Fibring as biporting subsumes asymmetric combinations

J. Rasga A. Sernadas C. Sernadas
Dep. Matemática, Instituto Superior Técnico, Universidade de Lisboa, Portugal
SQIG, Instituto de Telecomunicações, Lisboa, Portugal
{jfr,acs,css}@math.ist.utl.pt

February 18, 2013

Abstract

The transference of preservation results between importing (a logic combination mechanism that subsumes several asymmetrical mechanisms for combining logics like temporalization, modalization and globalization) and unconstrained fibring is investigated. For that purpose, a new (more convenient) formulation of fibring, called biporting, is introduced. The equivalence between fibring and biporting is established by showing that satisfaction and semantic consequence are preserved. In consequence, particular cases of importing, like temporalization, modalization and globalization are shown to be subsumed by fibring. As an illustration, the preservation of the finite model property by fibring is carried over to importing and then to globalization.

keywords: importing, temporalization, finite model property, fibring, biporting, combining logics.

MSC 2010: 03B62, 03B44, 03B45.

1 Introduction

Importing was recently proposed, see [21, 22], as an abstraction of asymmetric mechanisms for combining logics, including, among others, temporalization [10], modalization [9] and globalization [20]. Motivated by the importance of such mechanisms (they are used for example in model checking for verifying quantum protocols [15], in the specification of temporal evolving ontologies [1], and in hybridization of institutions [18]) we intend to investigate preservation results for importing since they can be adapted to most of the logic mechanisms it subsumes.

Preservation of properties from the logics to be combined, to the logic resulting from their combination is one of the main and most challenging goals behind the study of combination of logics [23, 2]. For instance, in fibring [4, 12, 3], and fusion [26, 17], a wide plethora of preservation results has been proved involving properties like soundness, completeness, consistency, interpolation, and decidability, among others [28, 13, 7, 16, 5].

Herein we start to address the question of reusing the preservation results proved for a logic combination mechanism in other different combination mechanisms. For that it is essential to intimately relate the combination mechanisms involved. We concentrate in this paper on importing (and, so, indirectly, in all the asymmetric mechanisms subsumed by importing like temporalization, modalization and globalization) and on unconstrained fibring.

Since fibring was proposed in [14], it was expected that a meaningful relationship with asymmetric mechanisms for combining logics (like temporalization and modalization) would be found. On one hand, it has been argued that, for instance, the fibring of two modal logics might be recovered by an infinite sequence of modalizations in alternate directions. On the other hand, it has been conjectured that suitably constrained fibring would subsume modalization.

We show the latter conjecture: how to recover an asymmetric combination mechanism from fibring. To this end we propose a new formulation of unconstrained fibring by looking at it as a kind of two-way importing. The key idea is to define unconstrained fibring as a kind of two simultaneous importings in opposite directions, that is, as a kind of import/export operation or, in short, biporting. Recall that importing relies on a specific connective ($\Lsh$) that brings the formulas from the imported logic into the importing one [21, 22]. This connective is interpreted as a relation between truth values (respecting the distinguished ones). For biporting we need, in addition, an export connective ($\Rsh$) with similar semantics. Biporting is shown to be equivalent, modulo the expected translation of formulas, to unconstrained fibring in a strong sense: there is a bijection between the models of fibring and the models of biporting respecting satisfaction of formulas with no schema variables. This equivalence capitalizes on the fact that both biporting and fibring are presented using the graph-theoretic account of logics and their combinations first proposed in [24, 25]. We recover asymmetric combinations like temporalization and modalization (particular cases of importing) by suitably constraining biporting. We show that importing is recovered at the language level by forgetting the export connective and at the semantic level by cocartesian lifting. Therefore, we obtain in a canonical way the models of importing from the models of biporting guided by the signature morphism.

These relationships are herein put to good use by establishing that globalization enjoys the finite model property under mild conditions capitalizing on a similar result for fibring, see [7].

For the convenience of the reader, a short summary of the graph-theoretic account of logics is provided in Section 2. The notion of biporting is introduced and illustrated in Section 3. The equivalence of biporting and unconstrained fibring is proved in Section 4, after recalling the graph-theoretic account of fibring. In Section 5, we show how to recover importing from fibring (formulated as biporting). In Section 6 we carry over to importing the preservation of the finite model property by fibring, see [7]. Finally, in Section 7, after a critical assessment of what was achieved we provide some clues on future work.

2 Graph-theoretic account of logics

We start by providing a brief overview of the graph-theoretic approach to describe logic systems and their combinations proposed in [24, 21] and followed with minor adaptations herein.

By a *multi-graph*, in short, an *m-graph*, we mean a tuple

$$G = (V, E, \mathsf{src}, \mathsf{trg})$$

where:

- V is a set (of *vertexes* or *nodes*);
- E is a set (of *m-edges*);
- $\mathsf{src} : E \to V^+$;
- $\mathsf{trg} : E \to V$;

where V^+ denotes the set of all finite non-empty sequences of V.

We may write either $e : s \to v$ or $e \in G(s, v)$ or $e \in E(s, v)$ whenever e is in E, $\mathsf{src}(e) = s$ and $\mathsf{trg}(e) = v$, and may write $G(\cdot, \cdot)$ or $E(\cdot, \cdot)$ for the collection of m-edges in E.

A *(propositional based) language signature* or, simply, a *signature*, Σ, is a tuple

$$(G, \diamond, \Pi)$$

where $G = (V, E, \mathsf{src}, \mathsf{trg})$ is an m-graph such that V is $\{\diamond\} \cup \Pi$, no m-edge in E has $\diamond$ as target, $\diamond$ only appears in the source of unary edges, and Π is non-empty.

The nodes in V play the role of language *sorts*, each sort in Π is a *propositions sort* (sort of schema formulas), and node $\diamond$ is the *concrete sort*. The m-edges play the role of *constructors* for building expressions of the available sorts.

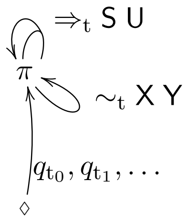

Figure 1: M-graph of the LTL signature described in Example 2.1.

As an example consider the signature for linear temporal logic.

Example 2.1 Let Q be a countable set $\{q_{t_0}, q_{t_1}, \ldots\}$ of propositional symbols. The *LTL signature over* Q (see [21]), denoted by Σ_Q^{LTL}, is an m-graph with the propositions sort π, the concrete sort $\diamond$, and the following m-edges:

- $q_{t_k} : \diamond \to \pi$ for each natural number k;
- $\sim_t, \mathsf{X}, \mathsf{Y} : \pi \to \pi$;

- $\Rightarrow_t, \mathsf{S}, \mathsf{U} : \pi\pi \to \pi$.

The m-edges $\sim_t$, $\Rightarrow_t$, S, U, X and Y represent the connectives negation, implication, the temporal operator since, the temporal operator until, the next temporal operator and the previous temporal operator, respectively. In the sequel we may denote the propositions sort π by π_t. For a graphical representation see Figure 1. ∇

Language

An expression over a signature is a morphism in the category induced by the m-graph of the signature. We follow [21] and assume that the category is generated by a reduction system where the reduction rules impose the properties that the category should satisfy. That is, given an m-graph G, the category G^+ with non-empty finite products induced by G is constructed in two steps. In the first we generate from the m-graph G a graph whose nodes are the finite non-empty sequences of nodes of G, and whose edges are the m-edges of G and edges $\mathsf{p}_i^{v_1 \dots v_n}$ for projections. From that graph we obtain $G^\dagger$ by iteratively enriching it with edges $\langle w_1, \dots, w_n \rangle$ for tuples. In the second step, a particular abstract reduction system induced by the m-graph G is defined and proved to be confluent, see [21] for more details. The objects of G^+ are then the finite non-empty sequences of nodes of V and the morphisms are the irreducible paths, according to that reduction system, on the graph $G^\dagger$. We denote the source and target maps in G^+ by src^+ and trg^+ respectively. The set of irreducible paths on $G^\dagger$, that is, of the morphisms of G^+, denoted by $\mathsf{IPaths}(G^\dagger)$, can be inductively characterized as follows: (i) $\epsilon_s \in \mathsf{IPaths}(G^\dagger)$, where ϵ_s is the empty path starting and ending at the finite non-empty sequence s of nodes of V; (ii) $\mathsf{p}_i^{v_1 \dots v_n} \in \mathsf{IPaths}(G^\dagger)$; (iii) $\langle w_1, \dots, w_n \rangle \in \mathsf{IPaths}(G^\dagger)$ whenever $w_1, \dots, w_n \in \mathsf{IPaths}(G^\dagger)$ and at least one w_i is not $\mathsf{p}_i^{v_1 \dots v_n}$; and (iv) $ew \in \mathsf{IPaths}(G^\dagger)$ whenever e is an m-edge of G and $w \in \mathsf{IPaths}(G^\dagger)$. The identity morphism in G^+ on an object s is the empty path ϵ_s on s, and given irreducible paths $w_1 : s1 \to s$ and $w_2 : s \to s_2$ their composition is the normal form, according to the reduction system, of the path $w_2 w_1$.

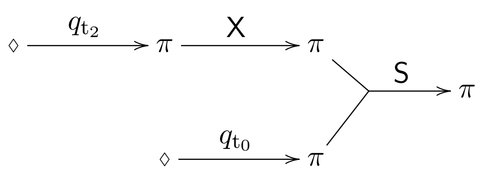

Figure 2: Formula $\mathsf{S}(\mathsf{X}q_{t_2}, q_{t_0})$.

A *formula over* $(G, \diamond, \Pi)$ is an irreducible path of $G^\dagger$, with target in Π. The *language over* Σ, denoted by $L(\Sigma)$, is the set of formulas over Σ. An irreducible path is said to be *concrete* whenever its source is a sequence of length one with a sort that can not be the target of m-edges of Σ (in our case this means that the source is $\diamond$). For instance, in the context of signature Σ_Q^{LTL} described in

Example 2.1, the formula $\mathsf{S}(\mathsf{X}q_{\mathrm{t}_2}, q_{\mathrm{t}_0})$, which is the irreducible path $\mathsf{S}\langle \mathsf{X}q_{\mathrm{t}_2}, q_{\mathrm{t}_0}\rangle$ represented in Figure 2, from $\diamond$ to π, is concrete.

A *schema formula* is a formula whose source has a sort that can be the target of m-edges of Σ. From now on, by a *schema variable* we mean either ϵ_s where s contains a sort that may be the target of m-edges of Σ (in our case that sort must be in Π) or we mean the projections $\mathsf{p}_i^{v_1 \dots v_n}$ with $v_1, \dots, v_n \in V$ where v_i is a sort that may be the target of m-edges of Σ.

Instantiation of a formula $w : s \to t$ by an expression w_0 with target s, both over the same signature, is the formula $w \circ w_0$, that is, the normal form of the path ww_0.

Semantics

Interpretation structures are also described using m-graphs, similarly to [24]. An interpretation structure for a signature contains an m-graph (the *operations m-graph*) where the nodes are the semantic values and the m-edges are operations on the values. However, this is not enough because we need to know how the semantic values are related to sorts and how operations are related to constructors, that is, we need to relate the operations m-graph with the signature m-graph, see Figure 3. For that we need an m-graph morphism. Contrarily to [24], herein we assume that this m-graph morphism is total. We recall first some relevant notions, terminology and notation.

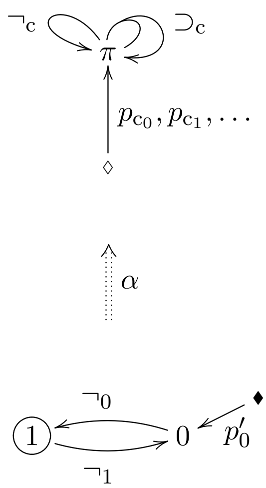

Figure 3: Part of an interpretation structure for classical logic (see Example 2.2).

Given two functions $v_1, v_2 : A \to B$ we write $v_1 \subseteq v_2$, if $\mathsf{dom}\, v_1 \subseteq \mathsf{dom}\, v_2$ and $v_2(a) = v_1(a)$ for every a in $\mathsf{dom}\, v_1$, and we write $v_1 = v_2$ if $v_1 \subseteq v_2$ and $v_2 \subseteq v_1$.

By an *m-graph (partial) morphism* $\mu : G_1 \to G_2$ we mean a pair of functions

$\mu^{\mathsf{v}} : V_1 \to V_2$ and $\mu^{\mathsf{e}} : E_1 \to E_2$, possibly partial, such that:

- $\mathsf{src}_2 \circ \mu^{\mathsf{e}} \subseteq \mu^{\mathsf{v}} \circ \mathsf{src}_1$;
- $\mathsf{trg}_2 \circ \mu^{\mathsf{e}} \subseteq \mu^{\mathsf{v}} \circ \mathsf{trg}_1$;

and by an *m-graph total morphism* we mean an m-graph morphism μ such that μ^{v} and μ^{e} are maps.

So, an *interpretation structure* I *for a signature* $(G, \diamond, \Pi)$ is a tuple

$$(G', \alpha, D, \blacklozenge)$$

such that G' is an m-graph (the *operations m-graph*), $\alpha : G' \to G$ is an m-graph total morphism (the *abstraction morphism*) such that $(\alpha^{\mathsf{v}})^{-1}(\diamond)$ is a singleton whose unique element is $\blacklozenge$ (the *concrete value*), and $D \subsetneq (\alpha^{\mathsf{v}})^{-1}(\Pi)$ is a non-empty set (of *designated values*).

In the sequel, when describing an interpretation structure, we name the m-edges in the operations m-graph in such a way that their names are hints of how they are mapped by α to the constructors. Moreover, when graphically representing an interpretation structure, we draw the designated values inside a circle, see Figure 3. We may use (Σ, I) when referring to an interpretation structure I for a signature Σ. Given $v_1, \ldots, v_n$ in V we denote the set $\{v'_1 \ldots v'_n \in V'^{+} : \alpha^{\mathsf{v}}(v'_1) = v_1, \ldots, \alpha^{\mathsf{v}}(v'_n) = v_n\}$ by $V'_{v_1 \ldots v_n}$, and given $U \subseteq V^+$ we denote the set $\cup_{s \in U} V'_s$ by V'_U. The elements of V'_Π are the *truth values*.

An *interpretation system* $\mathcal{I}$ is a pair $(\Sigma, \mathfrak{I})$ where Σ is a signature and $\mathfrak{I}$ is a class of interpretation structures for Σ. By a *consistent* interpretation system we mean an interpretation system with a non-empty set of interpretation structures.

As examples consider the interpretation systems for intuitionistic logic and for classical logic.

Example 2.2 *An interpretation system for classical propositional logic.*
Let P be a countable set $\{p_{\mathrm{c}_0}, p_{\mathrm{c}_1}, \ldots\}$ of propositional symbols. The interpretation system

$$(\Sigma^{\mathrm{c}}_P, \mathfrak{I}^{\mathrm{c}})$$

for classical propositional logic is such that: (i) Σ^{c}_P is the *signature over* P *for classical propositional logic*, i.e., is an m-graph with the propositions sort π, the concrete sort $\diamond$ and the following m-edges:

- $p_{\mathrm{c}_k} : \diamond \to \pi$ for each natural number k;
- $\neg_{\mathrm{c}} : \pi \to \pi$;
- $\supset_{\mathrm{c}} : \pi\pi \to \pi$.

In the sequel we may denote the propositions sort π also by π_{c}; and (ii) $\mathfrak{I}^{\mathrm{c}}$ is the class of all interpretation structures $(G', \alpha, D, \blacklozenge)$ for Σ^{c}_P induced by valuations $v : P \to \{0, 1\}$ for P, that is:

- G' is such that: $V' = \{0, 1\} \cup \{\blacklozenge\}$; $E' = \{p'_k : k \in \mathbb{N}\} \cup \{\neg_0, \neg_1\} \cup \{\supset_{a_1 a_2} : a_1, a_2 \in \{0, 1\}\}$; and src' and trg' are such that: $p'_k : \blacklozenge \to v(p_{\mathrm{c}_k})$ for each natural number k; $\neg_0 : 0 \to 1$; $\neg_1 : 1 \to 0$; $\supset_{00} : 0\,0 \to 1$; $\supset_{01} : 0\,1 \to 1$; $\supset_{10} : 1\,0 \to 0$; and $\supset_{11} : 1\,1 \to 1$;
- $\alpha : G' \to G$ is such that: $\alpha^{\mathsf{v}}(0) = \pi$; $\alpha^{\mathsf{v}}(1) = \pi$; $\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$; $\alpha^{\mathsf{e}}(p'_k) = p_{\mathrm{c}_k}$ for each natural number k; $\alpha^{\mathsf{e}}(\neg_a) = \neg_{\mathrm{c}}$; and $\alpha^{\mathsf{e}}(\supset_{a_1 a_2}) = \supset_{\mathrm{c}}$;
- $D = \{1\}$. ∇

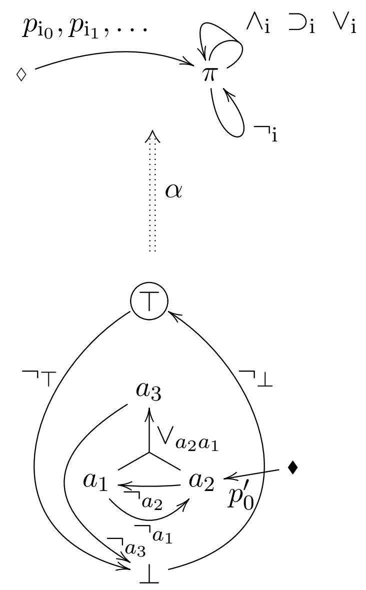

Figure 4: Part of an interpretation structure for intuitionistic logic (see Example 2.3).

Example 2.3 *An interpretation system for intuitionistic propositional logic.* Let P be a countable set $\{p_{\mathrm{i}_0}, p_{\mathrm{i}_1}, \ldots\}$ of propositional symbols. The interpretation system

$$(\Sigma^{\mathrm{i}}_P, \mathfrak{I}^{\mathrm{i}})$$

for intuitionistic propositional logic is such that: (i) Σ^{i}_P is the *signature over P for intuitionistic logic* (see [27]), i.e., is an m-graph with the propositions sort π, the concrete sort $\diamond$ and the following m-edges:

- $p_{\mathrm{i}_k} : \diamond \to \pi$ for each natural number k;
- $\neg_{\mathrm{i}} : \pi \to \pi$;
- $\wedge_{\mathrm{i}}, \vee_{\mathrm{i}}, \supset_{\mathrm{i}} : \pi\pi \to \pi$.

In the sequel we may denote the propositions sort π by π_{i}. For a graphical representation see the signature in Figure 4; and (ii) $\mathfrak{I}^{\mathrm{i}}$ is the class of all interpretation structures $(G', \alpha, D, \blacklozenge)$ for Σ^{i}_P induced by a Heyting algebra $(A, \sqcup, \sqcap, \sqsupset, \top, \bot)$ and a valuation v over the algebra (see [27]), that is:

- G' is such that: $V' = A \cup \{\blacklozenge\}$; $E' = \{p'_k : k \in \mathbb{N}\} \cup \{\neg_a : a \in A\} \cup \{\supset_{a_1 a_2}, \wedge_{a_1 a_2}, \vee_{a_1 a_2} : a_1, a_2 \in A\}$; and src' and trg' are such that: $p'_k : \blacklozenge \to v(p_{\mathrm{i}_k})$ for each natural number k; $\neg_a : a \to \sqsupset (a, \bot)$; $\supset_{a_1 a_2} : a_1\, a_2 \to \sqsupset (a_1, a_2)$; $\wedge_{a_1 a_2} : a_1\, a_2 \to a_1 \sqcap a_2$; and $\vee_{a_1 a_2} : a_1\, a_2 \to a_1 \sqcup a_2$;
- $\alpha : G' \to G$ is such that: $\alpha^{\mathsf{v}}(a) = \pi$ for all a in A; $\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$; $\alpha^{\mathsf{e}}(p'_k) = p_{\mathrm{i}_k}$ for each natural number k; $\alpha^{\mathsf{e}}(\neg_a) = \neg_{\mathrm{i}}$; $\alpha^{\mathsf{e}}(\supset_{a_1 a_2}) = \supset_{\mathrm{i}}$; $\alpha^{\mathsf{e}}(\wedge_{a_1 a_2}) = \wedge_{\mathrm{i}}$; and $\alpha^{\mathsf{e}}(\vee_{a_1 a_2}) = \vee_{\mathrm{i}}$.
- $D = \{\top\}$.

See in Figure 4 part of an interpretation structure in $\mathfrak{I}^{\mathrm{i}}$. ∇

In the sequel we need to refer to the total functor $\alpha^+ : G_1^+ \to G_2^+$ induced by an m-graph total morphism $\alpha : G_1 \to G_2$, defined as expected: $\alpha^+_{\mathsf{o}}(v'_1 \dots v'_n) = \alpha^{\mathsf{v}}(v'_1) \dots \alpha^{\mathsf{v}}(v'_n)$ and (i) $\alpha^+_{\mathsf{m}}(\epsilon_{s'}) = \epsilon_{\alpha^+_{\mathsf{o}}(s')}$; (ii) $\alpha^+_{\mathsf{m}}(\mathsf{p}_i^{s'}) = \mathsf{p}_i^{\alpha^+_{\mathsf{o}}(s')}$; (iii) $\alpha^+_{\mathsf{m}}(\langle w'_1, \dots, w'_n \rangle) = \langle \alpha^+_{\mathsf{m}}(w'_1), \dots, \alpha^+_{\mathsf{m}}(w'_n) \rangle$; (iv) $\alpha^+_{\mathsf{m}}(e'w') = \alpha^{\mathsf{e}}(e')\alpha^+_{\mathsf{m}}(w')$. We may refer either to α^+_{o} or to α^+_{m} simply by α^+.

Given a formula φ over a signature Σ and an interpretation structure $I = (G', \alpha, D, \blacklozenge)$ for Σ, we denote an irreducible path τ' of $G'^{\dagger}$ with $\alpha^+(\tau') = \varphi$ by a *path in I for φ*.

When τ' is a path in I for φ and its target is in D we write

$$I, \tau' \Vdash_{\Sigma} \varphi$$

and say that *path τ' of I satisfies* formula φ. Given s' in V'^{+} with $\alpha^+(s') = \mathsf{src}^+(\varphi)$, we say that φ is *satisfied from s' in I*, denoted by

$$I, s' \Vdash_{\Sigma} \varphi$$

whenever $I, \tau' \Vdash \varphi$ for every path τ' for φ in I starting at s'. Path satisfaction is easily extended to interpretation structures. We say that I *satisfies* φ, written

$$I \Vdash_{\Sigma} \varphi,$$

if $I, \tau' \Vdash_{\Sigma} \varphi$ for every path τ' in I for φ. Satisfaction is extended to sets of formulas as expected.

Entailment is defined on top of satisfaction as usual. We say that a set Γ of formulas over Σ *locally entails* φ within $(\Sigma, \mathfrak{I})$, denoted by

$$\Gamma \vDash^{\mathrm{l}}_{(\Sigma, \mathfrak{I})} \varphi$$

whenever the source of each formula in Γ coincides with the source of φ and

$$\text{if} \quad I, s' \Vdash_\Sigma \Gamma \quad \text{then} \quad I, s' \Vdash_\Sigma \varphi$$

for all s' in V'^+ whose image by α^+ is the source of φ.

A formula φ is *locally valid* within $(\Sigma, \mathfrak{I})$, denoted by

$$\vDash^{\mathrm{l}}_{(\Sigma,\mathfrak{I})} \varphi$$

whenever $\emptyset \vDash^{\mathrm{l}}_{(\Sigma,\mathfrak{I})} \varphi$.

Example 2.4 Consider the interpretation system $(\Sigma^{\mathrm{i}}_P, \mathfrak{I}^{\mathrm{i}})$ described in Example 2.3 for intuitionistic propositional logic, and let I in $\mathfrak{I}^{\mathrm{i}}$ be the interpretation structure partially described in Figure 4. Then

$$\vee_{a_2 a_1} \langle p'_0, \neg_{a_2} p'_0 \rangle$$

is a path in I for $p_{\mathrm{i}_0} \vee_{\mathrm{i}} (\neg_{\mathrm{i}} p_{\mathrm{i}_0})$. Hence

$$I, \ \vee_{a_2 a_1} \langle p'_0, \neg_{a_2} p'_0 \rangle \ \not\Vdash_{\Sigma^{\mathrm{i}}_P} \ p_{\mathrm{i}_0} \vee_{\mathrm{i}} (\neg_{\mathrm{i}} p_{\mathrm{i}_0})$$

since the target of $\vee_{a_2 a_1} \langle p'_0, \neg_{a_2} p'_0 \rangle$ is a_3 which is not in D. Therefore

$$I \ \not\Vdash_{\Sigma^{\mathrm{i}}_P} \ p_{\mathrm{i}_0} \vee_{\mathrm{i}} (\neg_{\mathrm{i}} p_{\mathrm{i}_0}).$$

and so

$$\not\vDash^{\mathrm{l}}_{(\Sigma^{\mathrm{i}}_P, \mathfrak{I}^{\mathrm{i}})} \ p_{\mathrm{i}_0} \vee_{\mathrm{i}} (\neg_{\mathrm{i}} p_{\mathrm{i}_0})$$

which is not a surprise since we are considering intuitionistic logic. ∇

When there is no ambiguity we may omit the reference to the signature and to the interpretation system in the satisfaction $\Vdash$ and entailment $\vDash^{\mathrm{l}}$ symbols respectively. Moreover, in order to simplify the presentation, we may write $\vDash$ instead of $\vDash^{\mathrm{l}}$, and may omit the qualification local when referring to entailment and validity. Observe that local entailment and global entailment (a set of formulas Γ *globally entails* a formula φ in $(\Sigma, \mathfrak{I})$ whenever if $I \Vdash_\Sigma \Gamma$ then $I \Vdash_\Sigma \varphi$ for all I in $\mathfrak{I}$) coincide for concrete formulas, since $I \Vdash_\Sigma \varphi$ if and only if $I, \blacklozenge \Vdash_\Sigma \varphi$ for every concrete formula φ.

3 Biporting

In this section we present a new account of unconstrained fibring, denoted by biporting. In biporting, the signature resulting from the combination is enriched with the constructors $\Lsh$ and $\Rsh$ playing the role of bridges between the two languages. This new account of unconstrained fibring, despite having a simpler semantics, is shown in Section 4 to be equivalent in terms of entailment with the original one, modulo the translation of formulas into the enriched language. It has the advantage of making explicit the two-way nature of fibring at the level of the language, and at the level of semantics (we leave deduction for a forthcoming paper). In fact, we were led to biporting after having proposed importing in [21] as a general way of combining logics subsuming, as proved in [21], well known asymmetric mechanisms like temporalization [10, 11], modalization [9] and exogenous enrichment[19, 6], when there is no sharing of connectives. In importing, the signature resulting from the combination is enriched with the importing connective ↰ for transforming expressions of the imported language into expressions of the importing one. So, very roughly, we can say that biporting is a simultaneous importing of one logic into the other.

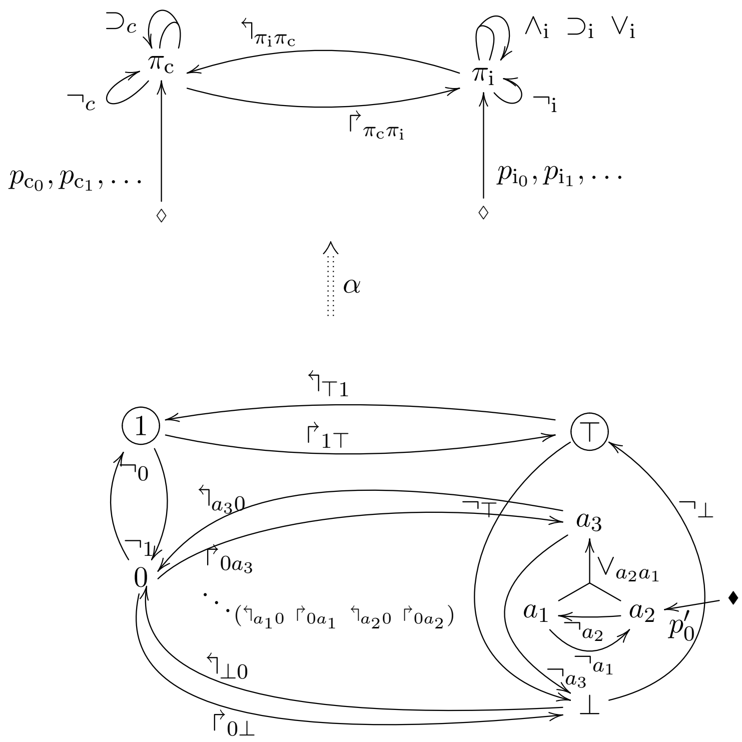

Figure 5: Part of the interpretation structure resulting from the biporting of the interpretation structure for classical logic partially described in Figure 3, and the interpretation structure for intuitionistic logic partially described in Figure 4.

Herein we consider an unconstrained version of biporting, that is, we define biporting only for interpretation systems not sharing connectives (we leave for a forthcoming paper the study of constrained biporting by sharing). We assume that:

- the pair of signatures of the interpretation systems involved in biporting is such that their sets of sorts are disjoint as well as their sets of m-edges;
- in the interpretation systems involved in biporting all the pairs with an interpretation structure (Σ_1, I_1) of one with an interpretation structure

(Σ_2, I_2) of the other are such that $(V_1')_{\Pi_1}$ and $(V_2')_{\Pi_2}$ are disjoint as well as E_1' and E_2'.

The *biporting* of interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$, denoted by

$$(\Sigma_1, \mathfrak{I}_1) \Lsh\!\Rsh (\Sigma_2, \mathfrak{I}_2)$$

is the interpretation system $(\Sigma_1 \Lsh\!\Rsh \Sigma_2, \mathfrak{I}_1 \Lsh\!\Rsh \mathfrak{I}_2)$ where

$$\Sigma_1 \Lsh\!\Rsh \Sigma_2$$

is the signature $((V, E, \mathsf{src}, \mathsf{trg}), \diamond, \Pi)$ with

- $V = V_1 \cup V_2$;
- E is the union of the sets
  - E_1;
  - E_2;
  - $\{\Rsh_{uv} : u \in \Pi_1, v \in \Pi_2\}$;
  - $\{\Lsh_{uv} : u \in \Pi_2, v \in \Pi_1\}$;
- src and trg are such that
  - $\mathsf{src}(e) = \mathsf{src}_i(e)$ and $\mathsf{trg}(e) = \mathsf{trg}_i(e)$ if e is in E_i for $i = 1, 2$;
  - $\mathsf{src}(\Rsh_{uv}) = u$ and $\mathsf{trg}(\Rsh_{uv}) = v$ for all $\Rsh_{uv}$ in $\{\Rsh_{uv} : u \in \Pi_1, v \in \Pi_2\}$;
  - $\mathsf{src}(\Lsh_{uv}) = u$ and $\mathsf{trg}(\Lsh_{uv}) = v$ for all $\Lsh_{uv}$ in $\{\Lsh_{uv} : u \in \Pi_2, v \in \Pi_1\}$;
- Π is $\Pi_1 \cup \Pi_2$;

and $\mathfrak{I}_1 \Lsh\!\Rsh \mathfrak{I}_2$ is the set $\{I_1 \Lsh\!\Rsh I_2 : I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}$ where

$$I_1 \Lsh\!\Rsh I_2$$

is the tuple $((V'_{\Lsh\Rsh}, E'_{\Lsh\Rsh}, \mathsf{src}'_{\Lsh\Rsh}, \mathsf{trg}'_{\Lsh\Rsh}), \alpha_{\Lsh\Rsh}, D_{\Lsh\Rsh}, \blacklozenge)$ such that

- $V'_{\Lsh\Rsh}$ is $V_1' \cup V_2'$;
- $E'_{\Lsh\Rsh}$ is the union of the sets
  - E_1';
  - E_2';
  - $\{\Lsh_{v_2' v_1'}, \Rsh_{v_1' v_2'} : v_1' \in D_1, v_2' \in D_2\}$;
  - $\{\Lsh_{v_2' v_1'}, \Rsh_{v_1' v_2'} : v_1' \in (V_1')_{\Pi_1} \setminus D_1, v_2' \in (V_2')_{\Pi_2} \setminus D_2\}$;
- $\mathsf{src}'_{\Lsh\Rsh}$ and $\mathsf{trg}'_{\Lsh\Rsh}$ are such that
  - $\mathsf{src}'_{\Lsh\Rsh}(e_i') = \mathsf{src}_i'(e_i')$ and $\mathsf{trg}'_{\Lsh\Rsh}(e_i') = \mathsf{trg}_i'(e_i')$ for $i = 1, 2$;
  - $\mathsf{src}'_{\Lsh\Rsh}(\Rsh_{v'v''}) = v'$ and $\mathsf{trg}'_{\Lsh\Rsh}(\Rsh_{v'v''}) = v''$;
  - $\mathsf{src}'_{\Lsh\Rsh}(\Lsh_{v'v''}) = v'$ and $\mathsf{trg}'_{\Lsh\Rsh}(\Lsh_{v'v''}) = v''$;

- $\alpha_{\Lsh\!\Rsh}$ is such that
  - $\alpha^{\mathsf{v}}_{\Lsh\!\Rsh}(v') = \alpha^{\mathsf{v}}_i(v')$ whenever $v' \in V'_i$ for $i = 1, 2$;
  - $\alpha^{\mathsf{e}}_{\Lsh\!\Rsh}(e') = \alpha^{\mathsf{e}}_i(e')$ whenever $e' \in E'_i$ for $i = 1, 2$;
  - $\alpha^{\mathsf{e}}_{\Lsh\!\Rsh}(\Lsh_{v'v''}) = \Lsh_{\alpha^{\mathsf{v}}_{\Lsh\!\Rsh}(v')\alpha^{\mathsf{v}}_{\Lsh\!\Rsh}(v'')}$;
  - $\alpha^{\mathsf{e}}_{\Lsh\!\Rsh}(\Rsh_{v'v''}) = \Rsh_{\alpha^{\mathsf{v}}_{\Lsh\!\Rsh}(v')\alpha^{\mathsf{v}}_{\Lsh\!\Rsh}(v'')}$;
- $D_{\Lsh\!\Rsh}$ is $D_1 \cup D_2$.

When Π_1 and Π_2 are singletons we omit the reference to the sorts in $\Lsh_{uv}$ and $\Rsh_{uv}$ and write simply $\Lsh$ and $\Rsh$ respectively.

Example 3.1 *Biporting intuitionistic logic with classical logic*
Let P_{c} and P_{i} be countable disjoint sets of propositional symbols, and $\hat{\mathfrak{I}}^{\mathrm{c}}$ and $\hat{\mathfrak{I}}^{\mathrm{i}}$ the classes of interpretation structures in $\mathfrak{I}^{\mathrm{c}}$ and $\mathfrak{I}^{\mathrm{i}}$ respectively, see Example 2.2 and Example 2.3, renamed such that the truth values and the m-edges of a structure in $\hat{\mathfrak{I}}^{\mathrm{c}}$ are indexed by c and the truth values and the m-edges of a structure in $\hat{\mathfrak{I}}^{\mathrm{i}}$ are indexed by i. Then, in the biporting

$$(\Sigma^{\mathrm{c}}_{P_{\mathrm{c}}}, \hat{\mathfrak{I}}^{\mathrm{c}}) \Lsh\!\Rsh (\Sigma^{\mathrm{i}}_{P_{\mathrm{i}}}, \hat{\mathfrak{I}}^{\mathrm{i}})$$

of the interpretation system $(\Sigma^{\mathrm{i}}_{P_{\mathrm{i}}}, \hat{\mathfrak{I}}^{\mathrm{i}})$ for intuitionistic propositional logic with the interpretation system $(\Sigma^{\mathrm{c}}_{P_{\mathrm{c}}}, \hat{\mathfrak{I}}^{\mathrm{c}})$ for classical propositional logic, the formula

$$p_{\mathrm{i}_0} \vee_{\mathrm{i}} \neg_{\mathrm{i}} p_{\mathrm{i}_0}$$

is not a validity in $(\Sigma^{\mathrm{c}}_{P_{\mathrm{c}}}, \hat{\mathfrak{I}}^{\mathrm{c}}) \Lsh\!\Rsh (\Sigma^{\mathrm{i}}_{P_{\mathrm{i}}}, \hat{\mathfrak{I}}^{\mathrm{i}})$. For example, the interpretation structure partially depicted in Figure 5 falsifies the formula (in order to simplify the presentation we have omitted the indexes in the truth values and m-edges since they are already disjoint). Hence there is no collapse as the one mentioned in [8]. ∇

4 Biporting is fibring

In this section we show that indeed biporting and unconstrained fibring are equivalent modulo a translation of formulas. Recall that herein we only consider unconstrained biporting, that is, biporting of interpretation systems not sharing any connective. Based on [25], we described (a set theoretic account of) the unconstrained fibring of logics presented using the graph-theoretic approach.

Unconstrained fibring

We assume that the signatures to be fibred do not have connectives with the same name. This is necessary since herein we consider a set theoretic account of unconstrained fibring and not a categorial account. Moreover, as the definition of fibring requires, see [25], we assume that the signatures have a unique propositions sort with the same name (assume it is π). For the uncontrained fibring of interpretation structures, we also assume that each pair of interpretation structures to be fibred have disjoint sets of truth values as well as disjoint sets of operations m-edges.

Following [25], the *unconstrained fibring* of interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$, denoted by

$$(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$$

is the interpretation system $(\Sigma_1 \star \Sigma_2, \mathfrak{I}_1 \star \mathfrak{I}_2)$ where

$$\Sigma_1 \star \Sigma_2$$

denoted by the *unconstrained fibring* of Σ_1 and Σ_2, is the signature $((V_\star, E_\star, \mathsf{src}_\star, \mathsf{trg}_\star), \diamond, \Pi_\star)$ such that

- $V_\star = \{\diamond, \pi\}$;
- $E_\star$, $\mathsf{src}_\star$ and $\mathsf{trg}_\star$ are such that $E_\star(s, v)$ is $E_1(s, v) \cup E_2(s, v)$;
- $\Pi_\star$ is $\{\pi\}$;

and

$$\mathfrak{I}_1 \star \mathfrak{I}_2$$

is the set $\{I_1 \star I_2 : I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}$ where

$$I_1 \star I_2$$

is the interpretation structure $((V'_\star, E'_\star, \mathsf{src}'_\star, \mathsf{trg}'_\star), \alpha_\star, D_\star, \blacklozenge)$ such that, denoting also the concrete value $\blacklozenge$ by $(\blacklozenge, \blacklozenge)$,

- $V'_\star$ is $\{\blacklozenge\} \cup ((V'_{1_\pi} \setminus D_1) \times (V'_{2_\pi} \setminus D_2)) \cup D_1 \times D_2$;
- $E'_\star$, $\mathsf{src}'_\star$ and $\mathsf{trg}'_\star$ are such that $E'_\star(s', v')$ is the union of the sets
  - $\{e'_{s'v'} : e'$ in $E'_1((s')_1, (v')_1)\}$;
  - $\{e'_{s'v'} : e'$ in $E'_2((s')_2, (v')_2)\}$;

  where $(s')_i$ is the sequence obtained by projecting each element (v'_1, v'_2) of the sequence s' to the ith-component, for i in $\{1, 2\}$;
- $\alpha_\star$ is such that
  - $\alpha^{\mathsf{v}}_\star(\blacklozenge) = \diamond$;
  - $\alpha^{\mathsf{v}}_\star(v') = \pi$ if v' in V' is not the concrete value;
  - $\alpha^{\mathsf{e}}_\star(e'_{s'v'}) = \alpha^{\mathsf{e}}_i(e')$ whenever e' is in E'_i for $i = 1, 2$;
- $D_\star$ is $D_1 \times D_2$.

Relating paths

Our goal now is to show that unconstrained fibring and biporting are equivalent in terms of entailment, modulo a translation of formulas.

Assume given interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ appropriate for unconstrained fibring as described above. We start by pointing out how to obtain interpretation systems suitable for biporting. First we have to consider signatures isomorphic to Σ_1 and Σ_2 appropriate for biporting. For that we rename the propositions sort in Σ_1 and the propositions sort in Σ_2 (assume both have the name π) in order to be different. We denote the signatures resulting from the renaming by

$$\Lsh\!\Rsh_1(\Sigma_1) \quad \text{and} \quad \Lsh\!\Rsh_2(\Sigma_2),$$

and assume without loss of generality that the propositions sort in $\Lsh\!\Rsh_1(\Sigma_1)$ is π_1 and the propositions sort in $\Lsh\!\Rsh_2(\Sigma_2)$ is π_2. Then we adapt each interpretation structure I_1 and I_2 in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively, in order to be an interpretation structure over $\Lsh\!\Rsh_1(\Sigma_1)$ and $\Lsh\!\Rsh_2(\Sigma_2)$ respectively. Observe that the only thing that needs to be done is to consider not the original propositions sort but its renaming. We denote the interpretation systems with those signatures and interpretation structures by $(\Lsh\!\Rsh_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\!\Rsh_1})$ and $(\Lsh\!\Rsh_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\!\Rsh_2})$.

In order to show that $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$ is equivalent in terms of entailment with $(\Lsh\!\Rsh_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\!\Rsh_1}) \Lsh\!\Rsh (\Lsh\!\Rsh_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\!\Rsh_2})$ modulo a translation of formulas, we have to relate paths of $\Sigma_1 \star \Sigma_2$ with paths of $\Lsh\!\Rsh_1(\Sigma_1) \Lsh\!\Rsh \Lsh\!\Rsh_2(\Sigma_2)$. For that, we need to refer to the signature m-graphs of $\Sigma_1 \star \Sigma_2$ and $\Lsh\!\Rsh_1(\Sigma_1) \Lsh\!\Rsh \Lsh\!\Rsh_2(\Sigma_2)$, denoted by $G_\star$ and $G_{\Lsh\!\Rsh}$ respectively, and to their sets of vertexes denoted by $V_\star$ and $V_{\Lsh\!\Rsh}$ respectively.

We start by relating the language sorts of $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$ with the language sorts of $(\Lsh\!\Rsh_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\!\Rsh_1}) \Lsh\!\Rsh (\Lsh\!\Rsh_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\!\Rsh_2})$, by the map $\cdot^{\Lsh\!\Rsh\mathsf{v}}$ from $V_\star$ to $\wp(V_{\Lsh\!\Rsh})$ such that $\diamond^{\Lsh\!\Rsh\mathsf{v}} = \{\diamond\}$ and $\pi^{\Lsh\!\Rsh\mathsf{v}} = \{\pi_1, \pi_2\}$.

Observe that, given a path in fibring that goes through π a corresponding path in biporting may go either from π_1 to π_2, or from π_1 to π_1, or from π_2 to π_2 or from π_2 to π_1. So, it may be necessary to introduce a bridge in the path in biporting in order to go from one such sort to the other. That is, given sorts u and v in $V_{\Lsh\!\Rsh}$ such that $\{u, v\}$ is either $\{\diamond\}$ or is contained in $\{\pi_1, \pi_2\}$, we define the irreducible path $(\downarrow_1)^u_v$ of $G^\dagger_{\Lsh\!\Rsh}$ as follows: if $u = v$ then $(\downarrow_1)^u_v$ is ϵ_u; otherwise if u is π_1 (and so v is π_2) then $(\downarrow_1)^u_v$ is $\Rsh_{\pi_1\pi_2}$ else $(\downarrow_1)^u_v$ is $\Lsh_{\pi_2\pi_1}$. Hence, given non-empty sequences $s = u_1 \dots u_m$ and $t = v_1 \dots v_m$ of sorts in $V_{\Lsh\!\Rsh}$ such that $\{u_i, v_i\}$ is either $\{\diamond\}$ or is contained in $\{\pi_1, \pi_2\}$ for $i = 1, \dots, m$, the *language bridge*, or simply the *bridge*, from s to t, also denoted by $\downarrow^s_t$, is the irreducible path of $G^\dagger_{\Lsh\!\Rsh}$ defined as follows: if $s = t$ then $\downarrow^s_t$ is ϵ_t; otherwise if m is 1 then $\downarrow^s_t$ is $(\downarrow_1)^s_t$ else $\downarrow^s_t$ is $\langle (\downarrow_1)^{u_1}_{v_1} \mathsf{p}_1^{u_1 \dots u_m}, \dots, (\downarrow_1)^{u_m}_{v_m} \mathsf{p}_m^{u_1 \dots u_m} \rangle$.

With the help of the maps above we can relate irreducible paths over the fibred signature with sets of irreducible paths over the biporting signature. Let

$\cdot^{\Lsh\Rsh\dagger}$ be the map from $\mathsf{IPaths}(G_\star^\dagger)$ to $\wp(\mathsf{IPaths}(G_{\Lsh\Rsh}^\dagger))$ inductively defined as follows:

- $(\epsilon_{v_1 \dots v_m})^{\Lsh\Rsh\dagger}$ is the set $\{\epsilon_{u_1 \dots u_m} : u_1 \in v_1^{\Lsh\Rsh\mathsf{v}}, \dots, u_m \in v_m^{\Lsh\Rsh\mathsf{v}}\}$;
- $(ew)^{\Lsh\Rsh\dagger}$ for e in $E_\star$, is $\{e{\downarrow}_{\mathsf{src}_{\Lsh\Rsh}(e)}^{\mathsf{trg}_{\Lsh\Rsh}^\dagger(w_0)} w_0 : w_0 \in w^{\Lsh\Rsh\dagger}\}$;
- $(\langle w_1, \dots, w_m \rangle)^{\Lsh\Rsh\dagger}$ is the set $\{\langle w_{10}, \dots, w_{m0} \rangle : w_{10} \in w_1^{\Lsh\Rsh\dagger}, \dots, w_{m0} \in w_m^{\Lsh\Rsh\dagger}, \mathsf{src}_{\Lsh\Rsh}^\dagger(w_{10}) = \dots = \mathsf{src}_{\Lsh\Rsh}^\dagger(w_{m0})\}$;
- $(\mathsf{p}_i^{v_1 \dots v_n})^{\Lsh\Rsh\dagger}$ is $\{\mathsf{p}_i^{u_1 \dots u_n} : u_1 \in v_1^{\Lsh\Rsh\mathsf{v}}, \dots, u_m \in v_m^{\Lsh\Rsh\mathsf{v}}\}$.

Observe that if $w_0 \in w^{\Lsh\Rsh\dagger}$ then $\mathsf{trg}_{\Lsh\Rsh}^+(w_0) \in \mathsf{trg}_\star^+(w)^{\Lsh\Rsh\mathsf{v}}$ and $\mathsf{src}_{\Lsh\Rsh}^+(w_0) \in \mathsf{src}_\star^+(w)^{\Lsh\Rsh\mathsf{v}}$, and if w in $\mathsf{IPaths}(G_\star^\dagger)$ is concrete then $w^{\Lsh\Rsh\dagger}$ is a singleton, as is stated in the next proposition. We omit its proof since it follows immediately by induction on w.

Proposition 4.1 Given a concrete w in $\mathsf{IPaths}(G_\star^\dagger)$ the set $w^{\Lsh\Rsh\dagger}$ is a singleton.

To show that $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$ is equivalent in terms of entailment with $({\Lsh\Rsh}_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\Rsh_1}) \Lsh\Rsh ({\Lsh\Rsh}_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\Rsh_2})$ we have also to establish a strong relationship between their semantic components. More specifically between the interpretation resulting from the fibring of two given interpretations and the interpretation resulting from their biporting. Let I_1 and I_2 be interpretation structures in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively, and denote by $G'_\star$ and $G'_{\Lsh\Rsh}$ the operations m-graphs of $(\Sigma_1, I_1) \star (\Sigma_2, I_2)$ and $({\Lsh\Rsh}_1(\Sigma_1), I_1^{\Lsh\Rsh_1}) \Lsh\Rsh ({\Lsh\Rsh}_2(\Sigma_2), I_2^{\Lsh\Rsh_2})$ respectively. Moreover, denote by $V'_\star$ and $V'_{\Lsh\Rsh}$ the sets of vertexes of $G'_\star$ and $G'_{\Lsh\Rsh}$ respectively.

We start by relating semantic values of $V'_\star$ with semantic values of $V'_{\Lsh\Rsh}$. We denote by $\cdot^{\Lsh\Rsh'_\mathsf{v}}$ the map from $V'_\star$ to $\wp(V'_{\Lsh\Rsh})$ defined as follows: $(\blacklozenge, \blacklozenge)^{\Lsh\Rsh'_\mathsf{v}} = \{\blacklozenge\}$, and $(v'_1, v'_2)^{\Lsh\Rsh'_\mathsf{v}} = \{v'_1, v'_2\}$. Bridges are also needed at the semantic level. We say that u' and v' in $V'_{\Lsh\Rsh}$ are *suitable for a bridge* whenever if $\alpha_{\Lsh\Rsh}^\mathsf{v}(u') = \alpha_{\Lsh\Rsh}^\mathsf{v}(v')$ then $u' = v'$ otherwise $\{\alpha_{\Lsh\Rsh}^\mathsf{v}(u'), \alpha_{\Lsh\Rsh}^\mathsf{v}(v')\} = \{\pi_1, \pi_2\}$ and either $u' \in D_1$ iff $v' \in D_2$ or $u' \in D_2$ iff $v' \in D_1$. Given such u' and v', we denote by $({\downarrow}'_1)_{v'}^{u'}$ the irreducible path of $G'^\dagger_{\Lsh\Rsh}$ defined as follows: if $u' = v'$ then $({\downarrow}'_1)_{v'}^{u'}$ is $\epsilon_{v'}$; otherwise if $\alpha_{\Lsh\Rsh}^\mathsf{v}(u') = \pi_1$ then $({\downarrow}'_1)_{v'}^{u'}$ is ${\Rsh}_{u'v'}$ else $({\downarrow}'_1)_{v'}^{u'}$ is ${\Lsh}_{u'v'}$. Moreover, given non-empty sequences $s' = u'_1 \dots u'_m$ and $t' = v'_1 \dots v'_m$ of values in $V'_{\Lsh\Rsh}$ suitable for a bridge, the *denotation bridge*, or simply the *bridge*, from s' to t', denoted by ${\downarrow}'^{s'}_{t'}$ is the irreducible path of $G'^\dagger_{\Lsh\Rsh}$ defined as follows: if $s' = t'$ then ${\downarrow}'^{s'}_{t'}$ is $\epsilon_{t'}$; otherwise if m is 1 then ${\downarrow}'^{s'}_{t'}$ is $({\downarrow}'_1)_{t'}^{s'}$ else ${\downarrow}'^{s'}_{t'}$ is $\langle ({\downarrow}'_1)_{v'_1}^{u'_1} \mathsf{p}_1^{u'_1 \dots u'_m}, \dots, ({\downarrow}'_1)_{v'_m}^{u'_m} \mathsf{p}_m^{u'_1 \dots u'_m} \rangle$.

We now relate the irreducible paths of $G'_\star$ with the irreducible paths of $G'_{\Lsh\Rsh}$ by defining the map $\cdot^{\Lsh\Rsh\dagger'}$ from $\mathsf{IPaths}(G'^\dagger_\star)$ to $\wp(\mathsf{IPaths}(G'^\dagger_{\Lsh\Rsh}))$ inductively as follows:

- $(\epsilon_{v'_1 \dots v'_m})^{\Lsh\Rsh\dagger'}$ is $\{\epsilon_{u'_1 \dots u'_m} : u'_1 \in v_1'^{\Lsh\Rsh'_\mathsf{v}}, \dots, u'_m \in v_m'^{\Lsh\Rsh'_\mathsf{v}}\}$;
- $(e'_{s'v'} w')^{\Lsh\Rsh\dagger'}$ for e' in $E'_1 \cup E'_2$, is the set $\{e'{\downarrow}'^{\mathsf{trg}'^+_{\Lsh\Rsh}(w'_0)}_{\mathsf{src}'_{\Lsh\Rsh}(e')} w'_0 : w'_0 \in w'^{\Lsh\Rsh\dagger'}\}$;

- $(\langle w'_1, \ldots, w'_m \rangle)^{{\Lsh\Rsh}_{\dagger}'}$ is $\{\langle w''_1, \ldots, w''_m \rangle : w''_1 \in w_1'^{{\Lsh\Rsh}_{\dagger}'}, \ldots, w''_m \in w_m'^{{\Lsh\Rsh}_{\dagger}'}$, and $\mathsf{src}^{\dagger}_{\Lsh\Rsh}(w''_1) = \ldots = \mathsf{src}^{\dagger}_{\Lsh\Rsh}(w''_m)\}$;

- $(\mathsf{p}_i^{v'_1 \ldots v'_m})^{{\Lsh\Rsh}_{\dagger}'} = \{\mathsf{p}_i^{u'_1 \ldots u'_m} : u'_1 \in v_1'^{{\Lsh\Rsh}'_{\mathsf{v}}}, \ldots, u'_m \in v_m'^{{\Lsh\Rsh}'_{\mathsf{v}}}\}$.

Observe that $\cdot^{{\Lsh\Rsh}_{\dagger}'}$ is such that given $w'_0 \in \mathsf{IPaths}(G'^{\dagger}_{\Lsh\Rsh})$ and $w' \in \mathsf{IPaths}(G'^{\dagger}_{\star})$ with $\mathsf{trg}'^{+}_{\star}(w') = v'_1 \ldots v'_m$ and $\mathsf{trg}'^{+}_{\Lsh\Rsh}(w'_0) = u'_1 \ldots u'_m$ if $w'_0 \in w'^{{\Lsh\Rsh}_{\dagger}'}$ then $u'_1 \in v_1'^{{\Lsh\Rsh}'_{\mathsf{v}}}, \ldots, u'_m \in v_m'^{{\Lsh\Rsh}'_{\mathsf{v}}}$, and similarly for $\mathsf{src}'^{+}_{\star}(w')$ and $\mathsf{src}'^{+}_{\Lsh\Rsh}(w'_0)$.

It is also necessary to relate semantic paths of the biported structure to semantic paths of the fibred structure. We start by associating to semantic values in $V'_{\Lsh\Rsh}$ a set of corresponding values in $V'_{\star}$. We write $\cdot^{\star'_{\mathsf{v}}}$ for the map from $V'_{\Lsh\Rsh}$ to $\wp(V'_{\star})$ such that: $\blacklozenge^{\star'_{\mathsf{v}}} = \{(\blacklozenge, \blacklozenge)\}$, and for v' in $V'_{\Lsh\Rsh} \setminus \{\blacklozenge\}$ if v' is in V'_1 then $v'^{\star'_{\mathsf{v}}} = \{(v', v'_2) : v'_2 \in D_2$ if $v' \in D'_1$; otherwise $v'_2 \in V'_2 \setminus (D_2 \cup \{\blacklozenge\})\}$, otherwise $v'^{\star'_{\mathsf{v}}} = \{(v'_1, v') : v'_1 \in D_1$ if $v' \in D'_2$; otherwise $v'_1 \in V'_1 \setminus (D_1 \cup \{\blacklozenge\})\}$. Moreover, given v'_1 and v'_2 in $V'_{\Lsh\Rsh}$ suitable for a bridge and such that if $\alpha^{\mathsf{v}}_{\Lsh\Rsh}(v'_1) = \alpha^{\mathsf{v}}_{\Lsh\Rsh}(v'_2)$ then $v'_1 = \blacklozenge$, we denote by $\{v'_1, v'_2\}^{()'_{\mathsf{v}}}$ the set $\{(v'_1, v'_2)\}$ if $\alpha^{\mathsf{v}}_{\Lsh\Rsh}(v'_1) = \pi_1$, the set $\{(v'_2, v'_1)\}$ if $\alpha^{\mathsf{v}}_{\Lsh\Rsh}(v'_1) = \pi_2$, and the set $\{(\blacklozenge, \blacklozenge)\}$ if $v'_1 = v'_2$. We denote by $\mathsf{IPaths}_{\star}(G^{\dagger}_{\Lsh\Rsh})$ and by $\mathsf{IPaths}_{\star}(G'^{\dagger}_{\Lsh\Rsh})$ the set of all irreducible paths of $G^{\dagger}_{\Lsh\Rsh}$ and $G'^{\dagger}_{\Lsh\Rsh}$ respectively, that do not start by a non-empty bridge and do not contain two consecutive non-empty bridges.

Hence, denote by $\cdot^{\star'}$ the map from $\mathsf{IPaths}_{\star}(G'^{\dagger}_{\Lsh\Rsh})$ to $\wp(\mathsf{IPaths}(G'^{\dagger}_{\star}))$ such that $\{\mathsf{trg}'^{+}_{\star}(w'_0) : w'_0 \in w'^{\star'}\} = (\mathsf{trg}'^{+}_{\Lsh\Rsh}(w'))^{\star'_{\mathsf{v}}}$ and $|w'^{\star'}| = |(\mathsf{trg}'^{+}_{\Lsh\Rsh}(w'))^{\star'_{\mathsf{v}}}|$, and similarly for src, inductively defined as follows:

- $(\epsilon_{v'_1 \ldots v'_m})^{\star'}$ is $\{\epsilon_{u'_1 \ldots u'_m} : u'_1 \in v_1'^{\star'_{\mathsf{v}}}, \ldots, u'_m \in v_m'^{\star'_{\mathsf{v}}}\}$;

- $(e' \downarrow^{\prime v'_1 \ldots v'_m}_{v''_1 \ldots v''_m} w')^{\star'}$ for e' in $E'_1 \cup E'_2$, is $\{e'_{u'_1 \ldots u'_m u'} w'_0 : w'_0 \in w'^{\star'}, u'_1 \in \{v'_1, v''_1\}^{()'_{\mathsf{v}}}, \ldots, u'_m \in \{v'_m, v''_m\}^{()'_{\mathsf{v}}}, \mathsf{trg}'^{+}_{\star}(w'_0) = u'_1 \ldots u'_m, u' \in (\mathsf{trg}'_{\Lsh\Rsh}(e'))^{\star'_{\mathsf{v}}}\}$;

- $(\langle w'_1, \ldots, w'_m \rangle)^{\star'}$ is $\{\langle w''_1, \ldots, w''_m \rangle : w''_1 \in w_1'^{\star'}, \ldots, w''_m \in w_m'^{\star'}, \mathsf{src}'^{+}_{\star}(w''_1) = \ldots = \mathsf{src}'^{+}_{\star}(w''_m)\}$;

- $(\mathsf{p}_i^{v'_1 \ldots v'_m})^{\star'}$ is the set $\{\mathsf{p}^{u'_1 \ldots u'_m} : u'_1 \in v_1'^{\star'_{\mathsf{v}}}, \ldots, u'_m \in v_m'^{\star'_{\mathsf{v}}}\}$.

As an abuse of notation we denote also by $\cdot^{\star'}$ the extension of this map to $\mathsf{IPaths}(G'^{\dagger}_{\Lsh\Rsh})$ such that if w' is in $\mathsf{IPaths}(G'^{\dagger}_{\Lsh\Rsh})$ but not in $\mathsf{IPaths}_{\star}(G'^{\dagger}_{\Lsh\Rsh})$ then $w'^{\star'}$ is $\emptyset$.

Equivalence of unconstrained fibring and biporting

We now show that a concrete formula in the fibred language is satisfied in an interpretation structure resulting from the unconstrained fibring of a pair of interpretation structures if and only if its translation is satisfied by the interpretation structure resulting from the biporting (of an appropriate renaming) of those structures.

Proposition 4.2 Given a concrete formula φ of $G^+_\star$,

$$(\Sigma_1, I_1) \star (\Sigma_2, I_2), \blacklozenge \Vdash \varphi \quad \text{iff} \quad ({\Lsh\Rsh}_1(\Sigma_1), I_1^{\Lsh\Rsh_1}) \mathbin{\Lsh\Rsh} ({\Lsh\Rsh}_2(\Sigma_2), I_2^{\Lsh\Rsh_2}), \blacklozenge \Vdash (\varphi)^{\Lsh\Rsh_\dagger}.$$

Proof: Observe that $(\varphi)^{\Lsh\Rsh_\dagger}$ is a singleton by Proposition 4.1. Without loss of generality denote $(\varphi)^{\Lsh\Rsh_\dagger}$ by ψ. So:

($\Rightarrow$) Let τ' be a path starting in $\blacklozenge$ for ψ, that is, such that $\alpha^+_{\Lsh\Rsh}(\tau') = \psi$. Then τ' is in $\mathsf{IPaths}_\star(G^\dagger_{\Lsh\Rsh})$ and each $\tau'' \in \tau'^{\star'}$ is a path for φ starting in $\blacklozenge$ by definition of $\cdot^{\star'}$. Observe that the set $(\tau')^{\star'}$ is not empty by definition of $\cdot^{\star'}$ since τ' is in $\mathsf{IPaths}_\star(G^+_{\Lsh\Rsh})$. Let τ'' in $G'^+_\star$ be a path in $\tau'^{\star'}$. Hence $(\Sigma_1, I_1) \star (\Sigma_2, I_2), \tau'' \Vdash \varphi$ and so $\mathsf{trg}'^+_\star(\tau'') \in D_\star$. On the other hand by definition of $\cdot^{\star'}$, $\mathsf{trg}'^+_\star(\tau'') \in (\mathsf{trg}'^+_{\Lsh\Rsh}(\tau'))^{\star'_{\mathsf{V}}}$. Then by definition of $\cdot^{\star'_{\mathsf{V}}}$ we have that $\mathsf{trg}'^+_{\Lsh\Rsh}(\tau') \in D_{\Lsh\Rsh}$ and so $\mathsf{trg}'^+_\star(\tau') \in D_{\Lsh\Rsh}$;

($\Leftarrow$) Let τ' be a path for φ. Observe that τ' starts at $\blacklozenge$. Then each τ'' in $(\tau')^{\Lsh\Rsh_\dagger{}'}$ is a path for a formula in $(\varphi)^{\Lsh\Rsh_\dagger}$, and by definition of $\cdot^{\Lsh\Rsh_\dagger{}'}$ starts also at $\blacklozenge$. Observe that $(\tau')^{\Lsh\Rsh_\dagger{}'}$ is not empty by definition of $\cdot^{\Lsh\Rsh_\dagger{}'}$. So, let τ'' be a path in $(\tau')^{\Lsh\Rsh_\dagger{}'}$. Hence $\mathsf{trg}'^+_{\Lsh\Rsh}(\tau'') \in D_{\Lsh\Rsh}$. On the other hand by definition of $\cdot^{\Lsh\Rsh_\dagger{}'}$, $\mathsf{trg}'^+_{\Lsh\Rsh}(\tau'') \in (\mathsf{trg}'^+_\star(\tau'))^{\Lsh\Rsh'_{\mathsf{V}}}$. Then by definition of $\cdot^{\Lsh\Rsh'_{\mathsf{V}}}$ and of $V_\star$ we have that $\mathsf{trg}'^+_\star(\tau') \in D_\star$. QED

As a straightforward corollary of Proposition 4.2 we can now establish that $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$ and $({\Lsh\Rsh}_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\Rsh_1}) \mathbin{\Lsh\Rsh} ({\Lsh\Rsh}_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\Rsh_2})$ are equivalent in terms of entailment modulo the mapping of formulas.

Proposition 4.3 Given a set $\Gamma \cup \{\varphi\}$ of concrete formulas in $G^+_\star$,

$$\Gamma \vDash_{(\Sigma_1,\mathfrak{I}_1)\star(\Sigma_2,\mathfrak{I}_2)} \varphi \quad \text{if and only if} \quad \Gamma^{\Lsh\Rsh_\dagger} \vDash_{({\Lsh\Rsh}_1(\Sigma_1),\mathfrak{I}_1^{\Lsh\Rsh_1})\Lsh\Rsh({\Lsh\Rsh}_2(\Sigma_2),\mathfrak{I}_2^{\Lsh\Rsh_2})} \varphi^{\Lsh\Rsh_\dagger}.$$

Proof: In fact: ($\Rightarrow$) Suppose $({\Lsh\Rsh}_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\Rsh_1}) \mathbin{\Lsh\Rsh} ({\Lsh\Rsh}_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\Rsh_2}), \blacklozenge \Vdash \Gamma^{\Lsh\Rsh_\dagger}$. Hence $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2), \blacklozenge \Vdash \Gamma$ by Proposition 4.2 and so $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2), \blacklozenge \Vdash \varphi$. Therefore $({\Lsh\Rsh}_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\Rsh_1}) \mathbin{\Lsh\Rsh} ({\Lsh\Rsh}_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\Rsh_2}), \blacklozenge \Vdash \varphi^{\Lsh\Rsh_\dagger}$ by Proposition 4.2; ($\Leftarrow$) The proof of this direction is similar to the proof of ($\Rightarrow$) so we omit it. QED

5 From biporting to importing

The goal of this section is to clarify the relationship between unconstrained fibring and asymmetric logic combination mechanisms like temporalization [10, 11], modalization [9] and exogenous enrichment [19, 6] when there is no sharing of connectives. For that we relate fibring, through biporting, with importing (which subsumes all those asymmetric mechanisms). We start by recalling importing, see [21].

Importing logics

Importing is a general mechanism of combining logics proposed in [21], subsuming well know asymmetric logic combination techniques like temporalization [10, 11], modalization [9] and exogenous enrichment [19, 6] when there are no sharing of connectives. In importing, the signature resulting from the combination is enriched with the importing connective ↰ for transforming expressions of the imported language into expressions of the importing one. Semantically, the interpretation structure resulting from importing one structure into another, has a copy of each structure plus the denotation of the importing connective, which relates the distinguished truth values of both structures, as well as the non-distinguished ones. As for biporting we define importing only for interpretation systems not sharing connectives and assume that:

- the pair of signatures of the interpretation systems involved in importing is such that their sets of sorts are disjoint as well as their sets of m-edges;
- in the interpretation systems involved in importing all the pairs with an interpretation structure (Σ_1, I_1) of one with an interpretation structure (Σ_2, I_2) of the other are such that $(V_1')_{\Pi_1}$ and $(V_2')_{\Pi_2}$ are disjoint as well as E_1' and E_2'.

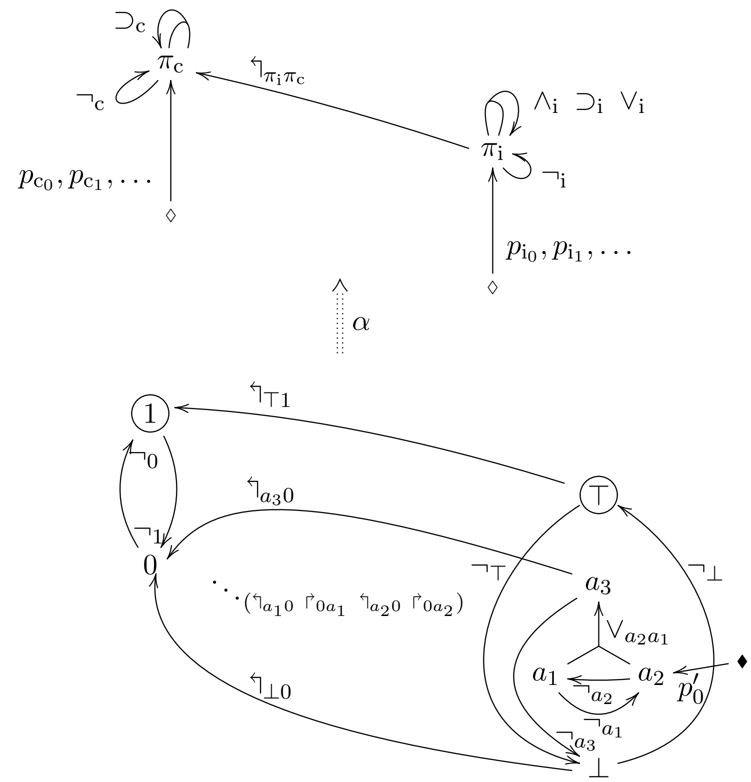

Figure 6: Part of the interpretation structure resulting from importing the structure for intuitionistic logic partially described in Figure 4, into the interpretation structure for classical logic partially described in Figure 3. The connectives and the propositions sorts were renamed in order for the signatures to be adequate for importing.

Importing an interpretation system $(\Sigma_1, \mathfrak{I}_1)$ *into an interpretation system* $(\Sigma_2, \mathfrak{I}_2)$, denoted by

$$(\Sigma_2, \mathfrak{I}_2)[(\Sigma_1, \mathfrak{I}_1)],$$

is the interpretation system $(\Sigma_2[\Sigma_1], \mathfrak{I}_2[\mathfrak{I}_1])$ where

$$\Sigma_2[\Sigma_1],$$

is the signature $((V, E, \mathsf{src}, \mathsf{trg}), \diamond, \Pi)$ with

- $V = V_1 \cup V_2$;
- E is $E_1 \cup E_2 \cup \{\Lsh_{uv} : u \in \Pi_2, v \in \Pi_1\}$;
- src and trg are such that
  - $\mathsf{src}(e) = \mathsf{src}_i(e)$ and $\mathsf{trg}(e) = \mathsf{trg}_i(e)$ if e is in E_i for $i = 1, 2$;
  - $\mathsf{src}(\Lsh_{uv}) = u$ and $\mathsf{trg}(\Lsh_{uv}) = v$;
- Π is $\Pi_1 \cup \Pi_2$;

and

$$\mathfrak{I}_2[\mathfrak{I}_1]$$

is the set of interpretation structures $\{I_2[I_1] : I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}$ where

$$I_2[I_1]$$

is the tuple $((V'_\Lsh, E'_\Lsh, \mathsf{src}'_\Lsh, \mathsf{trg}'_\Lsh), \alpha_\Lsh, D_\Lsh, \blacklozenge)$ such that

- $V'_\Lsh$ is $V'_1 \cup V'_2$;
- $E'_\Lsh$ is the union of the sets
  - E'_1;
  - E'_2;
  - $\{\Lsh_{v'_2 v'_1} : v'_1 \in D_1, v'_2 \in D_2\}$;
  - $\{\Lsh_{v'_2 v'_1} : v'_1 \in (V'_1)_{\Pi_1} \setminus D_1, v'_2 \in (V'_2)_{\Pi_2} \setminus D_2\}$;
- $\mathsf{src}'_\Lsh$ and $\mathsf{trg}'_\Lsh$ are such that
  - $\mathsf{src}'_\Lsh(e'_i) = \mathsf{src}'_i(e'_i)$ and $\mathsf{trg}'_\Lsh(e'_i) = \mathsf{trg}'_i(e'_i)$ for $i = 1, 2$;
  - $\mathsf{src}'_\Lsh(\Lsh_{v'v''}) = v'$ and $\mathsf{trg}'_\Lsh(\Lsh_{v'v''}) = v''$;
- $\alpha_\Lsh$ is such that
  - $\alpha^{\mathsf{v}}_\Lsh(v') = \alpha^{\mathsf{v}}_i(v')$ whenever $v' \in V'_i$ for $i = 1, 2$;
  - $\alpha^{\mathsf{e}}_\Lsh(e') = \alpha^{\mathsf{e}}_i(e')$ whenever $e' \in E'_i$ for $i = 1, 2$;
  - $\alpha^{\mathsf{e}}_\Lsh(\Lsh_{v'v''}) = \Lsh_{\alpha^{\mathsf{v}}_\Lsh(v')\alpha^{\mathsf{v}}_\Lsh(v'')}$;
- $D_\Lsh$ is $D_1 \cup D_2$.

In order to simplify the presentation, when Π_1 and Π_2 are singletons we may omit the reference to the sorts in $\Lsh_{uv}$ and simply write $\Lsh$. See Figure 6 for an example of an interpretation structure resulting from an importing.

Observe that the particular instances of importing introduced in [21] named $\Lsh$-temporalization and $\Lsh$-modalization, were shown in that paper to be equivalent with the well-known combination mechanisms temporalization [10, 11] and modalization [9], respectively.

As a more detailed example of importing, we describe $\Lsh$-globalization, introduced in [21], which is shown in that paper to be equivalent to globalization [20]. In this example, importing is applied to a more restrict class of interpretation systems, the class of g-appropriate interpretation systems.

An interpretation system $(\Sigma_1, \mathfrak{I}_1)$ is *g-appropriate* if the following conditions hold: (i) $(\Sigma_\emptyset, \{I_c\})$ and $(\Sigma_1, \mathfrak{I}_1)$ are appropriate for importing as described in the beginning of the section, where $(\Sigma_\emptyset, \{I_c\})$ is the interpretation system for classical propositional logic (introduced in Example 2.2) with no propositional variables; (ii) for every connective e of Σ_1 and $I_1 = ((V_1', E_1', \mathsf{src}_1', \mathsf{trg}_1'), \alpha_1, D_1, \blacklozenge)$ in $\mathfrak{I}_1$ there is e' in E_1' such that $\alpha_1^{\mathsf{e}}(e') = \mathsf{e}$; (iii) each structure I_1 in $\mathfrak{I}_1$ is deterministic in the sense that $(\alpha_1^+)^{-1}(\psi)$ is a singleton for every concrete formula ψ over Σ_1; and (iv) $\mathfrak{I}_1$ is non-empty, that is, $(\Sigma_1, \mathfrak{I}_1)$ is consistent.

Example 5.1 The $\Lsh$-*globalization* of a g-appropriate system $(\Sigma_1, \mathfrak{I}_1)$, denoted by

$$\mathrm{G}[(\Sigma_1, \mathfrak{I}_1)]$$

is the interpretation system $(\Sigma_\emptyset, \{I_c\})[(\Sigma_1, \vec{\mathfrak{I}}_1)]$ resulting from importing $(\Sigma_1, \vec{\mathfrak{I}}_1)$ into $(\Sigma_\emptyset, \{I_c\})$, where $\vec{\mathfrak{I}}_1$ is the collection of all interpretation structures

$$I_J = ((V_J', E_J', \mathsf{src}_J', \mathsf{trg}_J'), \alpha_J, D_J, \blacklozenge)$$

for Σ_1 where $J = \{I_k\}_{k<\beta}$ is a non-empty sequence of structures in $\mathfrak{I}_1$ and β is an ordinal, such that:

- $V_J' = \{v_0' \dots v_k' \dots : k < \beta, v_k' \in V_k' \text{ and } \alpha_0^{\mathsf{v}}(v_0') = \dots = \alpha_k^{\mathsf{v}}(v_k') = \dots\}$;
- $E_J' = \{e_0' \dots e_k' \dots : k < \beta, e_k' \in E_k' \text{ and } \alpha_0^{\mathsf{e}}(e_0') = \dots = \alpha_k^{\mathsf{e}}(e_k') = \dots\}$;
- $\mathsf{src}_J'(e_0' \dots e_k' \dots) = \mathsf{src}_0'(e_0') \dots \mathsf{src}_k'(e_k') \dots$;
- $\mathsf{trg}_J'(e_0' \dots e_k' \dots) = \mathsf{trg}_0'(e_0') \dots \mathsf{trg}_k'(e_k') \dots$;
- $\alpha_J^{\mathsf{v}}(v_0' \dots v_k' \dots) = \alpha_k^{\mathsf{v}}(v_k')$ and $\alpha_J^{\mathsf{e}}(e_0' \dots e_k' \dots) = \alpha_k^{\mathsf{e}}(e_k')$;
- $\blacklozenge$ is the sequence $\blacklozenge_0 \dots \blacklozenge_k \dots$;
- $D_J = \{v_0' \dots v_k' \dots : v_0' \in D_0, \dots, v_k' \in D_k, \dots\}$. ∇

Lifting

In this subsection we establish that the relationship between biporting and importing is universal in the sense of category theory. Moreover it is solely driven by their signatures relationship. In fact, the interpretation structures resulting from importing can be obtained from the structures resulting from the biporting, by the cocartesian lifting, through the forgetful functor, of a particular signature morphism defined in all sorts and connectives except for the export ↱ connective. We need first to introduce some notation.

A *signature partial morphism* h from signature Σ_1 to signature Σ_2 is an m-graph partial morphism from G_1 to G_2 such that h^{v} is a map, $h^{\mathsf{v}}(\diamond) = \diamond$, the restriction of h^{v} to Π_1 is one to one on its domain, and $h^{\mathsf{v}}(\Pi_1) \subseteq \Pi_2$. We denote by **Sig** the category of signatures and their morphisms.

An *interpretation structure morphism* from interpretation structure (Σ_1, I_1) to interpretation structure (Σ_2, I_2) is a pair (h, g) where h is a signature partial morphism from Σ_1 to Σ_2 and g is an m-graph partial morphism from G'_1 to G'_2 such that:

- g^{v} is a map;
- $g^{\mathsf{v}}(D_1) \subseteq D_2$ and $g^{\mathsf{v}}(V_1 \setminus D_1) \subseteq V_2 \setminus D_2$;
- $\alpha_2 \circ g \subseteq h \circ \alpha_1$.

We denote by **Int** the category of interpretation structures and their morphisms and by Fl the obvious forgetful functor from **Int** to **Sig**.

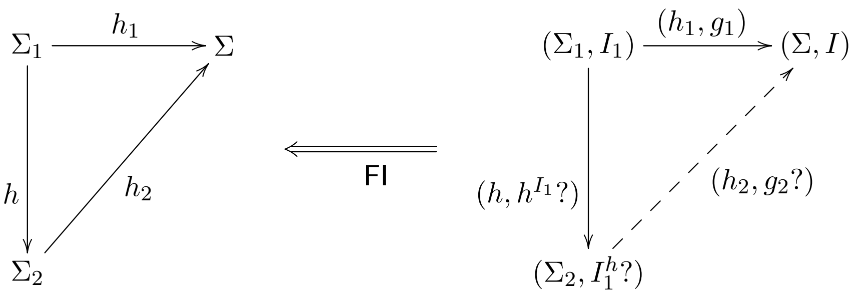

Figure 7: Cocartesian lifting from **Sig** to **Int**.

Proposition 5.2 Let $h : \Sigma_1 \to \Sigma_2$ be a signature morphism, (Σ_1, I_1) an interpretation structure with $I_1 = (G'_1, \alpha_1, D, \blacklozenge)$, and $G'_1 = (V', E'_1, \mathsf{src}'_1, \mathsf{trg}'_1)$. The *cocartesian lifting* (up to isomorphism) of (Σ_1, I_1) through h by Fl is the interpretation structure

$$(\Sigma_2, I_1^h)$$

where $I_1^h = (G_1'^h, \alpha_1^h, D, \blacklozenge)$ is the interpretation structure such that $G_1'^h$ is the m-graph $(V', E_1'^h, \mathsf{src}'_{1|E_1'^h}, \mathsf{trg}'_{1|E_1'^h})$ with:

- $E_1'^h = \{e_1' \in E_1' : h^{\mathsf{e}}(\alpha_1^{\mathsf{e}}(e_1'))\downarrow\}$;
- $\mathsf{src}'_{1|E_1'^h}, \mathsf{trg}'_{1|E_1'^h}$ are the restrictions of $\mathsf{src}'_1, \mathsf{trg}'_1$ to $E_1'^h$ respectively;
- $(\alpha_1^h)^{\mathsf{v}} = h^{\mathsf{v}} \circ \alpha_1^{\mathsf{v}}$;
- $(\alpha_2^h)^{\mathsf{e}} = h^{\mathsf{e}} \circ \alpha^{\mathsf{e}}_{1|E_1'^h}$ where $\alpha^{\mathsf{e}}_{1|E_1'^h}$ is the restriction of α_1^{e} to $E_1'^h$.

endowed with the *cocartesian morphism*

$$(h, h^{I_1}) : (\Sigma_1, I_1) \to (\Sigma_2, I_1^h)$$

where h^{I_1} is such that:

- $(h^{I_1})^{\mathsf{v}}$ is the identity on V';
- $(h^{I_1})^{\mathsf{e}}$ is such that $(h^{I_1})^{\mathsf{e}}(e_1') = e_1'$ if $e_1' \in E_1'^h$ otherwise $(h^{I_1})^{\mathsf{e}}(e_1')\uparrow$.

We omit the proof of the previous proposition since it follows in a canonical way.

We now show that, given interpretation structures (Σ_1, I_1) and (Σ_2, I_2), the structure resulting from the importing of (Σ_2, I_2) into (Σ_1, I_1) is exactly the structure resulting from the cocartesian lifting by FI of the biporting of (Σ_2, I_2) and (Σ_1, I_1) through the signature morphism

$$\sigma_{\Sigma_1,\Sigma_2}$$

from $\Sigma_1 \Lsh\!\!\Rsh \Sigma_2$ to $\Sigma_1[\Sigma_2]$ defined as follows: $\sigma^{\mathsf{v}}_{\Sigma_1,\Sigma_2}$ is the identity, $\sigma^{\mathsf{e}}_{\Sigma_1,\Sigma_2}(\Rsh_{uv})\uparrow$ and $\sigma^{\mathsf{e}}_{\Sigma_1,\Sigma_2}(e) = e$ otherwise.

The same holds for the structure resulting from the importing of (Σ_1, I_1) into (Σ_2, I_2) but now with respect to $\sigma_{\Sigma_2,\Sigma_1}$.

Proposition 5.3 Given interpretation structures (Σ_1, I_1) and (Σ_2, I_2), the cocartesian lifting of $(\Sigma_1 \Lsh\!\!\Rsh \Sigma_2, I_1 \Lsh\!\!\Rsh I_2)$ through $\sigma_{\Sigma_1,\Sigma_2}$ by FI is

$$(\mathsf{trg}(\sigma_{\Sigma_1,\Sigma_2}), I_1[I_2]).$$

We omit the proof of Proposition 5.3 since it follows immediately taking into account the definition of $I_1[I_2]$ and the definition in Proposition 5.2 of the structure resulting from cocartesian lifting.

As a visual aid, compare the structure partially depicted in Figure 6 resulting from an importing, and the structure partially depicted in Figure 5 resulting from the biporting of the structures involved in the importing.

Corollary 5.4 Given interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$, the interpretation system $(\Sigma_1, \mathfrak{I}_1)[(\Sigma_2, \mathfrak{I}_2)]$ is the tuple

$$(\mathsf{trg}(\sigma_{\Sigma_1,\Sigma_2}), \{I^{\sigma_{\Sigma_1,\Sigma_2}} : I \in \mathfrak{I}_1 \Lsh\!\!\Rsh \mathfrak{I}_2\}).$$

The proof of Corollary 5.4 follows immediately just by taking into account the definition of importing and Proposition 5.3.

Hence, the $\Lsh$-temporalization of an interpretation system $(\Sigma_1, \mathfrak{I}_1)$ with the interpretation system $(\Sigma_Q^{\mathrm{LTL}}, \mathfrak{I}^{\mathrm{LTL}})$ for linear temporal logic introduced in [21],

$$(\Sigma_Q^{\mathrm{LTL}}, \mathfrak{I}^{\mathrm{LTL}})[(\Sigma_1, \mathfrak{I}_1)]$$

is obtained from the biporting

$$(\Sigma_Q^{\mathrm{LTL}}, \mathfrak{I}^{\mathrm{LTL}}) \Lsh\!\Rsh (\Sigma_1, \mathfrak{I}_1)$$

of $(\Sigma_Q^{\mathrm{LTL}}, \mathfrak{I}^{\mathrm{LTL}})$ and $(\Sigma_1, \mathfrak{I}_1)$ and from the morphism $\sigma_{\Sigma_Q^{\mathrm{LTL}}, \Sigma_1}$, as described in Corollary 5.4. Similarly for $\Lsh$-modalization and for $\Lsh$-globalization.

Therefore, the relationship between unconstrained fibring and the asymmetric combination mechanisms of temporalization [10, 11], modalization [9] and globalization [20] when there is no sharing, is also clarified due to the equivalence between biporting and unconstrained fibring established in Proposition 4.3 and the equivalence, between $\Lsh$-temporalization, $\Lsh$-modalization, $\Lsh$-globalization, and temporalization, modalization and globalization respectively, established in [21].

6 Transferring the finite model property

We now show how the relationships between the combination mechanisms established in the previous section and in [21] can be put to good use to transfer the finite model property from fibring to globalization. More rigorously, we show first that a weak form of the finite model property holds in the interpretation system resulting from an importing whenever the finite model property holds in the interpretation system resulting from the unconstrained fibring of the given systems. So, sufficient conditions for unconstrained fibring to enjoy the finite model property guarantee that importing enjoys a weak form of it too. Using this result we obtain sufficient conditions for the interpretation system resulting from a globalization to enjoy the finite model property capitalizing on the conditions established in [7] for unconstrained fibring.

An interpretation system enjoys the *finite model property* whenever any falsifiable formula is not satisfied by a finite interpretation structure in the system, and enjoys the *finite model property with respect to a subset* of its language whenever any falsifiable formula in that subset is not satisfied by a finite interpretation structure in the system.

In order to simplify the presentation, for the rest of the section assume given once and for all, interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ appropriate for unconstrained fibring, that is, such that

- Σ_1 and Σ_2 do not have connectives with the same name and each one has a unique propositions sort both with the same name (assume it is π);
- each pair of interpretation structures in those systems have disjoint sets of truth values as well as disjoint sets of operations m-edges.

Let I_1 and I_2 be structures in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively. Denote by

- $E_\star$ and $E_{\Lsh\Rsh}$ the sets of m-edges of the signatures $\Sigma_\star$ and $\Sigma_{\Lsh\Rsh}$ of the interpretation structures $(\Sigma_1, I_1) \star (\Sigma_2, I_2)$ and $({\Lsh\Rsh}_1(\Sigma_1), I_1^{{\Lsh\Rsh}_1}) \Lsh\Rsh ({\Lsh\Rsh}_2(\Sigma_2), I_2^{{\Lsh\Rsh}_2})$;
- $V'_{I_1[I_2]}$, $E'_{I_1[I_2]}$ and $E_{\Sigma_1[\Sigma_2]}$ are the sets of values, operation m-edges and connectives, respectively, of $(\Sigma_1, I_1)[(\Sigma_2, I_2)]$.

As we show below in Proposition 6.2, biporting is a conservative extension of importing. First we establish an auxiliary proposition.

Proposition 6.1 Given an irreducible path w over $\Sigma_1[\Sigma_2]$, τ is an irreducible path in $I_1[I_2]$ for w if and only if τ is an irreducible path in $I_1 \Lsh\Rsh I_2$ for w.

Proof: ($\Rightarrow$) Assume that τ is an irreducible path in $I_1[I_2]$ for w. Observe that $V'_{I_1[I_2]} = V'_{\Lsh\Rsh}$, and $E'_{I_1[I_2]} \subseteq E'_{\Lsh\Rsh}$. Then it can be seen immediately by induction that τ is also an irreducible path in $I_1 \Lsh\Rsh I_2$ for w. ($\Leftarrow$) Assume that τ is an irreducible path in $I_1 \Lsh\Rsh I_2$ for w. The proof follows by induction on the irreducible path w:

(1) w is $\epsilon_{v_1 \dots v_m}$. Then τ is of the form $\epsilon_{v'_1 \dots v'_m}$ where $\alpha^{\mathsf{v}}_{\Lsh\Rsh}(v'_i) = v_i$ for $i = 1, \dots, m$. The thesis follows since $\alpha^{\mathsf{v}}_{I_1 \Lsh\Rsh I_2} = \alpha^{\mathsf{v}}_{I_1[I_2]}$;

(2) w is $\mathsf{p}_i^{v_1 \dots v_m}$. The proof is similar to the proof of case (1) so we omit it;

(3) w is ew_0 where e is in $E_{\Sigma_1[\Sigma_2]}$ and w_0 is an irreducible path over $\Sigma_1[\Sigma_2]$. Then $\alpha^+_{\Lsh\Rsh}(\tau) = ew_0$ and so, taking into account the definition of the functor $\alpha^+_{\Lsh\Rsh}$ in Section 2, τ is of the form $e'\tau_0$ where e' in $E'_{\Lsh\Rsh}$ is such that $\alpha^{\mathsf{e}}_{\Lsh\Rsh}(e') = e$ and τ_0 is an irreducible path in $I_1 \Lsh\Rsh I_2$ for w_0. So, by induction hypothesis, τ_0 is an irreducible path in $I_1[I_2]$ for w_0. Since all the m-edges of $E'_{\Lsh\Rsh} \setminus \{\Rsh_{u'v'} : \Rsh_{u'v'} \in E'_{\Lsh\Rsh}\}$ are also m-edges of $E'_{I_1[I_2]}$ and e' is not in $\{\Rsh_{u'v'} : \Rsh_{u'v'} \in E'_{\Lsh\Rsh}\}$ then e' is in $E'_{I_1[I_2]}$. So τ is an irreducible path in $I_1[I_2]$ for w;

(4) w is $\langle w_1, \dots, w_m \rangle$. The result follows immediately by induction hypothesis and the definition of $\alpha^+_{\Lsh\Rsh}$. QED

Proposition 6.2 Given a concrete formula φ in $L(\Sigma_1[\Sigma_2])$, $I_1[I_2] \Vdash \varphi$ if and only if $I_1 \Lsh\Rsh I_2 \Vdash \varphi$.

Proof: ($\Rightarrow$) Assume that $I_1[I_2] \Vdash \varphi$ and let τ be a path in $I_1 \Lsh\Rsh I_2$ for φ. Then τ is a path in $I_1[I_2]$ for φ by Proposition 6.1 and so $\mathsf{trg}'_{I_1[I_2]}(\tau) \in D_{I_1[I_2]}$. The thesis follows since $\mathsf{trg}'_{\Lsh\Rsh}(\tau) = \mathsf{trg}'_{I_1[I_2]}(\tau)$ and $D_{\Lsh\Rsh} = D_{I_1[I_2]}$. ($\Leftarrow$) The proof of this direction is similar to the proof of the other so we omit it. QED

Capitalizing on Proposition 4.1 we now show that the finite model property with respect to the formulas that are images of concrete formulas of unconstrained fibring is transferred from unconstrained fibring to biporting. We denote by $cL(\Sigma)$ the set of concrete formulas over a signature Σ.

Proposition 6.3 The interpretation system $({\Lsh\Rsh}_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\Rsh 1}) \Lsh\Rsh ({\Lsh\Rsh}_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\Rsh 2})$ enjoys the finite model property with respect to $(cL(\Sigma_1 \star \Sigma_2))^{\Lsh\Rsh\dagger}$ whenever $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$ enjoys the finite model property.

Proof: Let φ be a concrete formula in $L(\Sigma_1 \star \Sigma_1)$ such that there is an interpretation structure I in $\mathfrak{I}_1^{\Lsh\Rsh 1} \Lsh\Rsh \mathfrak{I}_2^{\Lsh\Rsh 2}$ with $I \not\Vdash (\varphi)^{\Lsh\Rsh\dagger}$. Observe that $(\varphi)^{\Lsh\Rsh\dagger}$ is a singleton by Proposition 4.1. Let I_1 in $\mathfrak{I}_1$ and I_2 in $\mathfrak{I}_2$ be such that I is $I_1^{\Lsh\Rsh 1} \Lsh\Rsh I_2^{\Lsh\Rsh 2}$. Hence $I_1 \star I_2 \not\Vdash \varphi$ by Proposition 4.2. Since $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$ enjoys the finite model property, let I_f be a finite interpretation structure in $\mathfrak{I}_1 \star \mathfrak{I}_2$ such that $I_f \not\Vdash \varphi$. Let I_{f_1} in $\mathfrak{I}_1$ and I_{f_2} in $\mathfrak{I}_2$ be finite interpretation structures such that I_f is $I_{f_1} \star I_{f_2}$. So $(I_{f_1})^{\Lsh\Rsh 1} \Lsh\Rsh (I_{f_2})^{\Lsh\Rsh 2} \not\Vdash (\varphi)^{\Lsh\Rsh\dagger}$ again by Proposition 4.2. The thesis follows since $(I_{f_1})^{\Lsh\Rsh 1} \Lsh\Rsh (I_{f_2})^{\Lsh\Rsh 2}$ is finite and $(\varphi)^{\Lsh\Rsh\dagger}$ is a singleton by Proposition 4.1. QED

The finite model property with respect to a set is transferred modulo a domain restriction from biporting to importing.

Proposition 6.4 Let L be a subset of the language of $(\Sigma_1, \mathfrak{I}_1) \Lsh\Rsh (\Sigma_2, \mathfrak{I}_2)$. The interpretation system $(\Sigma_1, \mathfrak{I}_1)[(\Sigma_2, \mathfrak{I}_2)]$ enjoys the finite model property with respect to $L \cap L(\Sigma_1[\Sigma_2])$ whenever $(\Sigma_1, \mathfrak{I}_1) \Lsh\Rsh (\Sigma_2, \mathfrak{I}_2)$ enjoys the finite model property with respect to L.

Proof: Let φ be a formula in $L \cap L(\Sigma_1[\Sigma_2])$ such that there is an interpretation structure I in $\mathfrak{I}_1[\mathfrak{I}_2]$ with $I \not\Vdash \varphi$. Let I_1 in $\mathfrak{I}_1$ and I_2 in $\mathfrak{I}_2$ be such that I is $I_1[I_2]$. Hence $I_1 \Lsh\Rsh I_2 \not\Vdash \varphi$ by Proposition 6.2. Since $(\Sigma_1, \mathfrak{I}_1) \Lsh\Rsh (\Sigma_2, \mathfrak{I}_2)$ enjoys the finite model property with respect to L and φ is in L let I_f be a finite interpretation structure in $\mathfrak{I}_1 \Lsh\Rsh \mathfrak{I}_2$ such that $I_f \not\Vdash \varphi$. Let I_{f_1} in $\mathfrak{I}_1$ and I_{f_2} in $\mathfrak{I}_2$ be finite interpretation structures such that I_f is $I_{f_1} \Lsh\Rsh I_{f_2}$. So $I_{f_1}[I_{f_2}] \not\Vdash \varphi$ again by Proposition 6.2. The thesis follows since $I_{f_1}[I_{f_2}]$ is finite. QED

So it is immediate to conclude the following corollary.

Corollary 6.5 The interpretation system $({\Lsh\Rsh}_1(\Sigma_1), \mathfrak{I}_1^{\Lsh\Rsh 1})[({\Lsh\Rsh}_2(\Sigma_2), \mathfrak{I}_2^{\Lsh\Rsh 2})]$ enjoys the finite model property with respect to $(cL(\Sigma_1 \star \Sigma_2))^{\Lsh\Rsh\dagger} \cap L({\Lsh\Rsh}_1(\Sigma_1)[{\Lsh\Rsh}_2(\Sigma_2)])$ whenever $(\Sigma_1, \mathfrak{I}_1) \star (\Sigma_2, \mathfrak{I}_2)$ enjoys the finite model property.

Capitalizing on the equivalence between globalization and $\Lsh$-globalization established in [21] and in Corollary 6.5 we now show that globalization enjoys the finite model property if that property holds in the interpretation system resulting from the fibring of $(\Sigma_\emptyset, \{I_c\})$ with the interpretation system being globalized. See the definition of $(\Sigma_1, \vec{\mathfrak{I}}_1)$ in Section 5.

Proposition 6.6 The globalization of a g-appropriate interpretation system $(\Sigma_1, \mathfrak{I}_1)$ enjoys the finite model property with respect to concrete formulas whenever the unconstrained fibring $(\Sigma_\emptyset, \{I_c\}) \star (\Sigma_1, \vec{\mathfrak{I}}_1)$ enjoys that property.

Proof: Assume that $(\Sigma_\emptyset, \{I_c\}) \star (\Sigma_1, \vec{\mathfrak{I}}_1)$ enjoys the finite model property. Let φ be a concrete formula in the language of the system resulting from the globalization of $(\Sigma_1, \mathfrak{I}_1)$ (see [21]). Recall the map $\cdot^{\Lsh g}$ defined in Section 5.1 of [21].

Observe that φ is a concrete formula in $L(\Sigma_\emptyset \star \Sigma_1)$ and that $(\varphi)^{\Lsh g}$ coincides with $(\varphi)^{\Lsh\Rsh\dagger}$ (in this proof we denote the unique element in the singleton $(\varphi)^{\Lsh\Rsh\dagger}$ also by $(\varphi)^{\Lsh\Rsh\dagger}$). Let V be a non-empty subset of interpretation structures in $\mathfrak{I}_1$ and suppose that $V \not\Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$. Then $I_{\mathrm{c}}[I_J] \not\Vdash (\varphi)^{\Lsh g}$ by Proposition 5.1 in [21] where J is a sequence with all the elements of V. Hence $I_{\mathrm{c}}[I_J] \not\Vdash (\varphi)^{\Lsh\Rsh\dagger}$. Since $(\varphi)^{\Lsh\Rsh\dagger}$ is in $(L(\Sigma_\emptyset \star \Sigma_1))^{\Lsh\Rsh\dagger} \cap L(\Sigma_\emptyset[\Sigma_1])$, $I \not\Vdash (\varphi)^{\Lsh\Rsh\dagger}$ for some finite interpretation structure I in $\{I_{\mathrm{c}}\}[\vec{\mathfrak{I}}_1]$ by Proposition 6.5. Hence I is of the form $I_{\mathrm{c}}[I_{J'}]$ where $I_{J'}$ is a finite structure. Let V' be the non-empty set of interpretation structures in $\mathfrak{I}_2$ with all the structures in J'. Then V' is a finite set of finite structures and $V' \not\Vdash \varphi$ by Proposition 5.1 in [21]. QED

Using Proposition 6.6 we now transfer the sufficient conditions, established in Theorem 4.5 of [7], for the finite model property to hold in fibring, to globalization. Observe that the interpretation system $(\Sigma_\emptyset, \{I_c\})$ has disjunction and strong negation as defined in [7].

Corollary 6.7 The globalization of a g-appropriate interpretation system $(\Sigma_1, \mathfrak{I}_1)$ enjoys the finite model property whenever $(\Sigma_1, \vec{\mathfrak{I}}_1)$ is a system with disjunction and strong negation.

7 Concluding remarks

We proposed a new formulation of fibring by looking at it as a kind of two-way importing. We proved a strong equivalence between fibring and this new formulation (biporting), that we used to prove the long standing conjecture that fibring subsumes asymmetric ways of combining logics, like temporalization, modalization, globalization and importing. It should be stressed that these results assume that no connectives are shared.

Furthermore, we were able to illustrate the usefulness of this subsumption by proving that importing preserves the finite model property whenever it is preserved by fibring. We expect to be able to obtain additional preservation results about importing (and its special cases, like temporalization, modalization, globalization, probabilization and quantization) from other known preservation results about fibring, such as soundness, completeness and Craig interpolation.

To this end, the work on relating fibring with biporting and importing should be carried over to the level of calculi. Another line of research that seems worthwhile pursuing should address the sharing of connectives in importing and biporting.

Acknowledgments

This work was partially supported by FCT and EU FEDER, namely via the projects PEst-OE/EEI/LA0008/2011 and AMDSC UTAustin/MAT/0057/2008, as well as under the MCL (Meet-Combination of Logics) and PQDR (Probabilistic, Quantum and Differential Reasoning) initiatives of SQIG at IT.

References

[1] F. Baader, S. Ghilardi, and C. Lutz. LTL over description logic axioms. *ACM Transactions on Computational Logic*, 13(3):21:1–21:32, 2012.

[2] P. Blackburn and M. de Rijke. Why combine logics? *Studia Logica*, 59(1):5–27, 1997.

[3] C. Caleiro, W. A. Carnielli, J. Rasga, and C. Sernadas. Fibring of logics as a universal construction. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, 2nd Edition*, volume 13, pages 123–187. Springer, 2005.

[4] C. Caleiro, A. Sernadas, and C. Sernadas. Fibring logics: Past, present and future. In S. Artemov, H. Barringer, A. S. d'Avila Garcez, L. C. Lamb, and J. Woods, editors, *We Will Show Them: Essays in Honour of Dov Gabbay, Volume One*, pages 363–388. King's College Publications, 2005.

[5] W. A. Carnielli, J. Rasga, and C. Sernadas. Preservation of interpolation features by fibring. *Journal of Logic and Computation*, 18(1):123–151, 2008.

[6] R. Chadha, P. Mateus, A. Sernadas, and C. Sernadas. Extending classical logic for reasoning about quantum systems. In D. Gabbay K. Engesser and D. Lehmann, editors, *Handbook of Quantum Logic and Quantum Structures: Quantum Logic*, pages 325–372. Elsevier, 2009.

[7] M. Coniglio, A. Sernadas, and C. Sernadas. Preservation by fibring of the finite model property. *Journal of Logic and Computation*, 21(2):375–402, 2011.

[8] L. F. del Cerro and A. Herzig. Combining classical and intuitionistic logic, or: intuitionistic implication as a conditional. In F. Baader and K. U. Schulz, editors, *Frontiers of Combining Systems (FroCos) 1996*, volume III of *Applied Logic Series*, pages 93–102. Kluwer Academic Publishers, 1996.

[9] R. A. S. Fajardo and M. Finger. Non-normal modalisation. In *Advances in Modal Logic*, volume 4, pages 83–95. King's College Publications, 2003.

[10] M. Finger and D. M. Gabbay. Adding a temporal dimension to a logic system. *Journal of Logic, Language and Information*, 1(3):203–233, 1992.

[11] M. Finger and M. A. Weiss. The unrestricted combination of temporal logic systems. *Logic Journal of the IGPL*, 10(2):165–189, 2002.

[12] D. Gabbay. *Fibring Logics*, volume 38 of *Oxford Logic Guides*. Oxford University Press, 1999.

[13] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Manydimensional modal logics: Theory and applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2003.

[14] D. M. Gabbay. Fibred semantics and the weaving of logics. I. Modal and intuitionistic logics. *The Journal of Symbolic Logic*, 61(4):1057–1120, 1996.

[15] S. J. Gay, R. Nagarajan, and N. Papanikolaou. Specification and verification of quantum protocols. In *Semantic Techniques in Quantum Computation*, pages 414–472. Cambridge Univ. Press, 2010.

[16] M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56(4):1469?1485, 1991.

[17] M. Kracht and F. Wolter. Simulation and transfer results in modal logic – a survey. *Studia Logica*, 59(2):149–177, 1997.

[18] M. A. Martins, A. Madeira, R. Diaconescu, and L. S. Barbosa. Hybridization of institutions. In *Algebra and Coalgebra in Computer Science*, volume 6859 of *Lecture Notes in Computer Science*, pages 283–297. Springer, 2011.

[19] P. Mateus and A. Sernadas. Weakly complete axiomatization of exogenous quantum propositional logic. *Information and Computation*, 204(5):771–794, 2006.

[20] P. Mateus, A. Sernadas, and C. Sernadas. Exogenous semantics approach to enriching logics. In G. Sica, editor, *Essays on the Foundations of Mathematics and Logic*, volume 1, pages 165–194. Polimetrica, 2005.

[21] J. Rasga, A. Sernadas, and C. Sernadas. Importing logics. *Studia Logica*, 100(3):545–581, 2012.

[22] J. Rasga, A. Sernadas, and C. Sernadas. Importing logics: Soundness and completeness preservation. *Studia Logica*, in print.

[23] A. Sernadas and C. Sernadas. Combining logic systems: Why, how, what for? *CIM Bulletin*, 15:9–14, December 2003.

[24] A. Sernadas, C. Sernadas, J. Rasga, and M. Coniglio. A graph-theoretic account of logics. *Journal of Logic and Computation*, 19(6):1281–1320, 2009.

[25] A. Sernadas, C. Sernadas, J. Rasga, and M. Coniglio. On graph-theoretic fibring of logics. *Journal of Logic and Computation*, 19:1321–1357, 2009.

[26] R. H. Thomason. Combinations of tense and modality. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 2, pages 135–165. Reidel, 1984.

[27] D. van Dalen. Intuitionistic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume III, pages 225–339. D. Reidel Publishing Company, 1986.

[28] A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *Journal of Symbolic Logic*, 66(1):414–439, 2001.