theories* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *of equality over the signature*

$$\Sigma = \langle \{1, 2\}, \emptyset, \emptyset, \alpha, \tau \rangle.$$

Since these are theories of equality, for each sort $\sigma \in \Sigma^S$ *there exists the symbol* $\cong_\sigma$ *such that* $\alpha(\cong_\sigma) = 2$ *and* $\tau(\cong_\sigma) = \sigma \times \sigma$*. Furthermore,* $\mathcal{T}_2$ *has the axioms:*

- $\mathsf{Ax1} : \exists x \exists y \, \neg(x \cong_1 y)$,
- $\mathsf{Ax2} : \exists x \exists y \exists z \, \neg(x \cong_2 y) \wedge \neg(x \cong_2 z) \wedge \neg(y \cong_2 z)$,

and $\mathcal{T}_1$ is axiomatized by $\forall x \forall y\ (x \cong_1 y)$. Hence, every model of $\mathcal{T}_1$ has cardinality at most one with respect to sort 1 and every model of $\mathcal{T}_2$ has cardinality at least two with respect to sort 1. Let φ denote the formula

$$(x \cong_1 x).$$

Observe that the theory $\mathcal{T}_2$ is smooth with respect to both sorts 1 and 2 and

$$\mathsf{witness}_{\mathcal{T}_2}(\varphi) := \varphi \wedge \bigwedge_{i=1}^{2}(w_i \cong_1 w_i) \wedge \bigwedge_{i=1}^{3}(z_i \cong_2 z_i)$$

is a witness function for $\mathcal{T}_2$ with respect to both sorts 1 and 2. Hence, the condition for the application of Theorem 5.1 requiring that $\mathcal{T}_2$ is polite is fulfilled. Moreover, it is immediate to see that checking whether a finite interpretation is a $\mathcal{T}_2$-model is decidable since $\mathcal{T}_2$ has only a finite number of axioms. Taking into account that $\mathsf{mincard}_{\mathcal{T}_2}(\varphi) = 3$, then by Algorithm 1,

$$\begin{aligned}
&\mathsf{s\text{-}witness}_{\mathcal{T}_2}(\varphi) = \varphi \wedge ((x \cong_1 x) \rightarrow (\gamma_2^1 \wedge \gamma_3^2)) \\
&= \varphi \wedge ((x \cong_1 x) \rightarrow (\neg(z_1 \cong_1 z_2) \wedge \neg(w_1 \cong_2 w_2) \wedge \neg(w_1 \cong_2 w_3) \wedge \neg(w_2 \cong_2 w_3))) \\
&= (x \cong_1 x) \wedge \neg(z_1 \cong_1 z_2) \wedge \neg(w_1 \cong_2 w_2) \wedge \neg(w_1 \cong_2 w_3) \wedge \neg(w_2 \cong_2 w_3).
\end{aligned}$$

Let Γ_1 be the formula $\mathtt{tt}$, Γ_2 the formula φ and Y the set $\mathsf{vars}(\mathsf{s\text{-}witness}(\Gamma_2))$ i.e. $\{\{x, z_1, z_2\}, \{w_1, w_2, w_3\}\}$. We now would like to check if there is $E \sqsubseteq Y^2$ such that $\Gamma_1 \wedge \delta_E^Y$ is $\mathcal{T}_1$-satisfiable and $\mathsf{s\text{-}witness}(\Gamma_2) \wedge \delta_E^Y$ is $\mathcal{T}_2$-satisfiable. Note that the δ_E^Y formulas satisfied by $\mathcal{T}_1$ are the ones induced by equivalence relations of the form

$$E = \{\{(x, z_1), (x, z_2), (z_1, z_2)\}, E_2\}$$

since all others would require the interpretation to have cardinality greater than one on the first sort. However, $\mathsf{s\text{-}witness}(\Gamma_2) \wedge \delta_E^Y$ is clearly not satisfiable. Hence, by Theorem 5.1, we conclude that φ is not satisfiable in $\mathcal{T}_1 \oplus \mathcal{T}_2$. In this simple case it is not difficult to see that this was the expected conclusion since there are no models that satisfy the theory resulting from the union of $\mathcal{T}_1$ and $\mathcal{T}_2$.

6 Conclusion and Further Research

In this paper we investigated the relationship between the notions of shininess, politeness and strong politeness, in the many-sorted case. We showed that, in the many-sorted case, a shiny theory with respect to a set of sorts (and equipped with a decidable quantifier-free satisfiability problem) is strongly polite with respect to the same set. On the other hand we were also able to prove that a strongly polite theory with respect to a set of sorts (with a decidable quantifier-free satisfiability problem) is shiny with respect to the same set. These results show that, contrarily to what was believed, the classes of shiny and strongly polite theories with a decidable quantifier-free satisfiability problem are in fact the same. Observe that these results provide new ways of proving that a theory satisfy the conditions of some of the Nelson-Oppen style combination theorems that have been proposed.

Capitalizing on the results related to the computation of the $\mathsf{mincard}$ function, we devised a colouring algorithm particularly suitable for applications where several graphs need to be coloured. This algorithm takes advantage of a preprocessing/query phase. We leave as future work to investigate its translation into other NP-complete problems (like for example Sat).

In the last section of the paper we proposed a Nelson-Oppen style method for deciding the satisfiability of quantifier-free formulas in the combination of an arbitrary many-sorted theory with a particular many-sorted polite theory. The sufficient conditions for the method to apply use the conditions for a polite theory to be shiny/strongly polite.

We intend to investigate in the future more general conditions that a theory should satisfy in order to be combined with an arbitrary theory by a Nelson-Oppen method. More concretely, we leave as future work the investigation of a class of theories strictly containing the shiny/strongly polite theories for which there exists a Nelson-Oppen method.

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