Importing Logics

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Abstract

The novel notion of importing logics is introduced, subsuming as special cases several kinds of asymmetric combination mechanisms, like temporalization [8, 9], modalization [7] and exogenous enrichment [13, 5, 12, 4, 1]. The graph-theoretic approach proposed in [15] is used, but formulas are identified with irreducible paths in the signature multi-graph instead of equivalence classes of such paths, facilitating proofs involving inductions on formulas. Importing is proved to be strongly conservative. Conservative results follow as corollaries for temporalization, modalization and exogenous enrichment.

keywords: combined logics, importing logics, temporalization, modalization, exogenous enrichment.

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1 Introduction

Temporalization was introduced by Gabbay and Finger in [8] and later on developed and extended, namely to modalization (see for instance [9, 7]). Temporalization and modalization are examples of asymmetric mechanisms for combining logics. More recently, another way of asymmetric combination (exogenous enrichment) was proposed and applied successfully to the problem of probabilizing or adding a quantum dimension to propositional logic [13, 5, 12, 4, 1].

We propose a new way of combining logics, named importing, subsuming temporalization and modalization (with no shared connectives), as well as exogenous enrichment. The combined language is endowed with an explicit constructor (importing connective) for transforming formulas of the imported logic into formulas of the importing one (differing in this aspect, and so in all its multiple effects, from parameterization [3]). Semantically, each model of the resulting logic is a pair composed of a model of the importing logic and a model of the imported logic plus the interpretation of the importing connective. The latter is a relation between the truth values of the two models respecting the distinguished ones in both directions. The conservative nature of importing is to be expected with the proposed semantics. Since the semantics of importing is quite different from the semantics of the subsumed combination mechanisms, detailed proofs are provided for their equivalence (module a trivial translation of formulas). In the case of temporalization and modalization the equivalence is obtained for validity. In the case of globalization (a simple example of exogenous enrichment) the equivalence is shown to hold for entailment.

We adopt the graph-theoretic account of logics proposed in [15] in assuming that the signatures and interpretation structures of the logics being considered are described using multi-graphs[1]. From any multi-graph we generate a category with non-empty finite products in order to avoid working with multicategories. This construction is based herein on the reduction system induced by the multi-graph, instead of the quotient technique used in [15]. According to this alternative method of generating such a category, morphisms are irreducible paths (they are unique due to Newman's Lemma [14] since we show that the reduction system is terminating and locally confluent). In this way we avoid the use of equivalence classes with all the advantages of this fact. Despite its simplicity, the category generated according to our method is isomorphic to the category generated by factorization in [15]. Finally, we provide an inductive characterization of the irreducible paths in order to facilitate proofs by induction.

For the convenience of the reader, in Section 2 we provide a summary of the graph-theoretic account of logics proposed in [15]. The details of the construction of the category (with irreducible paths as morphisms) induced by any given multi-graph are presented in the Appendix. Importing is defined and illustrated in Section 3. The strongly conservative nature of importing is established in Section 4. Section 5 establishes that both globalization (a simple but representative case of exogenous enrichment) and temporalization (a representative case of modalization), with no shared connectives, are subsumed by importing at the entailment level, strongly for the former and weakly for the latter. The strongly and weakly conservative natures of globalization and temporalization, respectively, follow as corollaries. Finally, in Section 6 we assess what was achieved and speculate on what lies ahead.

2 Graph-theoretic account of logics

Logic systems are presented using a variant of the graph-theoretic approach proposed in [15]. This approach was chosen since it can be used to describe uniformly all the components, that is, the language, the semantics and the deductive system, of a great variety of logics. Nevertheless, we do not follow [15] completely since we assume that formulas are irreducible paths and not equivalence classes and since herein a logic system may have more than one proposition sort (as is the case with the signature of logics resulting from importing).

[1]By a multi-graph we mean a graph where edges may have multiple sources, following the terminology of category theory.

2.1 Language

We start by introducing signatures (the language is generated from the signature in a free way). Following [15], a signature is to be seen as a multi-graph whose nodes are the sorts (indicating the relevant kinds of notions) and whose multi-edges are the language constructors. For instance, a signature for modal logic (see for example [10] and [2]) can be seen as a multi-graph with a node, named π, representing the notion of formula, and including a multi-edge $\sim$ from π to π for the negation connective, a multi-edge $\Rightarrow$ for the implication connective from $\pi\pi$ to π, and the multi-edge $\Diamond$ from π to π for the modal possibility connective. Propositional symbols are zero-ary constructors and should also be represented in the multi-graph. For this purpose we consider a special node, named !, and a multi-edge from ! to π for each propositional symbol. For a rigorous definition of the signature just described see Example 2.1 and for a graphical representation see Figure 1.

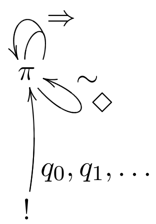

Figure 1: Multi-graph of the modal signature described in Example 2.1.

By a *multi-graph*, in short, an *m-graph*, we mean a tuple

$$G = (V, E, \mathsf{src}, \mathsf{trg})$$

where:

- V is a set (of *vertexes* or *nodes*);
- E is a set (of *multi-edges* or *m-edges*);
- $\mathsf{src} : E \to V^+$;
- $\mathsf{trg} : E \to V$;

where V^+ denotes the set of all finite non-empty sequences of V. In general, given a set S and $s \in S^+$, we denote by $|s|$ the length of s and, for each $i = 1, \dots, |s|$, we denote by $(s)_i$ the i-th element of s. Furthermore, given a map $f : S \to R$, we let f^+ be the map $\lambda\, s\, .\, f((s)_1) \dots f((s)_n) : S^+ \to R^+$. For the sake of simplicity, we tend to write f for f^+ when no confusion arises. We may write either $e : s \to v$ or $e \in G(s, v)$ or $e \in E(s, v)$ whenever e is in E, $\mathsf{src}(e) = s$ and $\mathsf{trg}(e) = v$, and may write $G(\cdot, \cdot)$ or $E(\cdot, \cdot)$ for the collection of m-edges in E.

We endow m-graphs with possibly partial morphisms, and introduce now some relevant terminology and notation from that paper. Given two functions

$v_1, v_2 : A \to B$ we write $v_1 \subseteq v_2$, if $\mathsf{dom}\, v_1 \subseteq \mathsf{dom}\, v_2$ and $v_2(a) = v_1(a)$ for every a in $\mathsf{dom}\, v_1$, and we write $v_1 = v_2$ if $v_1 \subseteq v_2$ and $v_2 \subseteq v_1$.

By an *m-graph (partial) morphism* $h : G_1 \to G_2$ we mean a pair of possibly partial functions $h^{\mathsf{v}} : V_1 \to V_2$ and $h^{\mathsf{e}} : E_1 \to E_2$ such that:

- $\mathsf{src}_2 \circ h^{\mathsf{e}} \subseteq h^{\mathsf{v}} \circ \mathsf{src}_1$;
- $\mathsf{trg}_2 \circ h^{\mathsf{e}} \subseteq h^{\mathsf{v}} \circ \mathsf{trg}_1$.

Observe that if the morphism is defined for an m-edge it is also defined for the sorts in its source and for the sorts in its target.

A *(propositional based) language signature* or, simply, a *signature*, is a tuple

$$\Sigma = (G, !, \Pi)$$

where $G = (V, E, \mathsf{src}, \mathsf{trg})$ is an m-graph such that V is $\{!\} \cup \Pi$, no m-edge in E has ! as target, ! only appears in the source of unary edges, ! is not in Π, and Π is non-empty. The nodes in V play the role of language *sorts*, each sort in Π is a *propositions sort*, and node ! is the *concrete sort*. The m-edges play the role of *constructors*. We do not impose that Π is a singleton since the signature of the logic resulting from importing has all the proposition sorts of the component logics. From now on, we denote by Σ_i the m-graph $(G_i, !, \Pi_i)$ where $G_i = (V_i, E_i, \mathsf{src}_i, \mathsf{trg}_i)$.

Example 2.1 Let Q be a countable set $\{q_0, q_1, \dots\}$ of propositional symbols. The *modal signature over* Q (see [2, 10]), denoted by $\Sigma_Q^{\Diamond}$, is an m-graph with the proposition sort π, the concrete sort !, and the following m-edges:

- $q_i : ! \to \pi$ for each natural number i;
- $\sim, \Diamond : \pi \to \pi$;
- $\Rightarrow : \pi\pi \to \pi$.

The m-edges $\sim$, $\Rightarrow$ and $\Diamond$ represent the connectives negation, implication and the modal operator of possibility, respectively. In the sequel we may denote the proposition sort π by π_{m}, and the m-edges $\sim$ and $\Rightarrow$ by $\sim_{\mathrm{m}}$ and $\Rightarrow_{\mathrm{m}}$ respectively. For a graphical representation see Figure 1. ∇

Example 2.2 Let P be a countable set $\{p_0, p_1, \dots\}$ of propositional symbols. The *signature over* P *for intuitionistic logic* (see [16]), denoted by $\Sigma_P^{\wedge,\vee}$, is an m-graph with the proposition sort π, the concrete sort ! and the following m-edges:

- $p_i : ! \to \pi$ for each natural number i;
- $\neg : \pi \to \pi$;
- $\wedge, \vee, \supset : \pi\pi \to \pi$.

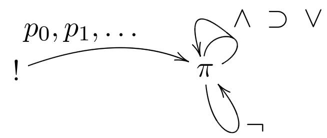

Figure 2: M-graph of the signature for intuitionistic logic described in Example 2.2.

The m-edges $\neg$, $\wedge$, $\vee$ and $\supset$ represent the connectives negation, conjunction, disjunction and implication, respectively. In the sequel we may denote the proposition sort π by π_i, and the m-edges $\wedge$, $\vee$ and $\supset$ by $\wedge_i$, $\vee_i$, and $\supset_i$ respectively. For a graphical representation see Figure 2. ∇

Example 2.3 Let Q be a countable set $\{q_0, q_1, \ldots\}$ of propositional symbols. The *LTL signature over* Q (based on [17]), denoted by Σ_Q^{LTL}, is an m-graph with the proposition sort π, the concrete sort !, and the following m-edges:

- $q_i : ! \to \pi$ for each natural number i;
- $\sim, \mathsf{X}, \mathsf{Y} : \pi \to \pi$;
- $\Rightarrow, \mathsf{S}, \mathsf{U} : \pi\pi \to \pi$.

The m-edges $\sim$, $\Rightarrow$, S, U, X and Y represent the connectives negation, implication, the temporal operator since, the temporal operator until, the next temporal operator and the previous temporal operator, respectively. In the sequel we may denote the proposition sort π by π_{ltl}, and the m-edges $\sim$ and $\Rightarrow$ by $\sim_{\mathrm{ltl}}$ and $\Rightarrow_{\mathrm{ltl}}$ respectively. For a graphical representation see Figure 3. ∇

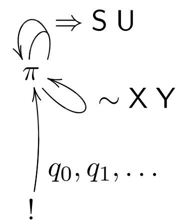

Figure 3: M-graph of the LTL signature described in Example 2.3.

Example 2.4 Let P be a countable set $\{p_0, p_1, \ldots\}$ of propositional symbols. The *signature over* P *for classical propositional logic*, denoted by Σ_P, is an m-graph with the proposition sort π, the concrete sort ! and the following m-edges:

- $p_i : ! \to \pi$ for each natural number i;
- $\neg : \pi \to \pi$;
- $\supset : \pi\pi \to \pi$.

The m-edges $\neg$ and $\supset$ represent the connectives negation and implication, respectively. In the sequel we may denote the proposition sort π by π_c, and the m-edges $\neg$ and $\supset$ by $\neg_c$ and $\supset_c$ respectively. ∇

A formula over a signature can be seen, intuitively, as a multi-path of the m-graph of the signature. For instance, in the context of the signature Σ_Q^{LTL} for LTL, the formula $\mathsf{X}(q_1) \Rightarrow (\sim \mathsf{X}(\sim q_1))$ can be seen as the multi-path depicted in Figure 4. Despite its simplicity, in order to cope in a nice and rigorous way with substitutions and instantiations and not have to deal with multi-categories, we enrich m-graphs with products and tuples and consider categories with non-empty finite products. We now provide a brief summary of the general way of obtaining such a category from an m-graph, developed in the Appendix.

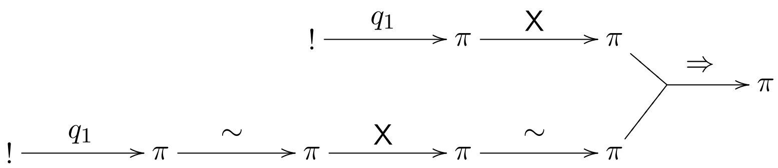

Figure 4: Formula $\mathsf{X}(q_1) \Rightarrow (\sim \mathsf{X}(\sim q_1))$.

Given an m-graph G, the category G^+ with non-empty finite products induced by G is constructed in two steps as detailed in the Appendix. In the first step, the graph $G^\dagger$ is obtained by enriching iteratively, see the Appendix, the m-graph G with edges for projections $\mathsf{p}_i^{v_1 \ldots v_n}$ and tuples $\langle w_1, \ldots, w_n \rangle$, and by considering as nodes the finite non-empty sequences of nodes of G. In the second step, the abstract reduction system $(\mathit{paths}(G^\dagger), \leadsto_G)$ induced by the m-graph G is defined and proved to be confluent. The morphisms of G^+ are then the irreducible paths of $G^\dagger$ according to this reduction system, and the objects of G^+ are the finite non-empty sequences of nodes of V. The identity morphism associated to an object s is the empty path ϵ_s on s, and given morphisms $w_2 : s_1 \to s_2$ and $w_1 : s_0 \to s_1$ their composition $w_2 \circ w_1$ is the morphism $\overset{\leadsto_G}{\mathsf{nf}}(w_2 w_1)$. When there is no ambiguity we will simply use nf. As we show in the Appendix, the reduction rules of $(\mathit{paths}(G^\dagger), \leadsto_G)$ are such that the category G^+ has non-empty finite products. Moreover, we inductively characterize the set $\mathsf{IPaths}(G^\dagger)$ of $\leadsto_G$-irreducible paths of the graph $G^\dagger$, that is, the set of morphisms of G^+.

A *formula over* $\Sigma = (G, !, \Pi)$ is a morphism of G^+, that is, a $\leadsto_G$-irreducible path of $G^\dagger$, with target in Π. The *language over* Σ, denoted by

$$L(\Sigma),$$

is the set of formulas over Σ. A formula is said to be *concrete* whenever its source is !. For instance, in the context of signature Σ_P described in Example 2.4, the formula $\supset \circ \langle \neg \circ p_1, p_2 \rangle$, that is, $\supset \langle \neg p_1, p_2 \rangle$, from ! to π, is a concrete formula, represented simply by

$$(\neg p_1) \supset p_2.$$

From now on, we may use interchangeably the simpler representation and the more rigorous one. For the sake of illustration, observe that the normal form of the path

$$\supset\langle \mathsf{p}_2^{\pi\pi}, \mathsf{p}_1^{\pi\pi}\rangle\langle p_2, \neg\, p_1\rangle$$

is the irreducible path $\supset\langle \neg\, p_1, p_2\rangle$ according to the reduction rules presented in the Appendix.

A *schema formula* is a formula whose source has a sort that may be the target of m-edges of the underlying m-graph. For instance, the formula

$$(\mathsf{p}_1^{\pi\pi} \supset (\mathsf{p}_1^{\pi\pi} \supset \mathsf{p}_1^{\pi\pi})) \supset \mathsf{p}_2^{\pi\pi} : \pi\pi \to \pi$$

over $\Sigma_P^{\wedge,\vee}$ is schematic. Traditionally, this formula is written with schema variables as follows

$$(\xi_1 \supset (\xi_1 \supset \xi_1)) \supset \xi_2.$$

Therefore, from now on, by a *schema variable* we mean either id_v where v is a sort that may be the target of m-edges of the underlying m-graph (which means, herein, that v is in Π) or the projections p_i^s where s contains at least one such sort. The *instantiation* of a formula $w : s \to t$ by an irreducible path w_0 with target s, both over a signature Σ, is the formula $w \circ w_0$, that is, $\mathsf{nf}(ww_0)$.

2.2 Semantics

We now recall the graph-theoretic approach to the semantics of a logic described in [15], with the relevant adaptations to the new context in which formulas are irreducible paths. An interpretation structure for a signature, includes an m-graph (the *operations m-graph*) where the nodes are semantic values and the m-edges are operations on the values. However, this is not enough because we need to know how the values are related to sorts and how operations are related to constructors, that is, we need to relate the operations m-graph with the signature m-graph, see Figure 5. Observe that we abstract the semantics into the syntax and not the other way around.

An *interpretation structure I for a signature* $(G, !, \Pi)$ is a tuple

$$(G', \alpha, D, !)$$

such that G' is an m-graph (the *operations m-graph*), $\alpha : G' \to G$ is an m-graph partial morphism (the *abstraction morphism*) such that α^{v} is total, $(\alpha^{\mathsf{v}})^{-1}(!)$ is a set containing ! (the *concrete value*), and $D \subseteq (\alpha^{\mathsf{v}})^{-1}(\Pi)$ is a non-empty set (of *designated values*). Observe that we use the same symbol for the concrete sort and for the concrete value since it will be clear from the context when we are referring to one or to the other. In the sequel, the names of the m-edges in the operations m-graph are hints of how they are mapped by α to the constructors, see Figure 5, and, in a graphical representation of an interpretation structure, designated values are represented inside a circle. As an abuse of notation we may use (Σ, I) when referring to an interpretation structure I for a signature Σ. In the sequel, the set $\{v'_1 \dots v'_n \in V'^{+} : \alpha^{\mathsf{v}}(v'_1) = v_1, \dots, \alpha^{\mathsf{v}}(v'_n) = v_n\}$ for $v_1, \dots, v_n$ in V is denoted by $V'_{v_1 \dots v_n}$, and given $U \subseteq V^+$ the set $\cup_{u \in U} V'_u$ is denoted by V'_U. The elements of V'_Π are the *truth values*.

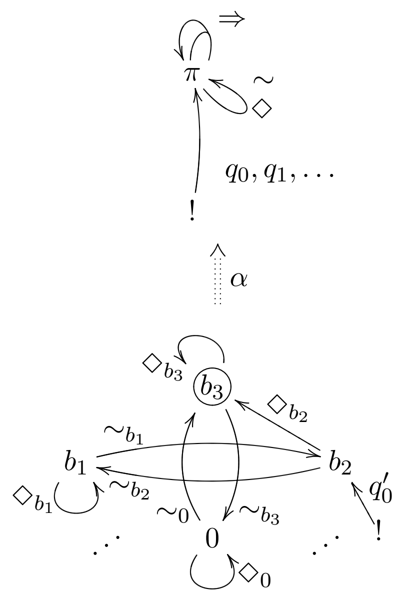

Figure 5: Part of an interpretation structure for modal logic T without the operation m-edges for ⇒ and for some of the propositional symbols (see Example 2.6).

An *interpretation system* $\mathcal{I}$ is a pair $(\Sigma, \mathfrak{I})$ where Σ is a signature and $\mathfrak{I}$ is a class of interpretation structures for Σ. We now describe several examples of interpretation systems useful in the rest of the paper.

Example 2.5 *An interpretation system for intuitionistic propositional logic.* The interpretation system $(\Sigma_P^{\wedge,\vee}, \mathfrak{I})$ for intuitionistic propositional logic is such that $\mathfrak{I}$ is the class of all interpretation structures $(G', \alpha, D, !)$ for $\Sigma_P^{\wedge,\vee}$ induced by a Heyting algebra $(A, \sqcup, \sqcap, \sqsupset, \top, \bot)$ and a valuation v over the algebra (see [16]), that is:

- G' is such that:

  $V' = A \cup \{!\}$;

  $E' = \{p_i' : i \in \mathbb{N}\} \cup \{\neg_a : a \in A\} \cup \{\supset_{a_1 a_2}, \wedge_{a_1 a_2}, \vee_{a_1 a_2} : a_1, a_2 \in A\}$;

  src' and trg' are such that:

  $p_i' : ! \to v(p_i)$ for each natural number i;

  $\neg_a : a \to \sqsupset(a, \bot)$;

  $\supset_{a_1 a_2} : a_1\, a_2 \to \sqsupset(a_1, a_2)$;

  $\wedge_{a_1 a_2} : a_1\, a_2 \to a_1 \sqcap a_2$;

  $\vee_{a_1 a_2} : a_1\, a_2 \to a_1 \sqcup a_2$.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(a) = \pi$ for all a in A;

  $\alpha^{\mathsf{v}}(!) = !$;

$\alpha^{\mathsf{e}}(p_i') = p_i$ for each natural number i;

$\alpha^{\mathsf{e}}(\neg_a) = \neg$;

$\alpha^{\mathsf{e}}(\supset_{a_1 a_2}) = \supset$;

$\alpha^{\mathsf{e}}(\wedge_{a_1 a_2}) = \wedge$;

$\alpha^{\mathsf{e}}(\vee_{a_1 a_2}) = \vee$.

- $D = \{\top\}$.

As an example, see in Figure 6 part of an interpretation structure in $\mathfrak{I}$. ∇

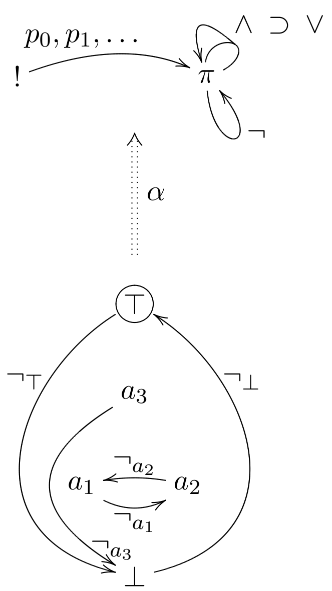

Figure 6: Part of an interpretation structure for intuitionistic logic (see Example 2.5).

Example 2.6 *An interpretation system for modal logic* T.
The interpretation system $(\Sigma_Q^\diamond, \mathfrak{I}_{\mathrm{T}})$ for modal logic T is such that $\mathfrak{I}_{\mathrm{T}}$ is the class of all interpretation structures $(G', \alpha, D, !)$ for $\Sigma_Q^\diamond$ induced by a boolean algebra with an operator $(B, +, -, 0, f_\diamond)$ where $b + f_\diamond(b) = f_\diamond(b)$ and by a valuation v over the algebra (see [2, 10]), that is:

- G' is such that:

  $V' = B \cup \{!\}$;

  $E' = \{q_i' : i \in \mathbb{N}\} \cup \{\sim_b, \diamond_b : b \in B\} \cup \{\Rightarrow_{b_1 b_2} : b_1 \in B \text{ and } b_2 \in B\}$;

  src' and trg' are such that:

  $q_i' : ! \to v(q_i)$ for each natural number i;

  $\sim_b : b \to -b$ for each b in B;

  $\Rightarrow_{b_1 b_2} : b_1\, b_2 \to ((-b_1) + b_2)$ for each b_1 and b_2 in B;

$\Diamond_b : b \to f_\Diamond(b)$ for each b in B.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(b) = \pi$;

  $\alpha^{\mathsf{v}}(!) = !$;

  $\alpha^{\mathsf{e}}(q'_i) = q_i$ for each natural number i;

  $\alpha^{\mathsf{e}}(\sim_b) = \sim$;

  $\alpha^{\mathsf{e}}(\Rightarrow_{b_1 b_2}) = \Rightarrow$;

  $\alpha^{\mathsf{e}}(\Diamond_b) = \Diamond$.

- $D = \{-0\}$.

As an example, see part of an interpretation structure for modal logic T in Figure 5. ∇

Example 2.7 *An interpretation system for linear temporal logic.*
The interpretation system $(\Sigma_Q^{\mathrm{LTL}}, \mathfrak{I}_{\mathrm{LTL}})$ for LTL is such that $\mathfrak{I}_{\mathrm{LTL}}$ is the class of all interpretation structures $(G', \alpha, D, !)$ for Σ_Q^{LTL}, induced by a strong linear Galois algebra (see [17]) $(B, \cap, \cup, \supset, 0, 1, \oplus, \ominus)$ and by a valuation v over the algebra, that is:

- G' is such that:

  $V' = B \cup \{!\}$;

  $E' = \{q'_i : i \in \mathbb{N}\} \cup \{\sim_b, \mathsf{X}_b, \mathsf{Y}_b : b \in B\} \cup \{\Rightarrow_{b_1 b_2}, \mathsf{S}_{b_1 b_2}, \mathsf{U}_{b_1 b_2} : b_1, b_2 \in B\}$;

  src' and trg' are such that:

  $q'_i : ! \to v(q_i)$ for each natural number i;
  $\sim_b : b \to (b \supset 0)$ for each b in B;
  $\Rightarrow_{b_1 b_2} : b_1\, b_2 \to b_1 \supset b_2$ for each b_1 and b_2 in B;
  $\mathsf{X}_b : b \to \oplus b$ for each b in B;
  $\mathsf{Y}_b : b \to \ominus b$ for each b in B;
  $\mathsf{S}_{b_1 b_2} : b_1\, b_2 \to \mu_b(b_2 \cup (b_1 \cap \ominus b))$ for each b_1 and b_2 in B;
  $\mathsf{U}_{b_1 b_2} : b_1\, b_2 \to \mu_b(b_2 \cup (b_1 \cap \oplus b))$ for each b_1 and b_2 in B.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(b) = \pi$;

  $\alpha^{\mathsf{v}}(!) = !$;

  $\alpha^{\mathsf{e}}(q'_i) = q_i$ for each natural number i;

  $\alpha^{\mathsf{e}}(\sim_b) = \sim$;

  $\alpha^{\mathsf{e}}(\Rightarrow_{b_1 b_2}) = \Rightarrow$;

  $\alpha^{\mathsf{e}}(\mathsf{X}_b) = \mathsf{X}$;

  $\alpha^{\mathsf{e}}(\mathsf{Y}_b) = \mathsf{Y}$;

$\alpha^{\mathsf{e}}(\mathsf{S}_{b_1 b_2}) = \mathsf{S}$;
$\alpha^{\mathsf{e}}(\mathsf{U}_{b_1 b_2}) = \mathsf{U}$.

- $D = \{b : ((\ominus(0 \supset 0)) \supset 0) \subseteq b\}$. ∇

Example 2.8 *An interpretation system for classical propositional logic.*
The interpretation system $(\Sigma_P, \mathfrak{I})$ for classical propositional logic is such that $\mathfrak{I}$ is the class of all interpretation structures $(G', \alpha, D, !)$ for Σ_P induced by valuations $v : P \to \{0,1\}$ for P, that is:

- G' is such that:

  $V' = \{0,1\} \cup \{!\}$;
  $E' = \{p'_i : i \in \mathbb{N}\} \cup \{\neg_0, \neg_1\} \cup \{\supset_{a_1 a_2} : a_1, a_2 \in \{0,1\}\}$;
  src' and trg' are such that:
    $p'_i : ! \to v(p_i)$ for each natural number i;
    $\neg_0 : 0 \to 1$;
    $\neg_1 : 1 \to 0$;
    $\supset_{00} : 0\,0 \to 1$;
    $\supset_{01} : 0\,1 \to 1$;
    $\supset_{10} : 1\,0 \to 0$;
    $\supset_{11} : 1\,1 \to 1$;

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(0) = \pi$;
  $\alpha^{\mathsf{v}}(1) = \pi$;
  $\alpha^{\mathsf{v}}(!) = !$;
  $\alpha^{\mathsf{e}}(p'_i) = p_i$ for each natural number i;
  $\alpha^{\mathsf{e}}(\neg_a) = \neg$;
  $\alpha^{\mathsf{e}}(\supset_{a_1 a_2}) = \supset$;

- $D = \{1\}$.

As an example, see in Figure 7 part of an interpretation structure in $\mathfrak{I}$. ∇

In the sequel we need to refer to the functor $h^+ = (h^+_{\mathsf{o}}, h^+_{\mathsf{m}})$ induced by an m-graph partial morphism h. In the Appendix we define the partial functor induced by an m-graph partial morphism.

We now introduce the notion of denotation of a formula, and the notion of entailment of a formula from a set of formulas. Intuitively, the denotation of a formula in the context of an interpretation structure $(G', \alpha, D, !)$, is the class of all morphisms of G'^+, i.e., all $\rightsquigarrow_{G'}$-irreducible paths of $G'^\dagger$, that are mapped by α^+ to that formula. More rigorously, given an interpretation structure $I = (G', \alpha, D, !)$ for a signature Σ, we say that a $\rightsquigarrow_{G'}$-irreducible path w' of $G'^\dagger$ is a *path in I for* a language path w if $\alpha^+(w') = w$. The *denotation* $[\![\varphi]\!]^I_\Sigma$ over I of a formula φ over Σ is given by $\{w' : \alpha^+(w') = \varphi\}$.

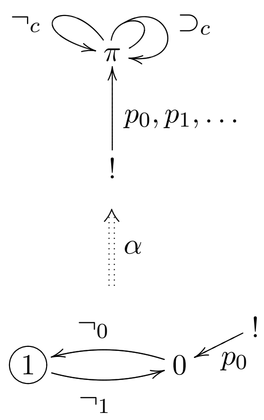

Figure 7: Part of an interpretation structure for classical logic (see Example 2.8).

Example 2.9 Consider the interpretation system $(\Sigma_Q^\diamond, \mathfrak{I}_{\mathrm{T}})$ described in Example 2.6 for modal logic T, and let I be the interpretation structure partially described in Figure 5. Then

$$\sim_{b_3}\diamond_{b_2}q_0'$$

is a path in I for $\sim\diamond q_0$, and so belongs to $[\![\sim\diamond q_0]\!]^I_{\Sigma_Q^\diamond}$. ∇

When τ' is a path in I for φ and its target is in D, for I in $\mathfrak{I}$ and φ over Σ, in the context of an interpretation system $(\Sigma, \mathfrak{I})$, we write

$$I, \tau' \Vdash_{(\Sigma,\mathfrak{I})} \varphi$$

and say that path τ' of I *satisfies* formula φ.

Example 2.10 Taking into account Example 2.9, we have that

$$I, \sim_{b_3}\diamond_{b_2}q_0' \not\Vdash_{(\Sigma_Q^\diamond,\mathfrak{I}_{\mathrm{T}})} \sim\diamond q_0$$

since $\mathsf{trg}'^{+}(\sim_{b_3}\diamond_{b_2}q_0') = 0 \notin D$. ∇

Path satisfaction is easily extended to interpretation structures. We say that I *satisfies* φ, written

$$I \Vdash_{(\Sigma,\mathfrak{I})} \varphi,$$

if $I, \tau' \Vdash_\Sigma \varphi$ for every path τ' in I for φ. Clearly, $\mathsf{trg}'^{+}([\![\varphi]\!]^I_\Sigma) \subseteq D$ iff $I \Vdash_\Sigma \varphi$.

Example 2.11 Taking into account Example 2.10 we can conclude that

$$I \not\Vdash_{(\Sigma_Q^\diamond,\mathfrak{I}_{\mathrm{T}})} \sim\diamond q_0.$$

since there is a path in I for $\sim\diamond q_0$ whose target is not designated. ∇

Satisfaction and denotation are extended to sets of formulas as expected. Entailment is defined on top of satisfaction as usual. We say that a set Γ of formulas over Σ *entails* in the context of an interpretation system $(\Sigma, \mathfrak{I})$ a formula φ over Σ, written $\Gamma \models_{(\Sigma,\mathfrak{I})} \varphi$, whenever for every $I \in \mathfrak{I}$ if $I \Vdash_{(\Sigma,\mathfrak{I})} \Gamma$ then $I \Vdash_{(\Sigma,\mathfrak{I})} \varphi$. Furthermore, we say that φ is *valid* in $(\Sigma, \mathfrak{I})$, if $\emptyset \models_{(\Sigma,\mathfrak{I})} \varphi$, that we may write simply as $\models_{(\Sigma,\mathfrak{I})} \varphi$.

Example 2.12 Hence

$$\not\models_{(\Sigma_Q^{\Diamond},\mathfrak{I}_{\mathrm{T}})} \sim\!\Diamond q_0$$

taking into account Example 2.11. ∇

When there is no ambiguity we may omit the reference to the interpretation system in the satisfaction $\Vdash$ and entailment $\models$ symbols.

3 Importing logics

In general terms, importing a logic system $\mathcal{L}_1$ into a logic system $\mathcal{L}_2$ produces a logic system, denoted by $\mathcal{L}_2[\mathcal{L}_1]$, such that: (i) the language of $\mathcal{L}_2[\mathcal{L}_1]$ consists of the language of $\mathcal{L}_2$ enriched, via the importing connective $\daleth$, with the formulas of $\mathcal{L}_1$ as monolithic elements (that is, like propositional symbols); (ii) its semantics consists of the interpretation structures of $\mathcal{L}_2$ enriched with the structures of $\mathcal{L}_1$, such that the distinguished truth-values of both structures are related via the denotation of the importing connective, as well as the non-distinguished truth-values; and (iii) its axioms are the axioms of $\mathcal{L}_2$ together with the theorems of $\mathcal{L}_1$, and its rules are the rules of $\mathcal{L}_2$. Herein we concentrate on importing from a semantical point view and so leave its proof theory for a future work. Importing subsumes several particular forms of logic combinations. In this section, after defining importing, we describe particular forms of this combination mechanism that we show in Section 5 to be equivalent to globalization [13] and to temporalization [8, 9] and modalization [7] when no connectives are shared.

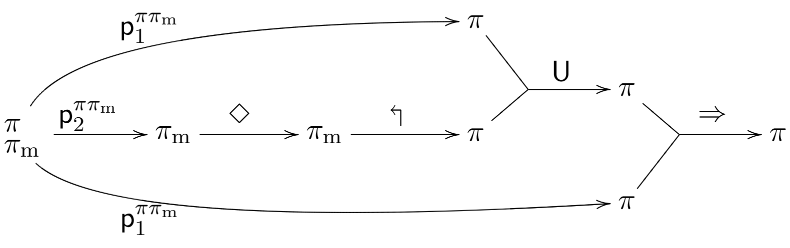

Figure 8: Formula $(\mathsf{p}_1^{\pi\pi_{\mathrm{m}}} \,\mathsf{U}\, (\daleth(\Diamond \mathsf{p}_2^{\pi\pi_{\mathrm{m}}}))) \Rightarrow \mathsf{p}_1^{\pi\pi_{\mathrm{m}}}$.

We consider an explicit connective $\daleth$ for transforming language formulas of the imported logic into formulas of the importing logic, but not in the other way around. For example, the formula

$$(\xi_1 \,\mathsf{U}\, (\daleth(\Diamond \xi_2))) \Rightarrow \xi_1,$$

that is,

$$(\mathsf{p}_1^{\pi\pi_{\mathrm{m}}} \,\mathsf{U}\, (\daleth(\Diamond \mathsf{p}_2^{\pi\pi_{\mathrm{m}}}))) \Rightarrow \mathsf{p}_1^{\pi\pi_{\mathrm{m}}} : \pi\pi_{\mathrm{m}} \to \pi,$$

depicted in Figure 8, is in the language induced by the signature $\Sigma^{\mathrm{LTL}}_Q[\Sigma^{\Diamond}_{Q'}]$, depicted in Figure 9, resulting from importing $\Sigma^{\Diamond}_{Q'}$ into Σ^{LTL}_Q after renaming the propositional connectives and the proposition sorts of the modal signature. For the sake of a lighter notation we may write 'φ' for $(\Lsh\,\varphi)$. Accordingly, the

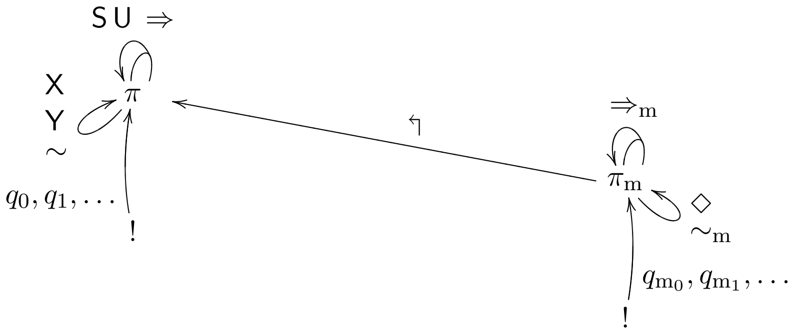

Figure 9: Signature $\Sigma^{\mathrm{LTL}}_Q[\Sigma^{\Diamond}_{Q'}]$.

formula above may be abbreviated by

$$(\xi_1 \,\mathsf{U}\, (\text{'}\Diamond\xi_2\text{'})) \Rightarrow \xi_1.$$

On the other hand, the expression

$$\Diamond(\xi \,\mathsf{U}\, q_{\mathrm{m}1}),$$

where ξ is id_π, does not belong to that language.

We define importing for a *suitably disjoint* pair Σ_1 and Σ_2 of signatures, that is, signatures where Π_1 and Π_2 are disjoint, as well as E_1 and E_2.

The *importing of a signature* Σ_1 *into a signature* Σ_2, where Σ_1 and Σ_2 are suitably disjoint, denoted by

$$\Sigma_2[\Sigma_1],$$

is the signature $((V, E, \mathsf{src}, \mathsf{trg}), !, \Pi)$ where

- $V = V_1 \cup V_2$;
- E is $E_1 \cup E_2 \cup \{\Lsh_{vu} : v \in \Pi_2, u \in \Pi_1\}$;
- src and trg are such that
  - $\mathsf{src}(e) = \mathsf{src}_i(e)$ and $\mathsf{trg}(e) = \mathsf{trg}_i(e)$ if e is in E_i for $i = 1, 2$;
  - $\mathsf{src}(\Lsh_{vu}) = u$ and $\mathsf{trg}(\Lsh_{vu}) = v$;
- Π is $\Pi_1 \cup \Pi_2$.

To simplify the presentation, when Π_1 and Π_2 are singletons we may omit the reference to the sorts in $\Lsh_{uv}$ and simply write $\Lsh$.

Example 3.1 Recall from Example 2.3 that Σ_Q^{LTL} is the signature of linear temporal propositional logic with Q as the set of propositional symbols. The ↰-*temporalization* of a signature Σ_1 suitably disjoint with Σ_Q^{LTL}, denoted by

$$\mathrm{LTL}[\Sigma_1],$$

is the signature resulting from the importing of Σ_1 into Σ_Q^{LTL}. See Figure 9 for a graphical representation of the signature resulting from the ↰-temporalization of the modal signature $\Sigma_Q^{\Diamond}$ (described in Example 2.1). For instance

$$(\xi_1 \,\mathsf{U}\, (\text{'}\Diamond\xi_2\text{'})) \Rightarrow \xi_1$$

where ξ_1 is $\mathsf{p}_1^{\pi\pi_\mathrm{m}}$ and ξ_2 is $\mathsf{p}_2^{\pi\pi_\mathrm{m}}$, is a schema formula over $\mathrm{LTL}(\Sigma_Q^{\Diamond})$. ∇

Example 3.2 Recall from Example 2.1 that $\Sigma_Q^{\Diamond}$ is the signature of modal propositional logic with Q as the set of propositional symbols. The ↰-*modalization* of a signature Σ_1 suitably disjoint with $\Sigma_Q^{\Diamond}$, denoted by

$$\mathrm{M}[\Sigma_1]$$

is the signature resulting from the importing of Σ_1 into $\Sigma_Q^{\Diamond}$. ∇

Example 3.3 Recall from Example 2.4 that Σ_P denotes the signature for classical propositional logic with P as the set of propositional symbols. In particular, $\Sigma_\emptyset$ is the classical propositional signature with no propositional symbols. The ↰-*globalization* of a signature Σ_1 suitably disjoint with $\Sigma_\emptyset$, denoted by

$$\mathrm{G}[\Sigma_1]$$

is the signature resulting from the importing of Σ_1 into $\Sigma_\emptyset$. ∇

With respect to the semantics of importing, for each model of the imported logic and each model of the importing logic, there is a model of the combined logic. This model contains a copy of each component model, and is such that the distinguished truth values of one logic are related, via the denotation of the importing ↰ connective, with the distinguished truth values of the other, and the non-distinguished truth values of one logic are related with the non-distinguished truth values of the other.

We assume that the structure being imported and the importing structure have disjoint sets of truth values and disjoint sets of operation m-edges. More rigorously, we work with *suitably disjoint* interpretation structures (Σ_1, I_1) and (Σ_2, I_2), that is, structures where Σ_1 and Σ_2 are suitably disjoint, $(V_1')_{\Pi_1}$ and $(V_2')_{\Pi_2}$ are disjoint, and E_1' and E_2' are disjoint as well.

Importing an interpretation structure (Σ_1, I_1) *into an interpretation structure* (Σ_2, I_2), where (Σ_1, I_1) and (Σ_2, I_2) are suitably disjoint, denoted by

$$(\Sigma_2, I_2)[(\Sigma_1, I_1)]$$

is $(\Sigma_2[\Sigma_1], I_2[I_1])$, where $I_2[I_1]$ is the tuple $((V'_{↰}, E'_{↰}, \mathsf{src}'_{↰}, \mathsf{trg}'_{↰}), \alpha_{↰}, D_{↰}, !)$ such that

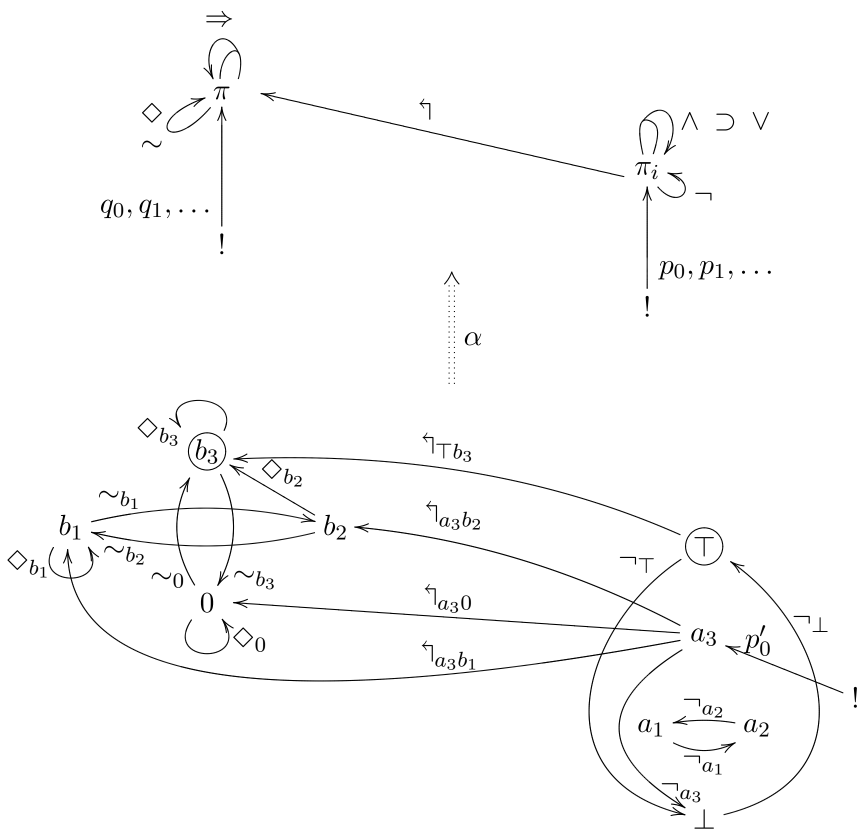

Figure 10: Part of the interpretation structure resulting from importing the structure partially described in Figure 6 for intuitionistic logic into the structure partially described in Figure 5 for modal logic T. This structure is in the $\Lsh$-modalization, see Example 3.7, of the intuitionistic interpretation system $(\Sigma_P^{\wedge,\vee}, \mathfrak{I})$ (described in Example 2.5). Note that, among others, the m-edges from $a_1, a_2, \bot$ to $b_1, b_2, 0$ corresponding to denotations of $\Lsh$ are not represented.

- $V'_{\Lsh}$ is $V'_1 \cup V'_2$;
- $E'_{\Lsh}$ is the union of the sets
  - E'_1;
  - E'_2;
  - $\{\Lsh_{v'_2 v'_1} : v'_1 \in D_1, v'_2 \in D_2\}$;
  - $\{\Lsh_{v'_2 v'_1} : v'_1 \in (V'_1)_{\Pi_1} \setminus D_1, v'_2 \in (V'_2)_{\Pi_2} \setminus D_2\}$;
- $\mathsf{src}'_{\Lsh}$ and $\mathsf{trg}'_{\Lsh}$ are such that
  - $\mathsf{src}'_{\Lsh}(e'_i) = \mathsf{src}'_i(e'_i)$ and $\mathsf{trg}'_{\Lsh}(e'_i) = \mathsf{trg}'_i(e'_i)$ for $i = 1, 2$;
  - $\mathsf{src}'_{\Lsh}(\Lsh_{v'v''}) = v'$ and $\mathsf{trg}'_{\Lsh}(\Lsh_{v'v''}) = v''$;
- $\alpha_{\Lsh}$ is such that

– $\alpha^{\mathsf{v}}_{\daleth}(v') = \alpha^{\mathsf{v}}_i(v')$ whenever $v' \in V'_i$ for $i = 1, 2$;
– $\alpha^{\mathsf{e}}_{\daleth}(e') = \alpha^{\mathsf{e}}_i(e')$ whenever $e' \in E'_i$ for $i = 1, 2$;
– $\alpha^{\mathsf{e}}_{\daleth}(\daleth_{v'v''}) = \daleth_{\alpha^{\mathsf{v}}_{\daleth}(v')\alpha^{\mathsf{v}}_{\daleth}(v'')}$;

- $D_{\daleth}$ is $D_1 \cup D_2$.

Importing an interpretation system into another, is simply the importing of all the structures of one into structures of the other, as long as the systems are *suitably disjoint*, that is, as long as all the pairs of an interpretation structure of one with an interpretation structure of the other, are suitably disjoint.

Importing an interpretation system $(\Sigma_1, \mathfrak{I}_1)$ *into an interpretation system* $(\Sigma_2, \mathfrak{I}_2)$, where $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ are suitably disjoint, denoted by

$$(\Sigma_2, \mathfrak{I}_2)[(\Sigma_1, \mathfrak{I}_1)],$$

is the interpretation system $(\Sigma_2[\Sigma_1], \mathfrak{I}_2[\mathfrak{I}_1])$ where $\mathfrak{I}_2[\mathfrak{I}_1]$ is the set of interpretation structures $\{(\Sigma_2, I_2)[(\Sigma_1, I_1)] : I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}$.

We now describe as examples several particular cases of importing, which are proven, in Section 5, to be equivalent with well-known logic combination techniques.

Example 3.4 The $\daleth$-*temporalization* of an interpretation system $(\Sigma_1, \mathfrak{I}_1)$, where $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma^{\mathrm{LTL}}_Q, \mathfrak{I}_{\mathrm{LTL}})$ are suitably disjoint, denoted by

$$\mathrm{LTL}[(\Sigma_1, \mathfrak{I}_1)]$$

is the interpretation system

$$(\Sigma^{\mathrm{LTL}}_Q, \mathfrak{I}_{\mathrm{LTL}})[(\Sigma_1, \mathfrak{I}_1)]$$

resulting from importing $(\Sigma_1, \mathfrak{I}_1)$ into $(\Sigma^{\mathrm{LTL}}_Q, \mathfrak{I}_{\mathrm{LTL}})$, recall $(\Sigma^{\mathrm{LTL}}_Q, \mathfrak{I}_{\mathrm{LTL}})$ in Example 2.7. ∇

In the sequel we denote the interpretation system for classical propositional logic (introduced in Example 2.8) with no propositional variables, i.e., with $P = \emptyset$, by

$$(\Sigma_\emptyset, \{I_c\})$$

where I_c is the unique interpretation structure that the system has by definition when $P = \emptyset$ (see Example 2.8).

Moreover, by a *consistent* interpretation system we mean an interpretation system with a non-empty set of interpretation structures.

In the next particular example of importing, named $\daleth$-globalization, importing is applied to a more restrict class of interpretation systems, the class of g-appropriate interpretation systems. An interpretation system $(\Sigma_1, \mathfrak{I}_1)$ is *g-appropriate* if the following conditions hold: (i) $(\Sigma_\emptyset, \{I_c\})$ and $(\Sigma_1, \mathfrak{I}_1)$ are suitably disjoint; (ii) for every e in E_1 and I_1 in $\mathfrak{I}_1$ there is e' in E'_1 such that $\alpha^{\mathsf{e}}_{I_1}(e') = e$; (iii) each structure I_1 in $\mathfrak{I}_1$ is deterministic in the sense that $(\alpha^+_{I_1})^{-1}(\psi)$ is a singleton for every concrete formula ψ over Σ_1; and (iv) $\mathfrak{I}_1$ is non-empty, that is, $(\Sigma_1, \mathfrak{I}_1)$ is consistent.

Example 3.5 The ↰-*globalization* of a g-appropriate interpretation system $(\Sigma_1, \mathfrak{I}_1)$, denoted by

$$\mathrm{G}[(\Sigma_1, \mathfrak{I}_1)]$$

is the interpretation system

$$(\Sigma_\emptyset, \{I_c\})[(\Sigma_1, \vec{\mathfrak{I}}_1)]$$

resulting from importing $(\Sigma_1, \vec{\mathfrak{I}}_1)$ into $(\Sigma_\emptyset, \{I_c\})$, where $\vec{\mathfrak{I}}_1$ is the collection of all interpretation structures

$$I_J = ((V'_J, E'_J, \mathsf{src}'_J, \mathsf{trg}'_J), \alpha_J, D_J, !)$$

for Σ_1, for each non-empty sequence $J = \{I_k\}_{k<\beta}$ of structures in $\mathfrak{I}_1$, where β is an ordinal, defined as follows:

- $V'_J = \{v'_0 \ldots v'_k \ldots : k < \beta, v'_k \in V'_k$ and $\alpha^{\mathsf{v}}_0(v'_0) = \ldots = \alpha^{\mathsf{v}}_k(v'_k) = \ldots\}$;
- $E'_J = \{e'_0 \ldots e'_k \ldots : k < \beta, e'_k \in E'_k$ and $\alpha^{\mathsf{e}}_0(e'_0) = \ldots = \alpha^{\mathsf{e}}_k(e'_k) = \ldots\}$;
- $\mathsf{src}'_J(e'_0 \ldots e'_k \ldots) = \mathsf{src}'_0(e'_0) \ldots \mathsf{src}'_k(e'_k) \ldots$;
- $\mathsf{trg}'_J(e'_0 \ldots e'_k \ldots) = \mathsf{trg}'_0(e'_0) \ldots \mathsf{trg}'_k(e'_k) \ldots$;
- $\alpha^{\mathsf{v}}_J(v'_0 \ldots v'_k \ldots) = \alpha^{\mathsf{v}}_k(v'_k)$ and $\alpha^{\mathsf{e}}_J(e'_0 \ldots e'_k \ldots) = \alpha^{\mathsf{e}}_k(e'_k)$;
- $D_J = \{v'_0 \ldots v'_k \ldots : v'_0 \in D_0, \ldots, v'_k \in D_k, \ldots\}$;
- ! is the sequence $!_0 \ldots !_k \ldots$. ∇

We omit the proof of the next proposition, Proposition 3.6, since it follows straightforwardly by induction.

Proposition 3.6 Given a g-appropriate interpretation system $(\Sigma_1, \mathfrak{I}_1)$, the set $(\alpha^+_{I_c[I_J]})^{-1}(\varphi)$ is a singleton for every concrete formula φ over $\Sigma_\emptyset[\Sigma_1]$ and non-empty sequence J of structures in $\mathfrak{I}_1$.

In the next example of ↰-modalization, we consider modalization by system T, that is, importing into system T. Other examples can be defined analogously by importing into other modal systems.

Example 3.7 The ↰-*modalization* of an interpretation system $(\Sigma_1, \mathfrak{I}_1)$, where $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma^{\Diamond}_Q, \mathfrak{I}_{\mathrm{T}})$ are suitably disjoint, denoted by

$$M_T[(\Sigma_1, \mathfrak{I}_1)]$$

is the interpretation system

$$(\Sigma^{\Diamond}_Q, \mathfrak{I}_{\mathrm{T}})[(\Sigma_1, \mathfrak{I}_1)]$$

resulting from importing $(\Sigma_1, \mathfrak{I}_1)$ into $(\Sigma^{\Diamond}_Q, \mathfrak{I}_{\mathrm{T}})$ (recall $(\Sigma^{\Diamond}_Q, \mathfrak{I}_{\mathrm{T}})$ in Example 2.6). See Figure 10 for a graphical description of part of an interpretation structure in the interpretation system resulting from the $\urcorner$-modalization of the intuitionistic interpretation system $(\Sigma_P^{\wedge,\vee}, \mathfrak{I})$ described in Example 2.5. Denote that interpretation structure by I'. Then, for example, the formula $\Diamond(\text{'}p_0\text{'})$ is not satisfied in I', represented by

$$I' \not\Vdash \Diamond\text{'}p_0\text{'}$$

since the target of the path $\Diamond_0 \urcorner_{a_3 0}\, p_0'$ in I' for $\Diamond\text{'}p_0\text{'}$ is not in D (as can be graphically seen in Figure 10). ∇

4 Conservativeness

Conservativeness is an important property for any logic combination mechanism, since for instance it guarantees that there are no unwanted collapses, as the ones mentioned in [6], of one logic system into another, when combining with no interaction consistent component logics. We start by introducing some relevant notions.

By *conservative extension* we mean the following, an interpretation system $(\Sigma, \mathfrak{I})$ is a *conservative extension* of another interpretation system $(\Sigma_1, \mathfrak{I}_1)$, whenever

$$\Gamma_1 \vDash_{(\Sigma,\mathfrak{I})} \varphi_1 \quad \text{if and only if} \quad \Gamma_1 \vDash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1,$$

for any set $\Gamma_1 \cup \{\varphi_1\}$ of formulas over Σ_1.

Proposition 4.1 Given suitably disjoint interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$, interpretation structures I_1 and I_2 in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively, and a formula φ_1 over Σ_1, τ is a path in $I_2[I_1]$ for φ_1 if and only if τ is a path in I_1 for φ_1.

Proof: The proof follows by induction on the irreducible path φ_1:

(a) φ_1 is $\mathsf{p}_i^{v_1\ldots v_n}$. ($\Rightarrow$) Let τ be an irreducible path of $G'^{\dagger}_{I_2[I_1]}$ such that $\alpha^+_{I_2[I_1]}(\tau) = \mathsf{p}_i^{v_1\ldots v_n}$. By definition of the functor induced by an m-graph morphism, see the Appendix, τ is of the form $\mathsf{p}_i^{v'_1\ldots v'_n}$ with $\alpha^{\mathsf{v}}_{I_2[I_1]}(v'_j) = v_j$ for $j = 1, \ldots, n$. Since $(V'_1)_{\Pi_1}$ and $(V'_2)_{\Pi_2}$ are disjoint, $v_1, \ldots, v_n$ are in V_1, and taking into account the definition of $\alpha^{\mathsf{v}}_{I_2[I_1]}$ we can conclude that v'_j is in V'_1 and $\alpha^{\mathsf{v}}_{I_1}(v'_j) = \alpha^{\mathsf{v}}_{I_2[I_1]}(v'_j) = v_j$ for $j = 1, \ldots, n$ and so τ is an irreducible path of $G'^{\dagger}_{I_1}$ with $\alpha^+_{I_1}(\tau) = \varphi_1$; ($\Leftarrow$) Let τ be such that $\alpha^+_{I_1}(\tau) = \mathsf{p}_i^{v_1\ldots v_n}$. Then τ is of the form $\mathsf{p}_i^{v'_1\ldots v'_n}$ with $\alpha^{\mathsf{v}}_{I_1}(v'_j) = v_j$ for $j = 1, \ldots, n$. Since $(V'_1)_{\Pi_1}$ and $(V'_2)_{\Pi_2}$ are disjoint, $v'_j \in V'^{+}_{I_2[I_1]}$ for $j = 1, \ldots, n$ and taking into account the definition of $\alpha^{\mathsf{v}}_{I_2[I_1]}$, the result follows since $\alpha^+_{I_2[I_1]}(v'_j) = \alpha^+_{I_1}(v'_j) = v_j$ for $j = 1, \ldots, n$;

(b) φ_1 is ϵ_{s_1}. The proof is similar to the proof of case (a) so we omit it;

(c) φ_1 is $e_1 w_1$. ($\Rightarrow$) Let τ be an irreducible path of $G'^{\dagger}_{I_2[I_1]}$ such that $\alpha^+_{I_2[I_1]}(\tau) = e_1 w_1$. By definition of the functor induced by an m-graph morphism, see the Appendix, τ is of the form $e'_1 w'_1$ with $\alpha^{\mathsf{e}}_{I_2[I_1]}(e'_1) = e_1$ and $\alpha^+_{I_2[I_1]}(w'_1) = w_1$. Since E'_1 and E'_2 are disjoint, $e_1 \in E_1$ and taking into account the definition of

$\alpha^{\mathsf{e}}_{I_2[I_1]}$ we can conclude that e'_1 is in E'_1 and $\alpha^{\mathsf{e}}_{I_1}(e'_1) = \alpha^{\mathsf{e}}_{I_2[I_1]}(e'_1) = e_1$. Moreover, by induction hypothesis, w'_1 is a path in I_1 for w_1. So $\alpha^{+}_{I_1}(e'_1 w'_1) = e_1 w_1$ as we wanted to show; ($\Leftarrow$) Let τ be such that $\alpha^{+}_{I_1}(\tau) = e_1 w_1$. Then τ is of the form $e'_1 w'_1$ with $\alpha^{\mathsf{e}}_{I_1}(e'_1) = e_1$ and $\alpha^{+}_{I_1}(w'_1) = w_1$. So by induction hypothesis w'_1 is a path in $I_2[I_1]$ with $\alpha^{+}_{I_2[I_1]}(w'_1) = w_1$, and $\alpha^{\mathsf{e}}_{I_2[I_1]}(e'_1) = e_1$ by definition of importing and of $\alpha^{\mathsf{e}}_{I_2[I_1]}$. Therefore τ is a path in $I_2[I_1]$ with $\alpha^{+}_{I_2[I_1]}(\tau) = \varphi_1$ as we wanted to show;

(d) φ_1 is $\langle w_1, \ldots, w_n \rangle$. The proof of this case is omitted since it is similar to the proof of case (c). QED

As was expected from Proposition 4.1 a similar result holds also for satisfaction.

Proposition 4.2 Given suitably disjoint interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$, and interpretation structures I_1 and I_2 in $\mathfrak{I}_1$ and $\mathfrak{I}_2$ respectively,

$$I_2[I_1] \Vdash_{(\Sigma_2,\mathfrak{I}_2)[(\Sigma_1,\mathfrak{I}_1)]} \varphi_1 \quad \text{if and only if} \quad I_1 \Vdash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1,$$

for any formula φ_1 over Σ_1.

Proof: In fact: ($\Rightarrow$) Assume that $I_2[I_1] \Vdash_{(\Sigma_2,\mathfrak{I}_2)[(\Sigma_1,\mathfrak{I}_1)]} \varphi_1$ and let τ be a path in I_1 for φ_1. Hence τ is a path in $I_2[I_1]$ for φ_1 by Proposition 4.1. So the target of τ is in $D_{I_2[I_1]}$. Since the target of τ is in $(V'_1)_{\Pi_1}$, $(V'_1)_{\Pi_1}$ and $(V'_2)_{\Pi_2}$ are disjoint, $D_{I_1} \subseteq (V'_1)_{\Pi_1}$, $D_{I_2} \subseteq (V'_2)_{\Pi_2}$, and $D_{I_2[I_1]} = D_{I_2} \cup D_{I_1}$ then the target of τ is in D_{I_1} as we wanted to show; ($\Leftarrow$) Assume that $I_1 \Vdash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1$ and let τ be a path in $I_2[I_1]$ for φ_1. Hence τ is a path in I_1 for φ_1 by Proposition 4.1. So the target of τ is in D_{I_1}. Since $D_{I_2[I_1]} = D_{I_2} \cup D_{I_1}$ then the target of τ is in $D_{I_2[I_1]}$ as we wanted to show. QED

As the following proposition, Proposition 4.3, shows, importing is conservative for the imported logic only if the importing logic is consistent.

Proposition 4.3 Given suitably disjoint interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ such that $(\Sigma_2, \mathfrak{I}_2)$ is consistent, then $(\Sigma_2, \mathfrak{I}_2)[(\Sigma_1, \mathfrak{I}_1)]$ is a conservative extension of $(\Sigma_1, \mathfrak{I}_1)$.

Proof: Let $\Gamma_1 \cup \{\varphi_1\}$ be a set of formulas over Σ_1. Then: ($\Rightarrow$) Assume that $\Gamma_1 \vDash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1$. Let $I_2[I_1]$ be an interpretation structure in $\mathfrak{I}_2[\mathfrak{I}_1]$ such that $I_2[I_1] \Vdash \Gamma_1$. Then $I_1 \Vdash \Gamma_1$ by Proposition 4.2. Hence $I_1 \Vdash \varphi_1$ and so $I_2[I_1] \Vdash \varphi_1$ by Proposition 4.2; ($\Leftarrow$) Assume that $\Gamma_1 \vDash_{(\Sigma_2,\mathfrak{I}_2)[(\Sigma_1,\mathfrak{I}_1)]} \varphi_1$. Let I_1 be an interpretation structure in $\mathfrak{I}_1$ such that $I_1 \Vdash \Gamma_1$. Since $(\Sigma_2, \mathfrak{I}_2)$ is consistent let I_2 be any model in $\mathfrak{I}_2$. Then $I_2[I_1] \Vdash \Gamma_1$ by Proposition 4.2, and so $I_2[I_1] \Vdash \varphi_1$. Therefore $I_1 \Vdash \varphi_1$ by Proposition 4.2. QED

5 Equivalences

We now show that the particular forms of importing described in Section 3 are equivalent with well known logic combination mechanisms. These equivalences with specific forms of importing offer a new and different perspective on those combination techniques. Moreover, the results established for importing may be immediately transferred, under mild conditions, for results about those combination mechanisms, as is the case of conservativeness proved in Section 4. Furthermore, it may be useful when dealing with one of those combination mechanisms, to use the simple and intuitive semantics for it offered by importing.

5.1 Globalization

Globalization is a particular case of exogenous enrichment introduced in [13] with the aim of probabilistic reasoning. Herein we abstract the main characteristics of globalization and define it in a more general context not restricted only to probabilistic reasoning.

In order to simplify the presentation assume given a g-appropriate interpretation system $(\Sigma_1, \mathfrak{I}_1)$. The *globalization* of $(\Sigma_1, \mathfrak{I}_1)$ produces a logic system, denoted by

$$G_{(\Sigma_1,\mathfrak{I}_1)}$$

such that the *language* of $G_{(\Sigma_1,\mathfrak{I}_1)}$, denoted by

$$L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$$

is the set of formulas inductively defined as follows:

- φ is in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ if φ is a formula over Σ_1 with target in Π_1;
- $\boxminus\varphi$ is in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ if φ is in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$;
- $\sqsupset\langle\varphi_1, \varphi_2\rangle$ is in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ if φ_1 and φ_2 are in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$;

satisfaction in $G_{(\Sigma_1,\mathfrak{I}_1)}$ is a binary relation $\Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ between subsets of interpretation structures in $\mathfrak{I}_1$ and formulas in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ inductively defined as follows:

- $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$ iff $V \Vdash_1 \varphi$, if φ is a formula over Σ_1 with target in Π_1;
- $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \boxminus\varphi$ iff $V \nVdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$;
- $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \sqsupset\langle\varphi_1, \varphi_2\rangle$ iff $V \nVdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_1$ or $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_2$;

moreover, entailment in $G_{(\Sigma_1,\mathfrak{I}_1)}$ is defined as expected, that is,

$$\Gamma \vDash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$$

iff $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$ whenever $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \Gamma$ for every non-empty subset V of interpretation structures in $\mathfrak{I}_1$.

Consider the map $\cdot^{\urcorner_g}$ from $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ to $L(\mathrm{G}[\Sigma_1])$ (recall $\urcorner$-globalization of a signature Σ_1, denoted by $\mathrm{G}[\Sigma_1]$, in Example 3.3) inductively defined as follows:

- $(\varphi)^{\urcorner_g}$ is φ if φ is a formula over Σ_1 with target in Π_1;
- $(\boxminus\varphi)^{\urcorner_g}$ is $\neg(\varphi)_{\urcorner_g}$;
- $(\sqsupset\langle\varphi_1, \varphi_2\rangle)^{\urcorner_g}$ is $\supset\langle(\varphi_1)_{\urcorner_g}, (\varphi_2)_{\urcorner_g}\rangle$;

where $\cdot_{\urcorner_g}$ is the map from $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ to $L(\mathrm{G}[\Sigma_1])$ such that $(\varphi)_{\urcorner_g}$ is $\urcorner\varphi$ if φ is a formula over Σ_1, otherwise $(\varphi)_{\urcorner_g}$ is $(\varphi)^{\urcorner_g}$.

We start by relating satisfaction in $G_{(\Sigma_1,\mathfrak{I}_1)}$ and satisfaction in $\mathrm{G}[(\Sigma_1,\mathfrak{I}_1)]$ (recall $\urcorner$-globalization of an interpretation system $(\Sigma_1,\mathfrak{I}_1)$, denoted by $\mathrm{G}[(\Sigma_1,\mathfrak{I}_1)]$, in Example 3.5).

Proposition 5.1 Given a concrete formula ψ in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$ and a non-empty subset V of interpretation structures in $\mathfrak{I}_1$,

$$V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \psi \quad \text{if and only if} \quad I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\psi)^{\urcorner_g}$$

where J is a sequence $\{I_k\}_{k<\beta}$ with all the elements of V, and β is an ordinal.

Proof: The proof follows by induction on ψ:

(1) ψ is a formula over Σ_1 with target in Π_1. Then $(\psi)^{\urcorner_g}$ is ψ. ($\Rightarrow$) Suppose that $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \psi$. Then $V \Vdash_1 \psi$ and so $I \Vdash_1 \psi$ for every interpretation structure I in V, that is, the target of every path in I for ψ is a designated value in D. Let τ be a path in $I_c[I_J]$ for ψ. Since ψ is a formula over Σ_1 then τ is simply a path over I_J, see Proposition 4.1. Hence, taking into account the definition of I_J in Example 3.5 denote by $\tau_0,\dots,\tau_k,\dots$ the paths in $I_0,\dots,I_k,\dots$ respectively for ψ induced by τ. Then, by the initial assumption, the target of $\tau_0,\dots,\tau_k,\dots$ are in $D_0,\dots,D_k,\dots$ respectively, and so the target of τ is in $D_J \subseteq D_{I_c[I_J]}$ as we wanted to show; ($\Leftarrow$) Assume that $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\psi)^{\urcorner_g}$, that is, $I_J \Vdash_J \psi$ by Proposition 4.2. We want to show that $V \Vdash_1 \psi$. Observe that every structure I in V has at least a path for ψ as a consequence of the properties imposed over $(\Sigma_1,\mathfrak{I}_1)$. Let $\tau_0,\dots,\tau_k,\dots$ be any paths in $I_0,\dots,I_k,\dots$ respectively, for ψ. Let τ be the path in I_J for ψ induced by those paths. Then τ is in D_J and so $\tau_0,\dots,\tau_k,\dots$ are in $D_0,\dots,D_k,\dots$ respectively, as we wanted to show;

(2) ψ is $\boxminus\varphi$. Then $(\psi)^{\urcorner_g}$ is $\neg(\varphi)_{\urcorner_g}$. Let τ be the unique path in $I_c[I_J]$ for $\neg(\varphi)_{\urcorner_g}$ according to Proposition 3.6. Then τ is of the form $\neg'\tau'$ where τ' is the unique path in $I_c[I_J]$ for $(\varphi)_{\urcorner_g}$. Observe that the target of τ and of τ' are both in I_c. ($\Rightarrow$) Suppose that $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \boxminus\varphi$. Then $V \nVdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$ and so $I_c[I_J] \nVdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi)^{\urcorner_g}$ by induction hypothesis. Hence the target of τ' is 0 and so the target of τ is in $D_{I_c[I_J]}$. This is enough to show our result since τ is the unique path for $\neg(\varphi)_{\urcorner_g}$; ($\Leftarrow$) Assume that $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} \neg(\varphi)_{\urcorner_g}$. Then $I_c[I_J] \nVdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi)_{\urcorner_g}$. Therefore $I_c[I_J] \nVdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi)^{\urcorner_g}$ and so $V \nVdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$ by induction hypothesis. Hence $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \boxminus\varphi$ by definition of satisfaction in $\Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}}$;

(3) ψ is $\sqsupset \langle\varphi_1,\varphi_2\rangle$. Then $(\psi)^{\urcorner_g}$ is $\supset\langle(\varphi_1)_{\urcorner_g},(\varphi_2)_{\urcorner_g}\rangle$. Let τ be the unique path in $I_c[I_J]$ for $\supset\langle(\varphi_1)_{\urcorner_g},(\varphi_2)_{\urcorner_g}\rangle$ according to Proposition 3.6. Then τ is of the form $\supset'\langle\tau_1',\tau_2'\rangle$ where τ_1' and τ_2' are the unique paths in $I_c[I_J]$ for $(\varphi_1)_{\urcorner_g}$ and $(\varphi_2)_{\urcorner_g}$ respectively. Observe that the target of τ, τ_1' and τ_2' are all in I_c. ($\Rightarrow$) Suppose that $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \sqsupset \langle\varphi_1,\varphi_2\rangle$. Then $V \nVdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_1$ or $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_2$. Hence $I_c[I_J] \nVdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_1)^{\urcorner_g}$ or $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_2)^{\urcorner_g}$ by induction hypothesis, and so $I_c[I_J] \nVdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_1)_{\urcorner_g}$ or $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_2)_{\urcorner_g}$. Therefore the target of τ_1' is 0 (note that τ_1' is the unique path for $(\varphi_1)_{\urcorner_g}$), the target of τ_2' is 1, and so the

target of τ is in $D_{I_c[I_J]}$. This is enough to show our result since τ is the unique path for $(\psi)^{\urcorner g}$; ($\Leftarrow$) Assume that $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\psi)^{\urcorner g}$. Then $I_c[I_J] \nVdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_1)_{\urcorner g}$ or $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_2)_{\urcorner g}$ since there is only one path in $I_c[I_J]$ for $(\psi)_{\urcorner g}$, for $(\varphi_1)_{\urcorner g}$ and for $(\varphi_2)_{\urcorner g}$. Therefore $I_c[I_J] \nVdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_1)^{\urcorner g}$ or $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_2)^{\urcorner g}$. So $V \nVdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_1$ or $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_2$ by induction hypothesis. Hence $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \sqsupset \langle\varphi_1, \varphi_2\rangle$ by definition of satisfaction in $\Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}}$. QED

The next result, Proposition 5.2, establishes the equivalence between globalization and $\urcorner$-globalization for concrete formulas.

Proposition 5.2 Given a set $\Gamma \cup \{\psi\}$ of concrete formulas in $L_{G_{(\Sigma_1,\mathfrak{I}_1)}}$,

$$\Gamma \vDash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \psi \quad \text{if and only if} \quad (\Gamma)^{\urcorner g} \vDash_{\mathrm{G}[(\Sigma_1,\mathfrak{I}_1)]} (\psi)^{\urcorner g}.$$

Proof: In fact: ($\Rightarrow$) Assume $\Gamma \vDash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$ and let I_J be an interpretation structure in $\vec{\mathfrak{I}}_1$ for a non-empty sequence J of structures in $\mathfrak{I}_1$ such that $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\Gamma)^{\urcorner g}$. Let V be the set with all the structures in J. Therefore $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \Gamma$ by Proposition 5.1 and so $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$. Hence $I_c[I_J] \Vdash_{\mathrm{G}[(\Sigma_1,\mathfrak{I}_1)]} (\varphi)^{\urcorner g}$ by Proposition 5.1; ($\Leftarrow$) Assume $(\Gamma)^{\urcorner g} \vDash_{\mathrm{G}[(\Sigma_1,\mathfrak{I}_1)]} (\psi)^{\urcorner g}$ and let V be a non-empty subset of interpretation structures in $\mathfrak{I}_1$ such that $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \Gamma$. Observe that such a non-empty subset exists since $(\Sigma_1, \mathfrak{I}_1)$ is consistent. Let I_J be an interpretation structure in $\vec{\mathfrak{I}}_1$ for a non-empty sequence J with all the structures in V. Then $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\Gamma)^{\urcorner g}$ by Proposition 5.1. Hence $I_c[I_J] \Vdash_{G[(\Sigma_1,\mathfrak{I}_1)]} (\varphi)^{\urcorner g}$ and so $V \Vdash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi$ by Proposition 5.1. QED

Results proved for importing hold in all its instances that satisfy their sufficient conditions. So, for instance in the case of conservativeness, we can show that globalization is conservative for concrete formulas relying on the equivalence between $\urcorner$-globalization and globalization proved in Proposition 5.2 and on the conservativeness result proved in Proposition 4.3 for importing and so for $\urcorner$-globalization.

Proposition 5.3 Given a set $\Gamma_1 \cup \{\varphi_1\}$ of concrete formulas over Σ_1,

$$\Gamma_1 \vDash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_1 \quad \text{if and only if} \quad \Gamma_1 \vDash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1,$$

Proof: In fact: ($\Leftarrow$) Assume that $\Gamma_1 \vDash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1$. Then $\Gamma_1 \vDash_{\mathrm{G}[(\Sigma_1,\mathfrak{I}_1)]} \varphi_1$ by Proposition 4.3, and so $(\Gamma_1)^{\urcorner g} \vDash_{\mathrm{G}[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_1)^{\urcorner g}$ since $(\varphi_1)^{\urcorner g} = \varphi_1$ and $(\Gamma_1)^{\urcorner g} = \Gamma_1$. So $\Gamma_1 \vDash_{G_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_1$ by the equivalence between globalization and $\urcorner$-globalization, see Proposition 5.2; ($\Rightarrow$) The proof follows similarly to the proof of the other direction so we omit it. QED

5.2 Temporalization

In this subsection we show that temporalization, see [8, 9], and $\urcorner$-temporalization, introduced in Section 3 in Example 3.1 and Example 3.4, are equivalent. We first recall temporalization, based on [8], in the case where no connectives are shared.

The *temporalization* of an interpretation system $(\Sigma_1, \Im_1)$ suitably disjoint with $(\Sigma_Q^{\rm LTL}, \Im_{\rm LTL})$ (recall $(\Sigma_Q^{\rm LTL}, \Im_{\rm LTL})$ in Example 3.4), produces a logic system, denoted by

$$T_{(\Sigma_1,\Im_1)}$$

such that the *language* of $T_{(\Sigma_1,\Im_1)}$, denoted by

$$L_{T_{(\Sigma_1,\Im_1)}}$$

is the set of formulas inductively defined as follows:

- φ is in $L_{T_{(\Sigma_1,\Im_1)}}$ if φ is a formula over Σ_1 with target in Π_1;
- $\sim\varphi$, $\mathsf{X}\varphi$ and $\mathsf{Y}\varphi$ are in $L_{T_{(\Sigma_1,\Im_1)}}$ if φ is in $L_{T_{(\Sigma_1,\Im_1)}}$;
- $\Rightarrow\langle\varphi_1, \varphi_2\rangle$, $\mathsf{S}\langle\varphi_1, \varphi_2\rangle$, $\mathsf{U}\langle\varphi_1, \varphi_2\rangle$ are in $L_{T_{(\Sigma_1,\Im_1)}}$ if φ_1 and φ_2 are in $L_{T_{(\Sigma_1,\Im_1)}}$;

and, taking into account the complete axiomatization of temporalization presented in [8], the valid formulas of $T_{(\Sigma_1,\Im_1)}$ are the valid formulas of the logic being imported together with the valid formulas of temporal logic possibly instantiated with formulas of the imported logic.

Consider the map $\cdot^{\urcorner_t}$ from $L_{T_{(\Sigma_1,\Im_1)}}$ to $L(\mathrm{LTL}[\Sigma_1])$ (recall $\mathrm{LTL}[\Sigma_1]$ in Example 3.1) inductively defined as follows:

- $(\varphi)^{\urcorner_t}$ is φ if φ is a formula over Σ_1 with target in Π_1;
- $(c\varphi)^{\urcorner_t}$ is $c(\varphi)_{\urcorner_t}$ for c in $\{\sim, \mathsf{X}, \mathsf{Y}\}$;
- $(c\langle\varphi_1, \varphi_2\rangle)^{\urcorner_t}$ is $c\langle(\varphi_1)_{\urcorner_t}, (\varphi_2)_{\urcorner_t}\rangle$ for c in $\{\Rightarrow, \mathsf{S}, U\}$;

where $\cdot_{\urcorner_t}$ is the map from $L_{T_{(\Sigma_1,\Im_1)}}$ to $L(\mathrm{LTL}[\Sigma_1])$ such that $(\varphi)_{\urcorner_t}$ is $\urcorner\varphi$ if φ is a formula over Σ_1, otherwise $(\varphi)_{\urcorner_t}$ is $(\varphi)^{\urcorner_t}$.

We now characterize the valid formulas of $\mathrm{LTL}[(\Sigma_1, \Im_1)]$.

Proposition 5.4 Given an interpretation system $(\Sigma_1, \Im_1)$ suitably disjoint with $(\Sigma_Q^{\rm LTL}, \Im_{\rm LTL})$, the valid formulas of $\mathrm{LTL}[(\Sigma_1, \Im_1)]$ are the valid formulas of $(\Sigma_1, \Im_1)$ together with the valid formulas of $(\Sigma_Q^{\rm LTL}, \Im_{\rm LTL})$ possibly instantiated with formulas of $(\Sigma_1, \Im_1)$.

Proof: We show that, given a formula φ over $\Sigma_Q^{\rm LTL}[\Sigma_1]$, $\vDash_{\mathrm{LTL}[(\Sigma_1,\Im_1)]} \varphi$ if and only if (i) $\vDash_{(\Sigma_1,\Im_1)} \varphi$ if φ is a formula over Σ_1; (ii) otherwise $\vDash_{(\Sigma_Q^{\rm LTL},\Im_{\rm LTL})} \varphi_t$ where φ_t is a formula of $(\Sigma_Q^{\rm LTL}, \Im_{\rm LTL})$ and φ_1 is a formula over $\Sigma_Q^{\rm LTL}[\Sigma_1]$ with no connectives in $E_{\rm LTL}$, such that $\varphi = \varphi_t \circ \varphi_1$. In fact:

($\Rightarrow$) Assume that $\vDash_{\mathrm{LTL}[(\Sigma_1,\Im_1)]} \varphi$. Consider two cases:

(i) φ is a formula over Σ_1. Let I_1 be an interpretation structure in $\Im_1$ and τ_1' a path in I_1 for φ. Let I_t be a model in $\Im_{\rm LTL}$. Then τ_1' is a path in $I_t[I_1]$ for φ by Proposition 4.1. Hence the target of τ_1' is a designated value in $D_{I_t[I_1]}$. Since $D_{I_t} \cap D_{I_1} = \emptyset$ and $D_{I_t[I_1]} = D_{I_t} \cup D_{I_1}$ then the target of τ_1' is in D_{I_1} as we wanted to show;

(ii) φ is not a formula over Σ_1. Let φ_t be a formula over $\Sigma_Q^{\rm LTL}$ and φ_1 a formula over $\Sigma_Q^{\rm LTL}[\Sigma_1]$ with no connectives of $E_{\rm LTL}$ such that φ is $\varphi_t \circ \varphi_1$. Let I_t be an interpretation structure in $\mathfrak{I}_{\rm LTL}$ and τ'_t a path in I_t for φ_t. Moreover let I_1 be an interpretation structure in $\mathfrak{I}_1$ and τ'_1 a path in $I_t[I_1]$ for φ_1 such that $\tau'_t \circ \tau'_1$ is a path in $I_t[I_1]$ for φ. Hence, the target of $\tau'_t \circ \tau'_1$, that is, the target of τ'_t is a designated value in $D_{I_t[I_1]}$. Since $D_{I_t} \cap D_{I_1} = \emptyset$ (note that $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_Q^{\rm LTL}, \mathfrak{I}_{\rm LTL})$ are suitably disjoint) and $D_{I_t[I_1]} = D_{I_t} \cup D_{I_1}$ then the target of τ'_t is in D_{I_t} as we wanted to show;

($\Leftarrow$) Consider two cases:

(i) assume that φ is a formula over Σ_1 such that $\vDash_{(\Sigma_1,\mathfrak{I}_1)} \varphi$ and let I be an interpretation structure in $\mathfrak{I}_{\rm LTL}[\mathfrak{I}_1]$ and τ a path in I for φ. Then I is of the form $I_t[I_1]$ and, by Proposition 4.1, τ is a path in I_1 for φ. Hence the target of τ is in D_1 and so is in $D_{I_t[I_1]}$;

(ii) let φ be a formula over $\Sigma_Q^{\rm LTL}[\Sigma_1]$, φ_1 a formula over $\Sigma_Q^{\rm LTL}[\Sigma_1]$ with no connectives of $E_{\rm LTL}$ and φ_t a non-empty formula over $\Sigma_Q^{\rm LTL}$ such that φ is $\varphi_t \circ \varphi_1$. Assume that $\vDash_{(\Sigma_Q^{\rm LTL},\mathfrak{I}_{\rm LTL})} \varphi_t$. Let I be an interpretation structure in $\mathfrak{I}_{\rm LTL}[\mathfrak{I}_1]$ and τ a path in I for φ. Observe that I is of the form $I_t[I_1]$. Let τ_t be a path in I_t for φ_t and τ_1 a path in I_1 for φ_1 such that τ is of the form $\tau_t \circ \tau_1$. Hence the target of τ_t is a designated value, and so the same happens for τ. QED

Corollary 5.5 Given an interpretation system $(\Sigma_1, \mathfrak{I}_1)$ suitably disjoint with $(\Sigma_Q^{\rm LTL}, \mathfrak{I}_{\rm LTL})$ and a formula ψ in $L_{T_{(\Sigma_1,\mathfrak{I}_1)}}$,

$$\vDash_{T_{(\Sigma_1,\mathfrak{I}_1)}} \psi \quad \text{if and only if} \quad \vDash_{{\rm LTL}[(\Sigma_1,\mathfrak{I}_1)]} (\psi)^{\urcorner_t}.$$

The proof of Corollary 5.5 follows immediately by Proposition 5.4 taking into account that the valid formulas of temporalization, see the description of temporalization in the beginning of the subsection, are, modulo the importing $\urcorner$ connective, the valid formulas of $\urcorner$-temporalization.

Taking into account the weak form of equivalence between temporalization and $\urcorner$-temporalization stated in Corollary 5.5, we can only transfer a weak form of conservativeness from importing, see Proposition 4.3, (more specifically from $\urcorner$-temporalization), to temporalization.

Proposition 5.6 Given a formula φ_1 over Σ_1,

$$\vDash_{T_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_1 \quad \text{if and only if} \quad \vDash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1,$$

Proof: In fact: ($\Leftarrow$) Assume that $\vDash_{(\Sigma_1,\mathfrak{I}_1)} \varphi_1$. Then $\vDash_{{\rm LTL}[(\Sigma_1,\mathfrak{I}_1)]} \varphi_1$ by Proposition 4.3, and so $\vDash_{{\rm LTL}[(\Sigma_1,\mathfrak{I}_1)]} (\varphi_1)^{\urcorner_t}$ since $(\varphi_1)^{\urcorner_t} = \varphi_1$. So $\vDash_{T_{(\Sigma_1,\mathfrak{I}_1)}} \varphi_1$ by the equivalence between temporalization and $\urcorner$-temporalization, see Corollary 5.5; ($\Rightarrow$) The proof follows similarly to the proof of the other direction so we omit it. QED

Modalization

The proof that modalization and $\urcorner$-modalization are equivalent follows in the same way as the proof that temporalization and $\urcorner$-temporalization are equivalent. That is, we would start by describing the language and the validities of

T-modalization, which are similar in terms of general structure to the validities of temporalization, see [7] and [8]. Then we would characterize the validities of $\urcorner$-modalization, which are also similar in terms of general structure to the validities of $\urcorner$-temporalization, and conclude that they coincide modulo the importing connective. The same happens for conservativeness.

6 Concluding remarks

A new general way of combining logics is defined subsuming particular asymmetric combination techniques, like temporalization [8, 9], modalization [7] and exogenous enrichment [13, 5, 12, 4, 1] as illustrated above, and many other such asymmetric mechanisms, like "propositionalization" of equational logic (importing equational logic into propositional logic) and "relevantization" (importing into relevance logic any given logic) not considered above. Thus, importing offers the possibility of obtaining results about all those combination techniques, just by proving them once for importing and invoking the specific subsumption results. That is precisely the case of conservativeness. We proved that importing is conservative and the result followed straightforwardly for all the subsumed combination mechanisms. Our definition of importing relies on an additional importing connective. The price paid at the language level is worthwhile given the advantages we gain at the semantic level.

A variant of the graph-theoretic approach proposed in [15] was adopted herein: instead of constructing the category induced by a multi-graph by factorization (obtaining morphisms as equivalence classes of paths) we built the category using a reduction system (obtaining morphisms as irreducible paths). In short, instead of imposing the required categorical equalities between paths we adopt a rewriting system implementing those equalities. This idea may be applicable in other situations where one needs to obtain a quotient structure and an inductive characterization of the set of equivalence classes is useful.

Given that importing subsumes several widely used asymmetric combination mechanisms and allows the proof for all those mechanisms of meta-properties like conservativeness, no one will question its import. However, we expect to reinforce the significance of importing by clarifying the relationship between it and fibring. To this end we are looking at fibring as a kind of two-way importing.

It should be stressed that we only addressed asymmetric combination mechanisms in the case where no connectives are shared. Constrained forms of importing, temporalization and modalization warrant attention for future work.

Further research on importing is needed in other directions, namely the definition of importing at the level of logic calculi. Afterwards, preservation of soundness and completeness can be investigated. Other transference results also deserve attention, even at the purely semantic level, like preservation of the finite model property that one expects to hold given that the models of the resulting logic are finitely built from pairs of models of the given logics. Therefore, one should expect an interesting sufficient condition for the preservation of decidability by importing.

Appendix

In order to improve the readability of the paper we concentrate in this section the constructions and the technical results related with obtaining the category with non-empty finite products induced by an m-graph, as well as the partial functor induced by an m-graph partial morphism.

The use of a reduction system to impose the product equivalences that morphisms should satisfy when defining the category with non-empty finite products induced by an m-graph morphism, is as far as we know new. It simplifies things since with this technique it is not necessary to consider equivalence classes. For instance, in the definition of maps whose domain is the set of morphisms. With this approach it is not necessary to guarantee that such a function gives the same value to all the elements of the equivalence class, since in this case a morphism is just an irreducible path.

Category with non-empty products induced by an m-graph

The generation of a category G^+ with finite non-empty products from an m-graph $G = (V, E, \mathsf{src}, \mathsf{trg})$ starts by transforming it to a graph and then enrich it with edges for projections and tuples. So, let $G_0^\dagger$ denote the graph $(V^+, E_0^\dagger, \mathsf{src}_0^\dagger, \mathsf{trg}_0^\dagger)$ with all the m-edges of G, plus additional edges of the form $\mathsf{p}_i^{v_1 \ldots v_m} : v_1 \ldots v_m \to v_i$ (to be used later on as projections) for $v_1, \ldots, v_m$ in V and a natural number m greater than or equal to 2. Graph $G_{k+1}^\dagger$ is obtained from graph $G_k^\dagger$ by adding edges of the form $\langle u_1, \ldots, u_m \rangle : s \to v_1 \ldots v_m$ (to be used later on for tupling) for any paths $u_1 : s \to v_1, \ldots, u_m : s \to v_m$ of $G_k^\dagger$ with $v_1, \ldots, v_m$ in V, for a natural number m greater than or equal to 2. Finally, we define graph $G^\dagger$ as $\cup_{k \in \mathbb{N}} G_k^\dagger$. In the sequel, given s in V^+ we denote by ϵ_s the empty path from s to s.

The properties that the category G^+ should satisfy in order to have products are described through an abstract reduction system, as defined below. An abstract reduction system is a pair $(S, \leadsto)$ where S is a set and $\leadsto$ is a binary relation on S. We introduce first some general useful notions about reduction systems, and then define what is the reduction system induced by an m-graph.

Given a binary relation $\leadsto$ over a set S, we denote by $\leadsto^+$ the transitive closure of $\leadsto$, and by $\overset{=}{\leadsto}^+$ the binary relation over S such that $x_1 \overset{=}{\leadsto}^+ x_2$ if either $x_1 = x_2$ or $x_1 \leadsto^+ x_2$. An element x in S is called $\leadsto$- *reducible* if there exists some other element x_1 in S such that $x \leadsto x_1$; otherwise it is called $\leadsto$-*irreducible* or in $\leadsto$- *normal form.* A $\leadsto$- *normal form of* x, is a $\leadsto$- *irreducible* element x' in S such that $x \overset{=}{\leadsto}^+ x'$. When an element x in S has a unique $\leadsto$-normal form, we denote that $\leadsto$-normal form either by $\overset{\leadsto}{x}$ or by $\overset{\leadsto}{\mathsf{nf}}(x)$ or simply by $\mathsf{nf}(x)$ when there is no ambiguity.

In order to properly define the reduction rules of the reduction system induced by an m-graph, we need first to define the notion of path context.

For that, assume fixed once and for all a fresh symbol x, and, given an m-graph G, denote by G_x the graph obtained from G by taking V^+ as the set of vertexes, taking the multi-edges as edges and adding an edge $x_{u_2}^{u_1} : u_1 \to u_2$ for each $u_1, u_2 \in V^+$. Build the graph $G_x^\dagger$ from G_x by adding the projections and tuplings like $G^\dagger$ was built above from G. Hence, by a *path context* or simply a *context* of $G^\dagger$ we mean a path of $(G_x)^\dagger$ where one and only one of these m-edges (now edges) occurs. We denote a generic path context by $yx_t^s z$. Moreover, given a path $w : s \to t$ of $G^\dagger$ we write ywz for the path of $G^\dagger$ obtained from the path context $yx_t^s z$ of $G^\dagger$ by replacing x_t^s by w.

The *reduction system induced by an m-graph G for a category with non-empty finite products*, is the pair $(paths(G^\dagger), \rightsquigarrow_G)$ where $\rightsquigarrow_G$ is the union of the binary relations $\rightsquigarrow_{G\mathsf{p}}$, $\rightsquigarrow_{G\langle\rangle}$ and $\rightsquigarrow_{G\epsilon}$ over $paths(G^\dagger)$ such that:

$$y\mathsf{p}_i^{v_1 \dots v_m}\langle u_1, \dots, u_m\rangle z \rightsquigarrow_{G\mathsf{p}} yu_i z$$

for any $v_1, \dots, v_m$ in V, natural number m greater than 1, i in $\{1, \dots, m\}$, context $yx_{v_i}^s z$ of $G^\dagger$, s in V^+, and for $j = 1, \dots, m$ path $u_j : s \to v_j$ of $G^\dagger$,

$$y\langle u_1, \dots, u_m\rangle az \rightsquigarrow_{G\langle\rangle} y\langle u_1 a, \dots, u_m a\rangle z$$

for any $v_1, \dots, v_m$ in V, natural number m greater than 1, context $yx_{v_1 \dots v_m}^s z$ of $G^\dagger$, edge $a : s \to t$ in $E^\dagger$, s and t in V^+, and for $j = 1, \dots, m$ path $u_j : t \to v_j$ of $G^\dagger$, and

$$y\langle \mathsf{p}_1^{v_1 \dots v_m}, \dots, \mathsf{p}_m^{v_1 \dots v_m}\rangle z \rightsquigarrow_{G\epsilon} y\epsilon_{v_1 \dots v_m} z$$

for any $v_1, \dots, v_m$ in V, natural number m greater than 1, and context $yx_{v_1 \dots v_m}^{v_1 \dots v_m} z$ of $G^\dagger$.

The reduction rules just presented are the key ingredient for G^+ (as defined below) to have non-empty finite products. Observe that rule $\rightsquigarrow_{G\mathsf{p}}$ imposes that edge p_j^s behaves like a projection, rule $\rightsquigarrow_{G\langle\rangle}$ is important for the universal property of the product, and rule $\rightsquigarrow_{G\epsilon}$ is important for expressing that certain tuples are equivalent to identities. The existence of arrows with this behavior is important for example when working with schematic formulas, that is, formulas open to instantiation, since when instantiating such a formula one wants to "distribute" the same value across all the occurrences of a variable, which is done straightforwardly using projections and tuples.

Example 6.1 Let $(G, !, \Pi)$ be the signature Σ_Q^{LTL} for linear temporal logic introduced in Example 2.3. Then

$$\begin{aligned} \Rightarrow\langle \mathsf{p}_2^{\pi\pi}, \mathsf{p}_1^{\pi\pi}\rangle\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle \quad &\rightsquigarrow_{G\langle\rangle} \quad \Rightarrow\langle \mathsf{p}_2^{\pi\pi}\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle, \mathsf{p}_1^{\pi\pi}\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle\rangle \\ &\rightsquigarrow_{G\mathsf{p}} \quad \Rightarrow\langle \mathsf{X}q_1, \mathsf{p}_1^{\pi\pi}\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle\rangle \\ &\rightsquigarrow_{G\mathsf{p}} \quad \Rightarrow\langle \mathsf{X}q_1, \mathsf{X}q_0\rangle \end{aligned}$$

and

$$\begin{aligned} \Rightarrow\langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi}\rangle\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle \quad &\rightsquigarrow_{G\epsilon} \quad \Rightarrow\epsilon_{\pi\pi}\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle \\ &= \quad \Rightarrow\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle \end{aligned}$$

are examples of reductions. Observe that $\Rightarrow\langle \mathsf{X}q_1, \mathsf{X}q_0\rangle$ and $\Rightarrow\langle \mathsf{X}q_0, \mathsf{X}q_1\rangle$, that is

$$((\mathsf{X}q_1) \Rightarrow (\mathsf{X}q_0)) \qquad \text{and} \qquad ((\mathsf{X}q_0) \Rightarrow (\mathsf{X}q_1))$$

are the only irreducible paths over $G^\dagger$. The other paths are not in normal form, that is, are not irreducible paths, according to the reduction system $(paths(G^\dagger), \rightsquigarrow_G)$. ∇

Reduction systems induced by m-graphs for categories with non-empty finite products enjoy some interesting properties. One of those properties is local confluence which states that if $w \rightsquigarrow_G w_1$ and $w \rightsquigarrow_G w_1'$ then there exists w' such that $w_1 \overset{=+}{\rightsquigarrow}_G w'$ and $w_1' \overset{=+}{\rightsquigarrow}_G w'$. See for instance [11] for information on general abstract reduction systems.

Proposition 6.2 The reduction system induced by an m-graph for a category with non-empty finite products is locally confluent and terminating.

So, by Newman's Lemma, see [11], we can conclude in Corollary 6.3 that the reduction system induced by an m-graph for a category with non-empty finite products, is confluent since by Proposition 6.2 it is locally confluent and terminating. Moreover every element has a unique $\rightsquigarrow_G$-normal form precisely because it is terminating and confluent, see [11]. As a side remark, the word problem is decidable for the reduction system induced by an m-graph for a category with non-empty finite products.

Corollary 6.3 The reduction system induced by an m-graph G for a category with non-empty finite products, is confluent and every path of $G^\dagger$ has a unique $\rightsquigarrow_G$-normal form.

We now provide an inductive characterization of the set of irreducible paths. Given an m-graph G, let

$$\mathsf{IPaths}(G^\dagger)$$

be the set of irreducible paths of $G^\dagger$ inductively defined as follows:

- ϵ_s is in $\mathsf{IPaths}(G^\dagger)$ for every s in V^+;
- $\mathsf{p}_i^{v_1 \dots v_n}$ is in $\mathsf{IPaths}(G^\dagger)$ for every $v_1, \dots, v_n$ in V, natural number n greater than 1 and i in $\{1, \dots, n\}$;
- $\langle w_1, \dots, w_n \rangle$ is in $\mathsf{IPaths}(G^\dagger)$ for path $\langle w_1, \dots, w_n \rangle$ of $G^\dagger$ whenever $w_1, \dots, w_n$ are in $\mathsf{IPaths}(G^\dagger)$ and either $|\mathsf{src}^\dagger(w_1)| \neq n$ or w_i is not $\mathsf{p}_i^{v_1 \dots v_n}$ for some i in $\{1, \dots, n\}$;
- ew is in $\mathsf{IPaths}(G^\dagger)$ for path ew of $G^\dagger$ whenever e is an m-edge in E and w is in $\mathsf{IPaths}(G^\dagger)$.

The envisaged category

$$G^+$$

is defined as follows:

- its objects are the vertexes of $G^\dagger$, that is, elements of V^+;
- its set of morphisms is $\mathsf{IPaths}(G^\dagger)$;
- $\mathsf{src}^+(w) = \mathsf{src}^\dagger(w)$, $\mathsf{trg}^+(w) = \mathsf{trg}^\dagger(w)$;
- id_s is ε_s;

- $w_2 \circ w_1$ is $\mathsf{nf}(w_2 w_1)$.

In fact, G^+ is a category with non-empty finite products.

Proposition 6.4 Given an m-graph G, G^+ is a category with non-empty finite products.

The category with non-empty finite products obtained through the reduction system, coincides with the category with non-empty finite products defined by factorization in [15]. It is enough to observe that, given an m-graph G and a path w of $G^\dagger$ the set $\{w' : \mathsf{nf}(w') = \mathsf{nf}(w)\}$ coincides with $[w]_{\Delta^{G^\ddagger}}$, where $\Delta^{G^\ddagger}$ is the congruence relation defined in [15] induced by the categorical laws at stake.

This construction of the category G^+ from any given m-graph G is used in Subsection 2.1 for building the language category Σ^+ from any given signature Σ.

Partial functor induced by an m-graph partial morphism

Each m-graph partial morphism $h : G' \to G$ canonically induces a partial functor h^+ from G'^+ to G^+, that is a functor $(h^+_{\mathsf{o}}, h^+_{\mathsf{m}})$ in **Cat** whose domain is a subcategory of G'^+ and its codomain is G^+. For example, the abstraction m-graph partial morphism, present in each interpretation structure, from the operations m-graph to the signature m-graph, induces a partial functor that maps semantic paths in its domain to language paths.

Each m-graph partial morphism $h : G_1 \to G_2$ canonically induces a partial functor $(h^+_{\mathsf{o}}, h^+_{\mathsf{m}})$ from G_1^+ to G_2^+, denoted by h^+, defined as follows:

- $h^+_{\mathsf{o}}(v_1 \dots v_n) =$
$$\begin{cases} h^{\mathsf{v}}(v_1) \dots h^{\mathsf{v}}(v_n) & \text{if } h^{\mathsf{v}}(v_1)\downarrow, \dots, h^{\mathsf{v}}(v_n)\downarrow \\ \text{undefined} & \text{otherwise;} \end{cases}$$

- $h^+_{\mathsf{m}}(\epsilon_{v_1 \dots v_n}) =$
$$\begin{cases} \epsilon_{h^+_{\mathsf{o}}(v_1 \dots v_n)} & \text{if } h^+_{\mathsf{o}}(v_1 \dots v_n)\downarrow \\ \text{undefined} & \text{otherwise;} \end{cases}$$

- $h^+_{\mathsf{m}}(\mathsf{p}_i^{v_1 \dots v_n}) =$
$$\begin{cases} \mathsf{p}_i^{h^+_{\mathsf{o}}(v_1 \dots v_n)} & \text{if } h^+_{\mathsf{o}}(v_1 \dots v_n)\downarrow \\ \text{undefined} & \text{otherwise;} \end{cases}$$

- $h^+_{\mathsf{m}}(ew) =$
$$\begin{cases} h^{\mathsf{e}}(e) h^+_{\mathsf{m}}(w) & \text{if } h^{\mathsf{e}}(e)\downarrow,\ h^+_{\mathsf{m}}(w)\downarrow \\ \text{undefined} & \text{otherwise;} \end{cases}$$

- $h^+_{\mathsf{m}}(\langle w_1 \dots w_m \rangle) =$
$$\begin{cases} \langle h^+_{\mathsf{m}}(w_1) \dots h^+_{\mathsf{m}}(w_m) \rangle & \text{if } h^+_{\mathsf{m}}(w_1)\downarrow, \dots, h^+_{\mathsf{m}}(w_m)\downarrow \\ \text{undefined} & \text{otherwise.} \end{cases}$$

By construction, h^+ preserves products in its domain. That is, either h^+ is undefined in all elements of the product cone (vertex plus projections) or it maps them onto a product cone.

This construction of the partial functor h^+ from any given m-graph partial morphism h is used in Subsection 2.2 for building the abstraction functor α^+ from any given abstraction morphism α.

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