Importing logics:
Soundness and completeness preservation

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Abstract

Importing subsumes several asymmetric ways of combining logics, including modalization and temporalization. A calculus is provided for importing, inheriting the axioms and rules from the given logics and including additional rules for lifting derivations from the imported logic. The calculus is shown to be sound and concretely complete with respect to the semantics of importing as proposed in [12].

Keywords: combined logics, importing logics, modalization, completeness preservation.

1 Introduction

Having in mind different fields of application, several asymmetric ways of combining logics have been reported in the literature, including temporalization [4], modalization [3], globalization [10], probabilization [2] and quantization [9]. We proposed in [12] importing as a general way of asymmetric combination of logics and showed that it subsumes such asymmetric combination mechanisms. Furthermore, in [11] we were able to recover fibring [6] as bidirectional importing. However, so far, importing has been developed only at the semantic level. Herein, we provide a calculus for importing, inheriting the axioms and rules from the given logics and including additional rules for lifting derivations from the imported logic, and prove its soundness and concrete completeness vis à vis the semantics proposed in [12].

As in our previous papers on importing we adopt the graph-theoretic account of language and semantics. This approach has the advantage of being applicable to a wider class of logics [13]. Herein, we present a novel graph-theoretic account of deduction, requiring a mild generalization of the notion of 2-category.

In Section 2, following [12], we provide for the convenience of the reader a short summary of the syntactic aspects of importing. In Section 3 we show how to set up a Hilbert calculus for importing, using the rules and axioms from the two given logics, and illustrate the construction for the cases of temporalization, modalization and importing into intuitionistic logic. Some technical details are left to the Appendix, concerning the generation of the generalized 2-category of derivations from the calculus as a 2-graph. In Section 4, after a short summary of the graph-theoretic models of importing defined in [12], we propose a local version of semantic entailment. Preservation of soundness, under the mild assumption of totality of the semantics of the two given logics, is proved in Section 5. Preservation of concrete completeness, under a mild assumption of fullness of the semantics of the two original logics, is established in Section 6. Finally, in Section 7 we assess what was achieved and speculate on what is still ahead.

2 Language

The language resulting from the importing contains the languages of both logics together with the formulas resulting from the instantiation of formulas of the importing logic by formulas of the logic being imported (see [12]). The graph-theoretic approach developed in [13] is followed and so signatures are presented using multi-graphs: the vertexes are the language sorts and the multi-edges are the language constructors. As an illustration, see Figure 1 for a graphical representation of a signature for the linear-time temporal logic (LTL).

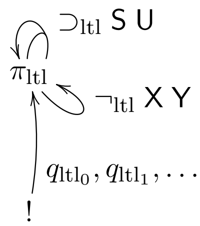

Figure 1: Multi-graph of the LTL signature.

By a *multi-graph*, in short, an *m-graph*, we mean a tuple

$$G = (V, E, \mathsf{src}, \mathsf{trg})$$

where V is a set (of *vertexes* or *nodes*), E is a set (of *m-edges*), $\mathsf{src} : E \to V^+$ and $\mathsf{trg} : E \to V$, with V^+ denoting the set of all finite non-empty sequences of V. We may write $e : s \to v$ for stating that m-edge e has source s and target v. By a *propositional based signature* or, simply, a *signature*, Σ, we mean a tuple

$$(G, !, \Pi)$$

where $G = (V, E, \mathsf{src}, \mathsf{trg})$ is an m-graph, Π is a non-empty set (of *propositions sorts*) contained in V, ! (the *concrete sort*) is in $V \setminus \Pi$, no m-edge in E has ! as target, and ! only appears in the source of unary edges. We now present some examples of signatures for modal logic [1, 7], linear-time temporal logic [4, 15] and intuitionistic logic [14], useful throughout the paper.

Example 2.1 *Signature for linear-time temporal logic.*
Let Q^{ltl} be a set $\{q_{\mathrm{ltl}_0}, q_{\mathrm{ltl}_1}, \dots\}$ of propositional symbols. The *signature for linear-time temporal logic over Q^{ltl}*, denoted by $\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}$, is an m-graph with the propositions sort π_{ltl}, the concrete sort !, and the m-edges: $q_{\mathrm{ltl}_j} : ! \to \pi_{\mathrm{ltl}}$ for each natural number j; $\neg_{\mathrm{ltl}}, \mathsf{X}, \mathsf{Y} : \pi_{\mathrm{ltl}} \to \pi_{\mathrm{ltl}}$; and $\supset_{\mathrm{ltl}}, \mathsf{S}, \mathsf{U} : \pi_{\mathrm{ltl}}\pi_{\mathrm{ltl}} \to \pi_{\mathrm{ltl}}$. For a graphical representation see Figure 1. ∇

Example 2.2 *Signature for modal logic.*
Let Q^{m} be a set $\{q_{\mathrm{m}_0}, q_{\mathrm{m}_1}, \dots\}$ of propositional symbols. The *modal signature over Q^m*, denoted by $\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}$, is an m-graph with the propositions sort π_{m}, the concrete sort !, and the m-edges: $q_{\mathrm{m}_j} : ! \to \pi_{\mathrm{m}}$ for each natural number j; $\neg_{\mathrm{m}}, \Diamond : \pi_{\mathrm{m}} \to \pi_{\mathrm{m}}$; and $\supset_{\mathrm{m}} : \pi_{\mathrm{m}}\pi_{\mathrm{m}} \to \pi_{\mathrm{m}}$. ∇

Example 2.3 *Signature for intuitionistic logic.*
Let Q^{i} be a countable set $\{q_{\mathrm{i}_0}, q_{\mathrm{i}_1}, \dots\}$ of propositional symbols. The *signature over Q^i for intuitionistic logic*, denoted by $\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}$, is an m-graph with the propositions sort π_{i}, the concrete sort ! and the m-edges: $q_{\mathrm{i}_j} : ! \to \pi_{\mathrm{i}}$ for each natural number j; $\neg_{\mathrm{i}} : \pi_{\mathrm{i}} \to \pi_{\mathrm{i}}$; and $\wedge_{\mathrm{i}}, \vee_{\mathrm{i}}, \supset_{\mathrm{i}} : \pi_{\mathrm{i}}\pi_{\mathrm{i}} \to \pi_{\mathrm{i}}$. ∇

As expected, formulas appear as m-paths over the signature m-graph ending at some $\pi \in \Pi$. Actually, it is more convenient to work in the corresponding graph enriched with tupling and projections. More concretely, let $G^\dagger$ be the graph induced by G having as nodes the finite sequences of nodes of G and as edges the m-edges of G together with edges $\mathsf{p}_j^{v_1\dots v_n}$, from $v_1 \dots v_n$ to v_j, for projections, and edges $\langle w_1, \dots, w_n\rangle$, from s to $v_1 \dots v_n$, for tuples, where $w_1, \dots, w_n$ are paths with the same source s and target $v_1, \dots, v_n$ respectively (for more details see [12]). Since many paths over $G^\dagger$ may collapse onto the same formula, for instance $\neg\, \mathsf{p}_1^{\pi\pi}\langle q_1, q_2\rangle$ and $\neg\, q_1$, it is convenient to work only with "irreducible" paths. The set of irreducible paths of $G^\dagger$ is inductively defined as follows:

- ϵ_s is an irreducible path;
- $\mathsf{p}_j^{v_1\dots v_n}$ is an irreducible path;
- $\langle w_1, \dots, w_n\rangle$ is an irreducible path whenever $w_1, \dots, w_n$ are irreducible paths and at least one w_j is not $\mathsf{p}_j^{v_1\dots v_n}$;
- ew is an irreducible path whenever e is an m-edge of G and w is an irreducible path.

The set of nodes of $G^\dagger$ together with its irreducible paths constitute a category, henceforth denoted by G^+, where composition of two irreducible paths is the irreducible path resulting from reducing the path obtained by concatenating them and identity at a given node is the empty path therein (for more details see [12]). In the sequel, given a signature $\Sigma = (G, !, \Pi)$, we may denote by Σ^+ the category G^+, by $\Sigma^\dagger$ the graph $G^\dagger$, and given a morphism w of Σ^+ from s_1 to s_2 we may denote its source s_1 by $\mathsf{src}^+(w)$ and its target s_2 by $\mathsf{trg}^+(w)$.

A *generalized formula* over a signature $(G, !, \Pi)$ is an irreducible path with target $\pi_1 \dots \pi_n$, for some $\pi_1, \dots, \pi_n$ in Π and natural number n, over the graph

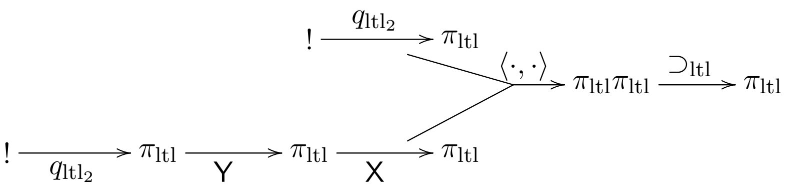

Figure 2: A temporal formula: $q_{\text{ltl}_2} \supset_{\text{ltl}} (\mathsf{X}(\mathsf{Y}q_{\text{ltl}_2}))$.

$G^\dagger$. An *expression* over Σ is an irreducible path over $G^\dagger$ and a *proper formula* is a generalized formula ending at $\pi \in \Pi$. We denote the set of generalized formulas over Σ by $L^\bullet(\Sigma)$ and the set of proper formulas of Σ, i.e. the *language* of Σ, by $L(\Sigma)$. We may refer to the elements of $L^\bullet(\Sigma)$ simply as *formulas.* An expression over Σ is said to be *concrete* whenever its source is ! and is said to be *schematic* if a sort different from ! occurs in its source. For instance, in the context of the signature $\Sigma^{\text{ltl}}_{Q^{\text{ltl}}}$ for linear temporal logic described in Example 2.1, the formula $\supset_{\text{ltl}}\langle q_{\text{ltl}_2}, \mathsf{XY}q_{\text{ltl}_2}\rangle$ from ! to π_{ltl}, see Figure 2, is a concrete formula, represented simply by

$$q_{\text{ltl}_2} \supset_{\text{ltl}} (\mathsf{X}(\mathsf{Y}q_{\text{ltl}_2})),$$

(in order to simplify the presentation, when writing irreducible paths we may write the language constructors in infix notation and so may not explicitly write the associated tuples), and the formula

$$\supset_{\text{ltl}}\langle \mathsf{p}_1^{\pi_{\text{ltl}}\pi_{\text{ltl}}}, \mathsf{XY}\mathsf{p}_2^{\pi_{\text{ltl}}\pi_{\text{ltl}}}\rangle$$

from $\pi_{\text{ltl}}\pi_{\text{ltl}}$ to π_{ltl} is schematic. Traditionally this formula is written with schema variables as follows:

$$\xi_1 \supset_{\text{ltl}} (\mathsf{X}(\mathsf{Y}\xi_2)).$$

From now on, we may use interchangeably the simpler traditional representation and the more rigorous one.

Given expressions w and w_0 in Σ^+, w_0 is *compatible* with w whenever $\mathsf{src}^+(w) = \mathsf{trg}^+(w_0)$. The *instantiation* of w by w_0, where w_0 is compatible with w, is the morphism $w \circ w_0$.

Importing a signature

Importing is an asymmetric combination technique in the sense that its language contains the formulas resulting from the instantiation of formulas of the importing logic by formulas of the logic being imported, but not formulas obtained in the other way around. One of the key characteristics of importing is that it makes explicit the bridge from the imported logic into the importing one. So, the signature resulting from the importing contains the constructors and sorts of both signatures and the added constructors $\Lsh_{vu}$ that are the only constructors that involve sorts of both components. As an illustration see in Figure 3 the signature resulting from importing the signature for linear temporal logic introduced in Example 2.1 into the signature for modal logic introduced in Example 2.2.

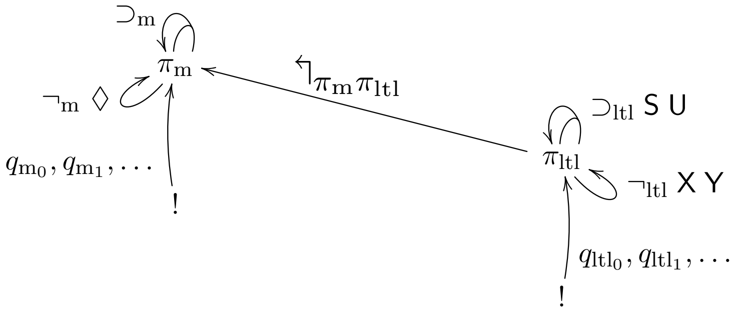

Figure 3: Importing the linear temporal signature into the modal signature.

The importing constructors act in formulas as “bridges” that transform formulas of the imported logic into formulas of the importing one (but not in the other way around). For example the formula

$$(\Diamond(\Lsh_{\pi_{\mathrm{m}}\pi_{\mathrm{ltl}}}(\xi_1 \,\mathsf{U}\, \xi_2))) \supset_{\mathrm{m}} (\Diamond(\Lsh_{\pi_{\mathrm{m}}\pi_{\mathrm{ltl}}}(\mathsf{X}\xi_2)))$$

where ξ_1 and ξ_2 are $\mathsf{p}_1^{\pi_{\mathrm{ltl}}\pi_{\mathrm{ltl}}}$ and $\mathsf{p}_2^{\pi_{\mathrm{ltl}}\pi_{\mathrm{ltl}}}$ respectively, is in the language induced by the signature, depicted in Figure 3, resulting from importing $\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}$ into $\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}$. When there is no ambiguity we may represent the imported formulas inside the host formula between quotes and omit the importing connective. For example, we may represent the formula above by

$$(\Diamond\text{'}\xi_1 \,\mathsf{U}\, \xi_2\text{'}) \supset_{\mathrm{m}} (\Diamond\text{'}\mathsf{X}\xi_2\text{'}).$$

Importing is defined for a *suitably disjoint* pair of signatures, that is, signatures $(G_1, !, \Pi_1)$ and $(G_2, !, \Pi_2)$ where $V_1 \setminus \{!\}$ and $V_2 \setminus \{!\}$ are disjoint, Π_1 and Π_2 are singletons, $\Lsh_{vu}$ is not in $E_1 \cup E_2$ for u and v in $\Pi_1 \cup \Pi_2$, and E_1 and E_2 are disjoint.

Importing a signature Σ_1 *into a signature* Σ_2, denoted by

$$\Sigma_2[\Sigma_1],$$

is, denoting the element of Π_1 by π_1 and the element of Π_2 by π_2, the signature

$$((V, E, \mathsf{src}, \mathsf{trg}), !, \{\pi_1, \pi_2\})$$

where

- $V = V_1 \cup V_2$;
- E is $E_1 \cup E_2 \cup \{\Lsh_{\pi_2\pi_1}\}$;
- src and trg are such that $\mathsf{src}(\Lsh_{\pi_2\pi_1}) = \pi_1$, $\mathsf{trg}(\Lsh_{\pi_2\pi_1}) = \pi_2$, and $\mathsf{src}(e) = \mathsf{src}_k(e)$ and $\mathsf{trg}(e) = \mathsf{trg}_k(e)$ if e is in E_k for $k = 1, 2$.

We now present some particular instances of importing. Each example is in fact a collection of instances of importing all over the same importing signature.

Example 2.4 By *adding a $\Lsh$-constructivist dimension* to a signature Σ_1 suitably disjoint with signature $\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}$ for intuitionistic logic introduced in Example 2.3, denoted by $\mathrm{I}[\Sigma_1]$, we mean the importing of Σ_1 into signature $\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}$. ∇

Example 2.5 The $\Lsh$-*modalization* of a signature Σ_1 suitably disjoint with signature $\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}$ for modal logic introduced in Example 2.2, denoted by $\mathrm{M}[\Sigma_1]$, is the importing of Σ_1 into signature $\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}$. See Figure 3 for a partial graphical representation of the signature $\mathrm{M}[\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}]$. ∇

Example 2.6 The $\Lsh$-*temporalization* of a signature Σ_1 suitably disjoint with signature $\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}$ for LTL, introduced in Example 2.1, denoted by $\mathrm{LTL}[\Sigma_1]$, is the importing of Σ_1 into signature $\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}$. ∇

3 Deduction

In this section we investigate what is importing in terms of deduction. For that, we need that the given deductive systems be described in a common way, and so we assume that they are Hilbert-style systems presented according to the graph-theoretic approach developed in [13].

Hence, a deductive system is described using a graph where the nodes are formulas and the edges are rules, either axiomatic or not. For instance, the rule depicted in Figure 4 and introduced in Example 3.3 for modal logic T, can be seen as an edge, from the schema formula $\xi_1 \supset_{\mathrm{m}} \xi_2$ to the schema formula $(\Diamond\xi_1) \supset_{\mathrm{m}} (\Diamond\xi_2)$, where ξ_1 is $\mathsf{p}_1^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}$ and ξ_2 is $\mathsf{p}_2^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}$. In the same vein, axiomatic rules are endo edges, that is, edges from a formula, the axiom, to itself. Multi-source edges are not needed since we make use of tupling in Σ^+.

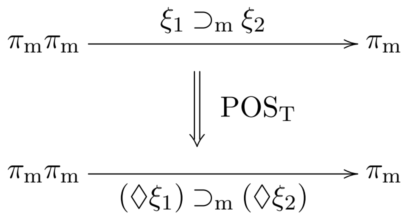

Figure 4: The possibility rule of modal logic T.

More rigorously, a *deductive system* is a pair (Σ, Δ) where Σ is a signature and Δ is a triple

$$(R, \mathsf{prem}, \mathsf{conc})$$

where R is a set (of *rules*), and $\mathsf{prem} : R \to L^{\bullet}(\Sigma)$ and $\mathsf{conc} : R \to L(\Sigma)$ are such that

$$\mathsf{src}^+ \circ \mathsf{prem} = \mathsf{src}^+ \circ \mathsf{conc}.$$

We may write $r : \psi \Rightarrow \varphi$ for stating that rule r has premise ψ and conclusion φ. An *axiom* is the source or the target formula of an endo-edge in R (such an endo-edge may be denoted by an *axiomatic rule*). When there is no ambiguity we may confuse an axiomatic rule with its associated axiom, and so, when presenting an axiomatic rule, we may simply present the target formula.

Example 3.1 *Deductive system for intuitionistic propositional logic.*
Consider the Hilbert axiomatization of intuitionistic logic proposed in [14]. That axiomatization can be represented as the deductive system $(\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}, \Delta^{\mathrm{i}})$, denoted by $\mathcal{D}^{\mathrm{i}}$, where:

- $\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}$ is the signature for intuitionistic logic described in Example 2.3;
- Δ^{i} contains the following axioms and rules:
  - $\mathsf{ax}_{\mathrm{i}_1} : \xi \supset_{\mathrm{i}} (\xi' \supset_{\mathrm{i}} \xi)$;
  - $\mathsf{ax}_{\mathrm{i}_2} : (\xi_1 \supset_{\mathrm{i}} \xi_2) \supset_{\mathrm{i}} ((\xi_1 \supset_{\mathrm{i}} (\xi_2 \supset_{\mathrm{i}} \xi_3)) \supset_{\mathrm{i}} (\xi_1 \supset_{\mathrm{i}} \xi_3))$;
  - $\mathsf{ax}_{\mathrm{i}_3} : \xi \supset_{\mathrm{i}} (\xi' \supset_{\mathrm{i}} (\xi \wedge_{\mathrm{i}} \xi'))$;
  - $\mathsf{ax}_{\mathrm{i}_4} : (\xi \wedge_{\mathrm{i}} \xi') \supset_{\mathrm{i}} \xi$;
  - $\mathsf{ax}_{\mathrm{i}_5} : (\xi \wedge_{\mathrm{i}} \xi') \supset_{\mathrm{i}} \xi'$;
  - $\mathsf{ax}_{\mathrm{i}_6} : \xi \supset_{\mathrm{i}} (\xi \vee_{\mathrm{i}} \xi')$;
  - $\mathsf{ax}_{\mathrm{i}_7} : \xi' \supset_{\mathrm{i}} (\xi \vee_{\mathrm{i}} \xi')$;
  - $\mathsf{ax}_{\mathrm{i}_8} : (\xi_1 \supset_{\mathrm{i}} \xi_3) \supset_{\mathrm{i}} ((\xi_2 \supset_{\mathrm{i}} \xi_3) \supset_{\mathrm{i}} ((\xi_1 \vee_{\mathrm{i}} \xi_2) \supset_{\mathrm{i}} \xi_3))$;
  - $\mathsf{ax}_{\mathrm{i}_9} : (\xi \supset_{\mathrm{i}} \xi') \supset_{\mathrm{i}} ((\xi \supset_{\mathrm{i}} (\neg_{\mathrm{i}} \xi')) \supset_{\mathrm{i}} (\neg_{\mathrm{i}} \xi))$;
  - $\mathsf{ax}_{\mathrm{i}_{10}} : \xi \supset_{\mathrm{i}} ((\neg_{\mathrm{i}} \xi) \supset_{\mathrm{i}} \xi')$;
  - $\mathsf{MP}_{\mathrm{i}} : \langle \xi, \xi \supset_{\mathrm{i}} \xi' \rangle \Rightarrow \xi'$;

  where ξ is $\mathsf{p}_1^{\pi_{\mathrm{i}}\pi_{\mathrm{i}}}$, ξ' is $\mathsf{p}_2^{\pi_{\mathrm{i}}\pi_{\mathrm{i}}}$, ξ_1 is $\mathsf{p}_1^{\pi_{\mathrm{i}}\pi_{\mathrm{i}}\pi_{\mathrm{i}}}$, ξ_2 is $\mathsf{p}_2^{\pi_{\mathrm{i}}\pi_{\mathrm{i}}\pi_{\mathrm{i}}}$ and ξ_3 is $\mathsf{p}_3^{\pi_{\mathrm{i}}\pi_{\mathrm{i}}\pi_{\mathrm{i}}}$. ∇

Example 3.2 *Deductive system for linear temporal logic.*
Consider the Hilbert axiomatization of LTL described in [15] using the following abbreviations: $(\widetilde{\oplus}\varphi)$ for $\neg_{\mathrm{ltl}} X(\neg_{\mathrm{ltl}}\varphi)$, $(\boxplus\varphi)$ for $\neg_{\mathrm{ltl}}(\mathsf{true}\,\mathsf{U}\,(\neg_{\mathrm{ltl}}\varphi))$, $(\widetilde{\ominus}\varphi)$ for $\neg_{\mathrm{ltl}} Y(\neg_{\mathrm{ltl}}\varphi)$, $(\boxminus\varphi)$ for $\neg_{\mathrm{ltl}}(\mathsf{true}\,\mathsf{S}\,(\neg_{\mathrm{ltl}}\varphi))$, $\varphi_1 \supset_{\boxplus} \varphi_2$ for $\boxplus(\varphi_1 \supset_{\mathrm{ltl}} \varphi_2)$, and $\varphi_1 \leftrightarrow_{\boxplus} \varphi_2$ for $(\varphi_1 \supset_{\boxplus} \varphi_2) \wedge (\varphi_2 \supset_{\boxplus} \varphi_1)$. This axiomatization can be represented as the deductive system $(\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}, \Delta^{\mathrm{ltl}})$, denoted by $\mathcal{D}^{\mathrm{ltl}}$, such that:

- $\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}$ is the LTL signature introduced in Example 2.1;
- Δ^{ltl} contains the first two axioms presented in Example 3.1 with the obvious adaptations to the LTL context, and
  - $\mathsf{ax}_{\mathrm{ltl}_c} : ((\neg_{\mathrm{ltl}} \xi_1) \supset_{\mathrm{ltl}} (\neg_{\mathrm{ltl}} \xi_2)) \supset_{\mathrm{ltl}} (\xi_2 \supset_{\mathrm{ltl}} \xi_1)$;
  - $\mathsf{ax}_{\mathrm{ltl}_1} : (\xi_1\,\mathsf{U}\,\xi_2) \leftrightarrow_{\boxplus} (\xi_2 \vee (\xi_1 \wedge \mathsf{X}(\xi_1\,\mathsf{U}\,\xi_2)))$;
  - $\mathsf{ax}_{\mathrm{ltl}_2} : (\xi_1\,\mathsf{S}\,\xi_2) \leftrightarrow_{\boxplus} (\xi_2 \vee (\xi_1 \wedge \mathsf{Y}(\xi_1\,\mathsf{S}\,\xi_2)))$;
  - $\mathsf{ax}_{\mathrm{ltl}_3} : (\xi\,\mathsf{U}\,\mathsf{false}) \supset_{\boxplus} \mathsf{false}$;
  - $\mathsf{ax}_{\mathrm{ltl}_4} : (\widetilde{\ominus}\mathsf{false})$;
  - $\mathsf{ax}_{\mathrm{ltl}_5} : (\boxplus\xi) \supset_{\mathrm{ltl}} \xi$;

– $\mathsf{ax}_{\mathrm{ltl}_6} : (\boxplus\xi) \supset_{\boxplus} \boxplus(\widetilde{\oplus}\xi));$
– $\mathsf{ax}_{\mathrm{ltl}_7} : \boxplus(\xi_1 \supset_{\mathrm{ltl}} \xi_2) \supset_{\boxplus} ((\boxplus\xi_1) \supset_{\mathrm{ltl}} (\boxplus\xi_2));$
– $\mathsf{ax}_{\mathrm{ltl}_8} : \boxminus(\xi_1 \supset_{\mathrm{ltl}} \xi_2) \supset_{\boxplus} ((\boxminus\xi_1) \supset_{\mathrm{ltl}} (\boxminus\xi_2));$
– $\mathsf{ax}_{\mathrm{ltl}_9} : \xi \supset_{\boxplus} \widetilde{\oplus}(Y\xi);$
– $\mathsf{ax}_{\mathrm{ltl}_{10}} : \xi \supset_{\boxplus} \widetilde{\ominus}(X\xi);$
– $\mathsf{ax}_{\mathrm{ltl}_{11}} : \boxplus(\xi \supset_{\mathrm{ltl}} (\widetilde{\oplus}\xi)) \supset_{\boxplus} \boxplus(\xi \supset_{\mathrm{ltl}} (\boxplus\xi));$
– $\mathsf{ax}_{\mathrm{ltl}_{12}} : \boxplus(\xi \supset_{\mathrm{ltl}} (\widetilde{\ominus}\xi)) \supset_{\mathrm{ltl}} \boxplus(\xi \supset_{\mathrm{ltl}} (\boxminus\xi));$
– $\mathsf{ax}_{\mathrm{ltl}_{13}} : (\boxplus\xi) \supset_{\mathrm{ltl}} \boxplus(\widetilde{\ominus}\xi);$
– $\mathsf{ax}_{\mathrm{ltl}_{14}} : (Y\xi) \supset_{\boxplus} (\widetilde{\ominus}\xi);$
– $\mathsf{ax}_{\mathrm{ltl}_{15}} : \widetilde{\ominus}(\xi_1 \supset_{\mathrm{ltl}} \xi_2) \leftrightarrow_{\boxplus} ((\widetilde{\ominus}\xi_1) \supset_{\mathrm{ltl}} (\widetilde{\ominus}\xi_2));$
– $\mathsf{ax}_{\mathrm{ltl}_{16}} : \widetilde{\oplus}(\xi_1 \supset_{\mathrm{ltl}} \xi_2) \leftrightarrow_{\boxplus} ((\widetilde{\oplus}\xi_1) \supset_{\mathrm{ltl}} (\widetilde{\oplus}\xi_2));$
– $\mathsf{ax}_{\mathrm{ltl}_{17a}} : (\widetilde{\oplus}\xi) \supset_{\boxplus} (\mathsf{X}\xi);$
– $\mathsf{ax}_{\mathrm{ltl}_{17b}} : (\mathsf{X}\xi) \supset_{\boxplus} (\widetilde{\oplus}\xi);$
– $\mathsf{MP}_{\mathrm{ltl}} : \langle \xi_1, \xi_1 \supset_{\mathrm{ltl}} \xi_2 \rangle \Rightarrow \xi_2;$
– $\mathsf{GEN}_{\boxplus} : \xi \Rightarrow (\boxplus\xi);$
– $\mathsf{GEN}_{\boxminus} : \xi \Rightarrow (\boxminus\xi);$

where ξ_1 is $\mathsf{p}_1^{\pi_{\mathrm{ltl}}\pi_{\mathrm{ltl}}}$, ξ_2 is $\mathsf{p}_2^{\pi_{\mathrm{ltl}}\pi_{\mathrm{ltl}}}$ and ξ is $\mathsf{id}_{\pi_{\mathrm{ltl}}}$. ∇

Example 3.3 *Deductive system for modal logic* T.
Consider the Hilbert axiomatization of modal logic T described in [1]. This axiomatization can be represented as the deductive system $(\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}, \Delta^{\mathrm{T}})$, denoted by $\mathcal{D}^{\mathrm{T}}$, where:

- $\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}$ is the modal signature introduced in Example 2.2;
- Δ^{T} contains the first two axioms presented in Example 3.1 with the obvious adaptations to the modal context, and:
  – $\mathsf{ax}_{\mathrm{T}_c} : ((\neg_{\mathrm{m}} \xi_1) \supset_{\mathrm{m}} (\neg_{\mathrm{m}} \xi_2)) \supset_{\mathrm{m}} (\xi_2 \supset_{\mathrm{m}} \xi_1);$
  – $\mathsf{ax}_{\mathrm{T}_1} : (\Diamond\mathsf{false}) \leftrightarrow \mathsf{false};$
  – $\mathsf{ax}_{\mathrm{T}_2} : \Diamond(\xi_1 \vee \xi_2) \leftrightarrow ((\Diamond\xi_1) \vee (\Diamond\xi_2));$
  – $\mathsf{ax}_{\mathrm{T}} : \xi \supset_{\mathrm{m}} (\Diamond\xi);$
  – $\mathsf{MP}_{\mathrm{T}} : \langle \xi_1, \xi_1 \supset_{\mathrm{m}} \xi_2 \rangle \Rightarrow \xi_2;$
  – $\mathrm{POS}_{\mathrm{T}} : (\xi_1 \supset_{\mathrm{m}} \xi_2) \Rightarrow ((\Diamond\xi_1) \supset_{\mathrm{m}} (\Diamond\xi_2));$

  where ξ_1 is $\mathsf{p}_1^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}$, ξ_2 is $\mathsf{p}_2^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}$ and ξ is $\mathsf{id}_{\pi_{\mathrm{m}}}$. ∇

Observe that a deductive system can be seen as having a 2-category flavor: rules are edges between formulas (which are morphisms in the language category induced by the signature). More precisely, as having a generalized 2-category flavor, since a generalized 2-category, see the Appendix, is a 2-category (see [8]) without the proviso that the source of the 2-cell source coincides with the source of its target, and similarly, that the target of the 2-cell source coincides with the target of its target. For example, MP_{i} in Example 3.1 could not be a 2-cell, since the target of its premise is $\pi_{\mathsf{i}}\pi_{\mathsf{i}}$ and the target of its conclusion is π_{i}.

In fact, as detailed in the Appendix, a deductive system (Σ, Δ) induces a generalized 2-category, denoted by

$$\Sigma^{\Delta},$$

where the objects are the expressions over Σ, and the set of generalized 2-cells is the quotient of the minimal set of paths of the graph containing the rules in Δ, 2-projections

$$\mathsf{P}_j^{\langle w_1,\ldots,w_n\rangle},$$

and 2-tuples

$$\bar{\langle}\delta_1, \ldots, \delta_n\bar{\rangle},$$

and closed under path vertical $\bullet_{\mathsf{v}}$ and horizontal $\bullet_{\mathsf{h}}$ compositions; with an equivalence relation $\approx$ for imposing that Σ^{Δ} is a generalized 2-category and has 2-products of objects with the same source, see the Appendix. We denote by

$$\overline{\mathsf{src}} \qquad \text{and} \qquad \overline{\mathsf{trg}}$$

the maps that assign to each generalized 2-cell in Σ^{Δ} its source and its target, respectively. Moreover given an expression w in Σ^{+}, the identity on w in Σ^{Δ}, denoted by

$$\mathsf{ID}_w$$

is $[\epsilon_w]_{\approx}$, and given appropriate generalized 2-cells $[\delta_1]_{\approx} : w_1 \to w_2$ and $[\delta_2]_{\approx} : w_3 \to w_4$, its vertical and horizontal composition in Σ^{Δ}, denoted respectively by

$$[\delta_2]_{\approx} \mathbin{\bar{\circ}_{\mathsf{v}}} [\delta_1]_{\approx} \qquad \text{and} \qquad [\delta_2]_{\approx} \mathbin{\bar{\circ}_{\mathsf{h}}} [\delta_1]_{\approx}$$

is $[\delta_2 \bullet_{\mathsf{v}} \delta_1]_{\approx}$ and $[\delta_2 \bullet_{\mathsf{h}} \delta_1]_{\approx}$ respectively. The horizontal composition is defined if and only if the source of w_3 coincides with the target of w_1 and the source of w_4 coincides with the target of w_2 (see Figure 5), in which case its horizontal

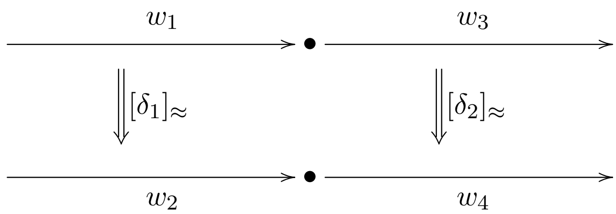

Figure 5: Generalized 2-cells "appropriate" for horizontal composition.

composition is a generalized 2-cell from $w_3 \circ w_1$ to $w_4 \circ w_2$. Similarly, the vertical composition is defined if and only if the target of $[\delta_1]_{\approx}$ coincides with the source of $[\delta_2]_{\approx}$, that is, w_2 coincides with w_3 (see Figure 6) in which case its vertical composition is a generalized 2-cell from w_1 to w_4.

In the sequel we represent a generalized 2-cell $[\delta]_{\approx}$ in Σ^{Δ} simply by δ. A *source-homogeneous* generalized 2-cell $\delta : w_1 \to w_2$ is a generalized 2-cell where

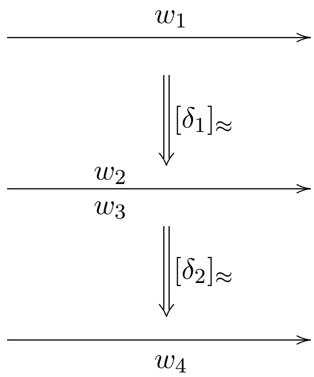

Figure 6: Generalized 2-cells “appropriate” for vertical composition.

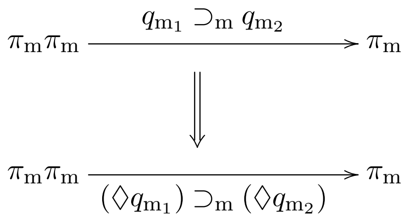

Figure 7: Instantiation of $\mathrm{POS_T}$ in Figure 4 by q_{m_1} and q_{m_2}.

the source of its source and the source of its target coincide, that is, $\mathsf{src}^+(w_1) = \mathsf{src}^+(w_2)$. All generalized 2-cells in Σ^Δ are source-homogeneous.

Instantiation of rules are naturally expressed in Σ^Δ using horizontal composition. For example, consider the instantiation of the rule $\mathrm{POS_T}$, depicted in Figure 4, where ξ_1 is instantiated by q_{m_1} and ξ_2 by q_{m_2}, see Figure 7. It is not difficult to see that

$$q_{\mathrm{m}_1} \supset_{\mathrm{m}} q_{\mathrm{m}_2} = (\xi_1 \supset_{\mathrm{m}} \xi_2) \circ \langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle$$

since

$$\begin{array}{rcl} q_{\mathrm{m}_1} \supset_{\mathrm{m}} q_{\mathrm{m}_2} & = & \supset_{\mathrm{m}}\langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle \\ & = & (\supset_{\mathrm{m}}\langle \mathsf{p}_1^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}, \mathsf{p}_2^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}\rangle) \circ \langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle \\ & = & (\xi_1 \supset_{\mathrm{m}} \xi_2) \circ \langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle \end{array}$$

and

$$(\Diamond q_{\mathrm{m}_1}) \supset_{\mathrm{m}} (\Diamond q_{\mathrm{m}_2}) = ((\Diamond \xi_1) \supset_{\mathrm{m}} (\Diamond \xi_2)) \circ \langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle$$

since

$$\begin{array}{rcl} (\Diamond q_{\mathrm{m}_1}) \supset_{\mathrm{m}} (\Diamond q_{\mathrm{m}_2}) & = & \supset_{\mathrm{m}}\langle \Diamond q_{\mathrm{m}_1}, \Diamond q_{\mathrm{m}_2}\rangle \\ & = & (\supset_{\mathrm{m}}\langle \Diamond\mathsf{p}_1^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}, \Diamond\mathsf{p}_2^{\pi_{\mathrm{m}}\pi_{\mathrm{m}}}\rangle) \circ \langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle \\ & = & ((\Diamond \xi_1) \supset_{\mathrm{m}} (\Diamond \xi_2)) \circ \langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle \end{array}$$

see Figure 8.

Henceforth, by the *instantiation* of a generalized source-homogeneous 2-cell b in Σ^Δ from w_1 to w_2, *by* w in Σ^+ with $\mathsf{trg}^+(w) = \mathsf{src}^+(w_1) = \mathsf{src}^+(w_2)$, denoted by

$$b * w,$$

we mean the generalized 2-cell $b \,\overline{\circ}_{\mathsf{h}}\, \mathsf{ID}_w$ from $w_1 \circ w$ to $w_2 \circ w$. So Σ^Δ contains all the instantiations of rules in Δ as well as their compositions.

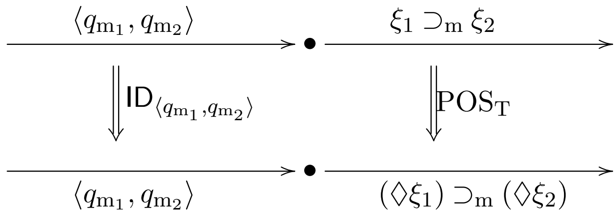

Figure 8: Another view of the generalized 2-cell $\mathrm{POS_T} * \langle q_{\mathrm{m}_1}, q_{\mathrm{m}_2}\rangle$ in Figure 7.

In the sequel we abbreviate the generalized 2-cell $\overline{\langle} \mathsf{P}_{j_1}^{\langle w_1,\dots,w_k\rangle}, \cdots, \mathsf{P}_{j_\ell}^{\langle w_1,\dots,w_k\rangle} \overline{\rangle}$ in Σ^Δ where $1 \le j_1, \dots, j_\ell \le k$ by $\mathsf{P}_{j_1,\dots,j_\ell}^{\langle w_1,\dots,w_k\rangle}$. As expected, by a tupling $\langle w\rangle$ of length one we mean w and by $\mathsf{P}_1^{\langle w\rangle}$ we mean ID_w.

Intuitively, a derivation is a tree labelled by formulas whose leaves are either hypothesis or axiom instances and such that the formula labelling each node is the conclusion of a rule instance from the formulas at its immediate predecessors in the tree. As a simple example, consider the derivation depicted in Figure 9 for deducing formula φ_3 from formulas φ_1, $\varphi_1 \supset \varphi_2$ and $\varphi_2 \supset \varphi_3$, in the context of a deductive system (Σ, Δ) with modus ponens. Observe that the first stage of this derivation is composed by the basic derivation

$$(\mathsf{MP} * \langle \varphi_1, \varphi_2\rangle) \,\overline{\circ}_{\mathsf{v}}\, \mathsf{P}_{1,2}^{\langle \varphi_1, \varphi_1\supset\varphi_2, \varphi_2\supset\varphi_3\rangle}$$

denoted by β_{11}, and by the basic derivation

$$(\mathsf{ID}_{\mathsf{id}_\pi} * (\varphi_2 \supset \varphi_3)) \,\overline{\circ}_{\mathsf{v}}\, \mathsf{P}_{3}^{\langle \varphi_1, \varphi_1\supset\varphi_2, \varphi_2\supset\varphi_3\rangle}$$

denoted by β_{12}, that is, is the generalized 2-cell

$$\overline{\langle} \beta_{11}, \beta_{12} \overline{\rangle}$$

from

$$\langle \varphi_1,\ \varphi_1 \supset \varphi_2,\ \varphi_2 \supset \varphi_3\rangle$$

to

$$\langle \varphi_2,\ \varphi_2 \supset \varphi_3\rangle,$$

and the second stage is the generalized 2-cell

$$\mathsf{MP} * \langle \varphi_2, \varphi_3\rangle$$

from $\langle \varphi_2,\ \varphi_2 \supset \varphi_3\rangle$ to φ_3.

More rigorously, by a *derivation* over a calculus (Σ, Δ) we mean a generalized 2-cell δ in Σ^Δ of the form:

$$\overline{\langle} \beta_{m1}, \dots, \beta_{mn_m} \overline{\rangle} \,\overline{\circ}_{\mathsf{v}} \dots \overline{\circ}_{\mathsf{v}}\, \overline{\langle} \beta_{11}, \dots, \beta_{1n_1} \overline{\rangle}$$

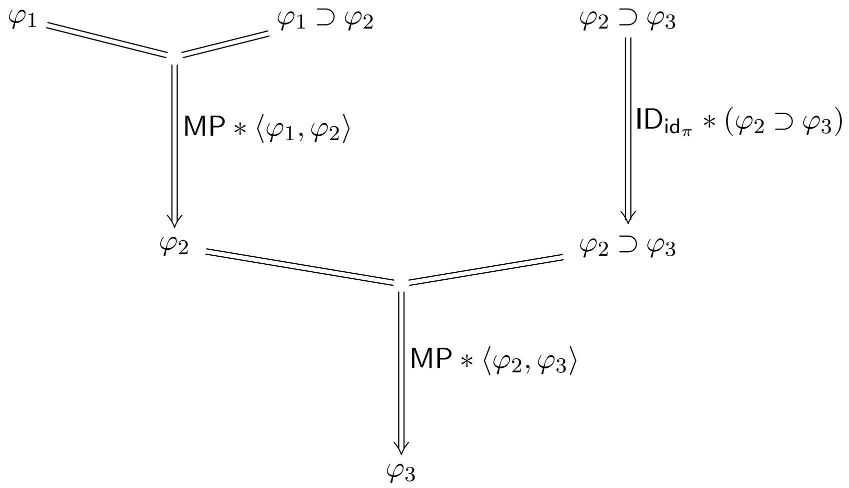

Figure 9: Deduction of φ_3 from φ_1, $\varphi_1 \supset \varphi_2$ and $\varphi_2 \supset \varphi_3$.

for non-zero natural numbers $m, n_1, \ldots, n_m$ with $n_m = 1$, where for $j = 1, \ldots, m$ and $k = 1, \ldots, n_j$ the β_{jk} are *basic derivations*, that is, are generalized 2-cells in Σ^Δ of the following form:

$$(b_{jk} * w_{jk}) \mathbin{\overline{\circ}_{\mathsf{v}}} \mathsf{P}^{\langle \varphi_{j1}, \ldots, \varphi_{j\ell_j} \rangle}_{j'_1, \ldots, j'_{\ell' jk}}$$

where b_{jk} is either a non-axiomatic rule or a generalized 2-cell identity (for vertical composition) over $\mathsf{id}_{\mathsf{trg}^+(w_{jk})}$, $\varphi_{j1}, \ldots, \varphi_{j\ell_j}$ are proper formulas and ℓ_j is non-zero. The basic derivation β_{jk} is said to be *axiomatic* if b_{jk} is a generalized 2-cell identity and w_{jk} is an axiom or an axiom instance. When b_{jk} is a non-axiomatic rule we may denote the basic derivation β_{jk} by *basic derivation over rule* b_{jk}. Observe that the conclusion of a derivation is a proper formula.

A derivation is said to be a *proof* if its premise is a tupling of axiom instances. The conclusion of a proof is said to be a *theorem* or a *concrete theorem* if it is a concrete formula. We write $\vdash_{(\Sigma,\Delta)} \varphi$ or $\vdash \varphi$ for stating that φ is a theorem. Furthermore, we write

$$\Gamma \vdash_{(\Sigma,\Delta)} \varphi$$

or $\Gamma \vdash \varphi$ when Γ is a set of proper formulas, $\mathsf{src}^+(\gamma) = \mathsf{src}^+(\varphi)$ for every $\gamma \in \Gamma$ and there is a derivation in Σ^Δ with conclusion φ and premise given by a tupling of elements of Γ and of axiom instances. In this situation we say that there is a derivation of φ from Γ. A derivation is *concrete* whenever all the formulas occurring in its steps are concrete.

In the sequel, by an *inference* we mean a generalized 2-cell in Σ^Δ with generalized formulas as source and target. The source of an inference is said to be its *antecedent* and its conclusion is said to be its *consequent.* Observe that every inference is source-homogeneous, that is, all formulas in the antecedent and in the consequent have the same sequence of sorts as source. An inference δ_1 in Σ^Δ is *compatible* with inference δ_2 in Σ^Δ if the antecedent of δ_2 coincides with the consequent of δ_1.

Importing a deductive system

We now define what is the importing of a deductive system into another. The goal is that the reasoning mechanism of the imported logic is present in the logic resulting from the combination but can only be applied to its expressions. In contrast, the reasoning mechanism of the importing logic is present in the logic resulting from the combination but is open to all expressions. This captures and generalizes the characteristic properties of some asymmetric techniques of combining logics like modalization and temporalization as developed in [4, 5, 3]. In fact, in [4, 5], the axioms of the deductive system resulting from the temporalization are the theorems of the imported logic together with the axioms of the importing one, and the rules are only the rules of the importing logic.

We assume that the deductive system being imported and the importing deductive system, say (Σ_1, Δ_1) and (Σ_2, Δ_2) respectively, are *suitably disjoint*, i.e., Σ_1 and Σ_2 are suitably disjoint, and R_1 and R_2 are disjoint. Observe that Π_1 and Π_2 are singletons since Σ_1 and Σ_2 are suitably disjoint.

Importing a deductive system (Σ_1, Δ_1) *into a deductive system* (Σ_2, Δ_2), denoted by

$$(\Sigma_2, \Delta_2)[(\Sigma_1, \Delta_1)],$$

is the deductive system $(\Sigma_2[\Sigma_1], \Delta_2[\Delta_1])$ where

$$\Delta_2[\Delta_1]$$

is the tuple $(R, \mathsf{prem}, \mathsf{conc})$ with

- $R = R_1 \cup R_2 \cup \{\mathsf{IMP}\} \cup \{\mathsf{REF}\}$;
- $\mathsf{prem}(r_k) = \mathsf{prem}_k(r_k)$ and $\mathsf{conc}(r_k) = \mathsf{conc}_k(r_k)$ if r_k is in R_k for $k = 1, 2$;
- $\mathsf{prem}(\mathsf{IMP}) = \mathsf{id}_{\pi_1}$ and $\mathsf{conc}(\mathsf{IMP}) = \Lsh_{\pi_2\pi_1}$;
- $\mathsf{prem}(\mathsf{REF}) = \Lsh_{\pi_2\pi_1}$ and $\mathsf{conc}(\mathsf{REF}) = \mathsf{id}_{\pi_1}$.

We now describe some specific instances of importing.

Example 3.4 Recall deductive system $\mathcal{D}^{\mathrm{T}}$ introduced in Example 3.3. The $\Lsh$-*modalization* by modal logic T of a deductive system $\mathcal{D}_1$ suitably disjoint with $\mathcal{D}^{\mathrm{T}}$, denoted by

$$\mathrm{M}_{\mathrm{T}}[\mathcal{D}_1],$$

is the deductive system resulting from importing $\mathcal{D}_1$ into $\mathcal{D}^{\mathrm{T}}$. See Figure 10 for a graphical description of part of the deductive system $\mathrm{M}_{\mathrm{T}}[\mathcal{D}^{\mathrm{ltl}}]$, where $\mathcal{D}^{\mathrm{ltl}}$ is the deductive system for linear temporal logic introduced in 3.2. Observe that

$$\varphi \vdash_{\mathrm{M}_{\mathrm{T}}[\mathcal{D}^{\mathrm{ltl}}]} \Diamond \text{'}\mathsf{X}(\mathsf{Y}\varphi)\text{'}$$

holds, for any formula φ over $\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}$. ∇

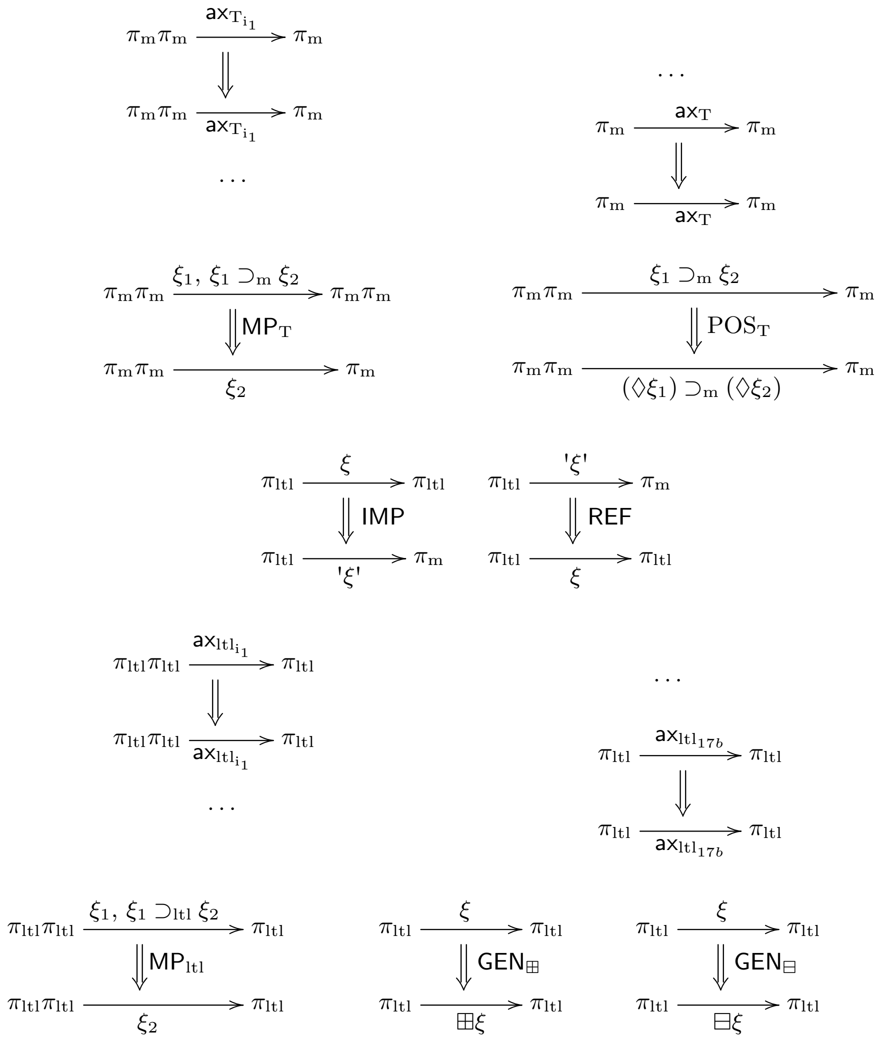

Figure 10: ↰-modalization of linear temporal logic by modal logic T.

Example 3.5 Recall deductive system $\mathcal{D}^{\mathrm{ltl}}$ introduced in Example 3.2. The ↰-*temporalization* of a deductive system $\mathcal{D}_1$ suitably disjoint with $\mathcal{D}^{\mathrm{ltl}}$, denoted by

$$\mathrm{LTL}[\mathcal{D}_1],$$

is the deductive system resulting from importing $\mathcal{D}_1$ into $\mathcal{D}^{\mathrm{ltl}}$. ∇

Example 3.6 By *adding a* ↰-*constructivist dimension* to a deductive system $\mathcal{D}_1$ suitably disjoint with $\mathcal{D}^{\mathrm{i}}$ for intuitionistic logic introduced in Example 3.1, denoted by

$$\mathrm{I}[\mathcal{D}_1],$$

we mean the importing of $\mathcal{D}_1$ into the deductive system $\mathcal{D}^{\mathrm{i}}$. ∇

Relationship with temporalization

Herein we relate $\urcorner$-temporalization, in terms of deductive consequence, with the well known temporalization combination mechanism introduced by Finger and Gabbay in [4], when no connectives are shared. In [12] a weak form of this result was proved for semantic entailment (in the global version). A similar result holds also for modalization. We first present a graph-theoretic version of temporalization when no connectives are shared.

The *temporalization* of a deductive system $\mathcal{D}_1$ suitably disjoint with $\mathcal{D}^{\mathrm{ltl}}$ (recall $\mathcal{D}^{\mathrm{ltl}}$ in Example 3.2 and Σ^{ltl} in Example 2.1), produces a deductive system $(\mathrm{T}[\Sigma_1], \mathrm{T}[\Delta_1])$, denoted by

$$\mathrm{T}[\mathcal{D}_1]$$

where $\mathrm{T}[\Sigma_1]$ is the signature Σ^{ltl} enriched with m-edges $\varphi_1 : ! \to \pi_{\mathrm{ltl}}$ for each concrete proper formula φ_1 over Σ_1, and $\mathrm{T}[\Delta_1]$ is the deductive system Δ^{ltl} enriched with the axiom $\varphi_1 : ! \to \pi_{\mathrm{ltl}}$ for each concrete proper theorem φ_1 over the deductive system (Σ_1, Δ_1). Observe that the difference between $\mathrm{LTL}[\Delta_1]$ and $\mathrm{T}[\Delta_1]$, in terms of the deductive system, is that $\mathrm{LTL}[\Delta_1]$, instead of having an axiom for each concrete proper theorem φ_1 in the deductive system (Σ_1, Δ_1), has the rules and axioms of Δ_1 together with the rules IMP and REF.

Consider the map $\cdot^{\urcorner_t}$ from $L(\mathrm{T}[\Sigma_1])$ to $L(\mathrm{LTL}[\Sigma_1])$ (recall $\mathrm{LTL}[\Sigma_1]$ in Example 2.6) inductively defined as follows:

- $(\varphi)^{\urcorner_t}$ is φ if φ is a concrete proper formula over Σ_1;
- $(c\varphi)^{\urcorner_t}$ is $c(\varphi)_{\urcorner_t}$ for c in $\{\neg_{\mathrm{ltl}}, \mathsf{X}, \mathsf{Y}\}$;
- $(c\langle\varphi_1, \varphi_2\rangle)^{\urcorner_t}$ is $c\langle(\varphi_1)_{\urcorner_t}, (\varphi_2)_{\urcorner_t}\rangle$ for c in $\{\Rightarrow_{\mathrm{ltl}}, \mathsf{S}, U\}$;

where $\cdot_{\urcorner_t}$ is the map from $L(\mathrm{T}[\Sigma_1])$ to $L(\mathrm{LTL}[\Sigma_1])$ such that:

- $(\varphi)_{\urcorner_t}$ is $'\varphi'$ if φ is a concrete proper formula over Σ_1;
- $(\varphi)_{\urcorner_t}$ is $(\varphi)^{\urcorner_t}$, otherwise.

In the next proposition, a derivation of $\varphi^{\urcorner_t}$ from $\Gamma^{\urcorner_t}$ in $\mathrm{LTL}[\mathcal{D}_1]$ is obtained from a derivation of φ from Γ in $\mathrm{T}[\mathcal{D}_1]$, by renaming the formulas in the given derivation according to $\cdot^{\urcorner_t}$, and by replacing the basic derivations where a theorem of (Σ_1, Δ_1) is used as an axiom, by its derivation. First we prove that renaming according to $\cdot^{\urcorner_t}$ transforms a derivation over $\mathrm{T}[\mathcal{D}_1]$ to a derivation over $\mathrm{LTL}[\mathcal{D}_1]$ modulo adding some additional hypothesis.

Proposition 3.7 Let $\overline{\langle}\beta_{m1}, \dots, \beta_{mn_m}\overline{\rangle} \mathbin{\bar{\circ}_{\mathsf{v}}} \dots \mathbin{\bar{\circ}_{\mathsf{v}}} \overline{\langle}\beta_{11}, \dots, \beta_{1n_1}\overline{\rangle}$ be a concrete derivation for $\Gamma \vdash_{\mathrm{T}[\mathcal{D}_1]} \varphi$, denoted by δ, where β_{ij} is $(b_{ij}{*}w_{ij}) \mathbin{\bar{\circ}_{\mathsf{v}}} \mathsf{P}^{\langle\varphi_{i1},\dots,\varphi_{ik_i}\rangle}_{a_{ij1},\dots,a_{ij\ell_{ij}}}$. Then,

$$\overline{\langle}(\beta_{m1})^{\urcorner_t}, \dots, (\beta_{mn_m})^{\urcorner_t}\overline{\rangle} \mathbin{\bar{\circ}_{\mathsf{v}}} \dots \mathbin{\bar{\circ}_{\mathsf{v}}} \overline{\langle}(\beta_{11})^{\urcorner_t}, \dots, (\beta_{1n_1})^{\urcorner_t}\overline{\rangle}$$

where, $(\beta_{ij})^{\urcorner_t}$, for $i = 1, \dots, m$ and $j = 1, \dots, n_i$, is

$$(b_{ij}{*}(w_{ij})^{\urcorner_t}) \mathbin{\bar{\circ}_{\mathsf{v}}} \mathsf{P}^{\langle(\varphi_{i1})^{\urcorner_t},\dots,(\varphi_{ik_i})^{\urcorner_t}\rangle}_{a_{ij1},\dots,a_{ij\ell_{ij}}},$$

is a concrete derivation, denoted by $(\delta)^{\natural_t}$, for $(\Gamma)^{\natural_t} \cup \{(\psi)^{\natural_t} : \psi$ is at step 1 of δ and is a concrete proper theorem over $\mathcal{D}_1\} \vdash_{\mathrm{LTL}[\mathcal{D}_1]} (\varphi)^{\natural_t}$.

Proof: The proof follows immediately by induction on the depth of the given derivation. It is enough to see that for any rule b in Δ^{ltl}, $(b{*}w)^{\natural_t}$ is $b{*}(w)^{\natural_t}$ since neither the source of b nor its target has a concrete proper formula over Σ_1 as sub-expression. The same happens if b is of the form $\mathsf{ID}_{\mathsf{id}_v}$. QED

Proposition 3.8 Given a set $\Gamma \cup \{\varphi\}$ of concrete proper formulas over $L(\mathrm{T}[\Sigma_1])$,

$$\Gamma \vdash_{\mathrm{T}[\mathcal{D}_1]} \varphi \quad \text{implies} \quad (\Gamma)^{\natural_t} \vdash_{\mathrm{LTL}[\mathcal{D}_1]} (\varphi)^{\natural_t}.$$

Proof: The proof follows immediately by Proposition 3.7 due to the transitivity of the consequence relation $\vdash_{\mathrm{LTL}[\mathcal{D}_1]}$ since $\vdash_{\mathrm{LTL}[\mathcal{D}_1]} (\varphi)^{\natural_t}$ for any concrete proper theorem φ over (Σ_1, Δ_1). QED

4 Semantics

Having in mind establishing the preservation of soundness and completeness by importing, we now provide for the convenience of the reader a brief summary of the graph-theoretic semantics of importing introduced in [12].

An interpretation, also called a model, over a signature, is an m-graph where the nodes are semantic values and the m-edges are operations on the values, together with functions to relate the semantic values with signature sorts and operations with constructors, see Figure 11. Herein we assume that these functions are total and consider a local version of the entailment introduced in [12].

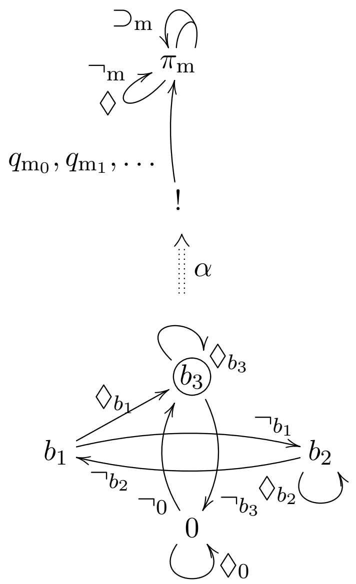

Figure 11: Part of an interpretation for modal logic T without the m-edges for $\supset_{\mathrm{m}}$ and the propositional symbols.

By an *m-graph morphism* $\alpha : G_1 \to G_2$ we mean a pair $\alpha^{\mathsf{v}} : V_1 \to V_2$ and $\alpha^{\mathsf{e}} : E_1 \to E_2$ of maps such that: $\mathsf{src}_2 \circ \alpha^{\mathsf{e}} = \alpha^{\mathsf{v}} \circ \mathsf{src}_1$ and $\mathsf{trg}_2 \circ \alpha^{\mathsf{e}} = \alpha^{\mathsf{v}} \circ \mathsf{trg}_1$. In the sequel we need to refer to the functor α^+ induced by an m-graph morphism α. An *interpretation for a signature* $(G, !, \Pi)$ is a tuple

$$(G', \alpha, D, !)$$

where G' is an m-graph (the *operations m-graph*), $\alpha : G' \to G$ is an m-graph morphism (the *abstraction morphism*) such that $(\alpha^{\mathsf{v}})^{-1}(!)$ is a set (of *concrete values*) containing !, and $D \subseteq (\alpha^{\mathsf{v}})^{-1}(\Pi)$ is a set (of *designated or distinguished values*). Observe that we use ! both for the concrete sort and for the concrete value since the context where they are employed will tell which is being used. We may use I^+ to refer to the category G'^+ of irreducible paths.

We say that a sequence s' of truth values in I^+ *abstracts* to the source of a language expression w in Σ^+ whenever $\alpha^+(s') = \mathsf{src}^+(w)$, and that a semantic expression (i.e., an irreducible path) w' in I^+ *abstracts* to an expression w in Σ^+ whenever $\alpha^+(w') = w$. We denote by $(\alpha^+)^{-1}(w)_{s'}$ the set of semantic irreducible paths in $(\alpha^+)^{-1}(w)$ that start by s'. When $(\alpha^+)^{-1}(w)_{s'}$ is a singleton we may confuse this set with its unique element.

An *interpretation system* $\mathcal{I}$ is a pair $(\Sigma, \mathfrak{I})$ where Σ is a signature and $\mathfrak{I}$ is a class of interpretations for Σ. An *interpretation system* $(\Sigma, \mathfrak{I})$ is *total* whenever all its interpretations are total, and an *interpretation* (Σ, I) is *total* whenever for any connective c in the signature Σ and s' in I^+ that abstracts to the source of c, there is an m-edge e' in I starting at s' that is abstracted to c.

Example 4.1 *An interpretation system for modal logic* T.
The interpretation system $(\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}, \mathfrak{I}^{\mathrm{T}})$ for modal logic T is such that $\mathfrak{I}^{\mathrm{T}}$ is the set of all interpretations for $\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}$ induced by the algebras for modal logic T (see [1, 7]), as defined in [12] (see [1, 7] as references for modal logic). ∇

Example 4.2 *An interpretation system for linear temporal logic.*
The interpretation system $(\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}, \mathfrak{I}^{\mathrm{ltl}})$ for LTL is such that $\mathfrak{I}^{\mathrm{ltl}}$ is the set of all interpretations for $\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}$ induced by strong linear Galois algebras (see [15]), as defined in [12].

Example 4.3 *An interpretation system for intuitionistic propositional logic.*
The interpretation system $(\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}, \mathfrak{I}^{\mathrm{i}})$ for intuitionistic propositional logic is such that $\mathfrak{I}^{\mathrm{i}}$ is the class of all interpretations for $\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}$ induced by a Heyting algebra and a valuation v over the algebra (see [14]), as defined in [12]. ∇

Satisfaction

An interpretation I is *non-deterministic* if it has distinct m-edges with the same source, that are mapped by the abstraction map to the same connective. Since choosing a unique denotation for all the non-deterministic connectives is equivalent to choosing a maximal deterministic sub-interpretation J of that interpretation, denoted by $J \leq I$, we define satisfaction not only with respect to I but also with respect to J. Observe that if I is already deterministic, its only maximal deterministic sub-interpretation is I.

So, given an interpretation I for a signature Σ, a formula φ over Σ, $J \leq I$ and a sequence of truth values s' in I that abstracts to the source of φ, we say that I, J and s' *satisfy* φ, written

$$I, J, s' \Vdash_\Sigma \varphi$$

whenever all the irreducible paths in J^+ starting at s' that abstract to φ, end at a distinguished truth value. Observe that there is at most one such irreducible path in J. In the sequel we assume that the abstraction map of J is β, and write $(\beta^+)^{-1}(\varphi)_{s'}\!\downarrow$ for stating that there is such a path. In that case we denote it by $(\beta^+)^{-1}(\varphi)_{s'}$. When there is no path in J^+ for φ starting at s' we write $(\beta^+)^{-1}(\varphi)_{s'}\!\uparrow$.

Entailment is defined on top of satisfaction as usual. We say that a set Γ of formulas over Σ *locally entails* within $(\Sigma, \mathfrak{I})$ a formula φ over Σ, all with the same source, denoted by

$$\Gamma \vDash^{\mathrm{l}}_{(\Sigma,\mathfrak{I})} \varphi$$

whenever $I, J, s' \Vdash_\Sigma \Gamma$ implies $I, J, s' \Vdash_\Sigma \varphi$, for all I in $\mathfrak{I}$, $J \leq I$ and s' in I^+ abstracted to the source of φ. Moreover we denote $\emptyset \vDash^{\mathrm{l}}_{(\Sigma,\mathfrak{I})} \varphi$ by $\vDash^{\mathrm{l}}_{(\Sigma,\mathfrak{I})} \varphi$ and say that the formula φ is *locally valid* with respect to $(\Sigma, \mathfrak{I})$.

When there is no ambiguity we may omit the reference to the signature and to the interpretation system in the satisfaction $\Vdash$ and entailment $\vDash^{\mathrm{l}}$ symbols respectively. We may also write $\vDash$ instead of $\vDash^{\mathrm{l}}$, and omit the qualification local.

Importing an interpretation system

Semantically, importing is defined at the level of models as explained in [12]. That is, for any given pair of interpretations of the component logics there is an interpretation in the importing, consisting of a faithful copy of each interpretation together with the denotation of the $\Lsh$ connective.

We assume that the interpretation being imported and the importing interpretation, say (Σ_1, I_1) and (Σ_2, I_2) respectively, are *suitably disjoint*, i.e., are interpretations where Σ_1 and Σ_2 are suitably disjoint, $V_1' \setminus (\alpha_1^{\mathsf{e}})^{-1}(!)$ and $V_2' \setminus (\alpha_2^{\mathsf{e}})^{-1}(!)$ are disjoint, $\Lsh_{v_2' v_1'}$ is not in $E_1' \cup E_2'$ for v_2' in V_2' and v_1' in V_1', and E_1' and E_2' are disjoint as well. Similarly for interpretation systems, i.e., that all the pairs with an interpretation of each system is suitably disjoint.

The *importing of an interpretation system* $(\Sigma_1, \mathfrak{I}_1)$ *into an interpretation system* $(\Sigma_2, \mathfrak{I}_2)$, denoted by

$$(\Sigma_2, \mathfrak{I}_2)[(\Sigma_1, \mathfrak{I}_1)],$$

is the interpretation system $(\Sigma_2[\Sigma_1], \mathfrak{I}_2[\mathfrak{I}_1])$ where $\mathfrak{I}_2[\mathfrak{I}_1]$ is the class of interpretations $\{I_2[I_1] : I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}$ over $\Sigma_2[\Sigma_1]$ such that

$$I_2[I_1]$$

is the tuple $((V', E', \mathsf{src}', \mathsf{trg}'), \alpha, D, !)$ with

- V' is $V_1' \cup V_2'$;
- $E' = E_1' \cup E_2' \cup \{\urcorner_{v_2' v_1'} : v_2' \in D_2, v_1' \in D_1\} \cup \{\urcorner_{v_2' v_1'} : v_2' \in \alpha_2^{-1}(\Pi_2) \setminus D_2, \alpha_1^{-1}(\Pi_1) \setminus D_1\}$;
- src' and trg' are such that $\mathsf{src}'(\urcorner_{v_2' v_1'}) = v_1'$, $\mathsf{trg}'(\urcorner_{v_2' v_1'}) = v_2'$, $\mathsf{src}'(e') = \mathsf{src}_k'(e')$ and $\mathsf{trg}'(e') = \mathsf{trg}_k'(e')$ for e' in E_k' and $k = 1, 2$;
- α is such that $\alpha^{\mathsf{v}}(v') = \alpha_k^{\mathsf{v}}(v')$ whenever v' is in V_k' for $k = 1, 2$, $\alpha^{\mathsf{e}}(e') = \alpha_k^{\mathsf{e}}(e')$ whenever e' is in E_k' for $k = 1, 2$ and $\alpha^{\mathsf{e}}(\urcorner_{v_2' v_1'}) = \urcorner_{\alpha^{\mathsf{v}}(v_2')\alpha^{\mathsf{v}}(v_1')}$;
- D is $D_1 \cup D_2$.

We recall some particular cases of importing described in [12], and introduce a new example.

Example 4.4 We denote by

$$\mathrm{LTL}[(\Sigma_1, \mathfrak{I}_1)]$$

the $\urcorner$-*temporalization* of an interpretation system $(\Sigma_1, \mathfrak{I}_1)$ suitably disjoint with $(\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}, \mathfrak{I}^{\mathrm{ltl}})$, and by

$$\mathrm{M_T}[(\Sigma_1, \mathfrak{I}_1)]$$

the $\urcorner$-*modalization* by modal logic T of $(\Sigma_1, \mathfrak{I}_1)$ suitably disjoint with $(\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}, \mathfrak{I}^{\mathrm{m}})$, as defined in [12]. By *adding a* $\urcorner$-*constructivist dimension* to $(\Sigma_1, \mathfrak{I}_1)$, suitably disjoint with $(\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}, \mathfrak{I}^{\mathrm{i}})$, denoted by

$$\mathrm{I}[(\Sigma_1, \mathfrak{I}_1)],$$

we mean the importing of $(\Sigma_1, \mathfrak{I}_1)$ into the interpretation system $(\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}, \mathfrak{I}^{\mathrm{i}})$ for intuitionistic logic introduced in Example 4.3. ∇

5 Preservation of soundness

In this section we show that soundness is preserved, under some conditions, by importing. First we need to introduce logic systems.

A *logic system* is a triple $(\Sigma, \Delta, \mathfrak{I})$ where (Σ, Δ) is a deductive system and $(\Sigma, \mathfrak{I})$ is an interpretation system. By a *total* logic system we mean a logic system whose underlying interpretation system is total. A logic system $(\Sigma, \Delta, \mathfrak{I})$ is *sound* whenever

$$\text{if} \quad \Gamma \vdash_{(\Sigma,\Delta)} \varphi \quad \text{then} \quad \Gamma \vDash_{(\Sigma,\mathfrak{I})} \varphi$$

for any set $\Gamma \cup \{\varphi\}$ of proper formulas over Σ, and is *complete* whenever

$$\text{if} \quad \Gamma \vDash_{(\Sigma,\mathfrak{I})} \varphi \quad \text{then} \quad \Gamma \vdash_{(\Sigma,\Delta)} \varphi$$

for any set $\Gamma \cup \{\varphi\}$ of proper formulas over Σ. Moreover, it is *concretely complete* whenever $\Gamma \cup \{\varphi\}$ is any set of concrete proper formulas over Σ.

Example 5.1 We denote by $\mathcal{L}^{\mathrm{T}}$ the logic system $(\Sigma^{\mathrm{m}}_{Q^{\mathrm{m}}}, \Delta^{\mathrm{T}}, \mathfrak{I}^{\mathrm{T}})$ for modal logic T, by $\mathcal{L}^{\mathrm{ltl}}$ the logic system $(\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}, \Delta^{\mathrm{ltl}}, \mathfrak{I}^{\mathrm{ltl}})$ for linear temporal logic and by $\mathcal{L}^{\mathrm{i}}$ the logic system $(\Sigma^{\mathrm{i}}_{Q^{\mathrm{i}}}, \Delta^{\mathrm{i}}, \mathfrak{I}^{\mathrm{i}})$ for intuitionistic logic. ∇

Importing a logic system into another is defined in terms of their semantic and deductive components, and so it is only applied to *suitably disjoint* logic systems, i.e., logic systems with suitably disjoint signatures, suitably disjoint deductive systems and suitably disjoint interpretation systems.

Hence, *importing a logic system* $(\Sigma_1, \Delta_1, \mathfrak{I}_1)$ *into a logic system* $(\Sigma_2, \Delta_2, \mathfrak{I}_2)$, denoted by

$$(\Sigma_2, \Delta_2, \mathfrak{I}_2)[(\Sigma_1, \Delta_1, \mathfrak{I}_1)],$$

is the logic system $(\Sigma_2[\Sigma_1], \Delta_2[\Delta_1], \mathfrak{I}_2[\mathfrak{I}_1])$.

Example 5.2 The ↰-*temporalization* of a logic system $\mathcal{L}_1$ suitably disjoint with $\mathcal{L}^{\mathrm{ltl}}$, denoted by

$$\mathrm{LTL}[\mathcal{L}_1]$$

is the logic system resulting from importing $\mathcal{L}_1$ into $\mathcal{L}^{\mathrm{ltl}}$. Moreoever, the ↰-*modalization* by modal logic T of logic system $\mathcal{L}_1$ suitably disjoint with $\mathcal{L}^{\mathrm{T}}$, denoted by

$$\mathrm{M}_{\mathrm{T}}[\mathcal{L}_1]$$

is the logic system resulting from importing $\mathcal{L}_1$ into $\mathcal{L}^{\mathrm{T}}$. By *adding a* ↰-*constructivist dimension* to $\mathcal{L}_1$, suitably disjoint with $\mathcal{L}^{\mathrm{i}}$, denoted by

$$\mathrm{I}[\mathcal{L}_1],$$

we mean the importing of $\mathcal{L}_1$ into the logic system $\mathcal{L}^{\mathrm{i}}$ for intuitionistic logic. ∇

Soundness

We now establish sufficient conditions for a logic system to be sound, and then investigate whether these conditions are preserved by importing.

Given a logic system $(\Sigma, \Delta, \mathfrak{I})$ and an interpretation I in $\mathfrak{I}$, an inference δ in Σ^{Δ} from $\langle \psi_1, \ldots, \psi_m \rangle$ to $\langle \varphi_1, \ldots, \varphi_n \rangle$ is *sound for* I whenever

$$I, J, s' \Vdash \{\psi_1, \ldots, \psi_m\} \quad \text{implies} \quad I, J, s' \Vdash \varphi_j$$

for all $J \leq I$, s' in I^+ that abstracts to the source of φ_j, and j in $\{1, \ldots, n\}$. The inference δ is said to be *sound* in $(\Sigma, \Delta, \mathfrak{I})$ whenever it is sound for all interpretations in $\mathfrak{I}$.

In order to prove that total logic systems with sound rules and valid axioms are sound, we show, under general conditions, that inference soundness is preserved by all the constructions (that is, instantiation, 2-tupling and composition) used in a derivation. We consider total logic systems since they are well behaved with respect to substitution, as we will see in the next technical results.

Proposition 5.3 Given a total interpretation I over a signature Σ, $J \le I$ with abstraction map β, an irreducible path w in Σ^+, and s' in I^+ that abstracts by β to the source of w, then $(\beta^+)^{-1}(w)_{s'}\downarrow$.

Proof: The proof follows by induction on w: (1) w is ϵ_s. Then $(\beta^+)^{-1}(w)_{s'}$ is $\epsilon_{s'}$ and so is defined; (2) w is p_j^s. Then $(\beta^+)^{-1}(w)_{s'}$ is $\mathsf{p}_j^{s'}$ and so is defined; (3) w is ew_0. Observe that $(\beta^+)^{-1}(w_0)_{s'}\downarrow$ by induction hypothesis, and that the target of $(\beta^+)^{-1}(w_0)_{s'}$ abstracts to the target of w_0 which coincides with the source of e. So $\beta^{-1}(e)_{\mathsf{trg}'^+((\beta^+)^{-1}(w_0)_{s'})}\downarrow$ since I is total. Hence $(\beta^+)^{-1}(ew_0)_{s'} = \beta^{-1}(e)_{\mathsf{trg}'^+((\beta^+)^{-1}(w_0)_{s'})}(\beta^+)^{-1}(w_0)_{s'}\downarrow$ is defined; (4) w is $\langle w_1, \ldots, w_n\rangle$. Observe that $(\beta^+)^{-1}(w_i)_{s'}$ is defined for $i = 1, \ldots, n$ by induction hypothesis. Hence $(\beta^+)^{-1}(w)_{s'} = \langle(\beta^+)^{-1}(w_1)_{s'}, \ldots, (\beta^+)^{-1}(w_n)_{s'}\rangle$ is also defined. QED

The following result states that the denotation of a composition is the composition of the denotations, and establishes its counterpart on satisfaction.

Proposition 5.4 Given a total interpretation I over a signature Σ, $J \le I$ with abstraction map β, irreducible paths w_1 and w_2 in Σ^+ with $\mathsf{src}^+(w_2) = \mathsf{trg}^+(w_1)$, and s' in I^+ that abstracts by β to the source of w_1, then

$$(\beta^+)^{-1}(w_2 \circ w_1)_{s'} = (\beta^+)^{-1}(w_2)_{\mathsf{trg}'^+((\beta^+)^{-1}(w_1)_{s'})} \circ (\beta^+)^{-1}(w_1)_{s'}.$$

Moreover,

$$I, J, s' \Vdash \varphi \circ w \quad \text{iff} \quad I, J, \mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'}) \Vdash \varphi.$$

Proof: The proof of the first assertion is omitted since it follows by a straightforward induction on w_1. We now concentrate on the proof of the second assertion.
($\Rightarrow$) Observe that, by the first assertion, $\mathsf{trg}'^+((\beta^+)^{-1}(\varphi)_{\mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'})})$ is $\mathsf{trg}'^+((\beta^+)^{-1}(\varphi)_{\mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'})} \circ (\beta^+)^{-1}(w)_{s'}) = \mathsf{trg}'^+((\beta^+)^{-1}(\varphi \circ w)_{s'}) \in D$ since $I, J, s' \Vdash \varphi \circ w$;
($\Leftarrow$) Observe that, by the first assertion, $\mathsf{trg}'^+((\beta^+)^{-1}(\varphi \circ w)_{s'}) = \mathsf{trg}'^+((\beta^+)^{-1}(\varphi)_{\mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'})} \circ (\beta^+)^{-1}(w)_{s'}) = \mathsf{trg}'^+((\beta^+)^{-1}(\varphi)_{\mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'})}) \in D$ since $I, J, \mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'}) \Vdash \varphi$. QED

We now prove that soundness is preserved by the constructions employed in derivations.

Proposition 5.5 The instantiation of an inference preserves soundness in total logic systems.

Proof: Let $(\Sigma, \Delta, \mathfrak{I})$ be a total logic system and δ a sound inference in Σ^Δ with antecedent $\langle\psi_1, \ldots, \psi_m\rangle$ and consequent $\langle\varphi_1, \ldots, \varphi_n\rangle$. Moreover, let w be an expression in Σ^+ compatible with the formulas in the antecedent and consequent of δ. We now show that $\delta * w$ is a sound inference. Let j be in $\{1, \ldots, n\}$, I be an interpretation in $\mathfrak{I}$, $J \le I$, and s' in I^+ that abstracts to the source of w such that $I, J, s' \Vdash \{\psi_1 \circ w, \ldots, \psi_m \circ w\}$. So $I, J, \mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'}) \Vdash \{\psi_1, \ldots, \psi_m\}$ by Proposition 5.4. Hence $I, J, \mathsf{trg}'^+((\beta^+)^{-1}(w)_{s'}) \Vdash \varphi_j$ by the soundness of δ. So by Proposition 5.4, $I, J, s' \Vdash \varphi_j \circ w$. QED

Proposition 5.6 The 2-tupling of inferences with a proper formula as consequent and with the same antecedent, preserves soundness.

Proof: Let $(\Sigma, \Delta, \mathfrak{I})$ be a logic system and $\beta_1, \ldots, \beta_n$ sound inferences in Σ^{Δ} with a proper formula φ_j as consequent for $j = 1, \ldots, n$ respectively, and with the same antecedent $\langle \psi_1, \ldots, \psi_m \rangle$. We now show that $\overline{\langle} \beta_1, \ldots, \beta_n \overline{\rangle}$ is a sound inference. Let j in $\{1, \ldots, n\}$, I be an interpretation in $\mathfrak{I}$, $J \leq I$, and s' in I^+ abstracting to the common source of the formulas in the antecedent of the inferences, such that $I, J, s' \Vdash \{\psi_1, \ldots, \psi_m\}$. Then $I, J, s' \Vdash \varphi_j$ since β_j is sound. QED

Proposition 5.7 The vertical composition of compatible inferences preserves soundness.

Proof: Let $(\Sigma, \Delta, \mathfrak{I})$ be a logic system and $\delta_1 : \langle \psi_1, \ldots, \psi_m \rangle \Rightarrow \langle \gamma_1, \ldots, \gamma_o \rangle$ and $\delta_2 : \langle \gamma_1, \ldots, \gamma_o \rangle \Rightarrow \langle \varphi_1, \ldots, \varphi_n \rangle$ sound inferences in Σ^{Δ}. We now show that $\delta_2 \mathbin{\overline{\circ}_{\mathsf{v}}} \delta_1$ is a sound inference. Let I be an interpretation in $\mathfrak{I}$, $J \leq I$ and s' in I^+ abstracting to the source of any formula in the antecedent of δ_1 such that $I, J, s' \Vdash \{\psi_1, \ldots, \psi_m\}$. Then $I, J, s' \Vdash \{\gamma_1, \ldots, \gamma_o\}$ by the soundness of δ_1, and so $I, J, s' \Vdash \{\varphi_1, \ldots, \varphi_n\}$ by the soundness of δ_2. QED

Proposition 5.8 Every derivation is sound in a total logic system where the rules are sound.

Proof: Let $\mathcal{L}$ be a total logic system and assume that δ is a derivation of the form $\overline{\langle} \beta_{m1}, \ldots, \beta_{mn_m} \overline{\rangle} \mathbin{\overline{\circ}_{\mathsf{v}}} \ldots \mathbin{\overline{\circ}_{\mathsf{v}}} \overline{\langle} \beta_{11}, \ldots, \beta_{1n_1} \overline{\rangle}$ with antecedent $\langle \psi_1, \ldots, \psi_m \rangle$. Let $((b_{xy} * \varphi_{xy}) \mathbin{\overline{\circ}_{\mathsf{v}}} \mathsf{P}^{\langle \vec{\varphi}_x \rangle}_{\vec{\jmath}_{xy}})$ be the basic derivation β_{xy}. Observe that $\mathsf{P}^{\langle \vec{\varphi}_x \rangle}_{\vec{\jmath}_{xy}}$ is a sound inference as well as any 2-cell identity (for vertical composition) over a proper formula. So, according to Proposition 5.5 and Proposition 5.7, each basic derivation β_{xy} is sound. Hence each step of the derivation is sound by Proposition 5.6 and so δ is sound by Proposition 5.7. QED

Theorem 5.9 (Soundness)
A total logic system is sound if and only if it has sound rules and valid axioms.

Proof: Let $\mathcal{L}$ be a total logic system. $(\leftarrow)$ Assume that δ is a derivation for $\Gamma \vdash \varphi$. Denote the antecedent of δ by $\langle \psi_1, \ldots, \psi_m \rangle$ where ψ_j is either in Γ or is an axiom. Observe that δ is sound by Proposition 5.8. Let I be an interpretation in $\mathfrak{I}$, $J \leq I$ and s' in I^+ abstracted to the source of φ such that $I, J, s' \Vdash \Gamma$. So $I, J, s' \Vdash \{\psi_1, \ldots, \ldots \psi_m\}$ taking into account that ψ_j is either in Γ or is an axiom instance, and that axioms are valid. Hence $I, J, s' \Vdash \varphi$ by the soundness of δ. $(\rightarrow)$ Let r be a rule in $\mathcal{L}$ from $\langle \psi_1, \ldots, \psi_m \rangle$ to φ, I an interpretation in $\mathfrak{I}$, $J \leq I$ and s' in I^+ abstracted to the source of φ. Consider two cases: (i) r is a non-axiomatic rule. Assume that $I, J, s' \Vdash \{\psi_1, \ldots, \psi_m\}$. Then $I, J, s' \Vdash \varphi$ since $\{\psi_1, \ldots, \psi_m\} \vdash \varphi$ and since $\mathcal{L}$ is sound; (ii) r is an axiom. Then $\vdash \varphi$ and so $I, J, s' \Vdash \varphi$ since $\mathcal{L}$ is sound. QED

Soundness preservation

The idea to show that soundness is preserved by importing, is to prove that the sufficient conditions for a logic to be sound (established in Theorem 5.9) are preserved by importing. It is immediate to prove that being total is preserved, so we concentrate now on preservation, by importing, of soundness of rules and validity of axioms.

In the sequel, given a suitably disjoint pair of total interpretations I_1 and I_2 over Σ_1 and Σ_2 respectively, k in $\{1,2\}$, and $J \leq (\Sigma_2, I_2)[(\Sigma_1, I_1)]$ with abstraction map β, we denote by $(\beta^+)_{\downarrow_k}$ the restriction of β^+ to Σ_k^+. Moreover we denote by $J_{\downarrow_k}$ the maximal sub-interpretation of J with $J_{\downarrow_k} \leq I_k$, and denote its abstraction map by $\beta_{\downarrow_k}$.

Proposition 5.10 Let w be an expression in Σ_k^+ and s' in I_k^+ abstracted by α_k to the source of w. Then $(\beta^+)^{-1}(w)_{s'} = ((\beta_{\downarrow_k})^+)^{-1}(w)_{s'}$. Moreover,

$$I_2[I_1], J, s' \Vdash \varphi \quad \text{if and only if} \quad I_k, J_{\downarrow_k}, s' \Vdash \varphi.$$

Proof: The proof of the first assertions follows by induction on w:

(1) w is ϵ_s. Then $(\beta^+)^{-1}(w)_{s'} = \epsilon_{s'} = ((\beta_{\downarrow_k})^+)^{-1}(w)_{s'}$;

(2) w is p_j^s. The proof of this case is similar to the proof of (1) so we omit it;

(3) w is ew_0. Therefore $(\beta^+)^{-1}(w)_{s'} = \beta^{-1}(e)_{\mathsf{trg}'^+((\beta^+)^{-1}(w_0)_{s'})}(\beta^+)^{-1}(w_0)_{s'} = (\beta_{\downarrow_k})^{-1}(e)_{\mathsf{trg}'^+(((\beta_{\downarrow_k})^+)^{-1}(w_0)_{s'})}((\beta_{\downarrow_k})^+)^{-1}(w_0)_{s'}$ which is $((\beta_{\downarrow_k})^+)^{-1}(w)_{s'}$;

(4) w is $\langle w_1, \ldots, w_m\rangle$. Hence $(\beta^+)^{-1}(w)_{s'} = \langle(\beta^+)^{-1}(w_1)_{s'}, \ldots, (\beta^+)^{-1}(w_m)_{s'}\rangle$ which by induction hypothesis is $\langle((\beta_{\downarrow_k})^+)^{-1}(w_1)_{s'}, \ldots, ((\beta_{\downarrow_k})^+)^{-1}(w_m)_{s'}\rangle = ((\beta_{\downarrow_k})^+)^{-1}(w)_{s'}$.

We now prove the second assertion. In fact $I_2[I_1], J, s' \Vdash \varphi$ if and only if $\mathsf{trg}'^+((\beta^+)^{-1}(\varphi)_{s'}) \in D$ if and only if $\mathsf{trg}'^+(((\beta_{\downarrow_k})^+)^{-1}(\varphi)_{s'}) \in D$ (by the first assertion) if and only if $I_k, J_{\downarrow_k}, s' \Vdash \varphi$. QED

Proposition 5.11 Soundness of inferences is preserved by importing when the given logic systems are total and suitably disjoint.

Proof: Let $(\Sigma_1, \Delta_1, \mathfrak{I}_1)$ and $(\Sigma_2, \Delta_2, \mathfrak{I}_2)$ be a suitably disjoint pair of total logic systems, k in $\{1,2\}$, δ be a sound inference in $\Sigma_k^{\Delta_k}$ from $\langle\psi_1, \ldots, \psi_m\rangle$ to $\langle\varphi_1, \ldots, \varphi_n\rangle$, I_1 and I_2 interpretations in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively, $J \leq I_2[I_1]$ and s' in $I_2[I_1]^+$ abstracted to the common source of the formulas in the antecedent of δ. Observe that $\psi_1, \ldots, \psi_m, \varphi_1, \ldots, \varphi_n$ are formulas of Σ_k^+, and s' is in I_k^+ and abstracts to the common source of the formulas in the antecedent of δ. Suppose that $I_2[I_1], J, s' \Vdash \{\psi_1, \ldots, \psi_m\}$ and let j be in $\{1, \ldots, n\}$. Then $I_k, J_{\downarrow_k}, s' \Vdash \{\psi_1, \ldots, \psi_m\}$ by Proposition 5.10 and so $I_k, J_{\downarrow_k}, s' \Vdash \varphi_j$ since δ is a sound inference in $\Sigma_k^{\Delta_k}$. Hence $I_2[I_1], J, s' \Vdash \varphi_j$ by Proposition 5.10. QED

Proposition 5.12 Validity is preserved by importing when the given logic systems are total and suitably disjoint.

Proof: Let $(\Sigma_1, \Delta_1, \mathfrak{I}_1)$ and $(\Sigma_2, \Delta_2, \mathfrak{I}_2)$ be a suitably disjoint pair of total logic systems, k in $\{1, 2\}$, φ a valid formula in Σ_k^+, I_1 and I_2 interpretations in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively, $J \leq I_2[I_1]$ and s' in $I_2[I_1]^+$ abstracted to the source of φ. Observe that s' is in I_k^+ and is also abstracted by α_k^+ to the source of φ. Then $I_k, J_{\downarrow_k}, s' \Vdash \varphi$ since φ is valid in $(\Sigma_k, \Delta_k, \mathfrak{I}_k)$. Hence $\mathsf{trg}_k'^+(((\beta_{\downarrow_k})^+)^{-1}(\varphi)_{s'}) \in D_k$ and so the thesis follows since $((\beta_{\downarrow_k})^+)^{-1}(\varphi)_{s'} = (\beta^+)^{-1}(\varphi)_{s'}$ by Proposition 5.10 and since $D_k \subseteq D_{I_2[I_1]}$. QED

Proposition 5.13 Rules IMP and REF are sound in the logic system resulting from importing when the given logic systems are total and suitably disjoint.

Proof: Let $(\Sigma_1, \Delta_1, \mathfrak{I}_1)$ and $(\Sigma_2, \Delta_2, \mathfrak{I}_2)$ be a suitably disjoint pair of total logic systems, I_1 and I_2 interpretations in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively, $J \leq I_2[I_1]$ and v_1' a truth value of I_1. (1) IMP is sound. Suppose that $I_2[I_1], J, v_1' \Vdash \mathsf{id}_{\pi_1}$. Then $\mathsf{trg}'^+((\beta^+)^{-1}(\mathsf{id}_{\pi_1})_{v_1'}) \in D_{I_2[I_1]}$, that is, $v_1' \in D_1$. Hence $\mathsf{trg}'^+((\beta^+)^{-1}(\daleth)_{v_1'}) \in D_{I_2[I_1]}$ by definition of $I_2[I_1]$, and so $I_2[I_1], J, v_1' \Vdash \daleth$; (2) REF is sound. Suppose that $I_2[I_1], J, v_1' \Vdash \daleth$. Then $\mathsf{trg}'^+((\beta^+)^{-1}(\daleth)_{v_1'}) \in D_{I_2[I_1]}$ and so $v_1' \in D_1$ by definition of $I_2[I_1]$. Hence $\mathsf{trg}'^+((\beta^+)^{-1}(\mathsf{id}_{\pi_1})_{v_1'}) \in D_{I_2[I_1]}$ and so $I_2[I_1], J, v_1' \Vdash \mathsf{id}_{\pi_1}$. QED

So we can now establish a sufficient condition for the preservation of soundness by importing.

Theorem 5.14 (Soundness preservation)
The logic system resulting from an importing is sound whenever the given logic systems are sound, total, and suitably disjoint.

Proof: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be a suitably disjoint pair of sound and total logic systems. Then their rules and axioms are sound and valid, by Theorem 5.9. Then by Proposition 5.11 and Proposition 5.13 all the rules of $\mathcal{L}_2[\mathcal{L}_1]$ are sound, and by Proposition 5.12 all the axioms of $\mathcal{L}_2[\mathcal{L}_1]$ are valid. Moreover as can be seen immediately by definition of importing, $\mathcal{L}_2[\mathcal{L}_1]$ is total. Hence $\mathcal{L}_2[\mathcal{L}_1]$ is sound by Proposition 5.9. QED

Corollary 5.15 The $\daleth$-modalization by modal logic T of a sound and total logic system suitably disjoint with $\mathcal{L}^{\mathrm{T}}$, is sound. Similarly for $\daleth$-temporalization and for adding a $\daleth$-constructivist dimension to a logic system.

6 Preservation of concrete completeness

In order to show that concrete completeness is preserved by importing we assume that the given logic systems have certain canonical interpretations. These canonical interpretations are such that, when combined, produce interpretations that guarantee that the logic system resulting from the importing is concretely complete.

In order to simplify the presentation, we assume fixed a suitably disjoint pair $(\Sigma_1, \Delta_1, \mathfrak{I}_1)$ and $(\Sigma_2, \Delta_2, \mathfrak{I}_2)$ of concretely complete logic systems, denoted by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, and assume fixed k in $\{1, 2\}$.

The *canonical interpretation for* Σ_k *induced by* $\mathcal{L}_2[\mathcal{L}_1]$ *and by a set* Γ *of concrete formulas in the language of* $\mathcal{L}_2[\mathcal{L}_1]$, denoted by

$$\mathbb{I}_{\Gamma_k},$$

is the interpretation $(\mathbb{G}', \alpha, \mathbb{D}, \mathsf{id}_!)$ defined as follows:

- $\mathbb{V}'$ is $\{w$ is a concrete expression over $\Sigma_2[\Sigma_1] : \mathsf{trg}^+(w) \in V_k\}$;
- $e_{w_1 \ldots w_m} \in \mathbb{E}'(w_1 \ldots w_m, e\langle w_1, \ldots, w_m\rangle)$ if and only if $w_1, \ldots, w_m$ is in $\mathbb{V}'$, e is in E_k and the source of e coincides with the target of $\langle w_1, \ldots, w_m\rangle$;
- $\mathbb{D} = \{\varphi$ is a formula in $\mathbb{V}' : \Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi\}$;
- $\alpha^{\mathsf{v}}(w)$ is the target of w and $\alpha^{\mathsf{e}}(e_{w_1 \ldots w_m})$ is e.

We now show that the rules of $\mathcal{L}_k$ are sound with respect to these canonical interpretations. That would mean that the given (concretely complete) logic systems can be enriched with these interpretations without affecting their entailments. We prove first some auxiliary results.

Proposition 6.1 The canonical interpretation $\mathbb{I}_{\Gamma_k}$ is total and deterministic.

Proof: It is enough to observe that for any elements $w_1, \ldots, w_n$ of $\mathbb{V}'$ and e in E_k with the source of e coinciding with the target of $\langle w_1, \ldots, w_n\rangle$, the set $\{e' \in \mathbb{E}' : \alpha^{\mathsf{e}}(e') = e$ and the source of e' is $w_1 \ldots w_n\}$ is, by definition of canonical interpretation, a singleton. QED

Observe that the unique maximal deterministic sub-interpretation of the canonical interpretation $\mathbb{I}_{\Gamma_k}$ coincides with it, since $\mathbb{I}_{\Gamma_k}$ is deterministic. So its abstraction map is α.

Proposition 6.2 Let $w_1, \ldots, w_m$ be expressions in $\mathbb{V}'$ and w an expression in Σ_k^+ with source coinciding with the target of $\langle w_1, \ldots, w_m\rangle$. Denote the target of $(\alpha^+)^{-1}(w)_{w_1 \ldots w_m}$ in $\mathbb{I}_{\Gamma_k}$ by $w'_1 \ldots w'_n$, then

$$\langle w'_1, \ldots, w'_n\rangle = w \circ \langle w_1, \ldots, w_m\rangle.$$

Proof: The proof is carried out by induction on w:

(1) w is $\epsilon_{v_1 \ldots v_m}$. Hence $(\alpha^+)^{-1}(w)_{w_1 \ldots w_m}$ is $\epsilon_{w_1 \ldots w_m}$ and so its target is $w_1 \ldots w_m$. The thesis follows since $\langle w_1, \ldots, w_m\rangle$ is $w \circ \langle w_1, \ldots, w_m\rangle$;

(2) w is p_j^s. The proof of this case is similar to the proof of (1) so we omit it;

(3) w is $\langle w_{01}, \ldots, w_{0n}\rangle$. Therefore $(\alpha^+)^{-1}(w)_{w_1 \ldots w_m}$ is $\langle(\alpha^+)^{-1}(w_{01})_{w_1 \ldots w_m}, \ldots, (\alpha^+)^{-1}(w_{0n})_{w_1 \ldots w_m}\rangle$. Since, for $j = 1, \ldots, m$, the target of $(\alpha^+)^{-1}(w_{0j})_{w_1 \ldots w_m}$ is a sequence with only one element, by induction hypothesis it is $w_{0j} \circ \langle w_1, \ldots, w_m\rangle$. Hence the target of $(\alpha^+)^{-1}(w)_{w_1 \ldots w_m}$ is $w_{01} \circ \langle w_1, \ldots, w_m\rangle \ldots w_{0n} \circ \langle w_1, \ldots, w_m\rangle$. The thesis follows since $\langle w_{01} \circ \langle w_1, \ldots, w_m\rangle, \ldots, w_{0n} \circ \langle w_1, \ldots, w_m\rangle\rangle$ is equal to $\langle w_{01}, \ldots, w_{0n}\rangle \circ \langle w_1, \ldots, w_m\rangle$;

(4) w is ew_0. Denote the target of $(\alpha^+)^{-1}(w_0)_{w_1 \ldots w_m}$ in $\mathbb{I}_{\Gamma_k}$ by $w'_{01} \ldots w'_{0n}$.

Hence the target of $(\alpha^+)^{-1}(w)_{w_1\dots w_m}$ is the target of $(\alpha^{\mathsf{e}})^{-1}(e)_{w'_{01}\dots w'_{0n}}$ which is $e\langle w'_{01},\dots,w'_{0n}\rangle$. By induction hypothesis $\langle w'_{01},\dots,w'_{0n}\rangle$ is $w_0 \circ \langle w_1,\dots,w_m\rangle$, and so the target of $(\alpha^+)^{-1}(w)_{w_1\dots w_m}$ is $e \circ (w_0 \circ \langle w_1,\dots,w_m\rangle)$ which is $w \circ \langle w_1,\dots,w_m\rangle$. QED

As an illustration, let c be a constructor of Σ_2 with source $\pi_2\pi_2$ and target π_2, and φ_1 and φ_2 concrete formulas in $\Sigma_2[\Sigma_1]^+$ with target π_2. Then

$$(\alpha^+)^{-1}(\langle \mathsf{p}_1^{\pi_2\pi_2}, c\rangle)_{\varphi_1\varphi_2}$$

is $\langle \mathsf{p}_1^{\varphi_1\varphi_2}, c_{\varphi_1\varphi_2}\rangle$ by definition of canonical interpretation, and its target is the sequence $\varphi_1\; c\langle\varphi_1,\varphi_2\rangle$. Moreover $\langle\varphi_1, c\langle\varphi_1,\varphi_2\rangle\rangle = \langle \mathsf{p}_1^{\pi_2\pi_2}, c\rangle \circ \langle\varphi_1,\varphi_2\rangle$.

The following result establishes the expected interconnection between derivation and satisfaction in a canonical interpretation.

Proposition 6.3 Given expressions $w_1,\dots,w_m$ in $\mathbb{V}'$ and an expression w in Σ_k^+ with source coinciding with the target of $\langle w_1,\dots,w_m\rangle$,

$$\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi \circ \langle w_1,\dots,w_m\rangle$$

if and only if

$$\mathbb{I}_{\Gamma_k}, \mathbb{I}_{\Gamma_k}, w_1\dots w_m \Vdash \varphi.$$

Proof:
($\Rightarrow$) Assume that $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi \circ \langle w_1,\dots,w_m\rangle$. Then $\varphi \circ \langle w_1,\dots,w_m\rangle$ is in $\mathbb{D}$. Observe that the target of $(\alpha^+)^{-1}(\varphi)_{w_1\dots w_m}$ is a sequence with a unique element equal to $\varphi \circ \langle w_1,\dots,w_m\rangle$ by Proposition 6.2. So $\mathbb{I}_{\Gamma_k}, \mathbb{I}_{\Gamma_k}, w_1\dots w_m \Vdash \varphi$;
($\Leftarrow$) Assume that $\mathbb{I}_{\Gamma_k}, \mathbb{I}_{\Gamma_k}, w_1\dots w_m \Vdash \varphi$. So the target of $(\alpha^+)^{-1}(\varphi)_{w_1\dots w_m}$ is in $\mathbb{D}$. Observe that the target of $(\alpha^+)^{-1}(\varphi)_{w_1\dots w_m}$ is a sequence with a unique element equal to $\varphi\circ\langle w_1,\dots,w_m\rangle$ by Proposition 6.2. So $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi\circ\langle w_1,\dots,w_m\rangle$ by definition of $\mathbb{D}$. QED

We say that an interpretation is a *structure for* a logic system if all the rules and axioms in the logic system are respectively sound for and satisfied by that interpretation. Recall the notion of a rule be sound for an interpretation in the beginning of Section 5.

Proposition 6.4 The interpretation $\mathbb{I}_{\Gamma_k}$ is a structure for $\mathcal{L}_k$.

Proof: (1) Let r be a non-axiomatic rule in $\mathcal{L}_k$ with $\langle\psi_1,\dots,\psi_m\rangle$ as premise γ as conclusion, and $w_1,\dots,w_m$ in $\mathbb{V}'$ such that the source of γ coincides with the target of $\langle w_1,\dots, w_m\rangle$. Assume that $\mathbb{I}_{\Gamma_k}, w_1\dots w_m \Vdash \{\psi_1,\dots,\psi_m\}$. Then, by Proposition 6.3, $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \psi_j \circ \langle w_1,\dots,w_m\rangle$ for $j = 1,\dots,m$. Hence $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \gamma\circ\langle w_1,\dots,w_m\rangle$ using rule r. Therefore, again by Proposition 6.3, we conclude $\mathbb{I}_{\Gamma_k}, \mathbb{I}_{\Gamma_k}, w_1\dots w_m \Vdash \gamma$.
(2) Let φ be an axiom of $\mathcal{L}_k$ and $w_1,\dots,w_m$ in $\mathbb{V}'$ such that the source of φ coincides with the target of $\langle w_1,\dots,w_m\rangle$. Note that $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi\circ\langle w_1,\dots,w_m\rangle$ and so $\varphi\circ\langle w_1,\dots,w_m\rangle$ is in $\mathbb{D}$, and that $\varphi\circ\langle w_1,\dots,w_m\rangle$ is the target of $(\alpha^+)^{-1}(\varphi)_{w_1\dots w_m}$ by Proposition 6.2. Hence $\mathbb{I}_{\Gamma_k}, \mathbb{I}_{\Gamma_k}, w_1\dots w_m \Vdash \varphi$ by Proposition 6.1. QED

We now study the properties of the interpretation $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]$, which is in the importing of $\mathcal{L}_1$ into $\mathcal{L}_2$ whenever they contain $\mathbb{I}_{\Gamma_1}$ and $\mathbb{I}_{\Gamma_2}$ respectively. The following result establishes that $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]$ is deterministic for all constructors except the importing constructor, and is total.

Proposition 6.5 Given $w_1, \ldots, w_n$ in $V'_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$ and a constructor c in $E_{\Sigma_2[\Sigma_1]} \setminus \{\urcorner\}$ with source coinciding with the target of $\langle w_1, \ldots, w_n\rangle$, the set $\{e' \in E'_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]} :$ the source of e' is $w_1 \ldots w_n$ and e' abstracts to $e\}$ is a singleton. Moreover $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]$ is total.

We omit the proof of this proposition since it follows immediately by definition of total interpretation, of importing and by Proposition 6.1.

We denote by

$$J_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$$

the maximal deterministic sub-interpretation of $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]$ with abstraction map $\beta_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$ such that $(\beta^{\mathsf{e}}_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]})^{-1}(\urcorner)_\varphi = \urcorner_{\urcorner\varphi\,\varphi}$ for every concrete proper formula φ in Σ_1^+. This sub-interpretation is well defined taking into account that $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi$ iff $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \urcorner\varphi$ for any concrete proper formula φ in Σ_1^+, due to IMP and REF.

Proposition 6.6 Let $w_1, \ldots, w_m$ be in $V'_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$ and w an irreducible path in $\Sigma_2[\Sigma_1]^+$ with source coinciding with the target of $\langle w_1, \ldots, w_n\rangle$. Denote the target of $(\beta^+_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]})^{-1}(w)_{w_1 \ldots w_m}$ by $w'_1 \ldots w'_n$, then

$$\langle w'_1, \ldots, w'_n\rangle = w \circ \langle w_1, \ldots, w_m\rangle.$$

We omit the proof of the previous proposition since it is identical to the proof of Proposition 6.2.

Proposition 6.7 Given expressions $w_1, \ldots, w_m$ in $V'_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$ and a formula φ in $\Sigma_2[\Sigma_1]^+$ with source coinciding with the target of $\langle w_1, \ldots, w_n\rangle$,

$$\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi \circ \langle w_1, \ldots, w_m\rangle$$

if and only if

$$\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}], J_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}, w_1 \ldots w_m \Vdash \varphi.$$

Proof:
($\Rightarrow$) Assume that $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi \circ \langle w_1, \ldots, w_m\rangle$. Observe that $\varphi \circ \langle w_1, \ldots, w_m\rangle$ is a concrete formula in $\Sigma_2[\Sigma_1]^+$ whose target is either in V_1 or in V_2. Then $\varphi \circ \langle w_1, \ldots, w_m\rangle$ is in $D_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$. Since the target of $(\beta^+_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]})^{-1}(\varphi)_{w_1 \ldots w_m}$ is a sequence with a unique element equal, by Proposition 6.6, to $\varphi \circ \langle w_1, \ldots, w_m\rangle$, we have that $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}], J_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}, w_1 \ldots w_m \Vdash \varphi$;
($\Leftarrow$) Assume that $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}], J_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}, w_1 \ldots w_m \Vdash \varphi$. Then the target of the path $(\beta^+_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]})^{-1}(\varphi)_{w_1 \ldots w_m}$ is in $D_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$. Since it is a sequence with a unique element equal, by Proposition 6.6, to $\varphi \circ \langle w_1, \ldots, w_m\rangle$, we have that $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi \circ \langle w_1, \ldots, w_m\rangle$ since $D_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}$ is the union of the sets of distinguished truth values of $\mathbb{I}_{\Gamma_1}$ and of $\mathbb{I}_{\Gamma_2}$. QED

Concrete completeness preservation

A logic system $\mathcal{L}_k$, for $k = 1, 2$, is *full for importing with respect to logic system* $\mathcal{L}_2[\mathcal{L}_1]$ whenever it contains the canonical interpretations induced by $\mathcal{L}_2[\mathcal{L}_1]$ and by any set Γ of concrete formulas in the language of $\mathcal{L}_2[\mathcal{L}_1]$.

It is immediate to show that concrete completeness is preserved by the importing of full logic systems.

Theorem 6.8 (Concrete completeness preservation)
The logic system resulting from importing logic system $\mathcal{L}_1$ into logic system $\mathcal{L}_2$ is concretely complete whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ are concretely complete and full for importing with respect to $\mathcal{L}_2[\mathcal{L}_1]$.

Proof: Let $\mathcal{L}_1 = (\Sigma_1, \Delta_1, \mathfrak{I}_1)$ and $\mathcal{L}_2 = (\Sigma_2, \Delta_2, \mathfrak{I}_2)$ be a suitably disjoint pair of concretely complete logic systems, full for importing with respect to $\mathcal{L}_2[\mathcal{L}_1]$, and let $\Gamma \cup \{\varphi\}$ be a set of concrete formulas over $\Sigma_2[\Sigma_1]$. Suppose that $\Gamma \not\vdash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi$. Then $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}], J_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}, \mathsf{id}_! \not\Vdash \varphi$ by Proposition 6.7. On the other hand $\Gamma \vdash_{\mathcal{L}_2[\mathcal{L}_1]} \gamma$ for every γ in Γ and so, by the same proposition, $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}], J_{\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]}, \mathsf{id}_! \Vdash \gamma$ for every γ in Γ. Since $\mathbb{I}_{\Gamma_k}$ is an interpretation for $\mathcal{L}_k$ by Proposition 6.4 and $\mathcal{L}_k$ is full for importing with respect to $\mathcal{L}_2[\mathcal{L}_1]$, for $k = 1, 2$, then the interpretation $\mathbb{I}_{\Gamma_2}[\mathbb{I}_{\Gamma_1}]$ is in logic system resulting from the importing, and so $\Gamma \not\vDash_{\mathcal{L}_2[\mathcal{L}_1]} \varphi$. QED

Observe that the enrichment of a complete logic system with the canonical interpretations that make it full for importing, does not change the entailment of the logic system. Hence, we enrich first the given logic systems with those interpretations, and only after that we do the importing.

Corollary 6.9 Let $\mathcal{L}_1$ be a concretely complete logic system $(\Sigma_1, \Delta_1, \mathfrak{I}_1)$ suitably disjoint with the logic system $\mathcal{L}^{\mathrm{ltl}}$ for linear temporal logic introduced in Example 4.2. Then $(\Sigma_1, \Delta_1, \mathfrak{I}_1 \cup \{\mathbb{I}_{\Gamma_1} : \Gamma \subseteq L(\Sigma^{\mathrm{ltl}}[\Sigma_1])\})$ and $(\Sigma^{\mathrm{ltl}}_{Q^{\mathrm{ltl}}}, \Delta^{\mathrm{ltl}}, \mathfrak{I}^{\mathrm{ltl}} \cup \{\mathbb{I}_{\Gamma_2} : \Gamma \subseteq L(\Sigma^{\mathrm{ltl}}[\Sigma_1])\})$ are equivalents in terms of entailment with $\mathcal{L}_1$ and $\mathcal{L}^{\mathrm{ltl}}$ respectively. Moreover the importing of the first into the latter is concretely complete.

Analogous corollaries can be established for importing involving the modal logic system and the intuitionistic logic system.

7 Outlook

We provided importing with a calculus canonically built from the calculi of the two given logics and proved, under mild conditions, that it is sound and concretely complete with respect to the semantics of importing proposed in [12]. To this end, we adopted the graph-theoretic account of syntax and semantics of logics first proposed in [13]. However, we presented herein for the first time how to define local entailment within the setting of the graph-theoretic semantics and developed a novel graph-theoretic account of Hilbert-style calculi. For illustrating purposes we analyzed temporalization [4], modalization [3] and adding a intuitionistic dimension to any given logic.

The graph-theoretic approach can be applied to a wide class of logics, even substructural ones and logics with partial semantics. However, our soundness preservation result assumes that our models are total. Note that all algebraic logics have total graph-theoretic models and, so, the totality assumption is not too restrictive. Furthermore, the assumption (presence of canonical models) needed for the completeness preservation result is quite mild.

Along this line of work on importing, one should look at extending the soundness preservation result to more exotic logics with partial models. In another direction, one should also check if importing is a conservative extension of both given logics. In fact, in [12] the result was obtained only for the imported logic and only for global entailment.

Appendix

For dealing with inference rules and derivations we need to work with morphisms between formulas. In fact, these morphisms live in a generalized 2-category (for more information on 2-categories see [8]), that we introduce now.

A *generalized 2-category* is a tuple

$$C = (C_0, C_1, C_2, \mathsf{src}, \mathsf{trg}, \mathsf{id}, \circ, \overline{\mathsf{src}}, \overline{\mathsf{trg}}, \mathsf{ID}, \overline{\circ}_\mathsf{v}, \overline{\circ}_\mathsf{h})$$

such that:

(i) $(C_0, C_1, \mathsf{src}, \mathsf{trg}, \mathsf{id}, \circ)$ is a category (the *base category*).

(ii) C_2 is a class (of the *generalized 2-cells*).

(iii) $(C_1, C_2, \overline{\mathsf{src}}, \overline{\mathsf{trg}}, \mathsf{ID}, \overline{\circ}_\mathsf{v})$ is a category (the *vertical meta category*).

(iv) $\overline{\circ}_\mathsf{h}$ (the *horizontal composition*) is a partial function from $C_2 \times C_2$ to C_2 such that whenever the horizontal compositions at hand are defined the following equalities hold:

  - $\overline{\mathsf{src}}(\delta_2 \,\overline{\circ}_\mathsf{h}\, \delta_1) = \overline{\mathsf{src}}(\delta_2) \circ \overline{\mathsf{src}}(\delta_1)$ and $\overline{\mathsf{trg}}(\delta_2 \,\overline{\circ}_\mathsf{h}\, \delta_1) = \overline{\mathsf{trg}}(\delta_2) \circ \overline{\mathsf{trg}}(\delta_1)$ (*compatibility of horizontal and base compositions*);
  - $\delta \,\overline{\circ}_\mathsf{h}\, \mathsf{ID}_{\mathsf{id}_A} = \delta$ and $\mathsf{ID}_{\mathsf{id}_A} \,\overline{\circ}_\mathsf{h}\, \delta = \delta$ (*unit of horizontal composition*);
  - $(\delta_3 \overline{\circ}_\mathsf{h} \delta_2) \overline{\circ}_\mathsf{h} \delta_1 = \delta_3 \overline{\circ}_\mathsf{h} (\delta_2 \overline{\circ}_\mathsf{h} \delta_1)$ (*associativity of horizontal composition*);
  - $(\delta_4 \,\overline{\circ}_\mathsf{h}\, \delta_3) \,\overline{\circ}_\mathsf{v}\, (\delta_2 \,\overline{\circ}_\mathsf{h}\, \delta_1) = (\delta_4 \,\overline{\circ}_\mathsf{v}\, \delta_2) \,\overline{\circ}_\mathsf{h}\, (\delta_3 \,\overline{\circ}_\mathsf{v}\, \delta_1)$ (*interchange law*).

In order to simplify the presentation, when $\overline{\mathsf{src}}(\delta) = f$ and $\overline{\mathsf{trg}}(\delta) = g$ we write $\delta : f \Rightarrow g$ or $\delta \in C_2(f, g)$. A generalized 2-category is *horizontally full* whenever $\mathsf{trg}(\overline{\mathsf{src}}(\delta_1)) = \mathsf{src}(\overline{\mathsf{src}}(\delta_2))$ and $\mathsf{trg}(\overline{\mathsf{trg}}(\delta_1)) = \mathsf{src}(\overline{\mathsf{trg}}(\delta_2))$ implies that $\delta_2 \,\overline{\circ}_\mathsf{h}\, \delta_1$ is defined.

Similarly to the canonical generation of the language category G^+ from a m-graph G, described in [12], a generalized 2-category can be canonically generated from a generalized 2-graph, as we describe now. First we define what is a generalized 2-graph and define the set of 2-paths of such a generalized 2-graph.

A *generalized 2-graph* H over a graph G is a graph with $G^+(\cdot,\cdot)$ as the set of vertexes. The set

$$\mathsf{2Pt}(H)$$

of *2-paths* of a 2-graph H and respective source $\mathsf{src}_{\mathsf{2Pt}(H)}$ and target $\mathsf{trg}_{\mathsf{2Pt}(H)}$ are inductively defined as follows:

- $\varepsilon_w \in \mathsf{2Pt}(H)$ where ε_w is the empty 2-path on w with

$$\begin{cases} \mathsf{src}_{\mathsf{2Pt}(H)}(\varepsilon_w) = w \\ \mathsf{trg}_{\mathsf{2Pt}(H)}(\varepsilon_w) = w; \end{cases}$$

- $e \in \mathsf{2Pt}(H)$ with

$$\begin{cases} \mathsf{src}_{\mathsf{2Pt}(H)}(e) = \mathsf{src}(e) \\ \mathsf{trg}_{\mathsf{2Pt}(H)}(e) = \mathsf{trg}(e) \end{cases}$$

  whenever e is an edge of H;

- $\delta_2 \bullet_{\mathsf{v}} \delta_1 \in \mathsf{2Pt}(H)$ with

$$\begin{cases} \mathsf{src}_{\mathsf{2Pt}(H)}(\delta_2 \bullet_{\mathsf{v}} \delta_1) = \mathsf{src}_{\mathsf{2Pt}(H)}(\delta_1) \\ \mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_2 \bullet_{\mathsf{v}} \delta_1) = \mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_2) \end{cases}$$

  whenever δ_1 and δ_2 are in $\mathsf{2Pt}(H)$ and $\mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_1) = \mathsf{src}_{\mathsf{2Pt}(H)}(\delta_2)$;

- $\delta_2 \bullet_{\mathsf{h}} \delta_1 \in \mathsf{2Pt}(H)$ with

$$\begin{cases} \mathsf{src}_{\mathsf{2Pt}(H)}(\delta_2 \bullet_{\mathsf{h}} \delta_1) = \mathsf{src}_{\mathsf{2Pt}(H)}(\delta_2) \circ \mathsf{src}_{\mathsf{2Pt}(H)}(\delta_1) \\ \mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_2 \bullet_{\mathsf{h}} \delta_1) = \mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_2) \circ \mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_1) \end{cases}$$

  whenever $\delta_1, \delta_2 \in \mathsf{2Pt}(H)$, $\mathsf{trg}^+(\mathsf{src}_{\mathsf{2Pt}(H)}(\delta_1)) = \mathsf{src}^+(\mathsf{src}_{\mathsf{2Pt}(H)}(\delta_2))$ and $\mathsf{trg}^+(\mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_1)) = \mathsf{src}^+(\mathsf{trg}_{\mathsf{2Pt}(H)}(\delta_2))$.

Observe that $\mathsf{2Pt}(H)$ induces the following 2-graph

$$H^{\dagger_2}$$

over G, defined as $\displaystyle\bigcup_{k\in\mathbb{N}} H_k^{\dagger_2}$ where:

- $H_0^{\dagger_2}$ is the 2-graph over G with all the edges of H taken as edges, plus additional edges of the form

$$\mathsf{P}_j^{\langle w_1,\ldots,w_n\rangle} : \langle w_1,\ldots,w_n\rangle \Rightarrow w_j$$

  (to be used later on as 2-projections) with $n > 1$.

- $H^{\dagger_2}_{k+1}$ is the 2-graph over G obtained from $H^{\dagger_2}_k$ by adding edges of the form
$$\overline{\langle}\delta_1,\ldots,\delta_m\overline{\rangle} : w \Rightarrow \langle w'_1,\ldots,w'_m\rangle$$
(to be used later on for tupling) for any 2-paths
$$\delta_j : w \Rightarrow w'_j$$
of $H^{\dagger_2}_k$ for $j = 1,\ldots,m$ with $m > 1$.

So, the envisaged horizontally full generalized 2-category with 2-products of objects with the same source, induced by a given 2-graph H, is the tuple:

$$G^H = (|G^+|, G^+(\cdot,\cdot), \mathsf{2Pt}(H^{\dagger_2})|_\approx, \mathsf{src}^+, \mathsf{trg}^+, \mathsf{id}, \circ, \overline{\mathsf{src}}, \overline{\mathsf{trg}}, \mathsf{ID}, \overline{\circ}_\mathsf{v}, \overline{\circ}_\mathsf{h})$$

where $\mathsf{2Pt}(H^{\dagger_2})|_\approx$ is the quotient set of $\mathsf{2Pt}(H^{\dagger_2})$ by $\approx$ defined as the least equivalence relation containing the following pairs:

- $\overline{\langle}\mathsf{P}_1^{\langle w_1,\ldots,w_n\rangle},\ldots,\mathsf{P}_n^{\langle w_1,\ldots,w_n\rangle}\overline{\rangle} \approx \varepsilon_{\langle w_1,\ldots,w_n\rangle}$;
- $\varepsilon_w \bullet_\mathsf{v} \delta \approx \delta$ and $\delta' \bullet_\mathsf{v} \varepsilon_w \approx \delta'$;
- $\varepsilon_{\mathsf{id}_s} \bullet_\mathsf{h} \delta \approx \delta$ and $\delta' \bullet_\mathsf{h} \varepsilon_{\mathsf{id}_s} \approx \delta'$;
- $\delta_1 \bullet_\mathsf{v} (\delta_2 \bullet_\mathsf{v} \delta_3) \approx (\delta_1 \bullet_\mathsf{v} \delta_2) \bullet_\mathsf{v} \delta_3$;
- $\delta_1 \bullet_\mathsf{h} (\delta_2 \bullet_\mathsf{h} \delta_3) \approx (\delta_1 \bullet_\mathsf{h} \delta_2) \bullet_\mathsf{h} \delta_3$;
- $(\delta_4 \bullet_\mathsf{h} \delta_3) \bullet_\mathsf{v} (\delta_2 \bullet_\mathsf{h} \delta_1) \approx (\delta_4 \bullet_\mathsf{v} \delta_2) \bullet_\mathsf{h} (\delta_3 \bullet_\mathsf{v} \delta_1)$;
- $\mathsf{P}_j^{\langle \mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1),\ldots,\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_n)\rangle} \bullet_\mathsf{v} \overline{\langle}\delta_1,\ldots,\delta_n\overline{\rangle} \approx \delta_j$;
- $\overline{\langle}\delta_1,\ldots,\delta_n\overline{\rangle} \approx \overline{\langle}\delta'_1,\ldots,\delta'_n\overline{\rangle}$ if $\delta_k \approx \delta'_k$ for $k = 1,\ldots,n$ and $\mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1) = \cdots = \mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_n)$;
- $\delta_2 \bullet_\mathsf{v} \delta_1 \approx \delta'_2 \bullet_\mathsf{v} \delta'_1$ whenever $\delta_k \approx \delta'_k$ for $k = 1, 2$ and $\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1) = \mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_2)$;
- $\delta_2 \bullet_\mathsf{h} \delta_1 \approx \delta'_2 \bullet_\mathsf{h} \delta'_1$ whenever $\delta_k \approx \delta'_k$ for $k = 1, 2$, $\mathsf{trg}^+(\mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1)) = \mathsf{src}^+(\mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_2))$ and $\mathsf{trg}^+(\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1)) = \mathsf{src}^+(\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_2))$;
- $\delta \approx \overline{\langle}\delta_1,\ldots,\delta_n\overline{\rangle}$ if $\mathsf{P}_k^{\langle w_1,\ldots,w_n\rangle} \bullet_\mathsf{v} \delta \approx \delta_k$ for $k = 1,\ldots,n$ and $\mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta) = \mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1) = \cdots = \mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_n)$ and $\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta) = \langle w_1,\ldots,w_n\rangle$;

and

- $\overline{\mathsf{src}}([\delta]) = \mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta)$ and $\overline{\mathsf{trg}}([\delta]) = \mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta)$;
- $\mathsf{ID}_w = [\varepsilon_w]$;
- $[\delta_2]\ \overline{\circ}_\mathsf{v}\ [\delta_1] = [\delta_2 \bullet_\mathsf{v} \delta_1]$ if $\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1) = \mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_2)$;

- $[\delta_2] \mathbin{\bar{\circ}_\mathsf{h}} [\delta_1] = [\delta_2 \bullet_\mathsf{h} \delta_1]$ if $\mathsf{trg}^+(\mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1)) = \mathsf{src}^+(\mathsf{src}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_2))$, $\mathsf{trg}^+(\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_1)) = \mathsf{src}^+(\mathsf{trg}_{\mathsf{2Pt}(H^{\dagger_2})}(\delta_2))$.

It is straightforward to verify that the tuple G^H is indeed a horizontally full generalized 2-category. Moreover, products in a generalized 2-category resulting from this construction are such that each finite tupling of morphisms in the base category, with the same source, is the vertex of a 2-product, as established without loss of generality for pairings as follows. Let w_1 and w_2 be morphisms in G^+ such that $\mathsf{src}^+(w_1) = \mathsf{src}^+(w_2)$. Then the triple

$$(\langle w_1, w_2\rangle, [\mathsf{P}_1^{\langle w_1,w_2\rangle}], [\mathsf{P}_2^{\langle w_1,w_2\rangle}])$$

is a product in the vertical meta category of G^H. Indeed, assume that $[\delta_1] : w \to w_1$ and $[\delta_2] : w \to w_2$ are 2-cells. Consider the 2-cell $[\bar{\langle}\delta_1, \delta_2\bar{\rangle}]$. Then

$$\begin{array}{rcl} [\mathsf{P}_k^{\langle w_1,w_2\rangle}] \mathbin{\bar{\circ}_\mathsf{v}} [\bar{\langle}\delta_1, \delta_2\bar{\rangle}] & = & [\mathsf{P}_k^{\langle w_1,w_2\rangle} \bullet_\mathsf{v} \bar{\langle}\delta_1, \delta_2\bar{\rangle}] \\ & = & [\delta_k]. \end{array}$$

Furthermore, assume that $[\delta] : w \to \langle w_1, w_2\rangle$ is a 2-cell such that

$$[\mathsf{P}_k^{\langle w_1,w_2\rangle}] \mathbin{\bar{\circ}_\mathsf{v}} [\delta] = [\delta_k]$$

for $k = 1, 2$. Then $\mathsf{P}_k^{\langle w_1,w_2\rangle} \bullet_\mathsf{v} \delta \approx \delta_k$ since $[\mathsf{P}_k^{\langle w_1,w_2\rangle}] \mathbin{\bar{\circ}_\mathsf{v}} [\delta] = [\mathsf{P}_k^{\langle w_1,w_2\rangle} \bullet_\mathsf{v} \delta]$. Thus, $\delta \approx \bar{\langle}\delta_1, \delta_2\bar{\rangle}$.

Since there is no risk of ambiguity, we avoid to use the equivalence class notation when referring to 2-cells in G^H. Moreover, we avoid using the qualification generalized when referring to generalized 2-cells or generalized 2-categories.

Observe that the set Δ in a deductive system (Σ, Δ) induces in an obvious way a 2-graph over Σ. From that 2-graph we generate, as described above, a horizontally full generalized 2-category

$$\Sigma^\Delta$$

with 2-products for objects (morphisms of Σ^+) with the same source, where rules, instantiated rules, proofs and their compositions live as 2-cells. Furthermore, since every rule in Δ is source-homogeneous, it is straightforward to verify that every 2-cell $\delta : \psi_1 \Rightarrow \psi_2$ of Σ^Δ is source-homogeneous, that is, $\mathsf{src}^+(\psi_1) = \mathsf{src}^+(\psi_2)$.

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