Completeness and Interpolation of Almost-Everywhere Quantification over Finitely Additive Measures

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Abstract

We give an axiomatization of first-order logic enriched with the almost-everywhere quantifier over finitely additive measures. Using an adapted version of the consistency property adequate for dealing with this generalized quantifier, we show that such a logic is both strongly complete and enjoys Craig interpolation, relying on a (countable) model existence theorem. We also discuss possible extensions of these results to the almost-everywhere quantifier over countably additive measures.

Keywords: generalized quantification, almost-everywhere quantifier, complete axiomatization, Craig interpolation.

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1 Introduction

Generalized quantifiers were first studied by Mostowski in [21], where an extension of first-order logic with numerical quantifiers was investigated, and by Fuhrken and Vaught in their works [10, 29] devoted to the quantifier *there exist uncountably many.* Generalizing these approaches, Lindström in [19] introduced the concept of generalized quantifier and proved general results about it. The topic was brought again to the attention of logicians a few years later when Keisler, relying on a refined construction of uncountable models, proved in [14] a concrete completeness theorem for first-order logic with the quantifier *there exist uncountably many*, and when Barwise published his works [5, 6] on abstract logics.

Several generalized quantifiers have been studied since then, based on discrete and continuous measure theory as well as probability theory. For instance, Keisler introduced in [16, 3] a logic with a *probabilistic quantifier* over first-order structures enriched with a probability measure, and Barwise, Kaufmann and Makkai studied in [4] an enrichment of first-order logic and of infinitary logic with a weak form of the second-order generalized quantifier *for almost* *all countable subsets* proposed by Shelah in [25] (see also [17, 26, 7, 27]) Outside pure mathematics, generalized quantifiers have also been considered in philosophy [1, 28], artificial intelligence [23, 11], as well as linguistic and natural languages [22, 2]. More recently, quantifier rank hierarchies for logics with generalized quantifiers have been studied [12, 18] capitalizing on the work of Lindström (see [19]).

In this paper we investigate first-order logic enriched with a generalized almost-everywhere quantifier (over finitely additive measures), expressing that a property holds everywhere except on a negligible set. We follow a qualitative approach in the sense that we do not explicitly use measure terms to characterize almost-everywhere. This qualitative approach complements the quantitative approaches developed for instance in [16, 11]. Observe that the logic with probabilistic quantifiers investigated in [16] does not have universal and existential quantifiers.

More concretely, we propose an axiomatization for first-order logic enriched with the almost-everywhere quantifier over finitely additive measures, and prove that it is strongly complete and enjoys Craig interpolation. The proofs rely on an adapted version of the consistency property over a finite collection of signatures, adequate for dealing with the almost-everywhere quantifier and for the proof of interpolation. The fact that the almost-everywhere quantifier is a particular case of a monotone quantifier, see [20, 30] and [13], can be noted in the definition of the consistency property. We show that the model existence theorem holds for consistency properties maximal with respect to a given collection of signatures. The notion of maximality had to be relativized since the consistency property used in the interpolation proof is not maximal in the usual sense.

This paper continues the study of almost-everywhere quantification started in [9], where a monadic second-order axiomatization of the almost-everywhere quantifier over supported structures (structures for which there is a largest null set), is presented and shown to be complete. However, unlike [9], we do not assume that our structures have to be supported. Observe that almost-everywhere quantification was first studied in the context of first-order logic in [1, 8] in a very restricted setting where it was applied only to implications and where no calculus was provided.

The paper is organized as follows: Section 2 introduces the language, the semantics and an axiomatization for first-order logic enriched with the almost-everywhere quantifier over finitely additive measures, $\mathscr{L}(\text{Æ})$. It then shows the validity of some formulas useful in the soundness and completeness proofs, as well as the soundness of useful entailments. Some auxiliary proof-theoretic results are also established. The section ends with the proof of the soundness of the given axiomatization. Section 3 gives an adapted definition of the consistency property suitable for proving the model existence theorem for $\mathscr{L}(\text{Æ})$. Capitalizing on this result, it is shown in Section 4 that $\mathscr{L}(\text{Æ})$ is strongly complete, and in Section 5 that $\mathscr{L}(\text{Æ})$ enjoys Craig interpolation. Finally, Section 6, briefly discusses some points and issues that appear when trying to find an axiomatization for the almost-everywhere quantifier over countably additive measures.

2 Language, Semantics and Deduction

We start by presenting the language, the semantics and an axiomatization for $\mathscr{L}(Æ)$. We denote first-order logic (with equality) by $\mathscr{L}$.

A *signature* Σ *for* $\mathscr{L}(Æ)$ is a first-order signature, that is, $\langle F, P, \tau \rangle$ where F is a set of function symbols, P is a set of relation symbols, and τ is a function assigning to each symbol a natural number corresponding to its *arity*. The elements of F with arity 0 are the *constants*. With respect to the language of $\mathscr{L}(Æ)$ we follow Keisler [15] for the choice of logical symbols of $\mathscr{L}$.

Definition 2.1 Assume fixed a given signature Σ for $\mathscr{L}(Æ)$ and a countably infinite set X of individual variables. *Terms* are generated in the usual way from Σ and X. The set of $\mathscr{L}(Æ)$ formulas over Σ is inductively defined as follows:

- $p(t_1, \ldots, t_n)$ is a formula for every p in P wth arity n and terms $t_1, \ldots, t_n$;
- $t_1 \cong t_2$ is a formula for all terms t_1 and t_2;
- $\neg\varphi$, $\forall x\varphi$, $\exists x\varphi$, Æ$x\varphi$, and Œ$x\varphi$ are formulas for every formula φ;
- $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \vee \varphi_2$ are formulas for all formulas φ_1 and φ_2. $\triangle$

Notation 2.2 Given a formula φ of $\mathscr{L}(Æ)$, a term t and a variable x, when t is free for x in φ, we denote by $[\varphi]^x_t$ the formula obtained by replacing the free occurrences of x in φ by t. $\diamond$

The following notation will be useful in the sequel, namely to simplify the presentation of deductive systems and the definition of the consistency property.

Notation 2.3 Given a formula φ of $\mathscr{L}(Æ)$, the formula $\varphi^\neg$ is inductively defined as follows:

- if φ is atomic then $\varphi^\neg$ is $\neg\varphi$;
- $(\neg\varphi_1)^\neg$ is φ_1;
- $(\varphi_1 \wedge \varphi_2)^\neg$ is $\neg\varphi_1 \vee \neg\varphi_2$;
- $(\varphi_1 \vee \varphi_2)^\neg$ is $\neg\varphi_1 \wedge \neg\varphi_2$;
- $(\forall x\varphi)^\neg$ is $\exists x(\neg\varphi)$;
- $(\exists x\varphi)^\neg$ is $\forall x(\neg\varphi)$;
- (Æ$x\varphi)^\neg$ is Œ$x(\neg\varphi)$;
- (Œ$x\varphi)^\neg$ is Æ$x(\neg\varphi)$. $\diamond$

The structures for $\mathscr{L}(\text{Æ})$ are first-order structures enriched with the collection of measure zero sets of a finitely additive measure over its domain. We impose that the domain has nonzero measure.

Definition 2.4 An (*interpretation*) *structure* I for $\mathscr{L}(\text{Æ})$ over a signature Σ is a tuple $\langle D, \cdot^F, \cdot^P, \mathcal{N}\rangle$ where:

- D is a non-empty set;
- $\langle D, \cdot^F, \cdot^P\rangle$ is a first-order structure, that is:
  - for each f in F with arity n, $f^F : D^n \to D$;
  - for each p in P with arity n, $p^P : D^n \to \{0, 1\}$;
- $\mathcal{N}$ is a non-empty set of subsets of D closed under finite union, with D not in $\mathcal{N}$. $\triangle$

The closure under finite union of the elements of $\mathcal{N}$ is required due to the finite additive character of underlying measures.

Definition 2.5 *Satisfaction* in a structure I for $\mathscr{L}(\text{Æ})$ over a signature Σ, given an assignment ρ over I, is defined in the usual way as for first-order logic, with the following extra clause:

$$I\rho \Vdash \text{Æ}x\varphi \text{ if there is } N \in \mathcal{N} \text{ such that } \left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N$$

where $|\varphi|^x_{I\rho}$ (*the extent of φ relative to x in I with assignment ρ*) is defined by[1]

$$|\varphi|^x_{I\rho} = \{d \in D \mid I\rho^x_d \Vdash \varphi\}.$$

Similarly for satisfaction of the quantifier $\text{Ȿ}x\varphi$ which is defined as for $\neg\text{Æ}x\neg\varphi$, that is,

$$I\rho \Vdash \text{Ȿ}x\varphi \text{ if } |\varphi|^x_{I\rho} \not\subseteq N \text{ for each } N \in \mathcal{N}.$$

Validity and *entailment* in $\mathscr{L}(\text{Æ})$ over Σ, denoted by $\vDash_{\mathscr{L}(\text{Æ}),\Sigma}$, are defined as expected, that is, $\Gamma \vDash_{\mathscr{L}(\text{Æ}),\Sigma} \varphi$ whenever $I \Vdash \Gamma$ implies $I \Vdash \varphi$ for every structure I for $\mathscr{L}(\text{Æ})$ over Σ. $\triangle$

When there is no ambiguity we may simply write $\vDash_\Sigma$ or $\vDash$ for entailment and omit the reference to $\mathscr{L}(\text{Æ})$. Intuitively $\text{Ȿ}x\varphi$ means that φ holds significantly with respect to x or that φ is relevant with respect to x.

For the sake of an example, consider an interpretation structure for $\mathscr{L}(\text{Æ})$ over a signature with only one binary predicate symbol e, and where the domain consists of the natural numbers, e is interpreted as the edge relation of a graph, and the null sets are the finite subsets of natural numbers. Then $\text{Æ}x\varphi$

[1] Throughout this paper, given a variable x in X, an assignment ρ over a structure I, and an element d in the domain of I, the assignment ρ^x_d takes x to d and behaves as ρ elsewhere.

means that the extent of φ is cofinite, i.e. its complement is finite. Also, $\text{SE}x\varphi$ means that the extent of φ is infinite. In this context, the sentence stating that there are infinitely many vertices with finite degree can be expressed by $\text{SE}x\text{Æ}y\neg e(x,y)$.

Observe that $\mathscr{L}(\text{Æ})$ is a conservative extension of first-order logic and so inherits its expressive power, see Proposition 2.7. We omit the proof of Lemma 2.6, since it follows straightforwardly by induction on φ.

Lemma 2.6 Given a first-order formula φ over a signature Σ, a structure $\langle D, \cdot^F, \cdot^P, \mathcal{N}\rangle$ for $\mathscr{L}(\text{Æ})$ over Σ, and an assignment ρ over the structure, $\langle D, \cdot^F, \cdot^P\rangle\rho \Vdash \varphi$ if and only if $\langle D, \cdot^F, \cdot^P, \mathcal{N}\rangle\rho \Vdash \varphi$. $\diamond$

In the proof of the following proposition we denote entailment in first-order logic over a signature Σ by $\vDash_{\mathscr{L},\Sigma}$.

Proposition 2.7 Entailment in $\mathscr{L}(\text{Æ})$ for first-order formulas coincides with entailment in first-order logic.

Proof. Let $\Gamma \cup \{\varphi\}$ be a set of first-order formulas over a signature Σ.
(1) Suppose that $\Gamma \vDash_{\mathscr{L}(\text{Æ}),\Sigma} \varphi$. Let $\langle D, \cdot^F, \cdot^P\rangle$ be a first-order structure over Σ, and ρ an assignment over $\langle D, \cdot^F, \cdot^P\rangle$ such that $\langle D, \cdot^F, \cdot^P\rangle\rho \Vdash \Gamma$. Observe that $\langle D, \cdot^F, \cdot^P, \{\emptyset\}\rangle$ is a structure for $\mathscr{L}(\text{Æ})$ over Σ. So $\langle D, \cdot^F, \cdot^P, \{\emptyset\}\rangle\rho \Vdash \Gamma$ by Lemma 2.6. Then $\langle D, \cdot^F, \cdot^P, \{\emptyset\}\rangle\rho \Vdash \varphi$ since $\Gamma \vDash_{\mathscr{L}(\text{Æ}),\Sigma} \varphi$. Hence $\langle D, \cdot^F, \cdot^P\rangle\rho \Vdash \varphi$ by Lemma 2.6, as we wanted to show;
(2) Suppose that $\Gamma \vDash_{\mathscr{L},\Sigma} \varphi$. Let $\langle D, \cdot^F, \cdot^P, \mathcal{N}\rangle$ be a structure for $\mathscr{L}(\text{Æ})$ over Σ, and ρ an assignment over $\langle D, \cdot^F, \cdot^P, \mathcal{N}\rangle$ such that $\langle D, \cdot^F, \cdot^P, \mathcal{N}\rangle\rho \Vdash \Gamma$. Then $\langle D, \cdot^F, \cdot^P\rangle\rho \Vdash \Gamma$ by Lemma 2.6. So $\langle D, \cdot^F, \cdot^P\rangle\rho \Vdash \varphi$ since $\Gamma \vDash_{\mathscr{L},\Sigma} \varphi$. Hence $\langle D, \cdot^F, \cdot^P, \mathcal{N}\rangle\rho \Vdash \varphi$ by Lemma 2.6, as we wanted to show. $\square$

The following proposition, which shows the validity of some formulas of $\mathscr{L}(\text{Æ})$ and establishes some entailments in this logic, is useful not only to better understand the logic at hand, but also when showing that the axiomatization for $\mathscr{L}(\text{Æ})$, proposed in Definition 2.9, is sound.

Proposition 2.8 In the context of $\mathscr{L}(\text{Æ})$, the following formulas are valid:

1. $(\forall x(\varphi \Rightarrow \psi)) \Rightarrow ((\text{Æ}x\varphi) \Rightarrow \text{Æ}x\psi)$;
2. $(\text{Æ}x(\varphi \Rightarrow \psi)) \Leftrightarrow (\varphi \Rightarrow \text{Æ}x\psi)$ where x does not occur free in φ;
3. $(\text{Æ}x\varphi) \Rightarrow \text{Æ}y[\varphi]^x_y$ where y does not occur free in φ and is free for x in φ;
4. $\text{Æ}x(x \cong x)$;
5. $((\text{Æ}x\varphi_1) \wedge (\text{Æ}x\varphi_2)) \Rightarrow \text{Æ}x(\varphi_1 \wedge \varphi_2)$;
6. $(\text{Æ}x\varphi) \Rightarrow \text{SE}x\varphi$;

7. $(\text{S\!\!E} x\varphi) \Rightarrow \exists x\varphi$;

8. $(\text{\AE} x\varphi) \Leftrightarrow \neg\, \text{S\!\!E} x \neg\, \varphi$;

9. $((\text{S\!\!E} x_1\varphi_1) \wedge (\text{\AE} x_2\varphi_2) \wedge (\text{\AE} x_3\varphi_3)) \Rightarrow \exists x([\varphi_1]^{x_1}_x \wedge [\varphi_2]^{x_2}_x \wedge [\varphi_3]^{x_3}_x)$ where x does not occur free in $\text{S\!\!E} x_1\varphi_1$, $\text{\AE} x_2\varphi_2$ and $\text{\AE} x_3\varphi_3$ and is free for x_1, x_2 and x_3 in φ_1, φ_2 and φ_3 respectively;

and the following entailments hold:

1. $\varphi, \varphi \Rightarrow \psi \models \psi$;

2. $\psi \Rightarrow \varphi \models \psi \Rightarrow \forall x\varphi$ whenever x does not occur free in ψ;

3. $\varphi \models \text{\AE} x\varphi$.

Proof. Let I be a structure for $\mathscr{L}(\text{\AE})$ and ρ an assignment over I. We just prove that the formulas are valid since the proof that the entailments hold is similar. In fact:

1. Suppose that $I\rho \Vdash \forall x(\varphi \Rightarrow \psi)$ and $I\rho \Vdash \text{\AE} x\varphi$. Let N in $\mathcal{N}$ be such that $\left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N$. Then

$$\begin{aligned}
\left(|\psi|^x_{I\rho}\right)^{\mathsf{c}} &= \{d \mid I\rho^x_d \not\Vdash \psi\}\\
&= \{d \mid I\rho^x_d \not\Vdash \psi \text{ and } I\rho^x_d \Vdash \varphi\} \cup \{d \mid I\rho^x_d \not\Vdash \psi \text{ and } I\rho^x_d \not\Vdash \varphi\}\\
&\subseteq \{d \mid I\rho^x_d \not\Vdash \varphi \Rightarrow \psi\} \cup \{d \mid I\rho^x_d \not\Vdash \varphi\}\\
&= \emptyset \cup \left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}}\\
&= \left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}}\\
&\subseteq N
\end{aligned}$$

and so $I\rho \Vdash \text{\AE} x\psi$.

2. Suppose that $I\rho \Vdash \text{\AE} x(\varphi \Rightarrow \psi)$ and $I\rho \Vdash \varphi$ where x does not occur free in φ. Let N in $\mathcal{N}$ be such that $\left(|\varphi \Rightarrow \psi|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N$. Then

$$\begin{aligned}
\left(|\psi|^x_{I\rho}\right)^{\mathsf{c}} &= \{d \mid I\rho^x_d \not\Vdash \psi\}\\
&= D \cap \{d \mid I\rho^x_d \not\Vdash \psi\}\\
&= \{d \mid I\rho^x_d \Vdash \varphi\} \cap \{d \mid I\rho^x_d \not\Vdash \psi\}\\
&= \{d \mid I\rho^x_d \Vdash \varphi \text{ and } I\rho^x_d \not\Vdash \psi\}\\
&= \{d \mid I\rho^x_d \not\Vdash \varphi \Rightarrow \psi\}\\
&= \left(|\varphi \Rightarrow \psi|^x_{I\rho}\right)^{\mathsf{c}}\\
&\subseteq N
\end{aligned}$$

and so $I\rho \Vdash \text{\AE} x\psi$. For the other direction suppose that $I\rho \Vdash \varphi \Rightarrow (\text{\AE} x\psi)$ where x does not occur free in φ. Consider two cases:

a) $I\rho \Vdash \varphi$. Let N in $\mathcal{N}$ be such that $\left(|\psi|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N$. Then $I\rho \Vdash \text{\AE} x(\varphi \Rightarrow$

ψ) since $\left(|\psi|^x_{I\rho}\right)^{\mathsf{c}} = \left(|\varphi \Rightarrow \psi|^x_{I\rho}\right)^{\mathsf{c}}$ as we showed above;

b) $I\rho \not\Vdash \varphi$. Let N be a set in $\mathcal{N}$. Then

$$\begin{aligned}\left(|\varphi \Rightarrow \psi|^x_{I\rho}\right)^{\mathsf{c}} &= \{d \mid I\rho^x_d \not\Vdash \varphi \Rightarrow \psi\} \\ &= \{d \mid I\rho^x_d \Vdash \varphi \text{ and } I\rho^x_d \not\Vdash \psi\} \\ &= \emptyset \\ &\subseteq N\end{aligned}$$

and so $I\rho \Vdash \text{Æ}x(\varphi \Rightarrow \psi)$;

3. Assume that $I\rho \Vdash \text{Æ}x\varphi$. Denote by N the set in $\mathcal{N}$ such that $\left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N$. The results follows since $|\varphi|^x_{I\rho} = |[\varphi]^x_y|^y_{I\rho}$ (observe that y does not occur free in φ and is free for x in φ).

4. Observe that $|x \cong x|^x_{I\rho} = D$ and so $\left(|x \cong x|^x_{I\rho}\right)^{\mathsf{c}} = \emptyset$. Since $\mathcal{N}$ is non empty then there is a set in it containing $\left(|x \cong x|^x_{I\rho}\right)^{\mathsf{c}}$. Therefore $I\rho \Vdash \text{Æ}x(x \cong x)$.

5. Assume that $I\rho \Vdash (\text{Æ}x\varphi_1) \wedge (\text{Æ}x\varphi_2)$. Denote by N_1 and N_2 the sets in $\mathcal{N}$ such that $\left(|\varphi_1|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N_1$ and $\left(|\varphi_2|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N_2$. Since $\left(|\varphi_1 \wedge \varphi_2|^x_{I\rho}\right)^{\mathsf{c}} = \left(|\varphi_1|^x_{I\rho} \cap |\varphi_2|^x_{I\rho}\right)^{\mathsf{c}} = \left(|\varphi_1|^x_{I\rho}\right)^{\mathsf{c}} \cup \left(|\varphi_2|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N_1 \cup N_2 \in \mathcal{N}$ because $\mathcal{N}$ is closed under finite union, we conclude that $I\rho \Vdash \text{Æ}x(\varphi_1 \wedge \varphi_2)$.

6. Suppose that $I\rho \Vdash \text{Æ}x\varphi$. Then $\left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N$ for some set N in $\mathcal{N}$. Observe that $(N)^{\mathsf{c}}$ is not contained in any set in $\mathcal{N}$, because otherwise D would be in $\mathcal{N}$. Since $(N)^{\mathsf{c}} \subseteq \left(\left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}}\right)^{\mathsf{c}} = |\varphi|^x_{I\rho}$, then there is no set in $\mathcal{N}$ in which $|\varphi|^x_{I\rho}$ is contained. Hence $I\rho \Vdash \text{Ꞩ}x\varphi$.

7. Suppose that $I\rho \Vdash \text{Ꞩ}x\varphi$. Observe that $|\varphi|^x_{I\rho}$ is not $\emptyset$ since otherwise it would be contained in any set N in $\mathcal{N}$. So $I\rho \Vdash \exists x\varphi$.

8. Observe that $I\rho \Vdash \text{Æ}x\varphi$ if and only if there is $N \in \mathcal{N}$ with $\left(|\varphi|^x_{I\rho}\right)^{\mathsf{c}} \subseteq N$ if and only if there is $N \in \mathcal{N}$ with $|\neg\varphi|^x_{I\rho} \subseteq N$ if and only if $I\rho \not\Vdash \text{Ꞩ}x \neg\varphi$ if and only if $I\rho \Vdash \neg\text{Ꞩ}x\neg\varphi$.

9. Suppose that $I\rho \Vdash \text{Ꞩ}x_1\varphi_1$, $I\rho \Vdash \text{Æ}x_2\varphi_2$ and $I\rho \Vdash \text{Æ}x_3\varphi_3$ where x does not occur free in $\text{Ꞩ}x_1\varphi_1$, $\text{Æ}x_2\varphi_2$ and $\text{Æ}x_3\varphi_3$ and is free for x_1, x_2 and x_3 in φ_1, φ_2 and φ_3 respectively, and let N_1 and N_2 in $\mathcal{N}$ be such that $\left(|\varphi_2|^{x_2}_{I\rho}\right)^{\mathsf{c}} \subseteq N_1$ and $\left(|\varphi_3|^{x_3}_{I\rho}\right)^{\mathsf{c}} \subseteq N_2$ and so $(N_1)^{\mathsf{c}} \subseteq |[\varphi_2]^{x_2}_x|^x_{I\rho}$ and $(N_2)^{\mathsf{c}} \subseteq |[\varphi_3]^{x_3}_x|^x_{I\rho}$. Observe that $|\varphi_1|^{x_1}_{I\rho} \not\subseteq N_1 \cup N_2$ since $I\rho \Vdash \text{Ꞩ}x_1\varphi_1$. Hence $|[\varphi_1]^{x_1}_x|^x_{I\rho} \cap (N_1 \cup N_2)^{\mathsf{c}} \neq \emptyset$ and so $|[\varphi_1]^{x_1}_x|^x_{I\rho} \cap |[\varphi_2]^{x_2}_x|^x_{I\rho} \cap |[\varphi_3]^{x_3}_x|^x_{I\rho} \neq \emptyset$. $\square$

Notice that removing the requirement that D is not in $\mathcal{N}$ in the definition of structure for $\mathscr{L}(Æ)$, Definition 2.4, affects the validity of the formula presented in item 6. The non-emptiness of $\mathcal{N}$ is important for the validity of items 2, 4 and 7.

We now provide an Hilbert-style axiomatization for $\mathscr{L}(Æ)$, denoted by $\mathcal{H}(Æ)$, obtained by enriching an axiomatization for $\mathscr{L}$ (the axioms and rules for first-order logic with equality in the axiomatization for $\mathscr{L}_{\omega_1\omega}$ proposed in [15]) with three axioms for the almost-everywhere quantifier.

Definition 2.9 The axiom system $\mathcal{H}(Æ)$ is composed of the axioms:

TAUT: All instances of propositional tautologies;

∀i: $(\forall x\varphi) \Rightarrow [\varphi]^x_t$ whenever t is free for x in φ;

Ne: $\neg\varphi \Leftrightarrow \varphi^{\neg}$;

Es: $\forall y\forall x((x \cong y) \Rightarrow (y \cong x))$;

Er: $\forall x(x \cong x)$;

Ea: $\forall x((\varphi \wedge (x \cong t)) \Rightarrow [\varphi]^x_t)$ whenever t is free for x in φ;

plus:

Æe: $(Æx\varphi) \Rightarrow \exists x\varphi$;

Ɛe: $(Ɛx\varphi) \Rightarrow \exists x\varphi$;

Ɛf: $((Ɛx_1\varphi_1) \wedge (Æx_2\varphi_2) \wedge (Æx_3\varphi_3)) \Rightarrow \exists x([\varphi_1]^{x_1}_x \wedge [\varphi_2]^{x_2}_x \wedge [\varphi_3]^{x_3}_x)$ where x does not occur free in $Ɛx_1\varphi_1$, $Æx_2\varphi_2$ and $Æx_3\varphi_3$ and is free for x_1, x_2 and x_3 in φ_1, φ_2 and φ_3 respectively;

and of the following inference rules:

MP: $\dfrac{\varphi_1 \quad \varphi_1\Rightarrow\varphi_2}{\varphi_2}$;

∀Gen: $\dfrac{\psi\Rightarrow\varphi}{\psi\Rightarrow\forall x\varphi}$ whenever x does not occur free in ψ. △

Given a signature Σ, the *derivation*, in the context of $\mathscr{L}(Æ)$ and Σ, of a formula φ from a set of formulas Γ, is defined in the usual way, and denoted by $\Gamma \vdash_\Sigma \varphi$. We say that a formula φ is a *theorem of $\mathscr{L}(Æ)$ in the context of* Σ whenever $\vdash_\Sigma \varphi$. When there is no ambiguity we omit the reference to the signature and simply say that φ *is derived* from Γ, denoted by $\Gamma \vdash \varphi$, or if it is the case, that φ is a *theorem*.

We start by noting that the *Deduction Theorem* still holds, under some conditions, in this enriched context of first-order logic with the quantifier Æ. For that we need to recall the following notions.

Given a set Γ of formulas of $\mathscr{L}(\text{Æ})$, $\gamma \in \Gamma$, and a derivation $\varphi_0 \ldots \varphi_n$ from the set of hypothesis Γ, a formula φ_i in the derivation is said to *depend* on the hypothesis γ if: either φ_i is γ; or φ_i is obtained by applying $\forall$Gen to a formula φ_j which depends on γ; or φ_i is obtained by applying MP to formulas φ_j and φ_k, and at least one of these depends on γ. An application of $\forall$Gen in a derivation is said to be an *essential generalization over a dependent of* γ if $\forall$Gen is applied to a formula that depends on γ and the variable being generalized occurs free in γ.

Proposition 2.10 Given a signature Σ and a set $\Gamma \cup \{\varphi, \psi\}$ of formulas of $\mathscr{L}(\text{Æ})$ over Σ, if there is a derivation for $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ where no essential generalizations were made over dependents of ψ, then $\Gamma \vdash_\Sigma \psi \Rightarrow \varphi$.

Proof.
Assume that there is a derivation for $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ where no essential generalizations were made over dependents of ψ. The proof follows by complete induction on the length of the derivation. Base. Then, the length of the derivation is 1 and one of the following cases occur:

(1) φ is ψ. Then $\Gamma \vdash_\Sigma \psi \Rightarrow \varphi$;

(2) φ either is in Γ or is an instance of an axiom. Then $\Gamma \vdash_\Sigma \varphi$. Since $\Gamma \vdash_\Sigma \varphi \Rightarrow (\psi \Rightarrow \varphi)$ then $\Gamma \vdash_\Sigma \psi \Rightarrow \varphi$.

Step. Assume that the length of the derivation is $n+1$ where n is not 0. The proof follows by case analysis:

(a) φ is the conclusion of an application of MP. Let φ' be such that φ results from the application of MP to $\varphi' \Rightarrow \varphi$ and φ'. Then, there are a derivation for $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi' \Rightarrow \varphi$ and a derivation for $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi'$, both with length less than $n+1$ and such that no essential generalizations were made over dependents of ψ. So, by the induction hypothesis $\Gamma \vdash_\Sigma \psi \Rightarrow (\varphi' \Rightarrow \varphi)$ and $\Gamma \vdash_\Sigma \psi \Rightarrow \varphi'$. Hence $\Gamma \vdash_\Sigma \psi \Rightarrow \varphi$ by TAUT and MP;

(b) φ is the conclusion of an application of $\forall$Gen. Then either (i) φ does not depend on ψ or (ii) the variable being generalized does not occur free in ψ. Assume (i) holds. Then $\Gamma \vdash_\Sigma \varphi$ and so the result follows by MP since $\Gamma \vdash_\Sigma \varphi \Rightarrow (\psi \Rightarrow \varphi)$. Assume now that (ii) holds and φ is $\varphi' \Rightarrow \forall x \varphi''$ where x does not occur free in φ'. Then by the induction hypothesis and then $\forall$Gen $\Gamma \vdash_\Sigma \psi \Rightarrow (\varphi' \Rightarrow \forall x \varphi'')$. □

Note that this form of the Deduction Theorem also holds for first-order logic, see [24], and is more general than the usual one found in the literature since neither ψ nor φ need to be a sentence. The difference between the proof of Proposition 2.10 and the proof of the usual form of the deduction theorem for first-order logic, is in case (b). In the proof of the usual form it follows immediately by applying the induction hypothesis and, then, since x does not occur free in ψ, $\forall$Gen.

In the sequel we also use the following admissible form of $\forall$Gen,

$$\text{if} \quad \Gamma \vdash_\Sigma \varphi \quad \text{then} \quad \Gamma \vdash_\Sigma \forall x \varphi.$$

Proposition 2.11 The following formulas are theorems of $\mathscr{L}(\text{Æ})$:

1. $(\forall x(\varphi \Rightarrow \psi)) \Rightarrow ((\text{Æ} x\varphi) \Rightarrow \text{Æ} x\psi)$;
2. $((\text{Æ} x\varphi_1) \wedge (\text{Æ} x\varphi_2)) \Rightarrow \text{Æ} x(\varphi_1 \wedge \varphi_2)$;
3. $(\varphi \Rightarrow \text{Æ} x\psi) \Rightarrow \text{Æ} x(\varphi \Rightarrow \psi)$ where x does not occur free in φ;
4. $(\text{Æ} x\varphi) \Rightarrow \text{Æ} y[\varphi]^x_y$ with y not occurring free in $\text{Æ} x\varphi$ and free for x in φ;
5. $(\text{SE} x\varphi) \Leftrightarrow \neg \text{Æ} x \neg \varphi$;
6. $\text{Æ} x(x \cong x)$.

Proof.
1. We show first that $\forall x(\varphi_1 \Rightarrow \varphi_2), \text{Æ} x\varphi_1, \neg \text{Æ} x\varphi_2 \vdash \bot$. Indeed:

1.	$\forall x(\varphi_1 \Rightarrow \varphi_2)$	Hyp
2.	$\text{Æ} x\varphi_1$	Hyp
3.	$\neg \text{Æ} x\varphi_2$	Hyp
4.	$\text{SE} x \neg \varphi_2$	Ne + TAUT + MP 3
5.	$\exists x(\varphi_1 \wedge \neg \varphi_2)$	SEf + TAUT + MP 2, 4
6.	$\neg \forall x(\varphi_1 \Rightarrow \varphi_2)$	Ne + TAUT + MP 5
7.	$\bot$	TAUT + MP 1, 6

and so $\vdash (\forall x(\varphi_1 \Rightarrow \varphi_2)) \Rightarrow ((\text{Æ} x\varphi_1) \Rightarrow \text{Æ} x\varphi_2)$.

2. Observe that

$$\text{Æ} x\varphi_1, \text{Æ} x\varphi_2, \neg\text{Æ} x(\varphi_1 \wedge \varphi_2) \vdash \bot$$

since

$$\text{Æ} x\varphi_1, \text{Æ} x\varphi_2, \neg\text{Æ} x(\varphi_1 \wedge \varphi_2) \vdash \exists x((\neg\varphi_1 \vee \neg\varphi_2) \wedge \varphi_1 \wedge \varphi_2)$$

by SEf, and

$$\exists x((\neg\varphi_1 \vee \neg\varphi_2) \wedge \varphi_1 \wedge \varphi_2) \vdash \bot.$$

3. We show first that

$$\neg \varphi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \bot \quad \text{and} \quad \text{Æ} x\psi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \bot$$

where x does not occur free in φ. For $\neg \varphi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \bot$ note that $\neg\varphi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \forall x(\neg\varphi)$ and $\neg\varphi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \exists x\varphi$. With respect to $\text{Æ} x\psi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \bot$ note that $\text{Æ} x\psi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \exists x(\varphi \wedge \neg \psi \wedge \psi)$. Hence $\varphi \Rightarrow \text{Æ} x\psi, \neg\text{Æ} x(\varphi \Rightarrow \psi) \vdash \bot$ and so $\vdash (\varphi \Rightarrow \text{Æ} x\psi) \Rightarrow \neg\text{Æ} x(\varphi \Rightarrow \psi)$.

4. We show first that $\text{Æ} x\varphi, \neg\text{Æ} y[\varphi]^x_y \vdash \bot$ where y does not occur free in $\text{Æ} x\varphi$ and is free for x in φ. Indeed:

1.	$\text{Æ} x\varphi$	Hyp
2.	$\neg\text{Æ} y[\varphi]^x_y$	Hyp
3.	$\text{SE} y\neg[\varphi]^x_y$	Ne + TAUT + MP 2
4.	$\exists y(\neg[[\varphi]^x_y]^y_y \wedge [\varphi]^x_y)$	SEf + TAUT + MP 1, 3
5.	$\bot$	TAUT + MP 4

and so $\vdash (\text{Æ}x\varphi) \Rightarrow \text{Æ}y[\varphi]^x_y$.

5. The result follows straightforwardly taking into account that $\text{Ȿ}x\varphi, \text{Æ}x \neg \varphi \vdash \bot$ by using ȾEf, and $\neg \text{Æ}x \neg \varphi, \neg \text{Ȿ}x\varphi \vdash \bot$ by using Ne and ȾEf.

6. Observe that $\neg \text{Æ}x(x \cong x) \vdash \bot$ since $\neg \text{Æ}x(x \cong x) \vdash \exists x \neg (x \cong x)$ by ȾEe. So $\vdash \text{Æ}x(x \cong x)$. □

As could be expected from the definition of $\mathcal{H}(\text{Æ})$ in Definition 2.9, there is a close relationship between the Æ quantifier and the ∀ quantifier, as we see in Proposition 2.12.

Proposition 2.12 The following formulas are theorems of $\mathscr{L}(\text{Æ})$:

1. $\forall x\varphi \Rightarrow \text{Æ}x\varphi$;
2. $(\text{Æ}x(\varphi \Rightarrow \psi)) \Rightarrow ((\text{Æ}x\varphi) \Rightarrow \text{Æ}x\psi)$.

Proof.
1. Indeed:

1.	$\forall x\varphi$	Hyp
2.	φ	∀i 1
3.	$\varphi \Rightarrow (x \cong x \Rightarrow \varphi)$	TAUT
4.	$x \cong x \Rightarrow \varphi$	MP $2, 3$
5.	$\forall x((x \cong x) \Rightarrow \varphi)$	∀Gen 4
6.	$\forall x((x \cong x) \Rightarrow \varphi) \Rightarrow (\text{Æ}x(x \cong x) \Rightarrow \text{Æ}x\varphi)$	Prop. 2.11, item 1
7.	$\text{Æ}x(x \cong x) \Rightarrow \text{Æ}x\varphi$	MP $5, 6$
8.	$\text{Æ}x(x \cong x)$	Prop. 2.11, item 6
9.	$\text{Æ}x\varphi$	MP $8, 7$

is a derivation for $\forall x\varphi \vdash \text{Æ}x\varphi$. Hence, by the Deduction Theorem, $\vdash \forall x\varphi \Rightarrow \text{Æ}x\varphi$.

2. Note that:

1.	$\text{Æ}x(\varphi \Rightarrow \psi)$	Hyp
2.	$\text{Æ}x\varphi$	Hyp
3.	$(\text{Æ}x(\varphi \Rightarrow \psi)) \Rightarrow (\text{Æ}x\varphi \Rightarrow (\text{Æ}x\varphi \wedge \text{Æ}x(\varphi \Rightarrow \psi)))$	TAUT
4.	$((\text{Æ}x\varphi) \wedge \text{Æ}x(\varphi \Rightarrow \psi)) \Rightarrow \text{Æ}x(\varphi \wedge (\varphi \Rightarrow \psi))$	Prop. 2.11, item 2
5.	$\text{Æ}x(\varphi \wedge (\varphi \Rightarrow \psi))$	MP* $1, 2, 3, 4$
6.	$(\varphi \wedge (\varphi \Rightarrow \psi)) \Rightarrow \psi$	TAUT
7.	$\forall x((\varphi \wedge (\varphi \Rightarrow \psi)) \Rightarrow \psi)$	∀Gen 6
8.	$\forall x((\varphi \wedge (\varphi \Rightarrow \psi)) \Rightarrow \psi) \Rightarrow (\text{Æ}x(\varphi \wedge (\varphi \Rightarrow \psi)) \Rightarrow \text{Æ}x\psi)$	Prop. 2.11, item 1
9.	$\text{Æ}x\psi$	MP* $7, 5, 8$

and so the result follows by applying the Deduction Theorem, twice. □

Note that some steps of the derivation in the proof of Proposition 2.12 may be in fact comprised of several steps. We compress them in just one step in order to make the proof more readable. With the same purpose, from now on, we allow ourselves to use derived rules in our derivations and assume that they consist of just one step.

We now turn our attention to an admissible rule of generalization for the Æ quantifier, called ÆGen rule.

Proposition 2.13 Given a set $\Gamma \cup \{\psi, \varphi\}$ of formulas of $\mathscr{L}(\text{Æ})$ over a signature Σ, if $\Gamma \vdash_\Sigma \psi \Rightarrow \varphi$ and x does not occur free in ψ then $\Gamma \vdash_\Sigma \psi \Rightarrow \text{Æ}x\varphi$.

Proof.
Assume that there is a derivation for $\Gamma \vdash_\Sigma \psi \Rightarrow \varphi$ with length α. Then:

	$\vdots$	
$\alpha.$	$\psi \Rightarrow \varphi$	
$\alpha + 1.$	$\psi \Rightarrow \forall x\varphi$	$\forall$Gen α
$\alpha + 2.$	$(\forall x\varphi) \Rightarrow \text{Æ}x\varphi$	Proposition 2.12, item 1
$\alpha + 3.$	$\psi \Rightarrow \text{Æ}x\varphi$	TAUT + MP $\alpha + 1, \alpha + 2$

is a derivation for $\Gamma \vdash_\Sigma \psi \Rightarrow \text{Æ}x\varphi$. □

We now show that a variant of rule ÆGen, not involving implication, is admissible. We will also denote it by ÆGen.

Proposition 2.14 Given a set $\Gamma \cup \{\varphi\}$ of formulas of $\mathscr{L}(\text{Æ})$ over a signature Σ, if $\Gamma \vdash_\Sigma \varphi$ then $\Gamma \vdash_\Sigma \text{Æ}x\varphi$.

Proof.
Let α be the length of a derivation for $\Gamma \vdash_\Sigma \varphi$. Then:

	$\vdots$	
$\alpha.$	φ	
$\alpha + 1.$	$\forall x\varphi$	$\forall$Gen, α
$\alpha + 2.$	$(\forall x\varphi) \Rightarrow \text{Æ}x\varphi$	Proposition 2.12, item 1
$\alpha + 3.$	$\text{Æ}x\varphi$	MP $\alpha + 1, \alpha + 2$

is a derivation for $\Gamma \vdash_\Sigma \text{Æ}x\varphi$ as we wanted to show. □

Capitalizing on the previous propositions we can show that the following formulas are theorems of $\mathscr{L}(\text{Æ})$.

Proposition 2.15 The following formulas are theorems of $\mathscr{L}(\text{Æ})$:

1. $(\text{Æ}x(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow \text{Æ}x\psi)$ where x does not occur free in φ;

2. $(\text{Æ}x\varphi) \Rightarrow \varphi$ where x does not occur free in φ;

3. $\text{Æ}z((\text{Æ}x\varphi) \Rightarrow [\varphi]^x_z)$ if z is free for x in φ and does not occur free in φ;

4. $(\text{Æ}x\varphi) \Rightarrow \text{SE}x\varphi$.

Proof.

1. In fact:

1.	$\text{Æ}x(\varphi \Rightarrow \psi)$	Hyp
2.	φ	Hyp
3.	$\text{Æ}x\varphi$	Proposition 2.14
4.	$(\text{Æ}x(\varphi \Rightarrow \psi)) \Rightarrow ((\text{Æ}x\varphi) \Rightarrow \text{Æ}x\psi)$	Proposition 2.12, item 2
5.	$\text{Æ}x\psi$	MP* 1, 3, 4

and so the result follows by applying the Deduction Theorem twice.

2. In fact:

1.	$(\text{Æ}x\varphi) \Rightarrow (\exists x\varphi)$	Æe
2.	$\forall x(((\neg\varphi) \wedge (x \cong x)) \Rightarrow (\neg\varphi))$	Ea
3.	$(\neg\varphi) \Rightarrow (\neg\varphi)$	Er + ∀i* + TAUT + MP* 2
4.	$(\neg\varphi) \Rightarrow \forall x(\neg\varphi)$	∀Gen 3
5.	$(\exists x\varphi) \Rightarrow \varphi$	Ne + TAUT* + MP* 4
6.	$(\text{Æ}x\varphi) \Rightarrow \varphi$	TAUT + MP* 1, 5

as we wanted to show.

3. Indeed

1.	$((\text{Æ}x\varphi) \Rightarrow \text{Æ}z[\varphi]^x_z) \Rightarrow \text{Æ}z((\text{Æ}x\varphi) \Rightarrow [\varphi]^x_z)$	Prop. 2.11, item 3
2.	$(\text{Æ}x\varphi) \Rightarrow \text{Æ}z[\varphi]^x_z$	Prop. 2.11, item 4
3.	$\text{Æ}z((\text{Æ}x\varphi) \Rightarrow [\varphi]^x_z)$	MP 1, 2

as we wanted to show.

4. Indeed:

1.	$\text{Æ}x\varphi$	Hyp
2.	$\neg\text{SE}x\varphi$	Hyp
3.	$(\neg\text{SE}x\varphi) \Leftrightarrow \text{Æ}x\neg\varphi$	Ne
4.	$\text{Æ}x\neg\varphi$	MP 2, 3
5.	$(\text{Æ}x\varphi) \wedge \text{Æ}x\neg\varphi$	TAUT + MP 1, 4
6.	$\text{Æ}x(\bot)$	Prop. 2.11 item 2 + TAUT + MP
7.	$\exists x(\bot)$	Æe + MP 6
8.	$\forall x(((\neg\bot) \wedge (x \cong x)) \Rightarrow (\neg\bot))$	Ea
9.	$(\neg\bot) \Rightarrow (\neg\bot)$	Er + ∀i* + TAUT + MP* 8
10.	$(\neg\bot) \Rightarrow \forall x(\neg\bot)$	∀Gen 9
11.	$(\exists x\bot) \Rightarrow \bot$	Ne + TAUT* + MP* 10
12.	$\bot$	MP 7, 11

and the result follows by contradiction. ◊

The next proposition is essential in the proof of the completeness theorem, Theorem 4.2. We omit its proof since it follows by a standard induction on the length of a derivation.

PROPOSITION 2.16 Given a signature Σ, a countable set $\mathcal{C}$ of symbols not in Σ, a set $\Gamma \cup \{\varphi\}$ of formulas of $\mathscr{L}$(Æ) over Σ and a derivation δ with formulas over the signature $\Sigma_{\mathcal{C}}$ obtained by enriching Σ with the elements of $\mathcal{C}$ as constants, for $\Gamma \vdash_{\Sigma_{\mathcal{C}}} \varphi$, then there exists a derivation with the same length as δ for $\Gamma \vdash_{\Sigma} \varphi$.$\diamond$

We end the section showing that the axiomatization $\mathcal{H}$(Æ) proposed in Definition 2.9 is sound.

THEOREM 2.17 Let $\Gamma \cup \{\varphi\}$ be a set of formulas of $\mathscr{L}$(Æ) over a signature Σ. If $\Gamma \vdash_{\Sigma} \varphi$ then $\Gamma \vDash_{\Sigma} \varphi$.

PROOF. The proof follows by complete induction on the length of the derivation. Base. The length is 1. Consider the following cases: (1) φ is in Γ. Then it is immediate to conclude that $\Gamma \vDash_{\Sigma} \varphi$; (2) φ is an instance of an axiom. Then taking into account that valid first-order formulas are also valid in $\mathscr{L}$(Æ), and taking into account Proposition 2.8, we can conclude that $\vDash_{\Sigma} \varphi$ and so that $\Gamma \vDash_{\Sigma} \varphi$. Step. Assume that the length of the derivation is m where m is greater than 1. Then φ is the conclusion of an application of a rule to a finite number $\varphi_1, \ldots, \varphi_m$ of premises. Hence, by the induction hypothesis $\Gamma \vDash_{\Sigma} \varphi_i$ for $i = 1, \ldots, m$ and so, since $\{\varphi_i : i = 1 \ldots, m\} \vDash_{\Sigma} \varphi$, taking into account Proposition 2.8, we can conclude that $\Gamma \vDash_{\Sigma} \varphi$. $\square$

3 The Model Existence Theorem

In the sequel, given a signature Σ and a set $\mathcal{C}$, we denote by $\Sigma_{\mathcal{C}}$ the signature obtained by adding $\mathcal{C}$, as constants, to Σ. Moreover by a *basic term over* Σ *and* $\mathcal{C}$ we mean either a constant symbol in $\Sigma_{\mathcal{C}}$ or a term of the form $f(c_1, \ldots, c_m)$ for $c_1, \ldots, c_m$ in $\mathcal{C}$ and an m-ary function symbol f in Σ. We denote by $T(\Sigma_{\mathcal{C}})$ and $L(\Sigma_{\mathcal{C}})$ the set of basic terms over Σ and $\mathcal{C}$, and the set of the sentences over $\Sigma_{\mathcal{C}}$, respectively.

The completeness and the interpolation proofs rely on the existence of a model for a particular set of sentences. In order to prove the existence of that model we adapt to $\mathscr{L}$(Æ) the notion of *consistency property* previously considered in the literature for instance in the context of $\mathscr{L}_{\omega_1\omega}$ (see [15]).

DEFINITION 3.1 A *consistency property* $\mathcal{S}$ *for* $\mathscr{L}$(Æ) *over signatures* $\Sigma_1, \ldots, \Sigma_n$ *and a countably infinite set* $\mathcal{C}$ of constant symbols not in the signatures, is a collection of sets

$$s = s_{(1)} \cup \ldots \cup s_{(n)}$$

where each $s_{(k)}$ is a subset of $L(\Sigma_{k\mathcal{C}})$ with a finite number of constants of $\mathcal{C}$, such that:

- if s' is a subset of s then s' is in $\mathcal{S}$;
- either φ or $\neg\varphi$ is not in s;

- if $\neg\varphi$ is in s then $s \cup \{\varphi^{\neg}\}$ is in $\mathcal{S}$;
- if $\varphi_1 \wedge \varphi_2$ is in s then $s \cup \{\varphi_i\}$ is in $\mathcal{S}$ for each i in $\{1, 2\}$;
- if $\varphi_1 \vee \varphi_2$ is in s then $s \cup \{\varphi_i\}$ is in $\mathcal{S}$ for some i in $\{1, 2\}$;
- if $\forall x\varphi$ is in s then for all c in $\mathcal{C}$, $s \cup \{[\varphi]^x_c\}$ is in $\mathcal{S}$;
- if $\exists x\varphi$ is in s then for some c in $\mathcal{C}$, $s \cup \{[\varphi]^x_c\}$ is in $\mathcal{S}$;
- if Æ$x\varphi$ is in s then $s \cup \{[\varphi]^x_c\}$ is in $\mathcal{S}$ for some c in $\mathcal{C}$;
- if Œ$x\varphi$ is in s then $s \cup \{[\varphi]^x_c\}$ is in $\mathcal{S}$ for some c in $\mathcal{C}$;
- if Œ$x_1\varphi_1$, Æ$x_2\varphi_2$ and Æ$x_3\varphi_3$ are in $s_{(k)}$ for some k in $\{1, \ldots, n\}$ then $s \cup \{[\varphi_1]^{x_1}_c \wedge [\varphi_2]^{x_2}_c \wedge [\varphi_3]^{x_3}_c\}$ is in $\mathcal{S}$ for some c in $\mathcal{C}$;
- if $c_1 \cong c_2$ is in s and c_1 and c_2 are in $\mathcal{C}$ then $s \cup \{c_2 \cong c_1\}$ is in $\mathcal{S}$;
- for each c in $\mathcal{C}$ and basic term t over Σ_k and $\mathcal{C}$, if $c \cong t$ and $[\varphi]^x_t$ are in s then $s \cup \{[\varphi]^x_c\}$ is in $\mathcal{S}$;
- for each basic term t over Σ_k and $\mathcal{C}$ there is c in $\mathcal{C}$ with $s \cup \{c \cong t\}$ in $\mathcal{S}$.$\triangle$

The consistency property is defined over a collection of signatures instead of just over one signature, since the proof of interpolation considers more than one signature and it is important to not mix symbols of the different signatures in the sentences used in the consistency property.

Not all consistency properties are necessarily closed under deduction, but maximal consistency properties are. A consistency property $\mathcal{S}$ for $\mathscr{L}$(Æ) over $\Sigma_1, \ldots, \Sigma_n$ and $\mathcal{C}$, is *maximal*, if

$$s \cup \{\varphi \vee \neg\varphi\} \text{ is in } \mathcal{S}$$

for every s in $\mathcal{S}$ and sentence φ in $L(\Sigma_{1\mathcal{C}}) \cup \ldots \cup L(\Sigma_{n\mathcal{C}})$.

Proposition 3.2 Let $\mathcal{S}$ be a maximal consistency property for $\mathscr{L}$(Æ) over $\Sigma_1, \ldots, \Sigma_n$ and $\mathcal{C}$, s in $\mathcal{S}$, k in $\{1, \ldots, n\}$, δ a derivation $\psi_1 \ldots \psi_m$ with formulas over $\Sigma_{k\mathcal{C}}$ and hypothesis in s, $x_1 \ldots x_a$ a sequence with no repetitions of the free variables of the formulas in δ, and $c_1 \ldots c_a$ a sequence with no repetitions of constants of $\mathcal{C}$ not occurring in δ. Then, for all $i = 1, \ldots, m$,

$$s \cup \{[\psi_i]^{\vec{x}_{\psi_i}}_{\vec{c}_{\psi_i}}\} \in \mathcal{S}$$

where $\vec{x}_{\psi_i}$ is the subsequence of $x_1 \ldots x_a$ with the free variables of ψ_i, and $\vec{c}_{\psi_i}$ is the subsequence of $c_1 \ldots c_a$ corresponding to $\vec{x}_{\psi_i}$.

PROOF.
The proof follows by complete induction on the length m of the derivation:

Base. Let the derivation be ψ. Consider the following cases (we omit the proof of the other cases since they follow either similarly or straightforwardly):

1. ψ is in s. Then ψ is a sentence and $s \cup \{\psi\}$ is s;

2. ψ is an instance of Ne. Assume that ψ is $(\neg\, \text{Ꞩ}x\varphi) \Leftrightarrow \text{Æ}x(\neg\,\varphi)$ and suppose by contradiction that $s \cup \{[\psi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is not in $\mathcal{S}$. Hence $s \cup \{\neg[\psi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is in $\mathcal{S}$ by the maximality of $\mathcal{S}$. Consider two cases:
(a) $s \cup \{\neg((\neg\, \text{Ꞩ}x[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}) \Rightarrow \text{Æ}x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}))\}$ is in $\mathcal{S}$. Therefore $s \cup \{\neg\, \text{Ꞩ}x[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}, \neg\, \text{Æ}x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi})\}$ is in $\mathcal{S}$, and so, by definition of the consistency property, $s \cup \{\text{Æ}x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}), \neg\, \text{Æ}x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}))\}$ is in $\mathcal{S}$, which is a contradiction;
(b) $s \cup \{\neg((\text{Æ}x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi})) \Rightarrow (\neg\, \text{Ꞩ}x[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}))\}$ is in $\mathcal{S}$. Therefore $s \cup \{\text{Æ}x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}), \text{Ꞩ}x[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is in $\mathcal{S}$, and so by definition of the consistency property there is c in $\mathcal{C}$ such that $s \cup \{\exists x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi} \wedge \neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi} \wedge [\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi})\}$ is in $\mathcal{S}$, which leads to a contradiction.

3. ψ is an instance of Æe, that is, ψ is of the form $(\text{Æ}x\varphi) \Rightarrow \exists x\varphi$. Suppose by contradiction that $s \cup \{[\psi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is not in $\mathcal{S}$. Then $s \cup \{\neg[\psi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is in $\mathcal{S}$ by the maximality of $\mathcal{S}$. Hence $s \cup \{\text{Æ}x[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}, \forall x\, \neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is in $\mathcal{S}$, and so, taking into account the condition for the almost-everywhere quantifier in the definition of the consistency property, let c in $\mathcal{C}$ be such that

$$s' = s \cup \{\text{Æ}x[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}, [[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}]^x_c, \forall x\, \neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}\},$$

is in $\mathcal{S}$. Therefore $s' \cup \{\neg[[\varphi]^{\vec{x}_\psi}_{\vec{c}_\psi}]^x_c\}$ is also in $\mathcal{S}$. Contradiction.

4. ψ is an instance of Ꞩf, that is, ψ is of the form $((\text{Ꞩ}x_1\varphi_1) \wedge (\text{Æ}x_2\varphi_2) \wedge (\text{Æ}x_3\varphi_3)) \Rightarrow \exists x([\varphi_1]^{x_1}_x \wedge [\varphi_2]^{x_2}_x \wedge [\varphi_3]^{x_3}_x)$ where x does not occur free in $\text{Ꞩ}x_1\varphi_1$, $\text{Æ}x_2\varphi_2$ and $\text{Æ}x_3\varphi_3$ and is free for x_1, x_2 and x_3 in φ_1, φ_2 and φ_3 respectively. Suppose by contradiction that $s \cup \{[\psi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is not in $\mathcal{S}$. Then, by the maximality of $\mathcal{S}$, $s \cup \{\neg[\psi]^{\vec{x}_\psi}_{\vec{c}_\psi}\}$ is in $\mathcal{S}$, and so $s \cup \{\text{Ꞩ}x_1\varphi_1, \text{Æ}x_2\varphi_2, \text{Æ}x_3\varphi_3, \forall x \neg([\varphi_1]^{x_1}_x \wedge [\varphi_2]^{x_2}_x \wedge [\varphi_3]^{x_3}_x)\}$ is in $\mathcal{S}$. Taking into account the condition for the almost-everywhere quantifier in the definition of the consistency property, let c in $\mathcal{C}$ be such that

$$s \cup \{\text{Ꞩ}x_1\varphi_1, \text{Æ}x_2\varphi_2, \text{Æ}x_3\varphi_3, [\varphi_1]^{x_1}_c, [\varphi_2]^{x_2}_c, [\varphi_3]^{x_3}_c, \forall x \neg([\varphi_1]^{x_1}_x \wedge [\varphi_2]^{x_2}_x \wedge [\varphi_3]^{x_3}_x)\}$$

is in $\mathcal{S}$, and denote this set by s'. This is a contradiction since $s' \cup \{\neg([[\varphi_1]^{x_1}_x]^x_c \wedge [[\varphi_2]^{x_2}_x]^x_c \wedge [[\varphi_3]^{x_3}_x]^x_c)\}$ is not in $\mathcal{S}$;

Step. Consider two cases:

1. ψ_m is obtained by MP from ψ_i and $(\neg\psi_i) \vee \psi_n$ in the derivation, for $i < n$. Then we can use the induction hypothesis to conclude that $s \cup \{[\psi_i]^{\vec{x}_{\psi_i}}_{\vec{c}_i}, \neg[\psi_i]^{\vec{x}_{\psi_i}}_{\vec{c}'_i} \vee [\psi_m]^{\vec{x}_{\psi_m}}_{\vec{c}'_m}\} \in \mathcal{S}$ where $\vec{c}'_i$ and $\vec{c}'_m$ agree on the common free variables of ψ_i and

ψ_m. Hence either $s \cup \{[\psi_i]^{\vec{x}_{\psi_i}}_{\vec{c}_i}, \neg[\psi_i]^{\vec{x}_{\psi_i}}_{\vec{c}_i}\} \in \mathcal{S}$ or $s \cup \{[\psi_i]^{\vec{x}_{\psi_i}}_{\vec{c}_i}, [\psi_m]^{\vec{x}_{\psi_m}}_{\vec{c}_m}\} \in \mathcal{S}$. So $s \cup \{[\psi_m]^{\vec{x}_{\psi_m}}_{\vec{c}_m}\} \in \mathcal{S}$;

2. ψ_m is of the form $\psi' \Rightarrow \forall x \psi''$ where x does not occur free in ψ' and is obtained by $\forall$Gen from $\psi' \Rightarrow \psi''$ in position i in the sequence. Suppose by contradiction that $s \cup \{[\psi']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_m}} \Rightarrow [\forall x \psi'']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_m}}\} \notin \mathcal{S}$. Then $s \cup \{[\psi']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_m}}, [\exists x \neg \psi'']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_m}})\} \in \mathcal{S}$ and let c in $\mathcal{C}$ be such that $s \cup \{[\psi']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_n}}, \neg[[\psi'']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_m}}]^x_c\}$ is in $\mathcal{S}$. Denote by s' the set $s \cup \{\neg[[\psi' \Rightarrow \psi'']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_m}}]^x_c\}$ which is in $\mathcal{S}$. Note that $\psi_1 \ldots \psi_i$ is a derivation for $\psi' \Rightarrow \psi''$ from s'. Then $s' \cup \{[\psi' \Rightarrow \psi'']^{\vec{x}_{\psi_i}}_{\vec{d}_{\psi_i}}\} \in \mathcal{S}$ by the induction hypothesis for any sequence $d_1 \ldots d_a$ denoted by $\vec{d}$ with no repetitions of constants of $\mathcal{C}$ not occurring in $\psi_1 \ldots \psi_i$. So in particular $s' \cup \{[[\psi' \Rightarrow \psi'']^{\vec{x}_{\psi_m}}_{\vec{c}_{\psi_m}}]^x_c\} \in \mathcal{S}$. Contradiction. □

Given a consistency property $\mathcal{S}$ over $\Sigma_1, \ldots, \Sigma_n$ and $\mathcal{C}$, and $\Gamma \cup \{\psi\}$ contained in $L(\Sigma_{1\mathcal{C}}) \cup \ldots \cup L(\Sigma_{n\mathcal{C}})$, we denote by

$$\Gamma \vdash_{\Sigma_1,\ldots,\Sigma_n,\mathcal{C}} \psi$$

a derivation $\psi_1 \ldots \psi_m$ for ψ from Γ where, for every $i = 1, \ldots, m$, each formula ψ_i is in $L(\Sigma_{k_i\mathcal{C}})$ for some k_i in $\{1, \ldots, n\}$. The following theorem is known as *the model existence theorem.*

Theorem 3.3 Every set of sentences in a maximal consistency property for $\mathscr{L}$(Æ) has a model.

Proof.
Let $\mathcal{S}$ be a maximal consistency property for $\mathscr{L}$(Æ) over $\Sigma_1, \ldots, \Sigma_n$ and $\mathcal{C}$, and s a set in $\mathcal{S}$.

Let $t_1, t_2, \ldots$ be an enumeration of all the basic terms in $T(\Sigma_{1\mathcal{C}}) \cup \cdots \cup T(\Sigma_{n\mathcal{C}})$, and $\varphi_1, \varphi_2, \ldots$ an enumeration of all the sentences in $L(\Sigma_{1\mathcal{C}}) \cup \cdots \cup L(\Sigma_{n\mathcal{C}})$. Consider the following sequence $s_0, s_1, \ldots$ of sets in $\mathcal{S}$ inductively defined as follows: s_0 is s, and given s_m in $\mathcal{S}$, s_{m+1} is a set in $\mathcal{S}$ containing: (i) s_m; (ii) $c \cong t_{m+1}$ for some $c \in \mathcal{C}$; and (iii) if $s_m \cup \{\varphi_{m+1}\}$ is in $\mathcal{S}$ then containing

- φ_{m+1};
- ψ_i for some i in $\{1, 2\}$ if φ_{m+1} is $\psi_1 \vee \psi_2$;
- $[\psi]^x_c$ for some c in $\mathcal{C}$ if φ_{m+1} is either $\exists x \psi$ or Æ$x\psi$ or Œ$x\psi$;

and denote by s_ω the set $\bigcup_{m<\omega} s_m$. Observe that every formula in s_ω is a sentence of $\mathscr{L}$(Æ) over some signature $\Sigma_{k\mathcal{C}}$ with a finite number of constants of C. Then, for every m,

$$\text{if } s_i \vdash_{\Sigma_1,\ldots,\Sigma_n,\mathcal{C}} \varphi_m \text{ for some natural number } i \text{ then } \varphi_m \in s_\omega, \qquad (\dagger)$$

and

$$\text{if } \varphi_m \notin s_\omega \text{ then } \neg\varphi_m \in s_\omega. \qquad (\ddagger)$$

In fact:

To show (†), consider the following cases: (i) $i < m$. Observe that $s_i \subseteq s_{m-1}$. Hence $s_{m-1} \vdash \varphi_m$ and so $s_{m-1} \cup \{\varphi_m\}$ is in $\mathcal{S}$ by Proposition 3.2. So $\varphi_m \in s_\omega$; (ii) $i \geq m$. Observe that $s_i \cup \{\varphi_m\}$ is in $\mathcal{S}$ by Proposition 3.2. Since $s_{m-1} \cup \{\varphi_m\}$ is a subset of $s_i \cup \{\varphi_m\}$, then $s_{m-1} \cup \{\varphi_m\}$ is in $\mathcal{S}$;

For the proof of (‡), suppose that $\varphi_m \notin s_\omega$ and that φ_i is $\neg\varphi_m$. Let j be the maximum of i and m. Observe that $s_{j-1} \cup \{\varphi_m\} \notin \mathcal{S}$ and that $s_{j-1} \cup \{\varphi_m \vee \neg\varphi_m\}$ is in $\mathcal{S}$, since $\mathcal{S}$ is maximal. Hence $s_{j-1} \cup \{\neg\varphi_m\}$ is in $\mathcal{S}$ by definition of the consistency property. Therefore $\neg\varphi_m$ is in s_ω.

Consider the binary relation $\sim_\omega$ on $\mathcal{C}$ such that $c_1 \sim_\omega c_2$ if and only if $c_1 \cong c_2 \in s_\omega$. We omit the proof that $\sim_\omega$ is a congruence relation since it follows straightforwardly.

In the sequel, given a basic term t, we denote by c_t the first constant of $\mathcal{C}$ in the enumeration $t_1, t_2, \ldots$ of all the basic terms, such that $c_t \cong t \in s_\omega$. The extension of c_t to all ground terms, denoted by c_t^*, is inductively defined for every ground term as follows: (i) $c_{c'}^*$ is $c_{c'}$; and (ii) $c_{f(t_1,\ldots,t_m)}^*$ is $c_{f(c_{t_1}^*,\ldots,c_{t_m}^*)}$ for every m-ary function symbol f and ground terms $t_1, \ldots, t_m$. In order to simplify the presentation we will also denote c_t^* by c_t.

Consider the following structure $(D, \cdot^F, \cdot^P, \mathcal{N})$, denoted by I_{s_ω}, where

- D is $\mathcal{C}_{\sim_\omega}$, that is, the set $\{[c]_{\sim_\omega} : c \in \mathcal{C}\}$;
- $f^F([c_1]_{\sim_\omega}, \ldots, [c_n]_{\sim_\omega}) = [c_{f(c_1,\ldots,c_n)}]_{\sim_\omega}$;
- $p^P([c_1]_{\sim_\omega}, \ldots, [c_n]_{\sim_\omega}) = 1$ if and only if $p(c_1, \ldots, c_n) \in s_\omega$;
- $\mathcal{N}$ is the collection of all the sets $\{[c]_{\sim_\omega} : \neg[\psi]_c^x \in s_\omega\}$ such that $\text{Æ}x\psi \in s_\omega$.

Then:

(a) $\cdot^F$ and $\cdot^P$ are well defined. We omit the proof of this case since it follows straightforwardly by definition of the consistency property.

(b) $\mathcal{N}$ is closed under finite union. Assume that $\text{Æ}x\psi_1$ and $\text{Æ}x\psi_2$ are in s_ω and let i, j and ℓ be such that φ_i is $\text{Æ}x\psi_1$, φ_j is $\text{Æ}x\psi_2$ and φ_ℓ is $\text{Æ}x(\psi_1 \wedge \psi_2)$. Denote the maximum of i, j and ℓ by m. Observe that $\text{Æ}x\psi_1$ and $\text{Æ}x\psi_2$ are in s_m. Hence $s_m \vdash_{\Sigma_1,\ldots,\Sigma_n,\mathcal{C}} \text{Æ}x(\psi_1 \wedge \psi_2)$ by Proposition 2.11 item 2. Thus $\text{Æ}x(\psi_1 \wedge \psi_2)$ is in s_ω by (†) and so $\{[c]_{\sim_\omega} : \neg[\psi_1 \wedge \psi_2]_c^x \in s_\omega\}$ is in $\mathcal{N}$ by definition of $\mathcal{N}$. Since

$$\{[c]_{\sim_\omega} : \neg([\psi_1]_c^x \wedge [\psi_2]_c^x) \in s_\omega\} = \{[c]_{\sim_\omega} : \neg[\psi_1]_c^x \in s_\omega\} \cup \{[c]_{\sim_\omega} : \neg[\psi_2]_c^x \in s_\omega\}$$

the result follows immediately.

(c) $\mathcal{N}$ is non-empty. Observe that $\text{Æ}x(x \cong x)$ is in s_ω by (†). So $\mathcal{N}$ contains at least the set $\{[c]_{\sim_\omega} : \neg[x \cong x]_c^x \in s_\omega\} = \emptyset$;

(d) $D \notin \mathcal{N}$. Let $\text{Æ}x\psi$ and φ_j be such that $\text{Æ}x\psi \in s_w$ and φ_j is $\text{Æ}x\psi$. Then, by definition of s_j, there is c in $\mathcal{C}$ such that $[\psi]_c^x$ is in s_j. Hence $\{[c]_{\sim_\omega} : \neg[\psi]_c^x \in s_\omega\}$ is not D since otherwise we will arrive to a contradiction.

Let ρ be an assignment over I_{s_ω}. Then:

(1) $[c_{f(c_{t_1},\ldots,c_{t_m})}]_{\sim_\omega} = [c_{f(t_1,\ldots,t_m)}]_{\sim_\omega}$ and $[c']_{\sim_\omega} = [c_{c'}]_{\sim_\omega}$ where $t_1,\ldots,t_m$ are basic terms and c' is in $\mathcal{C}$. Observe that (i) $c_{t_i} \cong t_i \in s_\omega$ for $i = 1,\ldots,n$. Therefore $f(c_{t_1},\ldots,c_{t_m}) \cong f(t_1,\ldots,t_m) \in s_\omega$. So $c \cong f(c_{t_1},\ldots,c_{t_m}) \in s_\omega$ if and only if $c \cong f(t_1,\ldots,t_m) \in s_\omega$; and (ii) $c' \cong c_{c'} \in s_\omega$ by definition of c_t;

(2) $[\![t]\!]^{I_{s_\omega}\rho} = [c_{[t]^{\vec{x}_t}_{\vec{c}_{\rho^{x_t}}}}]_{\sim_\omega}$ for every term t where $\vec{x}_t$ is a sequence $x_1,\ldots,x_m$ with the variables in t, and $\vec{c}_{\rho^{x_t}}$ is the sequence $c_{\rho(x_1)},\ldots,c_{\rho(x_m)}$ where $c_{\rho(x_i)}$ is a constant in $\mathcal{C}$ such that $c_{\rho(x_i)} \in \rho(x_i)$ for $i = 1,\ldots,n$. The proof follows straightforwardly by induction on t;

(3) $I_{s_\omega}\rho \Vdash \psi$ if and only if $[\psi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{x_\psi}}} \in s_\omega$, for any formula ψ in $L(\Sigma_{1\mathcal{C}})\cup\cdots\cup L(\Sigma_{n\mathcal{C}})$ where $\vec{x}_\psi$ is a sequence $x_1,\ldots,x_m$ with the free variables in ψ, and $\vec{c}_{\rho_{x_\psi}}$ is the sequence $c_{\rho(x_1)},\ldots,c_{\rho(x_m)}$ where $c_{\rho(x_i)}$ is a constant in $\mathcal{C}$ such that $c_{\rho(x_i)} \in \rho(x_i)$ for $i = 1,\ldots,n$. The proof follows by induction on ψ:

(i) ψ is $t_1 \cong t_2$. Observe that $I_{s_\omega}\rho \Vdash (t_1 \cong t_2)$ if and only if $[\![t_1]\!]^{I_{s_\omega}\rho} = [\![t_2]\!]^{I_{s_\omega}\rho}$ if and only if by (2) above $[c_{[t_1]^{\vec{x}_{t_1}}_{\vec{c}_{\rho^{x_{t_1}}}}}]_{\sim_\omega} = [c_{[t_2]^{\vec{x}_{t_2}}_{\vec{c}_{\rho^{x_{t_2}}}}}]_{\sim_\omega}$, i.e. if and only if, $c_{[t_1]^{\vec{x}_{t_1}}_{\vec{c}_{\rho^{x_{t_1}}}}} \sim_\omega c_{[t_2]^{\vec{x}_{t_2}}_{\vec{c}_{\rho^{x_{t_2}}}}}$ and since we have $c_{[t_1]^{\vec{x}_{t_1}}_{\vec{c}_{\rho^{x_{t_1}}}}} \sim_\omega [t_1]^{\vec{x}_{t_1}}_{\vec{c}_{\rho^{x_{t_1}}}}$ and $c_{[t_2]^{\vec{x}_{t_2}}_{\vec{c}_{\rho^{x_{t_2}}}}} \sim_\omega [t_2]^{\vec{x}_{t_2}}_{\vec{c}_{\rho^{x_{t_2}}}}$ then $[t_1]^{\vec{x}_{t_1}}_{\vec{c}_{\rho^{x_{t_1}}}} \sim_\omega [t_2]^{\vec{x}_{t_2}}_{\vec{c}_{\rho^{x_{t_2}}}}$, that is, $[t_1]^{\vec{x}_{t_1}}_{\vec{c}_{\rho^{x_{t_1}}}} \cong [t_2]^{\vec{x}_{t_2}}_{\vec{c}_{\rho^{x_{t_2}}}} \in s_\omega$. The statement follows since $[t_1]^{\vec{x}_{t_1}}_{\vec{c}_{\rho^{x_{t_1}}}} \cong [t_2]^{\vec{x}_{t_2}}_{\vec{c}_{\rho^{x_{t_2}}}} \in s_\omega$ if and only if $[t_1 \cong t_2]^{\vec{x}_\psi}_{\vec{c}_{\rho^{x_\psi}}} \in s_\omega$;

(ii) ψ is $\text{Æ}x\varphi$. Then: $(\rightarrow)$ assume that $I_{s_\omega}\rho \Vdash \text{Æ}x\varphi$ and let $\text{Æ}x\psi \in s_\omega$ be such that $\{[c]_{\sim_\omega} : [\psi]^x_c \in s_\omega\} \subseteq |\varphi|^x_{I_{s_\omega}\rho}$. Note that $|\varphi|^x_{I_{s_\omega}\rho} = \{[c]_{\sim_\omega} : I_{s_\omega}\rho^x_{[c]_{\sim_\omega}} \Vdash \varphi\}$ which is equal by the induction hypothesis to $\{[c]_{\sim_\omega} : [\varphi]^{\vec{x}_\varphi}_{\vec{c}_{(\rho^x_{[c]_{\sim_\omega}})^{\vec{x}_\varphi}}} \in s_\omega\} = \{[c]_{\sim_\omega} : [[\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}]^x_c \in s_\omega\}$. Hence if $[\psi]^x_c \in s_\omega$ then $[[\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}]^x_c \in s_\omega$ for all c in $\mathcal{C}$. Observe that $[\psi]^x_c$ has no free variables. We now show that $([\psi]^x_c \Rightarrow [[\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}]^x_c) \in s_\omega$. Consider two cases: (a) $[\psi]^x_c \in s_\omega$. Then the result follows immediately by (†); (b) $[\psi]^x_c \notin s_\omega$. Then $\neg[\psi]^x_c \in s_\omega$ by (‡), and so the result follows by (†). Hence $([\psi \Rightarrow [\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}]^x_c) \in s_\omega$ for all c in $\mathcal{C}$. So $\forall x(\psi \Rightarrow [\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}) \in s_\omega$. Hence $\text{Æ}x([\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}) \in s_\omega$, that is, $[\text{Æ}x\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{x_\psi}}} \in s_\omega$ as we wanted to show; $(\leftarrow)$ assume that $[\text{Æ}x\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{x_\psi}}} \in s_\omega$. Observe that $[\text{Æ}x\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{x_\psi}}} = \text{Æ}x\varphi$ since $\text{Æ}x\varphi$ is a sentence. Then $\{[c]_{\sim_\omega} : [\varphi]^x_c \notin s_\omega\} \in \mathcal{N}$. Note also that $\{[c]_{\sim_\omega} : [\varphi]^x_c \notin s_\omega\}$ is equal by the induction hypothesis to $\{[c]_{\sim_\omega} : I_{s_\omega}\rho^x_{[c]_{\sim_\omega}} \nVdash \varphi\} = \left(|\varphi|^x_{I_{s_\omega}\rho}\right)^{\mathsf{c}}$. Therefore $\left(|\varphi|^x_{I_{s_\omega}\rho}\right)^{\mathsf{c}} \in \mathcal{N}$ and so $I_{s_\omega}\rho \Vdash \text{Æ}x\varphi$;

(iii) ψ is $\exists x\varphi$. Then: $(\rightarrow)$ assume that $I_{s_\omega}\rho \Vdash \exists x\varphi$ and let $[c]_{\sim_\omega}$ be such that $I_{s_\omega}\rho^x_{[c]_{\sim_\omega}} \Vdash \varphi$. Then $[\varphi]^{\vec{x}_\varphi}_{\vec{c}_{(\rho^x_{[c]_{\sim_\omega}})^{\vec{x}_\varphi}}} \in s_\omega$ by the induction hypothesis. That is, $[[\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}]^x_c \in s_\omega$. Observe that $\forall x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}})$ is not in s_ω. Hence, by (‡), $\neg\forall x(\neg[\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}) \in s_\omega$ and so $[\exists x\varphi]^{\vec{x}_\psi}_{\vec{c}_{\rho^{\vec{x}_\psi}}}$ is in s_ω; ($\leftarrow$) the proof of this direction follows similarly taking into account the construction of s_ω.

The remaining cases follow similarly.

So, by (3), every subset of s_ω has a model and so s has a model. □

4 Completeness

Capitalizing on the model existence theorem, Theorem 3.3, we show in this section that $\mathscr{L}$(Æ) is strongly complete. Throughout this section we assume:

- a signature Σ with countably many function symbols;
- a countably infinite set $\mathcal{C}$ of constants not in Σ;

and define $\mathcal{S}_C$ to be the collection of all consistent sets of sentences of $\mathscr{L}$(Æ) over $\Sigma_\mathcal{C}$ with finitely many constants of $\mathcal{C}$.

Proposition 4.1 The collection $\mathcal{S}_C$ is a maximal consistency property over Σ and $\mathcal{C}$.

Proof.
(1) $\mathcal{S}_C$ is a consistency property over Σ and $\mathcal{C}$. We show only the properties relative to generalized quantifiers since the other properties follow similarly:

(a) Let s be a set in $\mathcal{S}_C$ and assume that Æ$x\varphi$ is in s. Suppose by contradiction that, for all c' in $\mathcal{C}$, $s \cup \{[\varphi]^x_{c'}\}$ is not in $\mathcal{S}$, that is, $s \cup \{[\varphi]^x_{c'}\} \vdash \bot$. Let c be a constant of $\mathcal{C}$ not in $s \cup \{\varphi\}$ (observe that $\mathcal{C}$ is infinite and the number of constants of $\mathcal{C}$ occurring in $s \cup \{\varphi\}$ is finite). Hence $s \cup \{[\varphi]^x_c\} \vdash \bot$ and so $s \vdash \neg[\varphi]^x_c$. Let $\varphi_1, \ldots, \varphi_n$ be a derivation for $s \vdash \neg[\varphi]^x_c$ and let y be a variable not occurring, neither bound nor free, in $\varphi_1, \ldots, \varphi_n$. Then $[\varphi_1]^c_y, \ldots, [\varphi_n]^c_y$ is a derivation for $s \vdash \neg[\varphi]^x_y$ and so $s \vdash \forall y \neg[\varphi]^x_y$ which contradicts the fact that s is consistent since $s \vdash \exists x\varphi$;

(b) Let s be a set in $\mathcal{S}_C$ and assume that Œ$x\varphi$ is in s. The proof of this case follows similarly to the proof of case (a) since (Œ$x\varphi$) $\Rightarrow \exists x\varphi$ is an axiom;

(c) Let s be a set in $\mathcal{S}_C$ and assume that Œ$x\varphi$, Æ$x\varphi_1$ and Æ$x\varphi_2$ are in s. Then $s \vdash \exists x(\varphi \wedge \varphi_1 \wedge \varphi_2)$ and so the proof follows similarly to the proof of (a) above and so is omitted;

(2) $\mathcal{S}_C$ is maximal. Let s be a set in $\mathcal{S}_C$ and φ a sentence, and suppose by contradiction that $s \cup \{\varphi \vee \neg\varphi\} \vdash \bot$. Since $\varphi \vee \neg\varphi$ is an instance of a propositional tautology we have that $s \vdash \varphi \vee \neg\varphi$, and so $s \vdash \bot$ contradicting the fact that s is in $\mathcal{S}_C$. □

Using the fact that $\mathcal{S}_C$ is a maximal consistency property, Proposition 4.1, and the model existence theorem, Theorem 3.3, we can now show that $\mathscr{L}$(Æ) is strongly complete.

THEOREM 4.2 Given a set $\Gamma \cup \{\varphi\}$ of sentences of $\mathscr{L}(\text{Æ})$ over a signature Σ, if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.

PROOF. Assume that $\Gamma \nvdash_\Sigma \varphi$. Then $\Gamma \cup \{\neg\varphi\} \nvdash_\Sigma \bot$ and so $\Gamma \cup \{\neg\varphi\} \nvdash_{\Sigma_\mathcal{C}} \bot$ by Proposition 2.16. So $\Gamma \cup \{\neg\varphi\}$ is in $\mathcal{S}_C$ since it is consistent and has no constants of $\mathcal{C}$. Since $\mathcal{S}_C$ is a maximal consistency property over Σ and $\mathcal{C}$ by Proposition 4.1, the set $\Gamma \cup \{\neg\varphi\}$ has a model by Theorem 3.3. Hence $\Gamma \nvDash_\Sigma \varphi$ as we wanted to show. □

5 Interpolation

We now show that $\mathscr{L}(\text{Æ})$ enjoys Craig interpolation for a wide class of formulas. Throughout the section we assume:

- signatures Σ_1 and Σ_2;
- a countable infinite set $\mathcal{C}$ of constants not in $\Sigma_1 \cup \Sigma_2$;

and define $\mathcal{S}_I$ to be the collection of all countable sets $s_{(1)} \cup s_{(2)}$ with a finite number of constants of $\mathcal{C}$ such that:

1. $s_{(i)}$ is a set of sentences over $\Sigma_{i\mathcal{C}}$, for $i = 1, 2$;
2. for all sentences θ_1 and θ_2 over $\Sigma_{1\mathcal{C}} \cap \Sigma_{2\mathcal{C}}$, if $s_{(1)} \vDash \theta_1$ and $s_{(2)} \vDash \theta_2$ then $\theta_1 \wedge \theta_2$ is consistent.

Observe that the notion of maximality of a consistency property over more than one signature, does not say anything with respect to mixed formulas, i.e., to formulas with connectives of different signatures. This is on purpose, since the consistency property introduced above would not be maximal if this notion would apply to all the formulas over $\Sigma_{1\mathcal{C}} \cup \Sigma_{1\mathcal{C}}$.

PROPOSITION 5.1 The collection $\mathcal{S}_I$ is a maximal consistency property over Σ_1, Σ_2 and $\mathcal{C}$.

PROOF.
(1) $\mathcal{S}_I$ is a consistency property for Σ_1, Σ_2 and $\mathcal{C}$. We only show that any subset of a set in $\mathcal{S}_I$ is also contained in $\mathcal{S}_I$, and that $\mathcal{S}$ satisfies the properties for the generalized quantifiers, and refer the reader to [15] for the proof of conditions similar to the remaining ones:

(a) Let s' be a subset of a set s in $\mathcal{S}_I$. Observe that $s'_{(1)} \subseteq s_{(1)}$ and $s'_{(2)} \subseteq s_{(2)}$. Suppose by contradiction that s' is not in $\mathcal{S}_I$ and so let θ'_1 and θ'_2 over $\Sigma_{1\mathcal{C}} \cap \Sigma_{2\mathcal{C}}$ be such that $s'_{(1)} \vDash \theta'_1$, $s'_{(2)} \vDash \theta'_2$ and $\theta'_1 \wedge \theta'_2$ is inconsistent. Then $s_{(1)} \vDash \theta'_1$ and $s_{(2)} \vDash \theta'_2$ which contradicts the fact that s is in $\mathcal{S}_I$;

(b) Let s be a set in $\mathcal{S}_I$ and assume that $\text{Æ}x\varphi$ is in s. Assume with no loss of generality that $\text{Æ}x\varphi$ is in $s_{(1)}$. Suppose by contradiction that $s \cup \{[\varphi]^x_{c'}\} \notin \mathcal{S}_I$ for all c' in $\mathcal{C}$ and let c be a constant of $\mathcal{C}$ not in $s \cup \{\varphi\}$ (observe that $\mathcal{C}$ is infinite and the number of constants of $\mathcal{C}$ occurring in $s \cup \{\varphi\}$ is finite), and let θ_1 and θ_2 be sentences over $\Sigma_{1\mathcal{C}} \cap \Sigma_{2\mathcal{C}}$ such that $s_{(1)} \cup \{[\varphi]^x_c\} \vDash \theta_1$, $s_{(2)} \vDash \theta_2$ and $\theta_1 \wedge \theta_2 \vDash \bot$. Hence $s_{(1)} \cup \{\exists x\varphi\} \vDash \theta_1$. Since $s_{(1)} \vDash \exists x\varphi$ then $s_{(1)} \vDash \theta_1$, which contradicts the fact that s is in $\mathcal{S}_I$;

(c) Let s be a set in $\mathcal{S}_I$ and assume that $\text{S\!E} x\varphi$ is in s. The proof of this case follows similarly to the proof of case (b) since $\vdash (\text{S\!E} x\varphi) \Rightarrow \exists x\varphi$;

(d) Let s be a set in $\mathcal{S}_I$ and assume that $\text{S\!E} x\varphi$, $\text{\AE} x\varphi_1$ and $\text{\AE} x\varphi_2$ are in s. Then $s \vdash \exists x(\varphi \wedge \varphi_1 \wedge \varphi_2)$ and so the proof follows similarly to the proof of (b) above and is omitted;

(2) $\mathcal{S}_I$ is maximal. Let s be a set in $\mathcal{S}_I$ and φ a sentence in $L(\Sigma_{1\mathcal{C}}) \cup L(\Sigma_{e\mathcal{C}})$. Suppose by contradiction that $s \cup \{\varphi \vee \neg\varphi\}$ is not in $\mathcal{S}_I$. Assume with no loss of generality that φ is in $s_{(1)}$ and let θ_1 and θ_2 over $\Sigma_{1\mathcal{C}} \cap \Sigma_{2\mathcal{C}}$ be such that $s_{(1)} \cup \{\varphi \vee \neg\varphi\} \vDash \theta_1$, $s_{(2)} \vDash \theta_2$ and $\theta_1 \wedge \theta_2$ is inconsistent. Since $s_{(1)} \vDash s_{(1)} \cup \{\varphi \vee \neg\varphi\}$, we have that $s_{(1)} \vDash \theta_1$ which contradicts the assumption that s is in $\mathcal{S}_I$. So $s \cup \{\varphi \vee \neg\varphi\}$ is in $\mathcal{S}_I$. $\square$

Finally, we can now establish that $\mathscr{L}(\text{\AE})$ enjoys interpolation.

THEOREM 5.2 Given a set $\Gamma \cup \{\varphi\}$ of sentences over a signature Σ, if $\Gamma \vDash_\Sigma \varphi$ then there exists a sentence θ such that every constant, function and relation symbol occurring in θ occurs both in Γ and in φ, and $\Gamma \vDash_\Sigma \theta$ and $\theta \vDash_\Sigma \varphi$.

PROOF. Let Σ_1 and Σ_2 be the signatures with the constant, function and relation symbols of Σ occurring in Γ and φ respectively. Observe that, by the model existence theorem, Theorem 3.3, each set in $\mathcal{S}_I$ has a model. Note also that $\Gamma \cup \{\neg\varphi\}$ satisfies all the conditions for belonging to $\mathcal{S}_I$ except possibly condition 2. So, if it satisfies that condition then it would have a model by Theorem 3.3. Since $\Gamma \cup \{\neg\,\varphi\}$ does not have a model (because $\Gamma \vDash_\Sigma \varphi$), then it does not satisfy condition 2. So, let θ_1 and θ_2 be sentences over $\Sigma_{1\mathcal{C}} \cap \Sigma_{2\mathcal{C}}$ such that $\Gamma \vDash \theta_1$, $\neg\varphi \vDash \theta_2$ and $\theta_1 \wedge \theta_2$ is inconsistent, that is, $\{\theta_1, \theta_2\} \vDash \bot$. Hence $\theta_1 \vDash \neg\theta_2$ and so $\theta_1 \vDash \varphi$. Let $c_1, \ldots, c_n$ be the constants of $\mathcal{C}$ in θ_1, $y_1, \ldots, y_n$ variables not in θ_1, and θ the sentence $(\forall y_1 \ldots \forall y_n [\ldots [\theta_1]^{c_1}_{y_1} \ldots]^{c_n}_{y_n})$ over $\Sigma_1 \cap \Sigma_2$. Then $\Gamma \vDash \theta$ and $\theta \vDash \varphi$. $\square$

6 Towards Countably Additive Measures

The question now is to extend the almost-everywhere quantifier to countably additive measure structures. In these structures the collection of zero measure sets is closed under countable union and so it seems necessary to go beyond first-order logic as the base logic. In fact, in first-order logic there is no way to even state that the null sets are closed under countable union. One plausible candidate to base logic is the infinitary logic $\mathscr{L}_{\omega_1\omega}$ where it is possible to consider infinite countable conjunctions. So, for the sake of argument, assume $\mathscr{L}_{\omega_1\omega}$ as base logic, and let $\mathscr{L}_{\omega_1\omega}(\text{\AE}_c)$ to be the logic such that:

- the language of $\mathscr{L}_{\omega_1\omega}(\text{Æ}_c)$ is the language of $\mathscr{L}_{\omega_1\omega}$ enriched with the almost-everywhere quantifier Æ_c;
- an interpretation structure I for $\mathscr{L}_{\omega_1\omega}(\text{Æ}_c)$ is a structure for $\mathscr{L}_{\omega_1\omega}$ enriched with a non-empty set $\mathcal{N}$ of subsets of the domain, closed under countable union, and such that the whole domain is not in $\mathcal{N}$;
- satisfaction of a formula whose main constructor is either Æ_c or Œ_c is defined as for $\mathscr{L}(\text{Æ})$ in Definition 2.5, and for the other formulas it is defined as in $\mathscr{L}_{\omega_1\omega}$.

REMARK 6.1 Observe that, given a structure I for $\mathscr{L}_{\omega_1\omega}(\text{Æ}_c)$ over a signature Σ and an assignment ρ over I, if $I\rho \Vdash \text{Œ}_c x\, \varphi$ and $\{d\}$ is contained in a null set for every $d \in |\varphi|^x_{I\rho}$, then $|\varphi|^x_{I\rho}$ is necessarily uncountable. △

PROPOSITION 6.2 Any structure for $\mathscr{L}_{\omega_1\omega}(\text{Æ}_c)$ that satisfies $\neg\exists x(\text{Œ}_c\, y(x \cong y))$ has an uncountable domain.

PROOF. Assume that $I \Vdash \neg\exists x(\neg\text{Æ}_c\, y\neg(x \cong y))$. Then every element of the domain is in a measure zero set of I. Suppose by contradiction that I has a countable domain D. Then D is the union of a countable number of measure zero sets and so is in $\mathcal{N}$. Contradiction. So I must have an uncountable domain. □

Taking into account Proposition 6.2 it is straightforward to establish a deep difference between $\mathscr{L}_{\omega_1\omega}(\text{Æ}_c)$ and $\mathscr{L}_{\omega_1\omega}$ since in $\mathscr{L}_{\omega_1\omega}$ the downward Löwenheim–Skolem theorem (which says that if a set of formulas has a model then it has a countable model) holds.

PROPOSITION 6.3 The logic $\mathscr{L}_{\omega_1\omega}(\text{Æ}_c)$ does not satisfy the downward Löwenheim–Skolem theorem.

PROOF. Let φ be the formula $\neg\exists x(\neg\text{Æ}_c\, y\neg(x \cong y))$, intuitively representing the semantic condition that all singleton subsets of the domain have measure zero. Clearly φ is satisfiable, since the usual Lebesgue measure on the real line has this property. However, it has no countable models, see Proposition 6.2. □

Observe that the reasoning in the proof of Proposition 6.3 above works just as well if we take φ to be $(\neg\exists x(\neg\text{Æ}_c\, y\neg p(x,y))) \wedge (\forall x(p(x,x)))$ and assume nothing at all about the interpretation of p.

So, proving the completeness of an axiomatics for $\mathscr{L}_{\omega_1\omega}(\text{Æ}_c)$, will involve the construction of a very specific canonical model of uncountable cardinality. Among the methods in the literature for constructing such a model, we emphasize as potentially useful, the sophisticated technique used by Keisler in the completeness proof of first-order logic enriched with the generalized quantifier *there exist uncountably many x such that $\varphi(x)$* in [14] (there, the canonical model is obtained as the uncountable union of countable models). However it is not clear at this point how to adapt his construction to our case in order to obtain an uncountable model fulfilling all the requirement needed, namely for instance the closure under countable union of the set of null sets. Another direction is to make Æ_c a second-order generalized quantifier, like for instance the quantifier *aa* considered in [4, 25]. The model theory of Borel structures developed in [26] seems also to be useful.

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