Appendix on *Interpolation via translations*: proofs as expected

Walter Carnielli[2,3] João Rasga[1,3] Cristina Sernadas[1,3]

[1] DM, IST, TU Lisbon, Portugal
[2] CLE and IFCH, UNICAMP, Brazil
[3] SQIG - Instituto de Telecomunicações, Portugal

In this appendix we give detailed proofs on some results that are just sketched, or not proven at all, due to space considerations, in our paper "Interpolation via translations" published in the Mathematical Logic Quarterly journal.

Theorem 3.2 A consequence system $(L, \vdash)$ enjoys Craig interpolation with respect to an L-function var, if (i) there is a Craig generalized translation schema from $(L, \vdash)$ to another consequence system $(L', \vdash')$ with respect to var and an L'-function var$'$; (ii) $(L', \vdash')$ enjoys Craig interpolation with respect to var$'$.

Proof: Let (h_1, h_2, h) be a Craig generalized translation schema from $(L, \vdash)$ to $(L', \vdash')$ via a consequence system $(L^\circ, \vdash^\circ)$, with respect to an L-function var and an L'-function var$'$. Assume that $(L', \vdash')$ enjoys Craig interpolation over var$'$. Suppose that $\Gamma \vdash \varphi$ and $\mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$. Then

$$h_1(\Gamma) \vdash' h_2(\varphi) \quad \text{and} \quad \mathrm{var}'(h_1(\Gamma)) \cap \mathrm{var}'(h_2(\varphi)) \neq \emptyset$$

using condition 1 and condition 5 of the definition of Craig generalized translation schema. Since $(L', \vdash')$ has Craig interpolation there is a non-empty finite set Ψ' of formulas of L' such that

$$h_1(\Gamma) \vdash' \Psi' \quad \text{and} \quad \Psi' \vdash' h_2(\varphi)$$

and $\mathrm{var}'(\Psi') \subseteq \mathrm{var}'(h_1(\Gamma)) \cap \mathrm{var}'(h_2(\varphi))$. So $h_1(\Gamma) \vdash^\circ \Psi'$ and $\Psi' \vdash^\circ h_2(\varphi)$. Using condition 2 of the definition of Craig generalized translation schema and the transitivity of the consequence relation the following holds

$$\Gamma \vdash^\circ \Psi', \quad \Psi' \vdash^\circ \varphi \quad \text{with} \quad \mathrm{var}'(\Psi') \subseteq \mathrm{var}'(h_1(\Gamma)) \cap \mathrm{var}'(h_2(\varphi)).$$

Finally, using condition 3, condition 4 and condition 6, $\Gamma \vdash h(\Psi')$ and $h(\Psi') \vdash \varphi$ and $\mathrm{var}(h(\Psi')) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi)$. Hence $h(\Psi')$ is a Craig interpolant in $(L, \vdash)$ for $\Gamma \vdash \varphi$. ◇

Proposition 3.3 Given a Maehara generalized translation schema $\langle h_1, h_2, h \rangle$ from the consequence system $(L, \vdash)$ to the consequence system $(L', \vdash')$ via the consequence system $(L^\circ, \vdash^\circ)$ with respect to an L-function var and an L'-function var$'$, the tuple

$$\langle h_1, h_2, h^- \rangle$$

where h^- is the restriction of h to the set $L'_{\text{var}'(h_1(L))\cap\text{var}'(h_2(L))}$, is a Craig generalized translation schema from $(L,\vdash)$ to $(L',\vdash')$ via $(L^\circ,\vdash^\circ)$ with respect to var and var$'$.

Proof: Let $\langle h_1, h_2, h\rangle$ be a Maehara generalized translation schema from $(L,\vdash)$ to $(L',\vdash')$ via $(L^\circ,\vdash^\circ)$ with respect to var and var$'$. Consider the tuple $\langle h_1, h_2, h^-\rangle$ where h^- is the restriction of h to the set $L'_{\text{var}'(h_1(L))\cap\text{var}'(h_2(L))}$. We now show that the conditions of a Craig generalized translation schema are satisfied by $\langle h_1, h_2, h^-\rangle$. Conditions 1. and 2. follow immediately; condition 3.: suppose $\Psi \vdash^\circ \varphi'$ where φ' is in $L'_{\text{var}'(h_1(L))\cap\text{var}'(h_2(L))}$. Then $\Psi \vdash h(\varphi')$ and so $\Psi \vdash h^-(\varphi')$ since $h^-(\varphi') = h(\varphi')$; condition 4. follows similarly to condition 3.; condition 5. follows immediately since $\langle h_1, h_2, h\rangle$ is a Maehara generalized translation schema; and condition 6. since $h^-(\Psi') = h(\Psi')$. $\diamond$

Lemma 5.1 The pair of maps h_1 and h_2 is such that $h_1(\Psi) \vdash^l_{\mathcal{D}_n} h_2(\varphi)$ whenever $\Psi \vdash^l_{\mathcal{D}_{n+ma}} \varphi$ and Ψ and $\{\varphi\}$ are contained in L_{n+ma}.

Proof: We show by complete induction on the depth of a sequent derivation that if $\Psi \to \Delta$ is a theorem in $\mathcal{D}_{n+ma}$ then $h_1(\Psi), \neg_\$ h_1(\Delta) \to p_\$$ is a theorem in $\mathcal{D}_n$. Let

$$\dfrac{\begin{array}{ccc} D_1 & & D_k \\ \Psi_1 \to \Delta_1 & \dots & \Psi_k \to \Delta_k \end{array}}{\Psi \to \Delta}\; r$$

be a sequent derivation in $\mathcal{D}_{n+ma}$ where k is greater than or equal to 0 ending with the application of rule r. Consider the following cases:

r is Init. Denote by δ the unique formula in Ψ and in Δ. Then

$$\dfrac{\dfrac{}{h_1(\delta) \to h_1(\delta)}\text{Init} \quad \dfrac{}{p_\$ \to p_\$}\text{Init}}{h_1(\delta), \neg_\$ h_1(\delta) \to p_\$}\text{L} \multimap$$

is a derivation for $h_1(\Psi), \neg_\$\ h_1(\Delta) \to$.

r is Cut. Denote by φ the formula to which r is applied. Then consider the derivation

$$\dfrac{\begin{array}{c} D_1^\circ \\ \neg_\$ h_1(\varphi), h_1(\Psi_1), \neg_\$ h_1(\Delta_1) \to p_\$ \end{array} \quad \dfrac{\begin{array}{c} D_2^\circ \\ h_1(\varphi), h_1(\Psi_2), \neg_\$ h_1(\Delta_2) \to p_\$ \end{array}}{h_1(\Psi_2), \neg_\$ h_1(\Delta_2) \to \neg_\$ h_1(\varphi)}\text{R} \multimap}{h_1(\Psi_1), h_1(\Psi_2), \neg_\$ h_1(\Delta_1), \neg_\$ h_1(\Delta_2) \to p_\$}\text{Cut}$$

where D_1° and D_2° exist by the induction hypothesis.

r is L$\bot$. The thesis follows since $h_1(\bot) \to p_\$$ is a theorem in $\mathcal{D}_n$.

r is R$\bot$. Denote by Δ' the multiset Δ without the formula $\bot$. The thesis follows due to the derivation

$$\dfrac{\begin{array}{c} D_1^\circ \\ h_1(\Psi), \neg_\$ h_1(\Delta') \to p_\$ \end{array} \quad \dfrac{}{p_\$ \to p_\$}\text{Init}}{h_1(\Psi), \neg_\$ h_1(\bot), \neg_\$ h_1(\Delta') \to p_\$}\text{L} \multimap$$

where D_1° exists by induction hypothesis.

r is L$\mathbf{1}$. Denote by Ψ' the multiset Ψ without the formula $\mathbf{1}$. The thesis follows due to the derivation

$$\dfrac{\overset{D_1^\circ}{h_1(\Psi'), \neg_\$ h_1(\Delta) \to p_\$} \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Psi), h_1(\mathbf{1}), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap$$

where D_1° exists by induction hypothesis.

r is R$\mathbf{1}$. The thesis follows due to the derivation

$$\dfrac{\dfrac{\dfrac{}{p_\$ \to p_\$}\,\text{Init}}{\to h_1(\mathbf{1})}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{\neg_\$ h_1(\mathbf{1}) \to p_\$}\,\text{L}\multimap$$

r is L$\sim$. Denote by $\sim \varphi$ the formula used by r and by Ψ' the multiset Ψ without that formula. Then the thesis follows since $h_1(\Psi'), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta) \to p_\$$ is a theorem by induction hypothesis.

r is R$\sim$. Denote by $\sim \varphi_1$ the formula used by r and by Δ' the multiset Δ without that formula. Then consider

$$\dfrac{\dfrac{\overset{D_1^\circ}{h_1(\varphi_1), h_1(\Psi), \neg_\$ h_1(\Delta') \to p_\$}}{h_1(\Psi), \neg_\$ h_1(\Delta') \to h_1(\sim \varphi_1)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Psi), \neg_\$ h_1(\sim \varphi_1), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap$$

where D_1° exists by the induction hypothesis.

r is L$\otimes_m$. Denote by $\varphi_1 \otimes_m \varphi_2$ the formula used by r and by Ψ' the multiset Ψ without that formula. Then consider the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_1^\circ}{h_1(\Psi'), h_1(\varphi_1), h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}}{h_1(\Psi'), h_1(\varphi_1), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Psi'), h_1(\varphi_1), \neg_\$\neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap}{h_1(\Psi'), \neg_\$\neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_1)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Psi'), \neg_\$\neg_\$ h_1(\varphi_1), \neg_\$\neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap}{h_1(\Psi'), h_1(\varphi_1 \otimes_m \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\otimes_n$$

where D_1° exists by the induction hypothesis.

r is L$\otimes_n$. We omit the proof of this case since it is very similar to the preceding case.

r is R$\otimes_m$. Denote by $\varphi_1 \otimes_m \varphi_2$ the formula used by r, by Δ' the multiset Δ without that formula, and by Δ'_1 and Δ'_2 the multisets of formulas of Δ' coming from the different premises of r and similarly for Ψ. Then consider the derivation

$$\dfrac{\dfrac{\dfrac{\overset{D_1^\circ}{h_1(\Psi_1), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta'_1) \to p_\$}}{h_1(\Psi_1), \neg_\$ h_1(\Delta'_1) \to \neg_\$\neg_\$ h_1(\varphi_1)}\,\text{R}\multimap \qquad \dfrac{\overset{D_2^\circ}{h_1(\Psi_2), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta'_2) \to p_\$}}{h_1(\Psi_2), \neg_\$ h_1(\Delta'_2) \to \neg_\$\neg_\$ h_1(\varphi_2)}\,\text{R}\multimap}{h_1(\Psi_1), \neg_\$ h_1(\Delta'_1), h_1(\Psi_2), \neg_\$ h_1(\Delta'_2) \to h_1(\varphi_1 \otimes_m \varphi_2)}\,\text{R}\otimes_n \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Psi_1), h_1(\Psi_2), \neg_\$ h_1(\varphi_1 \otimes_m \varphi_2), \neg_\$ h_1(\Delta'_1), \neg_\$ h_1(\Delta'_2) \to p_\$}\,\text{L}\multimap$$

where D_1° and D_2° exist by the induction hypothesis.

r is R$\otimes_n$. We omit the proof of this case since it is very similar to the preceding case.

r is L∇. Denote by $\varphi_1 \nabla \varphi_2$ the formula used by r, by Ψ' the multiset Ψ without that formula, and by Ψ'_1 and Ψ'_2 the multisets of formulas of Ψ' coming from the different premises of r and similarly for Δ. Then consider the derivation

$$
\dfrac{
\dfrac{
\dfrac{\genfrac{}{}{0pt}{}{D_1^\circ}{h_1(\Psi'_1), h_1(\varphi_1), \neg_\$ h_1(\Delta_1) \to p_\$}}{h_1(\Psi'_1), \neg_\$ h_1(\Delta_1) \to \neg_\$ h_1(\varphi_1)}\,\text{R}\multimap
\quad
\dfrac{\genfrac{}{}{0pt}{}{D_2^\circ}{h_1(\Psi'_2), h_1(\varphi_2), \neg_\$ h_1(\Delta_2) \to p_\$}}{h_1(\Psi'_2), \neg_\$ h_1(\Delta_2) \to \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap
}{h_1(\Psi'_1), \neg_\$ h_1(\Delta_1), h_1(\Psi'_2), \neg_\$ h_1(\Delta_2) \to (\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2))}\,\text{R}\otimes_n
\quad
\dfrac{}{p_\$ \to p_\$}\,\text{Init}
}{h_1(\Psi'_1), h_1(\Psi'_2), h_1(\varphi_1 \nabla \varphi_2), \neg_\$ h_1(\Delta_1), \neg_\$ h_1(\Delta_2) \to p_\$}\,\text{L}\multimap
$$

where D_1° and D_2° exist by the induction hypothesis.

r is R∇. Denote by $\varphi_1 \nabla \varphi_2$ the formula used by r and by Δ' the multiset Δ without that formula. Then consider the derivation

$$
\dfrac{
\dfrac{
\dfrac{\genfrac{}{}{0pt}{}{D_1^\circ}{h_1(\Psi), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to p_\$}}{h_1(\Psi), \neg_\$ h_1(\varphi_1) \otimes_n \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\otimes_n
}{h_1(\Psi), \neg_\$ h_1(\Delta') \to h_1(\varphi_1 \nabla \varphi_2)}\,\text{R}\multimap
\quad
\dfrac{}{p_\$ \to p_\$}\,\text{Init}
}{h_1(\Psi), \neg_\$ h_1(\varphi_1 \nabla \varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap
$$

where D_1° exists by the induction hypothesis.

r is L$\oplus_m$. Denote by $\varphi_1 \oplus_m \varphi_2$ the formula used by r and by Ψ' the multiset Ψ without that formula. Then consider the derivation

$$
\dfrac{
\dfrac{
\dfrac{\genfrac{}{}{0pt}{}{D_1^\circ}{h_1(\Psi'), h_1(\varphi_1), \neg_\$ h_1(\Delta) \to p_\$}}{h_1(\Psi'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_1)}\,\text{R}\multimap
\quad
\dfrac{}{p_\$ \to p_\$}\,\text{Init}
}{h_1(\Psi'), \neg_\$ \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap
\quad
\dfrac{
\dfrac{\genfrac{}{}{0pt}{}{D_2^\circ}{h_1(\Psi'), h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}}{h_1(\Psi'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap
\quad
\dfrac{}{p_\$ \to p_\$}\,\text{Init}
}{h_1(\Psi'), \neg_\$ \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap
}{h_1(\Psi'), h_1(\varphi_1 \oplus_m \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\oplus_n
$$

where D_1° and D_2° exist by the induction hypothesis.

r is L$\oplus_n$. We omit the proof of this case since it is very similar to the preceding case.

r is R$_i\oplus_m$. Denote by $\varphi_1 \oplus_m \varphi_2$ the formula used by r and by Δ' the multiset Δ without that formula. Then consider the derivation

$$
\dfrac{
\dfrac{
\dfrac{\genfrac{}{}{0pt}{}{D_1^\circ}{h_1(\Psi), \neg_\$ h_1(\varphi_i), \neg_\$ h_1(\Delta') \to p_\$}}{h_1(\Psi), h_1(\varphi_1), \neg_\$ h_1(\Delta') \to \neg_\$ \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap
}{h_1(\Psi), \neg_\$ \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to h_1(\varphi_1 \oplus_m \varphi_2)}\,\text{R}_i\oplus_n
\quad
\dfrac{}{p_\$ \to p_\$}\,\text{Init}
}{h_1(\Psi), \neg_\$ h_1(\varphi_1 \oplus_m \varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap
$$

where D_1° exists by the induction hypothesis.

r is R$_i\oplus_n$. We omit the proof of this case since it is very similar to the preceding case.

r is L$_i$&. Denote by $\varphi_1 \& \varphi_2$ the formula used by r and by Ψ' the multiset Ψ without that formula. Then consider the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Psi'), h_1(\varphi_i), \neg_\$ h_1(\Delta) \to p_\$}}{h_1(\Psi'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_i)}\ \text{R} \multimap}{h_1(\Psi'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_1) \oplus_n \neg_\$ h_1(\varphi_2)}\ \text{R}_i\oplus_n \qquad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{h_1(\Psi'), h_1(\varphi_1 \&_m \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\ \text{L} \multimap
$$

where D_1° exists by the induction hypothesis.

r is R&. Denote by $\varphi_1 \& \varphi_2$ the formula used by r and by Δ' the multiset Δ without that formula. Then consider the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Psi), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta') \to p_\$} \quad \dfrac{D_2^\circ}{h_1(\Psi), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to p_\$}}{h_1(\Psi), (\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$ h_1(\varphi_2)), \neg_\$ h_1(\Delta') \to p_\$}\ \text{L}\oplus_n}{h_1(\Psi), \neg_\$ h_1(\Delta') \to h_1(\varphi_1 \& \varphi_2)}\ \text{R} \multimap \qquad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{h_1(\Psi), \neg_\$ h_1(\varphi_1 \& \varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\ \text{L} \multimap
$$

where D_1° and D_2° exist by the induction hypothesis.

r is L$\multimap$. Denote by $\varphi_1 \multimap \varphi_2$ the formula used by r, by Ψ' the multiset Ψ without that formula and by Ψ'_1 and Ψ'_2 the multisets of formulas of Ψ' coming from the different premises of r. Then consider the derivation

$$
\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Psi'_1), \neg_\$ h_1(\varphi_1) \to p_\$}}{h_1(\Psi'_1) \to \neg_\$ \neg_\$ h_1(\varphi_1)}\ \text{R} \multimap \qquad \dfrac{\dfrac{\dfrac{D_2^\circ}{h_1(\Psi'_2), h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}}{h_1(\Psi'_2), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_2)}\ \text{R} \multimap \qquad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{h_1(\Psi'_2), \neg_\$ \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\ \text{L} \multimap}{h_1(\Psi'), h_1(\varphi_1 \multimap \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\ \text{L} \multimap
$$

where D_1° and D_2° exist by the induction hypothesis.

r is R$\multimap$. Denote by $\varphi_1 \multimap \varphi_2$ the formula used by r, and the only formula in Δ. Then consider the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Psi), h_1(\varphi_1), \neg_\$ h_1(\varphi_2) \to p_\$}}{h_1(\Psi), \neg_\$ h_1(\varphi_2) \to \neg_\$ h_1(\varphi_1)}\ \text{R} \multimap \qquad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{h_1(\Psi), \neg_\$ \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\varphi_2) \to p_\$}\ \text{L} \multimap}{h_1(\Psi), \neg_\$ \neg_\$ h_1(\varphi_1) \to \neg_\$ \neg_\$ h_1(\varphi_2)}\ \text{R} \multimap}{h_1(\Psi) \to h_1(\varphi_1 \multimap \varphi_2)}\ \text{R} \multimap \qquad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{h_1(\Psi), \neg_\$ h_1(\Delta) \to p_\$}\ \text{L} \multimap
$$

where D_1° and D_2° exist by the induction hypothesis. $\diamond$

Lemma 5.2 The map h is such that

- $\Gamma \vdash^l_{\mathcal{D}_{n+ma}} h(\psi)$ whenever $\Gamma \vdash^l_{\mathcal{D}_{n+_\$ ma}} \psi$
- $h(\Psi), \Delta \vdash^l_{\mathcal{D}_{n+ma}} \varphi$ whenever $\Psi, \Delta \vdash^l_{\mathcal{D}_{n+_\$ ma}} \varphi$

where $\Gamma \cup \Delta \cup \{\varphi\}$ is contained in L_{n+ma} and $\Psi \cup \{\psi\}$ is contained in L_n.

Proof: We show by complete induction on the depth of a sequent derivation that if Ψ and Δ are sets contained in $L_{n+_{\$}ma}$ and $\Psi \to \Delta$ is a theorem in $\mathcal{D}_{n+_{\$}ma}$ then

$$\overline{h}(\Psi) \to \overline{h}(\Delta)$$

is a theorem in $\mathcal{D}_{n+ma}$ where $\overline{h}$ is a map from $L_{n+_{\$}ma}$ to L_{n+ma} extending h by establishing an identity on the connectives of MALL. Let

$$\dfrac{\begin{array}{ccc} D_1 & & D_k \\ \Psi_1 \to \Delta_1 & \dots & \Psi_k \to \Delta_k \end{array}}{\Psi \to \Delta}\ r$$

be a derivation in $\mathcal{D}_{n+_{\$}ma}$ where k is greater than or equal to 0. Consider the following cases:

r is L$p_{\$}$. Note that $\overline{h}(\Psi)$ is $h(p_{\$})$, which is $\bot$, and Δ is empty. So, the thesis follows straightforwardly using rule L$\bot$ of $\mathcal{D}_{n+ma}$.

r is R$p_{\$}$. Note that $\overline{h}(p_{\$})$ is $\bot$, and denote by Δ' the multiset Δ without $p_{\$}$. So

$$\dfrac{\begin{array}{c} \mathcal{D}_1^{\circ} \\ \overline{h}(\Psi) \to \overline{h}(\Delta') \end{array}}{\overline{h}(\Psi) \to \overline{h}(p_{\$}), \overline{h}(\Delta')}\ \mathrm{R}\bot$$

is a derivation for $\overline{h}(\Psi) \to \overline{h}(\Delta)$ in $\mathcal{D}_{n+ma}$ where $\mathcal{D}_1^{\circ}$ exists by induction hypothesis.

r is Init. The proof of this case follows straightforwardly.

r is L$\mathbf{1}$. Note that $\overline{h}(\mathbf{1}) = \mathbf{1}$. Denote by Ψ' the multiset Ψ without the formula $\mathbf{1}$. Then

$$\dfrac{\begin{array}{c} D_1^{\circ} \\ \overline{h}(\Psi') \to \overline{h}(\Delta) \end{array}}{\overline{h}(\Psi'), \overline{h}(\mathbf{1}) \to \overline{h}(\Delta)}\ \mathrm{L}\mathbf{1}$$

is a derivation for $\overline{h}(\Psi) \to \overline{h}(\Delta)$ in $\mathcal{D}_{n+ma}$, where D_1° exists by induction hypothesis.

r is Cut, or R$\mathbf{1}$, or L$\bot$, or R$\bot$, or L$\otimes_m$, or R$\otimes_m$, or L$\oplus_m$, or R$\oplus_m$, or L∇, or R∇, or L&, or R&, or L$\otimes_n$, or R$\otimes_n$, or L$\oplus_n$, or R$\oplus_n$, or L$\multimap$ or R$\multimap$. The proofs of these cases are similar to the proof of L$\mathbf{1}$ so we omit them. $\diamond$

Lemma 5.3 The maps h_1, h_2 and h_a are such that

1. $\varphi \dashv\vdash^{l}_{\mathcal{D}_{n+_{\$}ma}} h_a(\varphi)$
2. $h_a(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+_{\$}ma}} h_1(\varphi)$
3. $h_2(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+_{\$}ma}} h_a(\varphi)$

for φ in L_{n+ma}.

Proof: The proofs follow by complete induction on the complexity of the formula:

1. $\varphi \dashv\vdash^{l}_{\mathcal{D}_{n+_{\$}ma}} h_a(\varphi)$

φ is $\bot$. The thesis follows straightforwardly by using Init since $h_a(\varphi)$ is φ;

φ is in $P_n \cup P_m$ or is $\mathbf{1}$. Note that $h_a(\varphi)$ is $\sim\sim \varphi$. The thesis follows since

$$\dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\,\text{Init}}{\varphi, \sim \varphi \to}\,\text{L}\sim}{\varphi \to \sim\sim \varphi}\,\text{R}\sim \qquad\qquad \dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\,\text{Init}}{\to \varphi, \sim \varphi}\,\text{R}\sim}{\sim\sim \varphi \to \varphi}\,\text{L}\sim$$

are derivations for $\varphi \dashv\vdash^{l}_{\mathcal{D}_{n+_{\$}ma}} h_a(\varphi)$.

φ is $\sim \varphi_1$. The thesis follows straightforwardly using the introduction rules for $\sim$.

φ is $\varphi_1 \otimes \varphi_2$ for $\otimes$ in $\{\otimes_m, \otimes_n\}$. Note that $h_a(\varphi)$ is $\sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)))$. Then consider the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_a(\varphi_1) \to \varphi_1}}{\sim\sim h_a(\varphi_1) \to \varphi_1}\,\sim^2 \quad \dfrac{\dfrac{D_2^\circ}{h_a(\varphi_2) \to \varphi_2}}{\sim\sim h_a(\varphi_2) \to \varphi_2}\,\sim^2}{\sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2) \to \varphi_1 \otimes \varphi_2}\,\text{R}\otimes}{(\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)) \to \varphi_1 \otimes \varphi_2}\,\text{L}\otimes}{\sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))) \to \varphi_1 \otimes \varphi_2}\,\sim^2$$

for $h_a(\varphi) \vdash_{\mathcal{D}_{n+_{\$}ma}} \varphi$ where D_1° and D_2° exist by induction hypothesis. The thesis follows since the derivation for $\varphi \vdash_{\mathcal{D}_{n+_{\$}ma}} h_a(\varphi)$ follows similarly.

φ is $\varphi_1 \oplus \varphi_2$ for $\oplus$ in $\{\oplus_m, \oplus_n\}$. Note that $h_a(\varphi)$ is $\sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)))$. Then consider the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\varphi_1 \to h_a(\varphi_1)}}{\varphi_1 \to\sim\sim h_a(\varphi_1)}\,\sim^2}{\varphi_1 \to (\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))}\,\text{R}_1\oplus \quad \dfrac{\dfrac{\dfrac{D_2^\circ}{\varphi_2 \to h_a(\varphi_2)}}{\varphi_1 \to\sim\sim h_a(\varphi_2)}\,\sim^2}{\varphi_2 \to (\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))}\,\text{R}_2\oplus}{\varphi_1 \oplus \varphi_2 \to (\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))}\,\text{L}\oplus}{\varphi_1 \oplus \varphi_2 \to\sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)))}\,\sim^2$$

for $\varphi \vdash_{\mathcal{D}_{n+_{\$}ma}} h_a(\varphi)$ where D_1° and D_2° exist by induction hypothesis. The thesis follows since the derivation for $h_a(\varphi) \vdash_{\mathcal{D}_{n+_{\$}ma}} \varphi$ follows similarly.

φ is $\varphi_1 \nabla \varphi_2$. We omit the proof of this case since it is similar to the case when φ is $\varphi_1 \otimes_m \varphi_2$.

φ is $\varphi_1 \& \varphi_2$. We omit the proof of this case since it is similar to the case when φ is $\varphi_1 \oplus_m \varphi_2$.

φ is $\varphi_1 \multimap \varphi_2$. Note that $h_a(\varphi)$ is $(\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2))$. Then consider the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\varphi_1 \to h_a(\varphi_1)}}{\varphi_1 \to\sim\sim h_a(\varphi_1)}\,\sim^2 \quad \dfrac{\dfrac{D_2^\circ}{h_a(\varphi_2) \to \varphi_2}}{\sim\sim h_a(\varphi_2) \to \varphi_2}\,\sim^2}{(\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2)), \varphi_1 \to \varphi_2}\,\text{L}\multimap}{(\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2)) \to \varphi_1 \multimap \varphi_2}\,\text{R}\multimap$$

for $h_a(\varphi) \vdash_{\mathcal{D}_{n+_{\$}ma}} \varphi$ where D_1° and D_2° exist by induction hypothesis. The thesis follows since the derivation for $\varphi \vdash_{\mathcal{D}_{n+_{\$}ma}} h_a(\varphi)$ follows similarly.

2. $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$

φ is $\bot$. Note that $h_a(\varphi)$ is $\bot$ and $h_1(\varphi) = p_\$$. The thesis follows since

$$\dfrac{\dfrac{}{\bot \to}\;\mathrm{L}\bot}{\bot \to p_\$}\;\mathrm{R}p_\$ \qquad \dfrac{\dfrac{}{p_\$ \to}\;\mathrm{L}p_\$}{p_\$ \to \bot}\;\mathrm{R}\bot$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$.

φ is $\mathbf{1}$. Note that $h_1(\varphi)$ is $p_\$ \multimap p_\$$ and $h_a(\mathbf{1})$ is $\sim\sim \mathbf{1}$. The thesis follows since

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_\$ \to}\;\mathrm{L}p_\$}{\mathbf{1}, p_\$ \to}\;\mathrm{L}\mathbf{1}}{\sim\sim \mathbf{1}, p_\$ \to}\;\sim^2}{\sim\sim \mathbf{1}, p_\$ \to p_\$}\;\mathrm{R}p_\$}{\sim\sim \mathbf{1} \to p_\$ \multimap p_\$}\;\mathrm{R}\multimap \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\to \mathbf{1}}\;\mathrm{R}\mathbf{1}}{\sim \mathbf{1} \to}\;\mathrm{L}\sim}{\sim \mathbf{1} \to p_\$}\;\mathrm{R}p_\$ \quad \dfrac{}{p_\$ \to}\;\mathrm{L}p_\$}{p_\$ \multimap p_\$, \sim \mathbf{1} \to}\;\mathrm{L}\multimap}{p_\$ \multimap p_\$ \to \sim\sim \mathbf{1}}\;\mathrm{R}\sim$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$.

φ is in $P_m \cup P_n$. Note that $h_1(\varphi)$ is φ and $h_a(\varphi) = \sim\sim \varphi$ The thesis follows since

$$\dfrac{\dfrac{}{\varphi \to \varphi}\;\mathrm{Init}}{\sim\sim \varphi \to \varphi}\;\sim^2 \qquad \dfrac{\dfrac{}{\varphi \to \varphi}\;\mathrm{Init}}{\varphi \to \sim\sim \varphi}\;\sim^2$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$.

φ is $\sim \varphi_1$. Note that $h_1(\varphi)$ is $\neg_\$ h_1(\varphi_1)$ and $h_a(\varphi) = \sim h_a(\varphi_1)$ The thesis follows since

$$\dfrac{\dfrac{\begin{array}{c} D_1^\circ \\ h_a(\varphi_1) \to h_1(\varphi_1)\end{array}}{\neg_\$ h_1(\varphi_1), h_a(\varphi_1) \to}\;\mathrm{L}\neg_\$}{\neg_\$ h_1(\varphi_1) \to \sim h_a(\varphi_1)}\;\mathrm{R}\sim \qquad \dfrac{\dfrac{\begin{array}{c} D_2^\circ \\ h_1(\varphi_1) \to h_a(\varphi_1)\end{array}}{\sim h_a(\varphi_1), h_1(\varphi_1) \to}\;\mathrm{L}\sim}{\sim h_a(\varphi_1) \to \neg_\$ h_1(\varphi_1)}\;\mathrm{R}\neg_\$$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1 \otimes \varphi_2$ for $\otimes$ in $\{\otimes_m, \otimes_n\}$. Note that $h_1(\varphi)$ is $(\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2))$ and $h_a(\varphi)$ is $\sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D_1^\circ \\ h_1(\varphi_1) \to h_a(\varphi_1)\end{array}}{h_1(\varphi_1), \sim h_a(\varphi_1) \to}\;\mathrm{L}\sim}{\neg_\$\neg_\$ h_1(\varphi_1), \sim h_a(\varphi_1) \to}\;\neg_\$^2}{\neg_\$\neg_\$ h_1(\varphi_1) \to \sim\sim h_a(\varphi_1)}\;\mathrm{R}\sim \quad \begin{array}{c} \vdots \\ \neg_\$\neg_\$ h_1(\varphi_2) \to \sim\sim h_a(\varphi_2)\end{array}}{\neg_\$\neg_\$ h_1(\varphi_1), \neg_\$\neg_\$ h_1(\varphi_2) \to (\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))}\;\mathrm{R}\otimes}{(\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2)) \to (\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))}\;\mathrm{L}\otimes_n}{(\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2)) \to \sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)))}\;\sim^2$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_2^\circ}{h_a(\varphi_1) \to h_1(\varphi_1)}}{h_a(\varphi_1), \neg_{\$} h_1(\varphi_1) \to}\,\mathrm{L}\neg_{\$}}{h_a(\varphi_1) \to \neg_{\$}\neg_{\$} h_1(\varphi_1)}\,\mathrm{R}\neg_{\$}}{\sim\sim h_a(\varphi_1) \to \neg_{\$}\neg_{\$} h_1(\varphi_1)}\,\sim^2 \qquad \dfrac{\vdots}{\sim\sim h_a(\varphi_2) \to \neg_{\$}\neg_{\$} h_1(\varphi_2)}}{\sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2) \to (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\,\mathrm{R}\otimes_n}{(\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)) \to (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\,\mathrm{L}\otimes}{\sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))) \to (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\,\sim^2
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+\$ma}} h_1(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1 \nabla \varphi_2$. Note that $h_1(\varphi)$ is $\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$} h_1(\varphi_2)))$ and $h_a(\varphi) = ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\varphi_1) \to h_a(\varphi_1)}}{\sim h_a(\varphi_1), h_1(\varphi_1) \to}\,\mathrm{L}\sim}{\sim h_a(\varphi_1) \to \neg_{\$} h_1(\varphi_1)}\,\mathrm{R}\neg_{\$} \qquad \dfrac{\dfrac{\dfrac{D_2^\circ}{h_1(\varphi_2) \to h_a(\varphi_2)}}{\sim h_a(\varphi_2), h_1(\varphi_2) \to}\,\mathrm{L}\sim}{\sim h_a(\varphi_2) \to \neg_{\$} h_1(\varphi_2)}\,\mathrm{R}\neg_{\$}}{\sim h_a(\varphi_1), \sim h_a(\varphi_2) \to (\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$} h_1(\varphi_2))}\,\mathrm{R}\otimes_n}{\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$} h_1(\varphi_2))), \sim h_a(\varphi_1), \sim h_a(\varphi_2) \to}\,\mathrm{L}\neg_{\$}}{\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$} h_1(\varphi_2))) \to \sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2)}\,\mathrm{R}\sim^2}{\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$} h_1(\varphi_2))) \to (\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))}\,\mathrm{R}\nabla
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_a(\varphi_1) \to h_1(\varphi_1)}}{\sim\sim h_a(\varphi_1) \to h_1(\varphi_1)}\,\sim^2}{\sim\sim h_a(\varphi_1), \neg_{\$} h_1(\varphi_1) \to}\,\mathrm{L}\neg_{\$} \qquad \dfrac{\vdots}{\sim\sim h_a(\varphi_2), \neg_{\$} h_1(\varphi_2) \to}}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)), \neg_{\$} h_1(\varphi_1), \neg_{\$} h_1(\varphi_2) \to}\,\mathrm{L}\nabla}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)), (\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$} h_1(\varphi_2)) \to}\,\mathrm{R}\otimes_n}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to \neg_{\$}((\neg_{\$} h_1(\varphi_1)) \otimes_n (\neg_{\$} h_1(\varphi_2)))}\,\mathrm{R}\neg_{\$}
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+\$ma}} h_1(\varphi)$ where D_1°, D_2° and D_3° exist by induction hypothesis.

φ is $\varphi_1 \oplus \varphi_2$ for $\oplus$ in $\{\oplus_m, \oplus_n\}$. Note that $h_1(\varphi)$ is $(\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))$ and $h_a(\varphi) = \sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\varphi_1) \to h_a(\varphi_1)}}{h_1(\varphi_1), \sim h_a(\varphi_1) \to}\,\mathrm{L}\sim}{\neg_{\$}\neg_{\$} h_1(\varphi_1), \sim h_a(\varphi_1) \to}\,\neg_{\$}^2}{\neg_{\$}\neg_{\$} h_1(\varphi_1) \to \sim\sim h_a(\varphi_1)}\,\mathrm{R}\sim}{\neg_{\$}\neg_{\$} h_1(\varphi_1) \to (\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))}\,\mathrm{R}_1\oplus \qquad \vdots}{(\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2)) \to (\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))}\,\mathrm{L}\oplus_n}{(\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2)) \to \sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)))}\,\sim^2
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_2^\circ}{h_1(\varphi_1) \to h_a(\varphi_1)}}{\sim\sim h_1(\varphi_1) \to h_a(\varphi_1)}\,\sim^2}{\sim\sim h_1(\varphi_1) \to \neg_{\$}\neg_{\$} h_a(\varphi_1)}\,\neg_{\$}^2}{\sim\sim h_a(\varphi_1) \to (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\,\mathrm{R}_1\oplus_n \qquad \vdots}{(\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)) \to (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\,\mathrm{L}\oplus}{\sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))) \to (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\,\sim^2
$$

are derivations for $h_a(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+\$ma}} h_1(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1\&\varphi_2$. Note that $h_1(\varphi)$ is $\neg_\$((\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$ h_1(\varphi_2)))$ and $h_a(\varphi) = ((\sim\sim h_a(\varphi_1))\&(\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\varphi_1)\to h_a(\varphi_1)\end{array}}{\sim h_a(\varphi_1), h_1(\varphi_1)\to}\,\mathrm{L}\sim}{\sim h_a(\varphi_1)\to\neg_\$ h_1(\varphi_1)}\,\mathrm{R}\neg_\$}{\sim h_a(\varphi_1)\to(\neg_\$ h_1(\varphi_1))\oplus_n(\neg_\$ h_1(\varphi_2))}\,\mathrm{R}_1\oplus_n}{\neg_\$((\neg_\$ h_1(\varphi_1))\oplus_n(\neg_\$ h_1(\varphi_2))),\sim h_a(\varphi_1)\to}\,\mathrm{L}\neg_\$}{\neg_\$((\neg_\$ h_1(\varphi_1))\oplus_n(\neg_\$ h_1(\varphi_2)))\to\sim\sim h_a(\varphi_1))}\,\mathrm{R}\sim \qquad \vdots}{\neg_\$((\neg_\$ h_1(\varphi_1))\oplus_n(\neg_\$ h_1(\varphi_2)))\to(\sim\sim h_a(\varphi_1))\&(\sim\sim h_a(\varphi_2))}\,\mathrm{R}\&$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_2^\circ\\ h_a(\varphi_1)\to h_1(\varphi_1)\end{array}}{\sim\sim h_a(\varphi_1)\to h_1(\varphi_1)}\,\sim^2}{\neg_\$ h_1(\varphi_1),\sim\sim h_a(\varphi_1)\to}\,\mathrm{L}\neg_\$}{\neg_\$ h_1(\varphi_1),(\sim\sim h_a(\varphi_1))\&(\sim\sim h_a(\varphi_2))\to}\,\mathrm{L}_1\& \qquad \vdots}{(\neg_\$ h_1(\varphi_1))\oplus_n(\neg_\$ h_1(\varphi_2)),(\sim\sim h_a(\varphi_1))\&(\sim\sim h_a(\varphi_2))\to}\,\mathrm{L}\oplus_n}{(\sim\sim h_a(\varphi_1))\&(\sim\sim h_a(\varphi_2))\to\neg_\$((\neg_\$ h_1(\varphi_1))\oplus_n(\neg_\$ h_1(\varphi_2)))}\,\mathrm{R}\neg_\$$$

are derivations for $h_a(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+\$ma}} h_1(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1 \multimap \varphi_2$. Note that $h_1(\varphi)$ is $(\neg_\$\neg_\$ h_1(\varphi_1)) \multimap (\neg_\$\neg_\$ h_1(\varphi_2))$ and $h_a(\varphi) = (\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2))$ The thesis follows since

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_a(\varphi_1)\to h_1(\varphi_1)\end{array}}{\sim\sim h_a(\varphi_1)\to h_1(\varphi_1)}\,\sim^2}{\sim\sim h_a(\varphi_1)\to\neg_\$\neg_\$ h_1(\varphi_1)}\,\neg_\${}^2 \qquad \dfrac{\dfrac{\begin{array}{c}D_2^\circ\\ h_1(\varphi_2)\to h_a(\varphi_2)\end{array}}{h_1(\varphi_2),\sim h_a(\varphi_2)\to}\,\mathrm{L}\sim}{\neg_\$\neg_\$ h_1(\varphi_2),\sim h_a(\varphi_2)\to}\,\neg_\${}^2}{(\neg_\$\neg_\$ h_1(\varphi_1))\multimap(\neg_\$\neg_\$ h_1(\varphi_2)),\sim\sim h_a(\varphi_1),\sim h_a(\varphi_2)\to}\,\mathrm{L}\multimap}{(\neg_\$\neg_\$ h_1(\varphi_1))\multimap(\neg_\$\neg_\$ h_1(\varphi_2)),\sim\sim h_a(\varphi_1)\to\sim\sim h_a(\varphi_2)}\,\mathrm{R}\sim}{(\neg_\$\neg_\$ h_1(\varphi_1))\multimap(\neg_\$\neg_\$ h_1(\varphi_2))\to(\sim\sim h_a(\varphi_1))\multimap(\sim\sim h_a(\varphi_2))}\,\mathrm{R}\multimap$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_3^\circ\\ h_1(\varphi_1)\to h_a(\varphi_1)\end{array}}{h_1(\varphi_1),\sim h_a(\varphi_1)\to}\,\mathrm{L}\sim}{\neg_\$\neg_\$ h_1(\varphi_1),\sim h_a(\varphi_1)\to}\,\neg_\${}^2}{\neg_\$\neg_\$ h_1(\varphi_1)\to\sim\sim h_a(\varphi_1)}\,\mathrm{R}\sim \qquad \dfrac{\dfrac{\begin{array}{c}D_4^\circ\\ h_a(\varphi_2)\to h_1(\varphi_2)\end{array}}{h_a(\varphi_2),\neg_\$ h_1(\varphi_2)\to}\,\mathrm{L}\neg_\$}{\sim\sim h_a(\varphi_2),\neg_\$ h_1(\varphi_2)\to}\,\sim^2}{(\sim\sim h_a(\varphi_1))\multimap(\sim\sim h_a(\varphi_2)),\neg_\$\neg_\$ h_1(\varphi_1),\neg_\$ h_1(\varphi_2)\to}\,\mathrm{L}\multimap}{(\sim\sim h_a(\varphi_1))\multimap(\sim\sim h_a(\varphi_2)),\neg_\$\neg_\$ h_1(\varphi_1)\to\neg_\$\neg_\$ h_1(\varphi_2)}\,\mathrm{R}\neg_\$}{(\sim\sim h_a(\varphi_1))\multimap(\sim\sim h_a(\varphi_2))\to(\neg_\$\neg_\$ h_1(\varphi_1))\multimap(\neg_\$\neg_\$ h_1(\varphi_2))}\,\mathrm{R}\multimap$$

are derivations for $h_a(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+\$ma}} h_1(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis.

3. $h_2(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+\$ma}} h_a(\varphi)$

φ is $\bot$. Note that $h_a(\varphi)$ is $\bot$ and $h_2(\varphi) = \neg_\$\neg_\$ p_\$$. The thesis follows since

$$
\dfrac{\dfrac{\dfrac{}{\bot \to}\;\mathrm{L}\bot}{\bot \to p_\$}\;\mathrm{R}p_\$}{\bot \to \neg_\$\neg_\$ p_\$}\;\neg^2_\$
\qquad\qquad
\dfrac{\dfrac{\dfrac{}{p_\$ \to}\;\mathrm{L}p_\$}{\neg_\$\neg_\$ p_\$ \to}\;\neg^2_\$}{\neg_\$\neg_\$ p_\$ \to \bot}\;\mathrm{R}\bot
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_2(\varphi)$.

φ is $\mathbf{1}$. Note that $h_1(\varphi)$ is $\neg_\$\neg_\$(p_\$ \multimap p_\$)$ and $h_a(\mathbf{1})$ is $\sim\sim \mathbf{1}$. The thesis follows since

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_\$ \to}\;\mathrm{L}p_\$}{\mathbf{1}, p_\$ \to}\;\mathrm{L}\mathbf{1}}{\sim\sim \mathbf{1}, p_\$ \to}\;\sim^2}{\sim\sim \mathbf{1}, p_\$ \to p_\$}\;\mathrm{R}p_\$}{\sim\sim \mathbf{1} \to p_\$ \multimap p_\$}\;\mathrm{R}\multimap}{\sim\sim \mathbf{1} \to \neg_\$\neg_\$(p_\$ \multimap p_\$)}\;\neg^2_\$
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\to \mathbf{1}}\;\mathrm{R}\mathbf{1}}{\sim \mathbf{1} \to}\;\mathrm{L}\sim}{\sim \mathbf{1} \to p_\$}\;\mathrm{R}p_\$ \quad \dfrac{}{p_\$ \to}\;\mathrm{L}p_\$}{p_\$ \multimap p_\$, \sim \mathbf{1} \to}\;\mathrm{L}\multimap}{\neg_\$\neg_\$(p_\$ \multimap p_\$), \sim \mathbf{1} \to}\;\neg^2_\$}{\neg_\$\neg_\$(p_\$ \multimap p_\$) \to \sim\sim \mathbf{1}}\;\mathrm{R}\sim
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_2(\varphi)$.

φ is in $P_m \cup P_n$. Note that $h_2(\varphi)$ is $\neg_\$\neg_\$\varphi$ and $h_a(\varphi) = \sim\sim \varphi$ The thesis follows since

$$
\dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\;\mathrm{Init}}{\sim\sim \varphi \to \varphi}\;\sim^2}{\sim\sim \varphi \to \neg_\$\neg_\$\varphi}\;\neg^2_\$
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\;\mathrm{Init}}{\varphi, \sim \varphi \to}\;\mathrm{L}\sim}{\neg_\$\neg_\$\varphi, \sim \varphi \to}\;\neg^2_\$}{\neg_\$\neg_\$\varphi \to \sim\sim \varphi}\;\mathrm{R}\sim
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_2(\varphi)$.

φ is $\sim \varphi_1$. Note that $h_2(\varphi)$ is $\neg_\$\neg_\$\neg_\$ h_1(\varphi_1)$ and $h_a(\varphi) = \sim h_a(\varphi_1)$ The thesis follows since

$$
\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_a(\varphi_1) \to h_1(\varphi_1) \end{matrix}}{\neg_\$\neg_\$\neg_\$ h_1(\varphi_1), h_a(\varphi_1) \to}\;\neg^3_\$}{\neg_\$\neg_\$\neg_\$ h_1(\varphi_1) \to \sim h_a(\varphi_1)}\;\mathrm{R}\sim
\qquad\qquad
\dfrac{\dfrac{\begin{matrix} D_2^\circ \\ h_1(\varphi_1) \to h_a(\varphi_1) \end{matrix}}{\sim h_a(\varphi_1), h_1(\varphi_1) \to}\;\mathrm{L}\sim}{\sim h_a(\varphi_1) \to \neg_\$\neg_\$\neg_\$ h_1(\varphi_1)}\;\neg^3_\$
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_2(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1 \otimes \varphi_2$ for $\otimes$ in $\{\otimes_m, \otimes_n\}$. Note that $h_2(\varphi)$ is $\neg_\$\neg_\$((\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2)))$ and $h_a(\varphi)$ is $\sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1(\varphi_1) \to h_a(\varphi_1) \end{matrix}}{h_1(\varphi_1), \sim h_a(\varphi_1) \to}\;\mathrm{L}\sim}{\neg_\$\neg_\$ h_1(\varphi_1), \sim h_a(\varphi_1) \to}\;\neg^2_\$}{\neg_\$\neg_\$ h_1(\varphi_1) \to \sim\sim h_a(\varphi_1)}\;\mathrm{R}\sim \quad \begin{matrix} \vdots \\ \neg_\$\neg_\$ h_1(\varphi_2) \to \sim\sim h_a(\varphi_2) \end{matrix}}{\neg_\$\neg_\$ h_1(\varphi_1), \neg_\$\neg_\$ h_1(\varphi_2) \to (\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))}\;\mathrm{R}\otimes}{(\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2)) \to (\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))}\;\mathrm{L}\otimes_n}{(\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2)), \sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))) \to}\;\mathrm{L}\sim}{\neg_\$\neg_\$((\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2))), \sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))) \to}\;\neg^2_\$}{\neg_\$\neg_\$((\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2))) \to \sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)))}\;\mathrm{R}\sim
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D_2^\circ \\ h_a(\varphi_1) \to h_1(\varphi_1) \end{array}}{h_a(\varphi_1), \neg_\$ h_1(\varphi_1) \to}\,\mathrm{L}\neg_\$}{h_a(\varphi_1) \to \neg_\$\neg_\$ h_1(\varphi_1)}\,\mathrm{R}\neg_\$}{\sim\sim h_a(\varphi_1) \to \neg_\$\neg_\$ h_1(\varphi_1)}\,\sim^2 \quad \begin{array}{c} \vdots \\ \sim\sim h_a(\varphi_2) \to \neg_\$\neg_\$ h_1(\varphi_2) \end{array}}{\sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2) \to (\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2))}\,\mathrm{R}\otimes_n}{(\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2)) \to (\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2))}\,\mathrm{L}\otimes}{\sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))) \to (\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2))}\,\sim^2}{\sim\sim ((\sim\sim h_a(\varphi_1)) \otimes (\sim\sim h_a(\varphi_2))) \to \neg_\$\neg_\$((\neg_\$\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$\neg_\$ h_1(\varphi_2)))}\,\neg_\$^2
$$

are derivations for $h_a(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+\$ma}} h_2(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1 \nabla \varphi_2$. Note that $h_2(\varphi)$ is $\neg_\$\neg_\$\neg_\$((\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2)))$ and $h_a(\varphi) = ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D_1^\circ \\ h_1(\varphi_1) \to h_a(\varphi_1) \end{array}}{\sim h_a(\varphi_1), h_1(\varphi_1) \to}\,\mathrm{L}\sim}{\sim h_a(\varphi_1) \to \neg_\$ h_1(\varphi_1)}\,\mathrm{R}\neg_\$ \quad \dfrac{\dfrac{\begin{array}{c} D_2^\circ \\ h_1(\varphi_2) \to h_a(\varphi_2) \end{array}}{\sim h_a(\varphi_2), h_1(\varphi_2) \to}\,\mathrm{L}\sim}{\sim h_a(\varphi_2) \to \neg_\$ h_1(\varphi_2)}\,\mathrm{R}\neg_\$}{\sim h_a(\varphi_1), \sim h_a(\varphi_2) \to (\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2))}\,\mathrm{R}\otimes_n}{\neg_\$((\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2))), \sim h_a(\varphi_1), \sim h_a(\varphi_2) \to}\,\neg_\$^3}{\neg_\$\neg_\$\neg_\$((\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2))) \to \sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2)}\,\mathrm{R}\sim^2}{\neg_\$\neg_\$\neg_\$((\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2))) \to (\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))}\,\mathrm{R}\nabla
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D_3^\circ \\ h_a(\varphi_1) \to h_1(\varphi_1) \end{array}}{\sim\sim h_a(\varphi_1) \to h_1(\varphi_1)}\,\sim^2}{\sim\sim h_a(\varphi_1), \neg_\$ h_1(\varphi_1) \to}\,\mathrm{L}\neg_\$ \quad \dfrac{\vdots}{\sim\sim h_a(\varphi_2), \neg_\$ h_1(\varphi_2) \to}}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\varphi_2) \to}\,\mathrm{L}\nabla}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)), (\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2)) \to}\,\mathrm{L}\otimes_n}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to \neg_\$\neg_\$\neg_\$((\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2)))}\,\neg_\$^3
$$

are derivations for $h_a(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{n+\$ma}} h_2(\varphi)$ where D_1°, D_2° and D_3° exist by induction hypothesis.

φ is $\varphi_1 \oplus \varphi_2$ for $\oplus$ in $\{\oplus_m, \oplus_n\}$. Note that $h_2(\varphi)$ is $\neg_\$\neg_\$((\neg_\$\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$\neg_\$ h_1(\varphi_2)))$ and $h_a(\varphi) = \sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D_1^\circ \\ h_1(\varphi_1) \to h_a(\varphi_1) \end{array}}{h_1(\varphi_1), \sim h_a(\varphi_1) \to}\,\mathrm{L}\sim}{\neg_\$\neg_\$ h_1(\varphi_1), \sim h_a(\varphi_1) \to}\,\neg_\$^2}{\neg_\$\neg_\$ h_1(\varphi_1) \to \sim\sim h_a(\varphi_1)}\,\mathrm{R}\sim}{\neg_\$\neg_\$ h_1(\varphi_1) \to (\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))}\,\mathrm{R}_1\oplus \quad \vdots}{(\neg_\$\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$\neg_\$ h_1(\varphi_2)) \to (\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))}\,\mathrm{L}\oplus_n}{(\neg_\$\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$\neg_\$ h_1(\varphi_2)), \sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))) \to}\,\mathrm{L}\sim}{\neg_\$\neg_\$((\neg_\$\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$\neg_\$ h_1(\varphi_2))), \sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))) \to}\,\neg_\$^2}{\neg_\$\neg_\$((\neg_\$\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$\neg_\$ h_1(\varphi_2))) \to \sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)))}\,\mathrm{R}\sim
$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_2^{\circ} \\ h_1(\varphi_1) \rightarrow h_a(\varphi_1) \end{matrix}}{\sim\sim h_1(\varphi_1) \rightarrow h_a(\varphi_1)}\;\sim^2}{\sim\sim h_1(\varphi_1) \rightarrow \neg_{\$}\neg_{\$} h_a(\varphi_1)}\;\neg_{\$}^2}{\sim\sim h_a(\varphi_1) \rightarrow (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\;\mathrm{R}_1\oplus_n \qquad \vdots}{(\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2)) \rightarrow (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\;\mathrm{L}\oplus}{\sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))) \rightarrow (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2))}\;\sim^2}{\sim\sim ((\sim\sim h_a(\varphi_1)) \oplus (\sim\sim h_a(\varphi_2))) \rightarrow \neg_{\$}\neg_{\$}((\neg_{\$}\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$}\neg_{\$} h_1(\varphi_2)))}\;\neg_{\$}^2$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_{\$}ma}} h_2(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1 \& \varphi_2$. Note that $h_2(\varphi)$ is $\neg_{\$}\neg_{\$}\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$} h_1(\varphi_2)))$ and $h_a(\varphi) = ((\sim\sim h_a(\varphi_1)) \& (\sim\sim h_a(\varphi_2)))$ The thesis follows since

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_1^{\circ} \\ h_1(\varphi_1) \rightarrow h_a(\varphi_1) \end{matrix}}{\sim h_a(\varphi_1), h_1(\varphi_1) \rightarrow}\;\mathrm{L}\sim}{\sim h_a(\varphi_1) \rightarrow \neg_{\$} h_1(\varphi_1)}\;\mathrm{R}\neg_{\$}}{\sim h_a(\varphi_1) \rightarrow (\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$} h_1(\varphi_2))}\;\mathrm{R}_1\oplus_n}{\neg_{\$}\neg_{\$}\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$} h_1(\varphi_2))), \sim h_a(\varphi_1) \rightarrow}\;\neg_{\$}^3}{\neg_{\$}\neg_{\$}\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$} h_1(\varphi_2))) \rightarrow \sim\sim h_a(\varphi_1))}\;\mathrm{R}\sim \qquad \vdots}{\neg_{\$}\neg_{\$}\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$} h_1(\varphi_2))) \rightarrow (\sim\sim h_a(\varphi_1)) \& (\sim\sim h_a(\varphi_2))}\;\mathrm{R}\&$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_2^{\circ} \\ h_a(\varphi_1) \rightarrow h_1(\varphi_1) \end{matrix}}{\sim\sim h_a(\varphi_1) \rightarrow h_1(\varphi_1)}\;\sim^2}{\neg_{\$} h_1(\varphi_1), \sim\sim h_a(\varphi_1) \rightarrow}\;\mathrm{L}\neg_{\$}}{\neg_{\$} h_1(\varphi_1), (\sim\sim h_a(\varphi_1)) \& (\sim\sim h_a(\varphi_2)) \rightarrow}\;\mathrm{L}_1\& \qquad \vdots}{(\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$} h_1(\varphi_2)), (\sim\sim h_a(\varphi_1)) \& (\sim\sim h_a(\varphi_2)) \rightarrow}\;\mathrm{L}\oplus_n}{(\sim\sim h_a(\varphi_1)) \& (\sim\sim h_a(\varphi_2)) \rightarrow \neg_{\$}\neg_{\$}\neg_{\$}((\neg_{\$} h_1(\varphi_1)) \oplus_n (\neg_{\$} h_1(\varphi_2)))}\;\neg_{\$}^3$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_{\$}ma}} h_2(\varphi)$ where D_1° and D_2° exist by induction hypothesis.

φ is $\varphi_1 \multimap \varphi_2$. Note that $h_2(\varphi)$ is $\neg_{\$}\neg_{\$}((\neg_{\$}\neg_{\$} h_1(\varphi_1)) \multimap (\neg_{\$}\neg_{\$} h_1(\varphi_2)))$ and $h_a(\varphi) = (\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2))$ The thesis follows since

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_1^{\circ} \\ h_a(\varphi_1) \rightarrow h_1(\varphi_1) \end{matrix}}{\sim\sim h_a(\varphi_1) \rightarrow h_1(\varphi_1)}\;\sim^2}{\sim\sim h_a(\varphi_1) \rightarrow \neg_{\$}\neg_{\$} h_1(\varphi_1)}\;\neg_{\$}{}^2 \qquad \dfrac{\dfrac{\begin{matrix} D_2^{\circ} \\ h_1(\varphi_2) \rightarrow h_a(\varphi_2) \end{matrix}}{h_1(\varphi_2), \sim h_a(\varphi_2) \rightarrow}\;\mathrm{L}\sim}{\neg_{\$}\neg_{\$} h_1(\varphi_2), \sim h_a(\varphi_2) \rightarrow}\;\neg_{\$}{}^2}{(\neg_{\$}\neg_{\$} h_1(\varphi_1)) \multimap (\neg_{\$}\neg_{\$} h_1(\varphi_2)), \sim\sim h_a(\varphi_1), \sim h_a(\varphi_2) \rightarrow}\;\mathrm{L}\multimap}{\neg_{\$}\neg_{\$}((\neg_{\$}\neg_{\$} h_1(\varphi_1)) \multimap (\neg_{\$}\neg_{\$} h_1(\varphi_2))), \sim\sim h_a(\varphi_1), \sim h_a(\varphi_2) \rightarrow}\;\neg_{\$}{}^2}{\neg_{\$}\neg_{\$}((\neg_{\$}\neg_{\$} h_1(\varphi_1)) \multimap (\neg_{\$}\neg_{\$} h_1(\varphi_2))), \sim\sim h_a(\varphi_1) \rightarrow \sim\sim h_a(\varphi_2)}\;\mathrm{R}\sim}{\neg_{\$}\neg_{\$}((\neg_{\$}\neg_{\$} h_1(\varphi_1)) \multimap (\neg_{\$}\neg_{\$} h_1(\varphi_2))) \rightarrow (\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2))}\;\mathrm{R}\multimap$$

and, denoting by D the derivation above for $h_a(\varphi) \vdash^l_{\mathcal{D}_{n+_{\$}ma}} h_1(\varphi)$,

$$\dfrac{\begin{matrix} D \\ (\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2)) \rightarrow (\neg_{\$}\neg_{\$} h_1(\varphi_1)) \multimap (\neg_{\$}\neg_{\$} h_1(\varphi_2)) \end{matrix}}{(\sim\sim h_a(\varphi_1)) \multimap (\sim\sim h_a(\varphi_2)) \rightarrow \neg_{\$}\neg_{\$}((\neg_{\$}\neg_{\$} h_1(\varphi_1)) \multimap (\neg_{\$}\neg_{\$} h_1(\varphi_2)))}\;\neg_{\$}{}^2$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_2(\varphi)$ where D_1° and D_2° exist by induction hypothesis. $\diamond$

Lemma 5.6 The pair of maps h_1 and h_2 is such that $h_1(\Psi) \vdash^g_{\mathcal{D}_n} h_2(\varphi)$ whenever $\Psi \vdash^g_{n+ma} \varphi$ and Ψ and $\{\varphi\}$ are contained in L_{n+ma}.

Proof: We show by complete induction on the depth of a sequent derivation that if $\Gamma \to \Delta$ is derivable in $\mathcal{D}_{n+ma}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ then $h_1(\Gamma), \neg_\$ h_1(\Delta) \to p_\$$ is derivable in $\mathcal{D}_n$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$. Let

$$\frac{\begin{array}{c}D_1\\ \Gamma_1 \to \Delta_1\end{array} \quad \ldots \quad \begin{array}{c}D_k\\ \Gamma_k \to \Delta_k\end{array}}{\Gamma \to \Delta}\ r$$

be a derivation in $\mathcal{D}_{n+ma}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ where k is greater than or equal to 0 and r is the justification of the last inference, which may be due to an hypothesis or to the application of a rule. Consider the following cases:

r is hypothesis. Denote by δ the unique formula in Δ. Note that Γ is empty and δ is in Ψ. Then

$$\frac{\dfrac{}{\to h_1(\delta)}\ \text{hypothesis} \qquad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{\neg_\$ h_1(\delta) \to p_\$}\ \text{L} \multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$;

r is Init. Denote by δ the unique formula in Γ and in Δ. Then

$$\frac{\dfrac{}{h_1(\delta) \to h_1(\delta)}\ \text{Init} \qquad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{h_1(\delta), \neg_\$ h_1(\delta) \to p_\$}\ \text{L} \multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$.

r is Cut. Denote by φ the formula to which r is applied. Then

$$\frac{\begin{array}{c}D_1^\circ\\ \neg_\$ h_1(\varphi), h_1(\Gamma_1), \neg_\$ h_1(\Delta_1) \to p_\$\end{array} \quad \dfrac{\begin{array}{c}D_2^\circ\\ h_1(\varphi), h_1(\Gamma_2), \neg_\$ h_1(\Delta_2) \to p_\$\end{array}}{h_1(\Gamma_2), \neg_\$ h_1(\Delta_2) \to \neg_\$ h_1(\varphi)}\ \text{R} \multimap}{h_1(\Gamma_1), h_1(\Gamma_2), \neg_\$ h_1(\Delta_1), \neg_\$ h_1(\Delta_2) \to p_\$}\ \text{Cut}$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° and D_2° exist by the induction hypothesis;

r is L$\perp$. The thesis follows since $h_1(\perp) \to p_\$$ is a theorem in $\mathcal{D}_n$.

r is R$\perp$. Denote by Δ' the multiset Δ without the formula $\perp$. Then

$$\frac{\begin{array}{c}D_1^\circ\\ h_1(\Gamma), \neg_\$ h_1(\Delta') \to p_\$\end{array} \quad \dfrac{}{p_\$ \to p_\$}\ \text{Init}}{h_1(\Gamma), \neg_\$ h_1(\perp), \neg_\$ h_1(\Delta') \to p_\$}\ \text{L} \multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by induction hypothesis;

r is L$\mathbf{1}$. Denote by Γ' the multiset Γ without the formula $\mathbf{1}$. Then

$$\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Gamma'), \neg_\$ h_1(\Delta) \to p_\$\end{array} \quad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma'), h_1(\mathbf{1}), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by induction hypothesis;

r is R$\mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{}{p_\$ \to p_\$}\,\text{Init}}{\to h_1(\mathbf{1})}\,\text{R}\multimap \quad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{\neg_\$ h_1(\mathbf{1}) \to p_\$}\,\text{L}\multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$;

r is L$\sim$. Denote by $\sim\varphi$ the formula used by r and by Γ' the multiset Γ without that formula. Then the thesis follows since $h_1(\Gamma'), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta) \to p_\$$ is a theorem in $\mathcal{D}_n$ by induction hypothesis.

r is R$\sim$. Denote by $\sim\varphi_1$ the formula used by r and by Δ' the multiset Δ without that formula. Then

$$\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\varphi_1), h_1(\Gamma), \neg_\$ h_1(\Delta') \to p_\$\end{array}}{h_1(\Gamma), \neg_\$ h_1(\Delta') \to h_1(\sim\varphi_1)}\,\text{R}\multimap \quad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma), \neg_\$ h_1(\sim\varphi_1), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by the induction hypothesis;

r is L$\otimes$ for $\otimes \in \{\otimes_m, \otimes_n\}$. Denote by $\varphi_1 \otimes \varphi_2$ the formula used by r and by Γ' the multiset Γ without that formula. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Gamma'), h_1(\varphi_1), h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$\end{array}}{h_1(\Gamma'), h_1(\varphi_1), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap \quad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma'), h_1(\varphi_1), \neg_\$\neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap}{h_1(\Gamma'), \neg_\$\neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_1)}\,\text{R}\multimap \quad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma'), \neg_\$\neg_\$ h_1(\varphi_1), \neg_\$\neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap}{h_1(\Gamma'), h_1(\varphi_1 \otimes \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\otimes_n$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by the induction hypothesis;

r is R$\otimes$ for $\otimes \in \{\otimes_m, \otimes_n\}$. Denote by $\varphi_1 \otimes \varphi_2$ the formula used by r, by Δ' the multiset Δ without that formula, and by Δ_1' and Δ_2' the multisets of formulas of Δ' coming from the different premises of r and similarly for Γ. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Gamma_1), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta_1') \to p_\$}}{h_1(\Gamma_1), \neg_\$ h_1(\Delta_1') \to \neg_\$\neg_\$ h_1(\varphi_1)}\,\text{R}\multimap \qquad \dfrac{\dfrac{D_2^\circ}{h_1(\Gamma_2), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta_2') \to p_\$}}{h_1(\Gamma_2), \neg_\$ h_1(\Delta_2') \to \neg_\$\neg_\$ h_1(\varphi_2)}\,\text{R}\multimap}{h_1(\Gamma_1), \neg_\$ h_1(\Delta_1'), h_1(\Gamma_2), \neg_\$ h_1(\Delta_2') \to h_1(\varphi_1 \otimes \varphi_2)}\,\text{R}\otimes_n \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma_1), h_1(\Gamma_2), \neg_\$ h_1(\varphi_1 \otimes \varphi_2), \neg_\$ h_1(\Delta_1'), \neg_\$ h_1(\Delta_2') \to p_\$}\,\text{L}\multimap
$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\, h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° and D_2° exist by induction hypothesis;

r is L∇. Denote by $\varphi_1 \nabla \varphi_2$ the formula used by r, by Γ' the multiset Γ without that formula, and by Γ_1' and Γ_2' the multisets of formulas of Γ' coming from the different premises of r and similarly for Δ. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Gamma_1'), h_1(\varphi_1), \neg_\$ h_1(\Delta_1) \to p_\$}}{h_1(\Gamma_1'), \neg_\$ h_1(\Delta_1) \to \neg_\$ h_1(\varphi_1)}\,\text{R}\multimap \qquad \dfrac{\dfrac{D_2^\circ}{h_1(\Gamma_2'), h_1(\varphi_2), \neg_\$ h_1(\Delta_2) \to p_\$}}{h_1(\Gamma_2'), \neg_\$ h_1(\Delta_2) \to \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap}{h_1(\Gamma_1'), \neg_\$ h_1(\Delta_1), h_1(\Gamma_2'), \neg_\$ h_1(\Delta_2) \to (\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2))}\,\text{R}\otimes_n \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma_1'), h_1(\Gamma_2'), h_1(\varphi_1 \nabla \varphi_2), \neg_\$ h_1(\Delta_1), \neg_\$ h_1(\Delta_2) \to p_\$}\,\text{L}\multimap
$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\, h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° and D_2° exist by induction hypothesis;

r is R∇. Denote by $\varphi_1 \nabla \varphi_2$ the formula used by r and by Δ' the multiset Δ without that formula. Then consider the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Gamma), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to p_\$}}{h_1(\Gamma), \neg_\$ h_1(\varphi_1) \otimes_n \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\otimes_n}{h_1(\Gamma), \neg_\$ h_1(\Delta') \to h_1(\varphi_1 \nabla \varphi_2)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma), \neg_\$ h_1(\varphi_1 \nabla \varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap
$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\, h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by induction hypothesis;

r is L$\oplus$ for $\oplus \in \{\oplus_m, \oplus_n\}$. Denote by $\varphi_1 \oplus \varphi_2$ the formula used by r and by Γ' the multiset Γ without that formula. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Gamma'), h_1(\varphi_1), \neg_\$ h_1(\Delta) \to p_\$}}{h_1(\Gamma'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_1)}\,\text{R}\multimap \quad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma'), \neg_\$\neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap \qquad \dfrac{\dfrac{\dfrac{D_2^\circ}{h_1(\Gamma'), h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}}{h_1(\Gamma'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap \quad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma'), \neg_\$\neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap}{h_1(\Gamma'), h_1(\varphi_1 \oplus \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\oplus_n
$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\, h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° and D_2° exist by induction hypothesis;

r is R$_i\oplus$ for $\oplus \in \{\oplus_m, \oplus_n\}$. Denote by $\varphi_1 \oplus \varphi_2$ the formula used by r and by Δ' the multiset Δ without that formula. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\Gamma), \neg_\$ h_1(\varphi_i), \neg_\$ h_1(\Delta') \to p_\$}}{h_1(\Gamma), \neg_\$ h_1(\Delta') \to \neg_\$\neg_\$ h_1(\varphi_i)}\,\text{R}\multimap}{h_1(\Gamma), \neg_\$ h_1(\Delta') \to h_1(\varphi_1 \oplus \varphi_2)}\,\text{R}_i\oplus_n \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma), \neg_\$ h_1(\varphi_1 \oplus \varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap
$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by the induction hypothesis;

r is $\text{L}_i\&$. Denote by $\varphi_1\&\varphi_2$ the formula used by r and by Γ' the multiset Γ without that formula. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Gamma'), h_1(\varphi_i), \neg_\$ h_1(\Delta) \to p_\$\end{array}}{h_1(\Gamma'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_i)}\,\text{R}\multimap}{h_1(\Gamma'), \neg_\$ h_1(\Delta) \to \neg_\$ h_1(\varphi_1) \oplus_n \neg_\$ h_1(\varphi_2)}\,\text{R}_i\oplus_n \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma'), h_1(\varphi_1 \&_m \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by induction hypothesis;

r is $\text{R}\&$. Denote by $\varphi_1\&\varphi_2$ the formula used by r and by Δ' the multiset Δ without that formula. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Gamma), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta') \to p_\$\end{array} \quad \begin{array}{c}D_2^\circ\\ h_1(\Gamma), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to p_\$\end{array}}{h_1(\Gamma), (\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$ h_1(\varphi_2)), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\oplus_n}{h_1(\Gamma), \neg_\$ h_1(\Delta') \to h_1(\varphi_1 \& \varphi_2)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma), \neg_\$ h_1(\varphi_1 \& \varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° and D_2° exist by induction hypothesis;

r is $\text{L}\multimap$. Denote by $\varphi_1 \multimap \varphi_2$ the formula used by r, by Γ' the multiset Γ without that formula, and by Γ_1' and Γ_2' the multisets of formulas of Γ' coming from the different premises of r and similarly for Δ. Then

$$\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Gamma_1'), \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta_1) \to\end{array}}{h_1(\Gamma_1'), \neg_\$ h_1(\Delta_1) \to \neg_\$ \neg_\$ h_1(\varphi_1)}\,\text{R}\neg_\$ \qquad \dfrac{\dfrac{\begin{array}{c}D_2^\circ\\ h_1(\Gamma_2'), h_1(\varphi_2), \neg_\$ h_1(\Delta_2) \to p_\$\end{array}}{h_1(\Gamma_2'), \neg_\$ h_1(\Delta_2) \to \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma_2'), \neg_\$ \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta_2) \to p_\$}\,\text{L}\multimap}{h_1(\Gamma'), h_1(\varphi_1 \multimap \varphi_2), \neg_\$ h_1(\Delta) \to p_\$}\,\text{L}\multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° and D_2° exist by induction hypothesis;

r is $\text{R}\multimap$. Denote by $\varphi_1 \multimap \varphi_2$ the formula used by r, and by Δ' the multiset Δ without that formula. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Gamma), h_1(\varphi_1), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to p_\$\end{array}}{h_1(\Gamma), \neg_\$ h_1(\varphi_2), \neg_\$ h_1(\Delta') \to \neg_\$ h_1(\varphi_1)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma), \neg_\$ h_1(\varphi_2), \neg_\$ \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap}{h_1(\Gamma), \neg_\$ \neg_\$ h_1(\varphi_1), \neg_\$ h_1(\Delta') \to \neg_\$ \neg_\$ h_1(\varphi_2)}\,\text{R}\multimap}{h_1(\Gamma), \neg_\$ h_1(\Delta') \to h_1(\varphi_1 \multimap \varphi_2)}\,\text{R}\multimap \qquad \dfrac{}{p_\$ \to p_\$}\,\text{Init}}{h_1(\Gamma), \neg_\$ h_1(\varphi_1 \multimap \varphi_2), \neg_\$ h_1(\Delta') \to p_\$}\,\text{L}\multimap$$

is a derivation in $\mathcal{D}_n$ for $h_1(\Gamma), \neg_\$\ h_1(\Delta) \to p_\$$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$, where D_1° exists by induction hypothesis. ◇

Lemma 5.7 The map h is such that

- $\Gamma \vdash^g_{\mathcal{D}_{n+ma}} h(\psi)$ whenever $\Gamma \vdash^g_{\mathcal{D}_{n+_\$ma}} \psi$
- $h(\Psi), \Delta \vdash^g_{\mathcal{D}_{n+ma}} \varphi$ whenever $\Psi, \Delta \vdash^g_{\mathcal{D}_{n+_\$ma}} \varphi$

where $\Gamma \cup \Delta \cup \{\varphi\}$ is contained in L_{n+ma} and $\Psi \cup \{\psi\}$ is contained in L_n.

Proof: We show by complete induction on the depth of a sequent derivation that if Γ and Δ are sets contained in $L_{n+_\$ma}$ and $\Gamma \to \Delta$ is derivable in $\mathcal{D}_{n+_\$ma}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ then

$$\overline{h}(\Gamma) \to \overline{h}(\Delta)$$

is derivable in $\mathcal{D}_{n+ma}$ from the set of hypothesis $\{\to \overline{h}(\psi) : \psi \in \Psi\}$ where $\overline{h}$ is a map from $L_{n+_\$ma}$ to L_{n+ma} extending h by establishing an identity on all the additional connectives. Let

$$\dfrac{\begin{array}{ccc} D_1 & & D_k \\ \Gamma_1 \to \Delta_1 & \ldots & \Gamma_k \to \Delta_k \end{array}}{\Gamma \to \Delta}\ r$$

be a derivation in $\mathcal{D}_{n+_\$m}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ where k is greater than or equal to 0. Consider the following cases:

r is hypothesis. Denote by δ the unique formula in Δ. Note that Γ is empty and δ is in Ψ. Then

$$\dfrac{}{\to \overline{h}(\delta)}\ \text{hypothesis}$$

is a derivation in $\mathcal{D}_{n+ma}$ for $\overline{h}(\Gamma) \to \overline{h}(\Delta)$ from the set $\{\to \overline{h}(\psi) : \psi \in \Psi\}$;

r is L$p_\$$. Note that $\overline{h}(\Gamma)$ is $h(p_\$)$, which is $\perp$, and Δ is empty. So, the thesis follows straightforwardly using rule L$\perp$ of $\mathcal{D}_{n+ma}$;

r is R$p_\$$. Note that $\overline{h}(p_\$)$ is $\perp$, and denote by Δ' the multiset Δ without $p_\$$. So

$$\dfrac{\begin{array}{c}\mathcal{D}_1^\circ \\ \overline{h}(\Gamma) \to \overline{h}(\Delta')\end{array}}{\overline{h}(\Gamma) \to \overline{h}(p_\$), \overline{h}(\Delta')}\ \text{R}\perp$$

is a derivation for $\overline{h}(\Gamma) \to \overline{h}(\Delta)$ in $\mathcal{D}_{n+ma}$ where $\mathcal{D}_1^\circ$ exists by induction hypothesis;

r is Init. The proof of this case follows straightforwardly;

r is L**1**. Note that $\overline{h}(\mathbf{1}) = \mathbf{1}$. Denote by Γ' the multiset Γ without the formula **1**. Then

$$\dfrac{\begin{array}{c}D_1^\circ \\ \overline{h}(\Gamma') \to \overline{h}(\Delta)\end{array}}{\overline{h}(\Gamma'), \overline{h}(\mathbf{1}) \to \overline{h}(\Delta)}\ \text{L}\mathbf{1}$$

is a derivation for $\overline{h}(\Gamma) \to \overline{h}(\Delta)$ in $\mathcal{D}_{n+ma}$, where D_1° exists by induction hypothesis;

r is Cut, or R**1**, or L$\perp$, or R$\perp$, or L$\otimes_m$, or R$\otimes_m$, or L$\oplus_m$, or R$\oplus_m$, or L∇, or R∇, or L&, or R&, or L$\otimes_n$, or R$\otimes_n$, or L$\oplus_n$, or R$\oplus_n$, or L$\multimap$, or R$\multimap$. The proofs of these cases are similar to the proof of L**1** so we omit them. $\diamond$

Lemma 5.11 The pair of maps h_1 and h_2 is such that $h_1(\Psi) \vdash^g_{\mathcal{D}_e} h_2(\varphi)$ whenever $\Psi \vdash^g_{\mathcal{D}_{w+m}} \varphi$ and Ψ and $\{\varphi\}$ are contained in L_{w+m}.

Proof: We show by complete induction on the depth of a sequent derivation that if $\Gamma \to \Delta$ is derivable in $\mathcal{D}_{w+m}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ then $h_1^l(\Gamma), \neg h_1(\Delta) \to$ is derivable in $\mathcal{D}_e$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$. Let

$$\dfrac{\begin{matrix} D_1 & & D_k \\ \Gamma_1 \to \Delta_1 & \ldots & \Gamma_k \to \Delta_k \end{matrix}}{\Gamma \to \Delta}\ r$$

be a derivation in $\mathcal{D}_{w+m}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ where k is greater than or equal to 0 ending with the application of rule r. Consider the following cases:

r is hypothesis. Denote by δ the unique formula in Δ. Note that Γ is empty and δ is in Ψ. Then

$$\dfrac{\dfrac{}{\to h_1(\delta)}\ \text{hypothesis}}{\neg h_1(\delta) \to}\ \text{L}\neg$$

is a derivation in $\mathcal{D}_e$ for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$;

r is Init. Denote by δ the unique formula in Γ and in Δ. Then

$$\dfrac{\dfrac{\dfrac{}{\to h_1(\delta)}\ \text{hypothesis}}{\neg h_1(\delta) \to}\ \text{L}\neg}{h_1^l(\delta), \neg h_1(\delta) \to}\ \text{Lw}$$

is a derivation in $\mathcal{D}_e$ for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$.

r is $\text{L}\mathbf{1}_m$. Note that $h_1^l(\mathbf{1}_m) = \mathbf{1} \wedge \mathbf{1}$ and denote by Γ' the multiset Γ without the formula $\mathbf{1}_m$. Then

$$\dfrac{\dfrac{\begin{matrix} D^\circ \\ h_1^l(\Gamma'), \neg h_1(\Delta) \to \end{matrix}}{h_1^l(\Gamma'), \mathbf{1}, \neg h_1(\Delta) \to}\ \text{L}\mathbf{1}}{h_1^l(\Gamma'), \mathbf{1} \wedge \mathbf{1}, \neg h_1(\Delta) \to}\ \text{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D° exists by induction hypothesis;

r is Rt for t in $\{\mathbf{1}_m, \mathbf{1}\}$. Note that $h_1(t) = \mathbf{1} \vee \bot$. Then

$$\dfrac{\dfrac{\dfrac{}{\to \mathbf{1}}\ \text{R}\mathbf{1}}{\to \mathbf{1} \vee \bot}\ \text{R}_1\vee}{\neg(\mathbf{1} \vee \bot) \to}\ \text{L}\neg$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$;

r is Lb for b in $\{\bot_m, \bot\}$. Note that $h_1^l(b) = \bot \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{}{\bot \to}\ \text{L}\bot}{\bot \wedge \mathbf{1} \to}\ \text{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$;

r is L$\mathbf{0}$. Note that $h_1^l(\mathbf{0}) = \mathbf{0} \wedge \mathbf{1}$. Denote by Γ' the multiset Γ without the formula $\mathbf{0}$. Then

$$\dfrac{\dfrac{}{h_1^l(\Gamma'), \mathbf{0}, \neg h_1(\Delta) \to}\ \mathrm{L}\mathbf{0}}{h_1^l(\Gamma'), \mathbf{0} \wedge \mathbf{1}, \neg h_1(\Delta) \to}\ \mathrm{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$;

r is R$\bot_m$. Note that $h_1(\bot_m) = \bot \vee \bot$ and denote by Δ' the multiset Δ without the formula $\bot_m$. Then

$$\dfrac{\dfrac{\dfrac{\begin{matrix} D^\circ \\ h_1^l(\Gamma), \neg h_1(\Delta') \to \end{matrix}}{h_1^l(\Gamma), \neg h_1(\Delta') \to \bot}\ \mathrm{R}\bot}{h_1^l(\Gamma), \neg h_1(\Delta') \to \bot \vee \bot}\ \mathrm{R}_1\vee}{h_1^l(\Gamma), \neg(\bot \vee \bot), \neg h_1(\Delta') \to}\ \mathrm{L}\neg$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D° exists by induction hypothesis;

r is Lw. Let φ be the formula introduced by r and Γ' the multiset Γ without that formula. Note that $h_1^l(\varphi) = h_1^{l\,-}(\varphi) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1^l(\Gamma'), \neg h_1(\Delta) \to \end{matrix}}{h_1^l(\Gamma'), \mathbf{1}, \neg h_1(\Delta) \to}\ \mathrm{L}\mathbf{1}}{h_1^l(\Gamma'), h_1^{l\,-}(\varphi) \wedge \mathbf{1}, \neg h_1(\Delta) \to}\ \mathrm{L}_2\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is Rw. Let φ be the formula in Δ. Note that $h_1(\varphi) = h_1^-(\varphi) \vee \bot$. Then

$$\dfrac{\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1^l(\Gamma) \to \end{matrix}}{h_1^l(\Gamma) \to \bot}\ \mathrm{R}\bot}{h_1^l(\Gamma) \to h_1^-(\varphi) \vee \bot}\ \mathrm{R}_2\vee}{h_1^l(\Gamma), \neg(h_1^-(\varphi) \vee \bot) \to}\ \mathrm{L}\neg$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is L$\sim$. Let $\sim \varphi_1$ be the formula introduced by r and Γ' the multiset Γ without that formula. Note that $h_1^l(\varphi) = (\neg\neg\neg h_1(\varphi_1)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1^l(\Gamma'), \neg h_1(\varphi_1), \neg h_1(\Delta) \to \end{matrix}}{h_1^l(\Gamma'), \neg\neg\neg h_1(\varphi_1), \neg h_1(\Delta) \to}\ \neg^2}{h_1^l(\Gamma'), (\neg\neg\neg h_1(\varphi_1)) \wedge \mathbf{1}, \neg h_1(\Delta) \to}\ \mathrm{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is R$\sim$. Let $\sim \varphi_1$ be the formula introduced by r and Δ' the multiset Δ without that formula. Note that $h_1(\varphi) = (\neg h_1^l(\varphi_1)) \vee \bot$. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1^l(\Gamma), h_1^l(\varphi_1), \neg h_1(\Delta') \to\end{array}}{h_1^l(\Gamma), \neg h_1(\Delta') \to \neg h_1^l(\varphi_1)}\,\text{R}\neg}{h_1^l(\Gamma), \neg h_1(\Delta') \to (\neg h_1^l(\varphi_1)) \vee \bot}\,\text{R}_1\vee}{h_1^l(\Gamma), \neg((\neg h_1^l(\varphi_1)) \vee \bot), \neg h_1(\Delta') \to}\,\text{L}\neg$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is L$\circ$ for $\circ$ in $\{\otimes, *\}$. Let $\varphi_1 \circ \varphi_2$ be the formula introduced by r and Γ' the multiset Γ without that formula. Note that $h_1^l(\varphi) = (h_1^l(\varphi_1) * h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1^l(\Gamma'), h_1^l(\varphi_1), h_1^l(\varphi_2), \neg h_1(\Delta) \to\end{array}}{h_1^l(\Gamma'), h_1^l(\varphi_1) * h_1^l(\varphi_2), \neg h_1(\Delta) \to}\,\text{L}*}{h_1^l(\Gamma'), (h_1^l(\varphi_1) * h_1^l(\varphi_2)) \wedge \mathbf{1}, \neg h_1(\Delta) \to}\,\text{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is R$\circ$ for $\circ$ in $\{\otimes, *\}$. Let $\varphi_1 \circ \varphi_2$ be the formula introduced by r, Δ' the multiset Δ without that formula and Δ_1' and Δ_2' the multisets of formulas of Δ' coming from the different premises of r and similarly for Γ. Note that $h_1(\varphi) = (\neg\neg h_1(\varphi_1) * \neg\neg h_1(\varphi_2)) \vee \bot$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1^l(\Gamma), \neg h_1(\varphi_1), \neg h_1(\Delta') \to\end{array}}{h_1^l(\Gamma), \neg h_1(\Delta') \to \neg\neg h_1(\varphi_1)}\,\text{L}\neg \quad \dfrac{\begin{array}{c}D_2^\circ\\ h_1^l(\Gamma), \neg h_1(\varphi_2), \neg h_1(\Delta') \to\end{array}}{h_1^l(\Gamma), \neg h_1(\Delta') \to \neg\neg h_1(\varphi_2)}\,\text{L}\neg}{h_1^l(\Gamma), \neg h_1(\Delta') \to \neg\neg h_1(\varphi_1) * \neg\neg h_1(\varphi_2)}\,\text{R}*}{h_1^l(\Gamma), \neg h_1(\Delta') \to (\neg\neg h_1(\varphi_1) * \neg\neg h_1(\varphi_2)) \vee \bot}\,\text{R}_1\vee}{h_1^l(\Gamma), \neg((\neg\neg h_1(\varphi_1) * \neg\neg h_1(\varphi_2)) \vee \bot), \neg h_1(\Delta') \to}\,\text{L}\neg$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is L∇. Let $\varphi_1 \nabla \varphi_2$ be the formula introduced by r, Γ' the multiset Γ without that formula, Γ_1' and Γ_2' the multisets of formulas of Γ' coming from the different premises of and similarly for Δ. Note that $h_1^l(\varphi) = ((\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1^l(\Gamma_1'), h_1^l(\varphi_1), \neg h_1(\Delta_1) \to\end{array}}{h_1^l(\Gamma_1'), \neg h_1(\Delta_1) \to \neg h_1^l(\varphi_1)}\,\text{R}\neg \quad \begin{array}{c}D_2^\circ\\ h_1^l(\Gamma_2'), h_1^l(\varphi_2), \neg h_1(\Delta_2) \to\end{array}}{h_1^l(\Gamma_1'), h_1^l(\Gamma_2'), (\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2), \neg h_1(\Delta_1), \neg h_1(\Delta_2) \to}\,\text{L}\supset}{h_1^l(\Gamma_1'), h_1^l(\Gamma_2'), ((\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}, \neg h_1(\Delta_1), \neg h_1(\Delta_2) \to}\,\text{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \to$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is R∇. Let $\varphi_1 \nabla \varphi_2$ be the formula introduced by r and note that $h_1(\varphi) = ((\neg\neg\neg h_1(\varphi_1)) \supset (\neg\neg h_1(\varphi_2))) \vee \bot$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1^l(\Gamma), \neg h_1(\varphi_1), \neg h_1(\varphi_2), \neg h_1(\Delta') \rightarrow}}{h_1^l(\Gamma), \neg\neg\neg h_1(\varphi_1), \neg h_1(\varphi_2), \neg h_1(\Delta') \rightarrow}\,\neg^2}{h_1^l(\Gamma), \neg\neg\neg h_1(\varphi_1), \neg h_1(\Delta') \rightarrow \neg\neg h_1(\varphi_2)}\,\mathrm{R}\neg}{h_1^l(\Gamma), \neg h_1(\Delta') \rightarrow (\neg\neg\neg h_1(\varphi_1)) \supset (\neg\neg h_1(\varphi_2))}\,\mathrm{R}\supset}{h_1^l(\Gamma), \neg h_1(\Delta') \rightarrow ((\neg\neg\neg h_1(\varphi_1)) \supset (\neg\neg h_1(\varphi_2))) \vee \bot}\,\mathrm{R}_1\vee}{h_1^l(\Gamma), \neg(((\neg\neg\neg h_1(\varphi_1)) \supset (\neg\neg h_1(\varphi_2))) \vee \bot), \neg h_1(\Delta') \rightarrow}\,\mathrm{L}\neg$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \rightarrow$ from the set of hypothesis $\{\rightarrow h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is L$\supset$. Let $\varphi_1 \supset \varphi_2$ be the formula introduced by r, Γ' the multiset Γ without that formula, and Γ'_1 and Γ'_2 the multisets of formulas of Γ' coming from the different premises of r. Note that $h_1^l(\varphi) = ((\neg\neg h_1(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1^l(\Gamma'_1), \neg h_1(\varphi_1) \rightarrow}}{h_1^l(\Gamma'_1) \rightarrow \neg\neg h_1(\varphi_1)}\,\mathrm{R}\neg \quad \dfrac{D_2^\circ}{h_1^l(\Gamma'_2), h_1^l(\varphi_2), \neg h_1(\Delta) \rightarrow}}{h_1^l(\Gamma'_1), h_1^l(\Gamma'_2), (\neg\neg h_1(\varphi_1)) \supset h_1^l(\varphi_2), \neg h_1(\Delta) \rightarrow}\,\mathrm{L}\supset}{h_1^l(\Gamma'_1), h_1^l(\Gamma'_2), ((\neg\neg h_1(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}, \neg h_1(\Delta) \rightarrow}\,\mathrm{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \rightarrow$ from the set of hypothesis $\{\rightarrow h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is R$\supset$. Let $\varphi_1 \supset \varphi_2$ be the formula introduced by r and denote by Δ' the multiset Δ without that formula. Note that $h_1(\varphi) = (h_1^l(\varphi_1) \supset (\neg\neg h_1(\varphi_2))) \vee \bot$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1^l(\Gamma), h_1^l(\varphi_1), \neg h_1(\varphi_2), \neg h_1(\Delta') \rightarrow}}{h_1^l(\Gamma), h_1^l(\varphi_1), \neg h_1(\Delta') \rightarrow \neg\neg h_1(\varphi_2)}\,\mathrm{R}\neg}{h_1^l(\Gamma), \neg h_1(\Delta') \rightarrow h_1^l(\varphi_1) \supset (\neg\neg h_1(\varphi_2))}\,\mathrm{R}\supset}{h_1^l(\Gamma), \neg h_1(\Delta') \rightarrow (h_1^l(\varphi_1) \supset (\neg\neg h_1(\varphi_2))) \vee \bot}\,\mathrm{R}_1\vee}{h_1^l(\Gamma), \neg((h_1^l(\varphi_1) \supset (\neg\neg h_1(\varphi_2))) \vee \bot), \neg h_1(\Delta') \rightarrow}\,\mathrm{L}\neg$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \rightarrow$ from the set of hypothesis $\{\rightarrow h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is L$_k\wedge$ for $k = 1, 2$. Let $\varphi_1 \wedge \varphi_2$ be the formula introduced by r and Γ' the multiset Γ without that formula. Note that $h_1^l(\varphi) = (h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1^l(\Gamma'), h_1^l(\varphi_k), \neg h_1(\Delta) \rightarrow}}{h_1^l(\Gamma'), h_1^l(\varphi_1) \wedge h_1^l(\varphi_2), \neg h_1(\Delta) \rightarrow}\,\mathrm{L}_k\wedge}{h_1^l(\Gamma'), (h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)) \wedge \mathbf{1}, \neg h_1(\Delta) \rightarrow}\,\mathrm{L}_1\wedge$$

is a derivation for $h_1^l(\Gamma), \neg h_1(\Delta) \rightarrow$ from the set of hypothesis $\{\rightarrow h_1(\psi) : \psi \in \Psi\}$ where D_1° exists by induction hypothesis.

r is R$\wedge$. Let $\varphi_1 \wedge \varphi_2$ be the formula introduced by r, and Δ' the multiset Δ without that formula. Note that $h_1(\varphi) = (\neg\neg h_1(\varphi_1) \wedge \neg\neg h_1(\varphi_2)) \vee \bot$. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1^l(\Gamma),\neg h_1(\varphi_1),\neg h_1(\Delta')\to\end{array}}{h_1^l(\Gamma),\neg h_1(\Delta')\to\neg\neg h_1(\varphi_1)}\,\mathrm{L}\neg \qquad \dfrac{\begin{array}{c}D_2^\circ\\ h_1^l(\Gamma),\neg h_1(\varphi_2),\neg h_1(\Delta')\to\end{array}}{h_1^l(\Gamma),\neg h_1(\Delta')\to\neg\neg h_1(\varphi_2)}\,\mathrm{L}\neg}{h_1^l(\Gamma),\neg h_1(\Delta')\to\neg\neg h_1(\varphi_1)\wedge\neg\neg h_1(\varphi_2)}\,\mathrm{R}\wedge}{h_1^l(\Gamma),\neg h_1(\Delta')\to(\neg\neg h_1(\varphi_1)\wedge\neg\neg h_1(\varphi_2))\vee\bot}\,\mathrm{R}_1\vee}{h_1^l(\Gamma),\neg((\neg\neg h_1(\varphi_1)\wedge\neg\neg h_1(\varphi_2))\vee\bot),\neg h_1(\Delta')\to}\,\mathrm{L}\neg
$$

is a derivation for $h_1^l(\Gamma),\neg h_1(\Delta)\to$ from the set of hypothesis $\{\to h_1(\psi):\psi\in\Psi\}$ where D_1° exists by induction hypothesis.

r is L$\vee$. Let $\varphi_1\vee\varphi_2$ be the formula introduced by r and Γ' the multiset Γ without that formula. Note that $h_1^l(\varphi)=(h_1^l(\varphi_1)\vee h_1^l(\varphi_2))\wedge\mathbf{1}$. Then

$$
\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1^l(\Gamma'),h_1^l(\varphi_1),\neg h_1(\Delta)\to\end{array}\quad \begin{array}{c}D_2^\circ\\ h_1^l(\Gamma'),h_1^l(\varphi_2),\neg h_1(\Delta)\to\end{array}}{h_1^l(\Gamma'),h_1^l(\varphi_1)\vee h_1^l(\varphi_2),\neg h_1(\Delta)\to}\,\mathrm{L}\vee}{h_1^l(\Gamma'),(h_1^l(\varphi_1)\vee h_1^l(\varphi_2))\wedge\mathbf{1},\neg h_1(\Delta)\to}\,\mathrm{L}_1\wedge
$$

is a derivation for $h_1^l(\Gamma),\neg h_1(\Delta)\to$ from the set of hypothesis $\{\to h_1(\psi):\psi\in\Psi\}$ where D_1° exists by induction hypothesis.

r is R$_k\vee$ for $k=1,2$. Let $\varphi_1\vee\varphi_2$ be the formula introduced by r. Note that $h_1(\varphi)=(\neg\neg h_1(\varphi_1)\vee\neg\neg h_1(\varphi_2))\vee\bot$. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1^l(\Gamma),\neg h_1(\varphi_k)\to\end{array}}{h_1^l(\Gamma)\to\neg\neg h_1(\varphi_k)}\,\mathrm{R}\neg}{h_1^l(\Gamma)\to\neg\neg h_1(\varphi_1)\vee\neg\neg h_1(\varphi_2)}\,\mathrm{R}_k\vee}{h_1^l(\Gamma)\to(\neg\neg h_1(\varphi_1)\vee\neg\neg h_1(\varphi_2))\vee\bot}\,\mathrm{L}_1\vee}{h_1^l(\Gamma),\neg((\neg\neg h_1(\varphi_1)\vee\neg\neg h_1(\varphi_2))\vee\bot)\to}\,\mathrm{L}\neg
$$

is a derivation for $h_1^l(\Gamma),\neg h_1(\Delta)\to$ from the set of hypothesis $\{\to h_1(\psi):\psi\in\Psi\}$ where D_1° exists by induction hypothesis. $\diamond$

Lemma 5.12 The maps h_1, h_2 and h_a are such that

1. $h_2(\varphi)\dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$ and $h_a(\varphi)\dashv\vdash^l_{\mathcal{D}_{w+m}} h_1(\varphi)$
2. $\varphi\dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$

for φ in L_{w+m}.

Proof:
1. $h_1(\varphi)\dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, $h_2(\varphi)\dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$ and $h_a(\varphi)\dashv\vdash^l_{\mathcal{D}_{w+m}} h_1^l(\varphi)$:

The proof follows by complete induction on the complexity of the formula:

φ is $\bot$ or $\bot_m$. Note that $h_1(\varphi)$ is $\bot\vee\bot$, $h_2(\varphi)$ is $\neg\neg(\bot\vee\bot)$, $h_a(\varphi)$ is φ and $h_1^l(\varphi)$ is $\bot\wedge\mathbf{1}$. Then

$$
\dfrac{\dfrac{}{\varphi\to}\,\mathrm{L}\varphi}{\varphi\to\bot\vee\bot}\,\mathrm{Rw}
\qquad\qquad
\dfrac{\dfrac{\dfrac{}{\bot\to}\,\mathrm{L}\bot\quad\dfrac{}{\bot\to}\,\mathrm{L}\bot}{\bot\vee\bot\to}\,\mathrm{L}\vee}{\bot\vee\bot\to\varphi}\,\mathrm{Rw}
$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$\dfrac{\dfrac{}{\varphi \to}\,\mathrm{L}\varphi}{\varphi \to \neg\neg(\bot \vee \bot)}\,\mathrm{Rw} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot \quad \dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot \vee \bot \to}\,\mathrm{L}\vee}{\neg\neg(\bot \vee \bot) \to}\,\neg^2}{\neg\neg(\bot \vee \bot) \to \varphi}\,\mathrm{Rw}$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and

$$\dfrac{\dfrac{}{\varphi \to}\,\mathrm{L}\varphi}{\varphi \to \bot \wedge \mathbf{1}}\,\mathrm{Rw} \qquad \dfrac{\dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot \wedge \mathbf{1} \to}\,\mathrm{L}_1\wedge}{\bot \wedge \mathbf{1} \to \varphi}\,\mathrm{Rw}$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h^l_1(\varphi)$;

φ is $\mathbf{0}$. Note that $h_1(\varphi)$ is $\mathbf{0} \vee \bot$, $h_2(\varphi)$ is $\neg\neg(\mathbf{0} \vee \bot)$, $h_a(\mathbf{0})$ is $\mathbf{0}$, and $h^l_1(\varphi)$ is $\mathbf{0} \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{}{\mathbf{0} \to}\,\mathrm{L}\mathbf{0} \quad \dfrac{}{\bot \to}\,\mathrm{L}\bot}{\mathbf{0} \vee \bot \to}\,\mathrm{L}\vee}{\mathbf{0} \vee \bot \to \mathbf{0}}\,\mathrm{Rw} \qquad \dfrac{}{\mathbf{0} \to \mathbf{0} \vee \bot}\,\mathrm{L}\mathbf{0}$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{0} \to}\,\mathrm{L}\mathbf{0} \quad \dfrac{}{\bot \to}\,\mathrm{L}\bot}{\mathbf{0} \vee \bot \to}\,\mathrm{L}\vee}{\neg\neg(\mathbf{0} \vee \bot) \to}\,\neg^2}{\neg\neg(\mathbf{0} \vee \bot) \to \mathbf{0}}\,\mathrm{Rw} \qquad \dfrac{}{\mathbf{0} \to \neg\neg(\mathbf{0} \vee \bot)}\,\mathrm{L}\mathbf{0}$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and

$$\dfrac{\dfrac{}{\mathbf{0} \to \mathbf{0}}\,\mathrm{L}\mathbf{0}}{\mathbf{0} \wedge \mathbf{1} \to \mathbf{0}}\,\mathrm{L}_1\wedge \qquad \dfrac{}{\mathbf{0} \to \mathbf{0} \wedge \mathbf{1}}\,\mathrm{L}\mathbf{0}$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h^l_1(\varphi)$;

φ is $\mathbf{1}$ or $\mathbf{1}_m$. Note that $h_1(\varphi)$ is $\mathbf{1} \vee \bot$, $h_2(\varphi)$ is $\neg\neg(\mathbf{1} \vee \bot)$, $h_a(\varphi)$ is φ and $h^l_1(\varphi)$ is $\mathbf{1} \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{}{\to \varphi}\,\mathrm{R}\varphi}{\mathbf{1} \vee \bot \to \varphi}\,\mathrm{Lw} \qquad \dfrac{\dfrac{\dfrac{}{\to \mathbf{1}}\,\mathrm{R}\mathbf{1}}{\to \mathbf{1} \vee \bot}\,\mathrm{R}_1\vee}{\varphi \to \mathbf{1} \vee \bot}\,\mathrm{Lw}$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$\dfrac{\dfrac{}{\to \varphi}\,\mathrm{R}\varphi}{\neg\neg(\mathbf{1} \vee \bot) \to \varphi}\,\mathrm{Lw} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\to \mathbf{1}}\,\mathrm{R}\mathbf{1}}{\to \mathbf{1} \vee \bot}\,\mathrm{R}_1\vee}{\to \neg\neg(\mathbf{1} \vee \bot)}\,\neg^2}{\varphi \to \neg\neg(\mathbf{1} \vee \bot)}\,\mathrm{Lw}$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and

$$\dfrac{\dfrac{}{\to \varphi}\,\mathrm{R}\varphi}{\mathbf{1} \wedge \mathbf{1} \to \varphi}\,\mathrm{Lw} \qquad \dfrac{\dfrac{\dfrac{}{\to \mathbf{1}}\,\mathrm{R}\mathbf{1} \quad \dfrac{}{\to \mathbf{1}}\,\mathrm{R}\mathbf{1}}{\to \mathbf{1} \wedge \mathbf{1}}\,\mathrm{R}\wedge}{\varphi \to \mathbf{1} \wedge \mathbf{1}}\,\mathrm{Lw}$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h^l_1(\varphi)$;

φ is in P. Note that $h_1(\varphi)$ is $\varphi \vee \bot$, $h_2(\varphi)$ is $\neg\neg(\varphi \vee \bot)$, $h_a(\varphi)$ is $\sim\sim \varphi$, and $h^l_1(\varphi)$ is $\varphi \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\,\mathrm{Init}}{\varphi, \sim \varphi \to}\,\mathrm{L}\sim \quad \dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot, \sim \varphi \to}\,\mathrm{Lw}}{\varphi \vee \bot, \sim \varphi \to}\,\mathrm{L}\vee}{\varphi \vee \bot \to \sim\sim \varphi}\,\mathrm{R}\sim \qquad \dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\,\mathrm{Init}}{\varphi \to \varphi \vee \bot}\,\mathrm{R}_1\vee}{\sim\sim \varphi \to \varphi \vee \bot}\,\sim^2$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\text{Init}}{\varphi, \sim \varphi \to}\text{L}\sim \quad \dfrac{\dfrac{}{\bot \to}\text{L}\bot}{\bot, \sim \varphi \to}\text{Lw}}{\varphi \vee \bot, \sim \varphi \to}\text{L}\vee}{\neg\neg(\varphi \vee \bot), \sim \varphi \to}\neg^2}{\neg\neg(\varphi \vee \bot) \to \sim\sim \varphi}\text{R}\sim \qquad \dfrac{\dfrac{\dfrac{}{\varphi \to \varphi}\text{Init}}{\varphi \to \varphi \vee \bot}\text{R}_1\vee}{\sim\sim \varphi \to \neg\neg(\varphi \vee \bot)}\sim^2 + \neg^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and

$$\dfrac{\dfrac{}{\varphi \to \varphi}\text{Init}}{\varphi \wedge \mathbf{1} \to \varphi}\text{L}_1\wedge \qquad \dfrac{\dfrac{}{\varphi \to \varphi}\text{Init} \quad \dfrac{\dfrac{}{\to \mathbf{1}}\text{R}\mathbf{1}}{\varphi \to \mathbf{1}}\text{Lw}}{\varphi \to \varphi \wedge \mathbf{1}}\text{R}\wedge$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h^l_1(\varphi)$;

φ is $\sim \varphi_1$. Note that $h_1(\varphi)$ is $(\neg h^l_1(\varphi_1)) \vee \bot$, $h_2(\varphi)$ is $\neg\neg((\neg h^l_1(\varphi_1)) \vee \bot)$, $h_a(\varphi)$ is $\sim h_a(\varphi_1)$, and $h^l_1(\varphi)$ is $(\neg h_2(\varphi_1)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} D_1^\circ \\ h_a(\varphi_1) \to h^l_1(\varphi_1) \end{array}}{\neg h^l_1(\varphi_1), h_a(\varphi_1) \to}\text{L}\neg \quad \dfrac{\dfrac{}{\bot \to}\text{L}\bot}{\bot, h_a(\varphi_1) \to}\text{Lw}}{(\neg h^l_1(\varphi_1)) \vee \bot, h_a(\varphi_1) \to}\text{L}\vee}{(\neg h^l_1(\varphi_1)) \vee \bot \to \sim h_a(\varphi_1)}\text{R}\sim$$

and

$$\dfrac{\dfrac{\begin{array}{c} D_2^\circ \\ h^l_1(\varphi_1) \to h_a(\varphi_1) \end{array}}{\sim h_a(\varphi_1) \to \neg h^l_1(\varphi_1)}\text{L}\sim +\text{R}\neg}{\sim h_a(\varphi_1) \to (\neg h^l_1(\varphi_1)) \vee \bot}\text{R}_1\vee$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D_3^\circ \\ h_a(\varphi_1) \to h^l_1(\varphi_1) \end{array}}{\neg h^l_1(\varphi_1), h_a(\varphi_1) \to}\text{L}\neg \quad \dfrac{\dfrac{}{\bot \to}\text{L}\bot}{\bot, h_a(\varphi_1) \to}\text{Lw}}{(\neg h^l_1(\varphi_1)) \vee \bot, h_a(\varphi_1) \to}\text{L}\vee}{\dfrac{\neg\neg((\neg h^l_1(\varphi_1)) \vee \bot), h_a(\varphi_1) \to}{\neg\neg((\neg h^l_1(\varphi_1)) \vee \bot) \to \sim h_a(\varphi_1)}\text{R}\sim}\neg^2$$

and

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} D_4^\circ \\ h^l_1(\varphi_1) \to h_a(\varphi_1) \end{array}}{\sim h_a(\varphi_1) \to \neg h^l_1(\varphi_1)}\text{L}\sim +\text{R}\neg}{\sim h_a(\varphi_1) \to (\neg h^l_1(\varphi_1)) \vee \bot}\text{R}_1\vee}{\sim h_a(\varphi_1) \to \neg\neg((\neg h^l_1(\varphi_1)) \vee \bot)}\neg^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and

$$\dfrac{\dfrac{\begin{array}{c} D_5^\circ \\ h_a(\varphi_1) \to h_2(\varphi_1) \end{array}}{\neg h_2(\varphi_1) \to \sim h_a(\varphi_1)}\text{L}\neg + \text{R}\sim}{(\neg h_2(\varphi_1)) \wedge \mathbf{1} \to \sim h_a(\varphi_1)}\text{L}_1\wedge$$

and

$$\dfrac{\dfrac{\overset{D_6^\circ}{h_2(\varphi_1) \to h_a(\varphi_1)}}{\sim h_a(\varphi_1) \to \neg h_2(\varphi_1)}\ \mathrm{L}\sim+\mathrm{R}\neg \qquad \dfrac{\dfrac{}{\to \mathbf{1}}\ \mathrm{R}\mathbf{1}}{\sim h_a(\varphi_1) \to \mathbf{1}}\ \mathrm{Lw}}{\sim h_a(\varphi_1) \to (\neg h_2(\varphi_1)) \wedge \mathbf{1}}\ \mathrm{R}\wedge$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_1^l(\varphi)$, and D_1°, D_2°, D_3°, D_4°, D_5°, D_6° exist by induction hypothesis;

φ is $\varphi_1 \circ \varphi_2$ for $\circ$ in $\{\otimes, *\}$. Note that $h_2(\varphi)$ is $\neg\neg((h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot)$, $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2))$, $h_1(\varphi)$ is $(h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot$, and $h_1^l(\varphi)$ is $(h_1^l(\varphi_1) * h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_1^\circ}{h_2(\varphi_1) \to h_a(\varphi_1)} \quad \overset{D_2^\circ}{h_2(\varphi_2) \to h_a(\varphi_2)}}{h_2(\varphi_1), h_2(\varphi_2) \to h_a(\varphi_1) \circ h_a(\varphi_2)}\ \mathrm{R}\circ}{h_2(\varphi_1), h_2(\varphi_2), \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{L}\sim}{h_2(\varphi_1) * h_2(\varphi_2), \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{L}* \qquad \dfrac{\dfrac{}{\bot \to}\ \mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{Lw}}{(h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot, \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{L}\vee}{\neg\neg((h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot), \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \neg^2}{\neg\neg((h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot) \to \sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2))}\ \mathrm{R}\sim$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_3^\circ}{h_a(\varphi_1) \to h_2(\varphi_1)} \quad \overset{D_4^\circ}{h_a(\varphi_2) \to h_2(\varphi_2)}}{h_a(\varphi_1), h_a(\varphi_2) \to h_2(\varphi_1) * h_2(\varphi_2)}\ \mathrm{R}*}{h_a(\varphi_1) \circ h_a(\varphi_2) \to h_2(\varphi_1) * h_2(\varphi_2)}\ \mathrm{L}\circ}{\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to h_2(\varphi_1) * h_2(\varphi_2)}\ \sim^2}{\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to (h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot}\ \mathrm{R}_1\vee}{\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to \neg\neg((h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot)}\ \neg^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_5^\circ}{h_1^l(\varphi_1) \to h_a(\varphi_1)} \quad \overset{D_6^\circ}{h_1^l(\varphi_2) \to h_a(\varphi_2)}}{h_1^l(\varphi_1), h_1^l(\varphi_2) \to h_a(\varphi_1) \circ h_a(\varphi_2)}\ \mathrm{R}\circ}{h_1^l(\varphi_1) * h_1^l(\varphi_2) \to h_a(\varphi_1) \circ h_a(\varphi_2)}\ \mathrm{L}*}{(h_1^l(\varphi_1) * h_1^l(\varphi_2)) \wedge \mathbf{1} \to h_a(\varphi_1) \circ h_a(\varphi_2)}\ \mathrm{L}_1\wedge}{(h_1^l(\varphi_1) * h_1^l(\varphi_2)) \wedge \mathbf{1} \to \sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2))}\ \sim^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_7^\circ}{h_a(\varphi_1) \to h_1^l(\varphi_1)} \quad \overset{D_8^\circ}{h_a(\varphi_2) \to h_1^l(\varphi_2)}}{h_a(\varphi_1), h_a(\varphi_2) \to h_1^l(\varphi_1) * h_1^l(\varphi_2)}\ \mathrm{R}*}{h_a(\varphi_1) * h_a(\varphi_2) \to h_1^l(\varphi_1) * h_1^l(\varphi_2)}\ \mathrm{L}* \qquad \dfrac{\dfrac{}{\to \mathbf{1}}\ \mathrm{R}\mathbf{1}}{h_a(\varphi_1) * h_a(\varphi_2) \to \mathbf{1}}\ \mathrm{Lw}}{h_a(\varphi_1) * h_a(\varphi_2) \to (h_1^l(\varphi_1) * h_1^l(\varphi_2)) \wedge \mathbf{1}}\ \mathrm{R}\wedge}{\sim\sim (h_a(\varphi_1) * h_a(\varphi_2)) \to (h_1^l(\varphi_1) * h_1^l(\varphi_2)) \wedge \mathbf{1}}\ \sim^2$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_1^l(\varphi)$, and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_9^\circ}{h_2(\varphi_1) \to h_a(\varphi_1)} \quad \overset{D_{10}^\circ}{h_2(\varphi_2) \to h_a(\varphi_2)}}{h_2(\varphi_1), h_2(\varphi_2) \to h_a(\varphi_1) \circ h_a(\varphi_2)}\ \mathrm{R}\circ}{h_2(\varphi_1), h_2(\varphi_2), \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{L}\sim \qquad \dfrac{\dfrac{}{\bot \to}\ \mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{Lw}}{h_2(\varphi_1) * h_2(\varphi_2), \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{L}*}{(h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot, \sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to}\ \mathrm{L}\vee}{(h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot \to \sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2))}\ \mathrm{R}\sim$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_{11}^{\circ} & D_{12}^{\circ} \\ h_a(\varphi_1) \to h_2(\varphi_1) & h_a(\varphi_2) \to h_2(\varphi_2) \end{matrix}}{h_a(\varphi_1), h_a(\varphi_2) \to h_2(\varphi_1) * h_2(\varphi_2)}\,\mathrm{R}*}{h_a(\varphi_1) \circ h_a(\varphi_2) \to h_2(\varphi_1) * h_2(\varphi_2)}\,\mathrm{L}\circ}{\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to h_2(\varphi_1) * h_2(\varphi_2)}\,\sim^2}{\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to (h_2(\varphi_1) * h_2(\varphi_2)) \vee \bot}\,\mathrm{R}_1\vee$$

are derivations for $h_1(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$, and D_1°, D_2°, D_3°, D_4°, D_5°, D_6°, D_7°, D_8°, D_9°, D_{10}°, D_{11}°, D_{12}° exist by induction hypothesis;

φ is $\varphi_1 \supset \varphi_2$. Note that $h_2(\varphi)$ is $\neg\neg((h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot)$, $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2))$, $h_1(\varphi)$ is $(h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot$, and $h_1^l(\varphi)$ is $(h_2(\varphi_1) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_1^{\circ} & D_2^{\circ} \\ h_a(\varphi_1) \to h_1^l(\varphi_1) & h_2(\varphi_2) \to h_a(\varphi_2) \end{matrix}}{h_1^l(\varphi_1) \supset h_2(\varphi_2), h_a(\varphi_1) \to h_a(\varphi_2)}\,\mathrm{L}\supset}{h_1^l(\varphi_1) \supset h_2(\varphi_2) \to h_a(\varphi_1) \supset h_a(\varphi_2)}\,\mathrm{R}\supset}{h_1^l(\varphi_1) \supset h_2(\varphi_2), \sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to}\,\mathrm{L}\sim \quad \dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to}\,\mathrm{Lw}}{(h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot, \sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to}\,\mathrm{L}\vee}{\neg\neg((h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot), \sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to}\,\neg^2}{\neg\neg((h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot) \to \sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2))}\,\mathrm{R}\sim$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_3^{\circ} & D_4^{\circ} \\ h_1^l(\varphi_1) \to h_a(\varphi_1) & h_a(\varphi_2) \to h_2(\varphi_2) \end{matrix}}{h_a(\varphi_1) \supset h_a(\varphi_2), h_1^l(\varphi_1) \to h_2(\varphi_2)}\,\mathrm{L}\supset}{h_a(\varphi_1) \supset h_a(\varphi_2) \to h_1^l(\varphi_1) \supset h_2(\varphi_2)}\,\mathrm{R}\supset}{\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to h_1^l(\varphi_1) \supset h_2(\varphi_2)}\,\sim^2}{\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to (h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot}\,\mathrm{R}_1\vee}{\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to \neg\neg((h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot)}\,\neg^2$$

are derivations for $h_2(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_5^{\circ} & D_6^{\circ} \\ h_a(\varphi_1) \to h_2(\varphi_1) & h_1^l(\varphi_2) \to h_a(\varphi_2) \end{matrix}}{h_2(\varphi_1) \supset h_1^l(\varphi_2), h_a(\varphi_1) \to h_a(\varphi_2)}\,\mathrm{L}\supset}{h_2(\varphi_1) \supset h_1^l(\varphi_2) \to h_a(\varphi_1) \supset h_a(\varphi_2)}\,\mathrm{R}\supset}{(h_2(\varphi_1) \supset h_1^l(\varphi_2)) \wedge \mathbf{1} \to h_a(\varphi_1) \supset h_a(\varphi_2)}\,\mathrm{L}_1\wedge}{(h_2(\varphi_1) \supset h_1^l(\varphi_2)) \wedge \mathbf{1} \to \sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2))}\,\sim^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_7^{\circ} & D_8^{\circ} \\ h_2(\varphi_1) \to h_a(\varphi_1) & h_a(\varphi_2) \to h_1^l(\varphi_2) \end{matrix}}{h_a(\varphi_1) \supset h_a(\varphi_2), h_2(\varphi_1) \to h_1^l(\varphi_2)}\,\mathrm{L}\supset}{h_a(\varphi_1) \supset h_a(\varphi_2) \to h_2(\varphi_1) \supset h_1^l(\varphi_2)}\,\mathrm{R}\supset \quad \dfrac{\dfrac{}{\to \mathbf{1}}\,\mathrm{R}\mathbf{1}}{h_a(\varphi_1) \supset h_a(\varphi_2) \to \mathbf{1}}\,\mathrm{Lw}}{h_a(\varphi_1) \supset h_a(\varphi_2) \to (h_2(\varphi_1) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}}\,\mathrm{R}\wedge}{\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to (h_2(\varphi_1) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}}\,\sim^2$$

are derivations for $h_a(\varphi) \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_1^l(\varphi)$, and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_9^\circ}{h_a(\varphi_1) \to h_1^l(\varphi_1)} \quad \overset{D_{10}^\circ}{h_2(\varphi_2) \to h_a(\varphi_2)}}{h_1^l(\varphi_1) \supset h_2(\varphi_2), h_a(\varphi_1) \to h_a(\varphi_2)}\,\mathrm{L}\supset}{h_1^l(\varphi_1) \supset h_2(\varphi_2) \to h_a(\varphi_1) \supset h_a(\varphi_2)}\,\mathrm{R}\supset}{h_1^l(\varphi_1) \supset h_2(\varphi_2), \sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to}\,\mathrm{L}\sim \quad \dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to}\,\mathrm{Lw}}{\dfrac{(h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot, \sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to}{(h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot \to \sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2))}\,\mathrm{R}\sim}\,\mathrm{L}\vee
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_{11}^\circ}{h_1^l(\varphi_1) \to h_a(\varphi_1)} \quad \overset{D_{12}^\circ}{h_a(\varphi_2) \to h_2(\varphi_2)}}{h_a(\varphi_1) \supset h_a(\varphi_2), h_1^l(\varphi_1) \to h_2(\varphi_2)}\,\mathrm{L}\supset}{h_a(\varphi_1) \supset h_a(\varphi_2) \to h_1^l(\varphi_1) \supset h_2(\varphi_2)}\,\mathrm{R}\supset}{\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to h_1^l(\varphi_1) \supset h_2(\varphi_2)}\,\sim^2}{\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to (h_1^l(\varphi_1) \supset h_2(\varphi_2)) \vee \bot}\,\mathrm{R}_1\vee
$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and D_1°, D_2°, D_3°, D_4°, D_5°, D_6°, D_7°, D_8°, D_9°, D_{10}°, D_{11}°, D_{12}° exist by induction hypothesis;

φ is $\varphi_1 \wedge \varphi_2$. Note that $h_1(\varphi)$ is $(h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot$, $h_2(\varphi)$ is $\neg\neg((h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot)$, $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2))$, and $h_1^l(\varphi)$ is $(h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_1^\circ}{h_2(\varphi_1) \to h_a(\varphi_1)}}{h_2(\varphi_1) \wedge h_2(\varphi_2) \to h_a(\varphi_1)}\,\mathrm{L}_1\wedge \quad \dfrac{\overset{D_2^\circ}{h_2(\varphi_2) \to h_a(\varphi_2)}}{h_2(\varphi_1) \wedge h_2(\varphi_2) \to h_a(\varphi_2)}\,\mathrm{L}_2\wedge}{h_2(\varphi_1) \wedge h_2(\varphi_2) \to h_a(\varphi_1) \wedge h_a(\varphi_2)}\,\mathrm{R}\wedge}{h_2(\varphi_1) \wedge h_2(\varphi_2), \sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to}\,\mathrm{L}\sim \quad \dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to}\,\mathrm{Lw}}{\dfrac{\dfrac{(h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot, \sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to}{\neg\neg((h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot), \sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to}\,\neg^2}{\neg\neg((h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot) \to \sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2))}\,\mathrm{R}\sim}\,\mathrm{L}\vee
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_3^\circ}{h_a(\varphi_1) \to h_2(\varphi_1)}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_2(\varphi_1)}\,\mathrm{L}_1\wedge \quad \dfrac{\overset{D_4^\circ}{h_a(\varphi_2) \to h_2(\varphi_2)}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_2(\varphi_2)}\,\mathrm{L}_2\wedge}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_2(\varphi_1) \wedge h_2(\varphi_2)}\,\mathrm{R}\wedge}{\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to h_2(\varphi_1) \wedge h_2(\varphi_2)}\,\sim^2}{\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to (h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot}\,\mathrm{R}_1\vee}{\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to \neg\neg((h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot)}\,\neg^2
$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_5^\circ}{h_1^l(\varphi_1) \to h_a(\varphi_1)}}{h_1^l(\varphi_1) \wedge h_1^l(\varphi_2) \to h_a(\varphi_1)}\,\mathrm{L}_1\wedge \quad \dfrac{\overset{D_6^\circ}{h_1^l(\varphi_2) \to h_a(\varphi_2)}}{h_1^l(\varphi_1) \wedge h_1^l(\varphi_2) \to h_a(\varphi_2)}\,\mathrm{L}_2\wedge}{h_1^l(\varphi_1) \wedge h_1^l(\varphi_2) \to h_a(\varphi_1) \wedge h_a(\varphi_2)}\,\mathrm{R}\wedge}{(h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)) \wedge \mathbf{1} \to h_a(\varphi_1) \wedge h_a(\varphi_2)}\,\mathrm{L}_1\wedge}{(h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)) \wedge \mathbf{1} \to \sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2))}\,\sim^2
$$

and

$$
\dfrac{\dfrac{\dfrac{\overset{D_7^\circ}{h_a(\varphi_1) \to h_1^l(\varphi_1)}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_1^l(\varphi_1)}\,\mathrm{L}_1\wedge \quad \dfrac{\overset{D_8^\circ}{h_a(\varphi_2) \to h_1^l(\varphi_2)}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_1^l(\varphi_2)}\,\mathrm{L}_2\wedge}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)}\,\mathrm{R}\wedge \quad \dfrac{\dfrac{}{\to \mathbf{1}}\,\mathrm{R}\mathbf{1}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to \mathbf{1}}\,\mathrm{Lw}}{\dfrac{h_a(\varphi_1) \wedge h_a(\varphi_2) \to (h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)) \wedge \mathbf{1}}{\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to (h_1^l(\varphi_1) \wedge h_1^l(\varphi_2)) \wedge \mathbf{1}}\,\sim^2}\,\mathrm{R}\wedge
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_1^l(\varphi)$, and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_9^\circ}{h_2(\varphi_1) \to h_a(\varphi_1)}}{h_2(\varphi_1) \wedge h_2(\varphi_2) \to h_a(\varphi_1)}\,\mathrm{L}_1\wedge \quad \dfrac{\overset{D_{10}^\circ}{h_2(\varphi_2) \to h_a(\varphi_2)}}{h_2(\varphi_1) \wedge h_2(\varphi_2) \to h_a(\varphi_2)}\,\mathrm{L}_2\wedge}{h_2(\varphi_1) \wedge h_2(\varphi_2) \to h_a(\varphi_1) \wedge h_a(\varphi_2)}\,\mathrm{R}\wedge}{h_2(\varphi_1) \wedge h_2(\varphi_2), \sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to}\,\mathrm{L}\sim \quad \dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to}\,\mathrm{Lw}}{\dfrac{(h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot, \sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to}{(h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot \to \sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2))}\,\mathrm{R}\sim}\,\mathrm{L}\vee
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_{11}^\circ}{h_a(\varphi_1) \to h_2(\varphi_1)}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_2(\varphi_1)}\,\mathrm{L}_1\wedge \quad \dfrac{\overset{D_{12}^\circ}{h_a(\varphi_2) \to h_2(\varphi_2)}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_2(\varphi_2)}\,\mathrm{L}_2\wedge}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to h_2(\varphi_1) \wedge h_2(\varphi_2)}\,\mathrm{R}\wedge}{\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to h_2(\varphi_1) \wedge h_2(\varphi_2)}\,\sim^2}{\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to (h_2(\varphi_1) \wedge h_2(\varphi_2)) \vee \bot}\,\mathrm{R}_1\vee
$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and D_1°, D_2°, D_3°, D_4°, D_5°, D_6°, D_7°, D_8°, D_9°, D_{10}°, D_{11}°, D_{12}° exist by induction hypothesis;

φ is $\varphi_1 \vee \varphi_2$. Note that $h_2(\varphi)$ is $\neg\neg((h_2(\varphi_1) \vee h_2(\varphi_2)) \vee \bot)$, $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2))$, $h_1(\varphi)$ is $(h_2(\varphi_1) \vee h_2(\varphi_2)) \vee \bot$, and $h_1^l(\varphi)$ is $(h_1^l(\varphi_1) \vee h_1^l(\varphi_2)) \wedge \mathbf{1}$. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_1^\circ}{h_2(\varphi_1) \to h_a(\varphi_1)}}{h_2(\varphi_1) \to h_a(\varphi_1) \vee h_a(\varphi_2)}\,\mathrm{R}_1\vee \quad \dfrac{\overset{D_2^\circ}{h_2(\varphi_2) \to h_a(\varphi_2)}}{h_2(\varphi_2) \to h_a(\varphi_1) \vee h_a(\varphi_2)}\,\mathrm{R}_2\vee}{h_2(\varphi_1) \vee h_2(\varphi_2) \to h_a(\varphi_1) \vee h_a(\varphi_2)}\,\mathrm{L}\vee}{h_2(\varphi_1) \vee h_2(\varphi_2), \sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to}\,\mathrm{L}\sim \quad \dfrac{\dfrac{}{\bot \to}\,\mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to}\,\mathrm{Lw}}{\dfrac{\dfrac{(h_2(\varphi_1) \vee h_2(\varphi_2)) \vee \bot, \sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to}{\neg\neg((h_2(\varphi_1) \vee h_2(\varphi_2)) \vee \bot), \sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to}\,\neg^2}{\neg\neg((h_2(\varphi_1) \vee h_2(\varphi_2)) \vee \bot) \to \sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2))}\,\mathrm{R}\sim}\,\mathrm{L}\vee
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_3^\circ}{h_a(\varphi_1) \to h_2(\varphi_1)}}{h_a(\varphi_1) \to h_2(\varphi_1) \vee h_2(\varphi_2)}\,\mathrm{R}_1\vee \quad \dfrac{\overset{D_4^\circ}{h_a(\varphi_2) \to h_2(\varphi_2)}}{h_a(\varphi_2) \to h_2(\varphi_1) \vee h_2(\varphi_2)}\,\mathrm{R}_2\vee}{h_a(\varphi_1) \vee h_a(\varphi_2) \to h_2(\varphi_1) \vee h_2(\varphi_2)}\,\mathrm{L}\vee}{\sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to h_2(\varphi_1) \vee h_2(\varphi_2)}\,\sim^2}{\sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to (h_2(\varphi_1) \vee h_2(\varphi_2)) \vee \bot}\,\mathrm{R}_1\vee}{\sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to \neg\neg((h_2(\varphi_1) \vee h_2(\varphi_2)) \vee \bot)}\,\neg^2
$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_5^\circ}{h_1^l(\varphi_1)\to h_a(\varphi_1)}}{h_1^l(\varphi_1)\to h_a(\varphi_1)\vee h_a(\varphi_2)}\ \mathrm{R}_1\vee \qquad \dfrac{\overset{D_6^\circ}{h_1^l(\varphi_2)\to h_a(\varphi_2)}}{h_1^l(\varphi_2)\to h_a(\varphi_1)\vee h_a(\varphi_2)}\ \mathrm{R}_2\vee}{h_1^l(\varphi_1)\vee h_1^l(\varphi_2)\to h_a(\varphi_1)\vee h_a(\varphi_2)}\ \mathrm{L}\vee}{(h_1^l(\varphi_1)\vee h_1^l(\varphi_2))\wedge \mathbf{1}\to h_a(\varphi_1)\vee h_a(\varphi_2)}\ \mathrm{L}_1\wedge}{(h_1^l(\varphi_1)\vee h_1^l(\varphi_2))\wedge\mathbf{1}\to{\sim\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))}\ {\sim}^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_7^\circ}{h_a(\varphi_1)\to h_1^l(\varphi_1)}}{h_a(\varphi_1)\to h_1^l(\varphi_1)\vee h_1^l(\varphi_2)}\ \mathrm{R}_1\vee \qquad \dfrac{\overset{D_8^\circ}{h_a(\varphi_2)\to h_1^l(\varphi_2)}}{h_a(\varphi_2)\to h_1^l(\varphi_1)\vee h_1^l(\varphi_2)}\ \mathrm{R}_2\vee}{h_a(\varphi_1)\vee h_a(\varphi_2)\to h_1^l(\varphi_1)\vee h_1^l(\varphi_2)}\ \mathrm{L}\vee \qquad \dfrac{\dfrac{}{\to\mathbf{1}}\ \mathrm{R}\mathbf{1}}{h_a(\varphi_1)\vee h_a(\varphi_2)\to\mathbf{1}}\ \mathrm{Lw}}{h_a(\varphi_1)\vee h_a(\varphi_2)\to(h_1^l(\varphi_1)\vee h_1^l(\varphi_2))\wedge\mathbf{1}}\ \mathrm{R}\wedge}{{\sim\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))\to(h_1^l(\varphi_1)\vee h_1^l(\varphi_2))\wedge\mathbf{1}}\ {\sim}^2$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_1^l(\varphi)$, and

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_9^\circ}{h_2(\varphi_1)\to h_a(\varphi_1)}}{h_2(\varphi_1)\to h_a(\varphi_1)\vee h_a(\varphi_2)}\ \mathrm{R}_1\vee \qquad \dfrac{\overset{D_{10}^\circ}{h_2(\varphi_2)\to h_a(\varphi_2)}}{h_2(\varphi_2)\to h_a(\varphi_1)\vee h_a(\varphi_2)}\ \mathrm{R}_2\vee}{h_2(\varphi_1)\vee h_2(\varphi_2)\to h_a(\varphi_1)\vee h_a(\varphi_2)}\ \mathrm{L}\vee}{h_2(\varphi_1)\vee h_2(\varphi_2),{\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))\to}\ \mathrm{L}\sim \qquad \dfrac{\dfrac{}{\bot\to}\ \mathrm{L}\bot}{\bot,{\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))\to}\ \mathrm{Lw}}{\dfrac{(h_2(\varphi_1)\vee h_2(\varphi_2))\vee\bot,{\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))\to}{(h_2(\varphi_1)\vee h_2(\varphi_2))\vee\bot\to{\sim\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))}\ \mathrm{R}\sim}\ \mathrm{L}\vee$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_{11}^\circ}{h_a(\varphi_1)\to h_2(\varphi_1)}}{h_a(\varphi_1)\to h_2(\varphi_1)\vee h_2(\varphi_2)}\ \mathrm{R}_1\vee \qquad \dfrac{\overset{D_{12}^\circ}{h_a(\varphi_2)\to h_2(\varphi_2)}}{h_a(\varphi_2)\to h_2(\varphi_1)\vee h_2(\varphi_2)}\ \mathrm{R}_2\vee}{h_a(\varphi_1)\vee h_a(\varphi_2)\to h_2(\varphi_1)\vee h_2(\varphi_2)}\ \mathrm{L}\vee}{{\sim\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))\to h_2(\varphi_1)\vee h_2(\varphi_2)}\ {\sim}^2}{{\sim\sim}\,(h_a(\varphi_1)\vee h_a(\varphi_2))\to(h_2(\varphi_1)\vee h_2(\varphi_2))\vee\bot}\ \mathrm{R}_1\vee$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and D_1°, D_2°, D_3°, D_4°, D_5°, D_6°, D_7°, D_8°, D_9°, D_{10}°, D_{11}°, D_{12}° exist by induction hypothesis;

φ is $\varphi_1\nabla\varphi_2$. Note that $h_2(\varphi)$ is $\neg\neg(((\neg h_2(\varphi_1))\supset h_2(\varphi_2))\vee\bot)$, $h_a(\varphi)$ is ${\sim\sim}\,(({\sim\sim}\,h_a(\varphi_1))\nabla({\sim\sim}\,h_a(\varphi_2)))$, $h_1(\varphi)$ is $((\neg h_2(\varphi_1))\supset h_2(\varphi_2))\vee\bot$, and $h_1^l(\varphi)$ is $((\neg h_1^l(\varphi_1))\supset h_1^l(\varphi_2))\wedge\mathbf{1}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_1^\circ}{h_2(\varphi_1)\to h_a(\varphi_1)}}{{\sim}\,h_a(\varphi_1),h_2(\varphi_1)\to}\ \mathrm{L}\sim}{{\sim}\,h_a(\varphi_1)\to\neg h_2(\varphi_1)}\ \mathrm{R}\neg \qquad \dfrac{\overset{D_2^\circ}{h_2(\varphi_2)\to h_a(\varphi_2)}}{h_2(\varphi_2),{\sim}\,h_a(\varphi_2)\to}\ \mathrm{L}\sim}{(\neg h_2(\varphi_1))\supset h_2(\varphi_2),{\sim}\,h_a(\varphi_1),{\sim}\,h_a(\varphi_2)\to}\ \mathrm{L}\supset}{(\neg h_2(\varphi_1))\supset h_2(\varphi_2)\to{\sim\sim}\,h_a(\varphi_1),{\sim\sim}\,h_a(\varphi_2)}\ \mathrm{R}\,{\sim}^2}{(\neg h_2(\varphi_1))\supset h_2(\varphi_2)\to({\sim\sim}\,h_a(\varphi_1))\nabla({\sim\sim}\,h_a(\varphi_2))}\ \mathrm{R}\nabla}{(\neg h_2(\varphi_1))\supset h_2(\varphi_2),{\sim}\,(({\sim\sim}\,h_a(\varphi_1))\nabla({\sim\sim}\,h_a(\varphi_2)))\to}\ \mathrm{L}\sim \qquad \dfrac{\dfrac{}{\bot\to}\ \mathrm{L}\bot}{\bot,{\sim}\,(h_a(\varphi_1)\nabla h_a(\varphi_2))\to}\ \mathrm{Lw}}{((\neg h_2(\varphi_1))\supset h_2(\varphi_2))\vee\bot,{\sim}\,(({\sim\sim}\,h_a(\varphi_1))\nabla({\sim\sim}\,h_a(\varphi_2)))\to}\ \mathrm{L}\vee}{\neg\neg(((\neg h_2(\varphi_1))\supset h_2(\varphi_2))\vee\bot),{\sim}\,(({\sim\sim}\,h_a(\varphi_1))\nabla({\sim\sim}\,h_a(\varphi_2)))\to}\ \neg^2}{\neg\neg(((\neg h_2(\varphi_1))\supset h_2(\varphi_2))\vee\bot)\to{\sim\sim}\,(({\sim\sim}\,h_a(\varphi_1))\nabla({\sim\sim}\,h_a(\varphi_2)))}\ \mathrm{R}\sim$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_3^\circ}{h_a(\varphi_1) \to h_2(\varphi_1)}}{\sim\sim h_a(\varphi_1), \neg h_2(\varphi_1) \to}\ \sim^2 + \mathrm{L}\neg \qquad \dfrac{\overset{D_4^\circ}{h_a(\varphi_2) \to h_2(\varphi_2)}}{\sim\sim h_a(\varphi_2) \to h_2(\varphi_2)}\ \sim^2}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)), \neg h_2(\varphi_1) \to h_2(\varphi_2)}\ \mathrm{L}\nabla}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to ((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)}\ \mathrm{R}\supset}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to ((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)) \vee \bot}\ \mathrm{R}_1\vee}{\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to ((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)) \vee \bot}\ \sim^2}{\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to \neg\neg(((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)) \vee \bot)}\ \neg^2
$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$,

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_5^\circ}{h_1^l(\varphi_1) \to h_a(\varphi_1)}}{\sim h_a(\varphi_1) \to \neg h_1^l(\varphi_1)}\ \mathrm{R}\neg + \mathrm{L}\sim \qquad \dfrac{\overset{D_6^\circ}{h_1^l(\varphi_2) \to h_a(\varphi_2)}}{h_1^l(\varphi_2) \to\sim\sim h_a(\varphi_2)}\ \sim^2}{(\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2), \sim h_a(\varphi_1) \to\sim\sim h_a(\varphi_2)}\ \mathrm{R}}{(\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2) \to\sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2)}\ \mathrm{R}\sim}{(\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2) \to (\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))}\ \mathrm{R}\nabla}{((\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1} \to (\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))}\ \mathrm{L}_1\wedge}{((\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1} \to\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)))}\ \sim^2
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_7^\circ}{h_a(\varphi_1) \to h_1^l(\varphi_1)}}{\sim\sim h_a(\varphi_1), \neg h_1^l(\varphi_1) \to}\ \sim^2 + \mathrm{L}\neg \qquad \dfrac{\overset{D_8^\circ}{h_a(\varphi_2) \to h_1^l(\varphi_2)}}{\sim\sim h_a(\varphi_2) \to h_1^l(\varphi_2)}\ \sim^2}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)), \neg h_1^l(\varphi_1) \to h_1^l(\varphi_2)}\ \mathrm{L}\nabla}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to (\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2)}\ \mathrm{R}\supset \qquad \dfrac{\dfrac{}{\to \mathbf{1}}\ \mathrm{R}\mathbf{1}}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to \mathbf{1}}\ \mathrm{Lw}}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to ((\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}}\ \mathrm{R}\wedge}{\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))) \to ((\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2)) \wedge \mathbf{1}}\ \sim^2
$$

are derivations for $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_1^l(\varphi)$, and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_9^\circ}{h_2(\varphi_1) \to h_a(\varphi_1)}}{\sim h_a(\varphi_1), h_2(\varphi_1) \to}\ \mathrm{L}\sim}{\sim h_a(\varphi_1) \to \neg h_2(\varphi_1)}\ \mathrm{R}\neg \qquad \dfrac{\overset{D_{10}^\circ}{h_2(\varphi_2) \to h_a(\varphi_2)}}{h_2(\varphi_2), \sim h_a(\varphi_2) \to}\ \mathrm{L}\sim}{(\neg h_2(\varphi_1)) \supset h_2(\varphi_2), \sim h_a(\varphi_1), \sim h_a(\varphi_2) \to}\ \mathrm{L}\supset}{(\neg h_2(\varphi_1)) \supset h_2(\varphi_2) \to\sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2)}\ \mathrm{R}\sim^2}{(\neg h_2(\varphi_1)) \supset h_2(\varphi_2) \to (\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))}\ \mathrm{R}\nabla}{(\neg h_2(\varphi_1)) \supset h_2(\varphi_2), \sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))) \to}\ \mathrm{L}\sim \qquad \dfrac{\dfrac{}{\bot \to}\ \mathrm{L}\bot}{\bot, \sim (h_a(\varphi_1)\nabla h_a(\varphi_2)) \to}\ \mathrm{Lw}}{((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)) \vee \bot, \sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))) \to}\ \mathrm{L}\vee}{((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)) \vee \bot \to\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)))}\ \mathrm{R}\sim
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{D_{11}^\circ}{h_a(\varphi_1) \to h_2(\varphi_1)}}{\sim\sim h_a(\varphi_1), \neg h_2(\varphi_1) \to}\ \sim^2 + \mathrm{L}\neg \qquad \dfrac{\overset{D_{12}^\circ}{h_a(\varphi_2) \to h_2(\varphi_2)}}{\sim\sim h_a(\varphi_2) \to h_2(\varphi_2)}\ \sim^2}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)), \neg h_2(\varphi_1) \to h_2(\varphi_2)}\ \mathrm{L}\nabla}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to ((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)}\ \mathrm{R}\supset}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to ((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)) \vee \bot}\ \mathrm{R}_1\vee}{\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to ((\neg h_2(\varphi_1)) \supset h_2(\varphi_2)) \vee \bot}\ \sim^2
$$

are derivations for $h_1(\varphi) \dashv\vdash^l_{\mathcal{D}_{w+m}} h_a(\varphi)$, and D_1°, D_2°, D_3°, D_4°, D_5°, D_6°, D_7°, D_8°, D_9°, D_{10}°, D_{11}°, D_{12}° exist by induction hypothesis;

2. $\varphi \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$:

The proof follows by complete induction on the complexity of the formula:

φ is $\bot$ or $\bot_m$ or $\mathbf{0}$ or $\mathbf{1}$ or $\mathbf{1}_m$. The thesis follows straightforwardly since $h_a(\varphi) = \varphi$ using Init;

φ is in $P_w \cup P_m$. Note that $h_a(\varphi)$ is $\sim\sim \varphi$. The thesis follows straightforwardly by applying the rules for $\sim$;

φ is $\varphi_1 \circ \varphi_2$ for $\circ$ in $\{\otimes, *\}$. Note that $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\varphi_1 \to h_a(\varphi_1)} \quad \dfrac{D_2^\circ}{\varphi_2 \to h_a(\varphi_2)}}{\varphi_1, \varphi_2 \to h_a(\varphi_1) \circ h_a(\varphi_2)}\,\mathrm{R}\circ}{\varphi_1 \circ \varphi_2 \to h_a(\varphi_1) \circ h_a(\varphi_2)}\,\mathrm{L}\circ}{\varphi_1 \circ \varphi_2 \to \sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2))}\,\sim^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_a(\varphi_1) \to \varphi_1} \quad \dfrac{D_4^\circ}{h_a(\varphi_2) \to \varphi_2}}{h_a(\varphi_1), h_a(\varphi_2) \to \varphi_1 \circ \varphi_2}\,\mathrm{R}\circ}{h_a(\varphi_1) \circ h_a(\varphi_2) \to \varphi_1 \circ \varphi_2}\,\mathrm{L}\circ}{\sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2)) \to \varphi_1 \circ \varphi_2}\,\sim^2$$

are derivations for $\varphi \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \supset \varphi_2$. Note that $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_a(\varphi_1) \to \varphi_1} \quad \dfrac{D_2^\circ}{\varphi_2 \to h_a(\varphi_2)}}{\varphi_1 \supset \varphi_2, h_a(\varphi_1) \to h_a(\varphi_2)}\,\mathrm{L}\supset}{\varphi_1 \supset \varphi_2 \to h_a(\varphi_1) \supset h_a(\varphi_2)}\,\mathrm{R}\supset}{\varphi_1 \supset \varphi_2 \to \sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2))}\,\sim^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{\varphi_1 \to h_a(\varphi_1)} \quad \dfrac{D_4^\circ}{h_a(\varphi_2) \to \varphi_2}}{h_a(\varphi_1) \supset h_a(\varphi_2), \varphi_1 \to \varphi_2}\,\mathrm{L}\supset}{h_a(\varphi_1) \supset h_a(\varphi_2) \to \varphi_1 \supset \varphi_2}\,\mathrm{R}\supset}{\sim\sim (h_a(\varphi_1) \supset h_a(\varphi_2)) \to \varphi_1 \supset \varphi_2}\,\sim^2$$

are derivations for $\varphi \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \wedge \varphi_2$. Note that $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\varphi_1 \to h_a(\varphi_1)}}{\varphi_1 \wedge \varphi_2 \to h_a(\varphi_1)}\,\mathrm{L}_1\wedge \quad \dfrac{\dfrac{D_2^\circ}{\varphi_2 \to h_a(\varphi_2)}}{\varphi_1 \wedge \varphi_2 \to h_a(\varphi_2)}\,\mathrm{L}_2\wedge}{\varphi_1 \wedge \varphi_2 \to h_a(\varphi_1) \wedge h_a(\varphi_2)}\,\mathrm{R}\wedge}{\varphi_1 \wedge \varphi_2 \to \sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2))}\,\sim^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_a(\varphi_1) \to \varphi_1}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to \varphi_1}\,\mathrm{L}_1\wedge \quad \dfrac{\dfrac{D_4^\circ}{h_a(\varphi_2) \to \varphi_2}}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to \varphi_2}\,\mathrm{L}_2\wedge}{h_a(\varphi_1) \wedge h_a(\varphi_2) \to \varphi_1 \wedge \varphi_2}\,\mathrm{R}\wedge}{\sim\sim (h_a(\varphi_1) \wedge h_a(\varphi_2)) \to \varphi_1 \wedge \varphi_2}\,\sim^2$$

are derivations for $\varphi \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \vee \varphi_2$. Note that $h_a(\varphi)$ is $\sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\varphi_1 \to h_a(\varphi_1)}}{\varphi_1 \to h_a(\varphi_1) \vee h_a(\varphi_2)}\,\mathrm{R}_1\vee \quad \dfrac{\dfrac{D_2^\circ}{\varphi_2 \to h_a(\varphi_2)}}{\varphi_2 \to h_a(\varphi_1) \vee h_a(\varphi_2)}\,\mathrm{R}_2\vee}{\varphi_1 \vee \varphi_2 \to h_a(\varphi_1) \vee h_a(\varphi_2)}\,\mathrm{L}\vee}{\varphi_1 \vee \varphi_2 \to \sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2))}\,\sim^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_a(\varphi_1) \to \varphi_1}}{h_a(\varphi_1) \to \varphi_1 \vee \varphi_2}\,\mathrm{R}_1\vee \quad \dfrac{\dfrac{D_4^\circ}{h_a(\varphi_2) \to \varphi_2}}{h_a(\varphi_2) \to \varphi_1 \vee \varphi_2}\,\mathrm{R}_2\vee}{h_a(\varphi_1) \vee h_a(\varphi_2) \to \varphi_1 \vee \varphi_2}\,\mathrm{L}\vee}{\sim\sim (h_a(\varphi_1) \vee h_a(\varphi_2)) \to \varphi_1 \vee \varphi_2}\,\sim^2$$

are derivations for $\varphi \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \nabla \varphi_2$. Note that $h_a(\varphi)$ is $\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\varphi_1 \to h_a(\varphi_1)}}{\varphi_1 \to \sim\sim h_a(\varphi_1)}\,\sim^2 \quad \dfrac{\dfrac{D_2^\circ}{\varphi_2 \to h_a(\varphi_2)}}{\varphi_2 \to \sim\sim h_a(\varphi_2)}\,\sim^2}{\varphi_1\nabla\varphi_2 \to \sim\sim h_a(\varphi_1), \sim\sim h_a(\varphi_2)}\,\mathrm{L}\nabla}{\varphi_1\nabla\varphi_2 \to (\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))}\,\mathrm{R}\nabla}{\varphi_1\nabla\varphi_2 \to \sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)))}\,\sim^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_a(\varphi_1) \to \varphi_1}}{\sim\sim h_a(\varphi_1) \to \varphi_1}\,\sim^2 \quad \dfrac{\dfrac{D_4^\circ}{h_a(\varphi_2) \to \varphi_2}}{\sim\sim h_a(\varphi_2) \to \varphi_2}\,\sim^2}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to \varphi_1, \varphi_2}\,\mathrm{L}\nabla}{(\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)) \to \varphi_1\nabla\varphi_2}\,\mathrm{R}\nabla}{\sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2))) \to \varphi_1\nabla\varphi_2}\,\sim^2$$

are derivations for $\varphi \dashv\vdash^{l}_{\mathcal{D}_{w+m}} h_a(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\sim \varphi_1$. Note that $h_a(\varphi)$ is $\sim h_a(\varphi_1)$. We omit the proof of this case since it follows straightforwardly. $\diamond$

Lemma 5.15 The pair of maps h_1 and h_2 is such that $h_1(\Psi) \vdash^{l}_{\mathcal{D}_i} h_2(\varphi)$ whenever $\Psi \vdash^{l}_{\mathcal{D}_{i+c}} \varphi$, and Ψ and $\{\varphi\}$ are contained in L_{i+c}.

Proof: We show by complete induction on the depth of a sequent derivation that if $\Psi \to \Delta$ is a theorem in $\mathcal{D}_{i+c}$ then $h_1(\Psi), \neg_i h_1(\Delta) \to$ is a theorem in $\mathcal{D}_i$. Let

$$\dfrac{\dfrac{D_1}{\Psi_1 \to \Delta_1} \quad \ldots \quad \dfrac{D_k}{\Psi_k \to \Delta_k}}{\Psi \to \Delta}\, r$$

be a derivation in $\mathcal{D}_{i+c}$ where k is greater than or equal to 0. Consider the following cases:

r is Ax. Then

$$\dfrac{\overline{h_1(\delta) \to h_1(\delta)}\ \text{Ax}}{h_1(\delta), \neg_i h_1(\delta) \to}\ \text{L}\neg_i$$

is a derivation for $h_1(\Psi), \neg_i h_1(\Delta) \to$.

r is L$\bot_c$ or L$\bot_i$. Then $\bot_i \to$ is a theorem in $\mathcal{D}_i$.

r is Rw, Rc, Lw, Lc. Straightforward by the induction hypothesis.

r is R$\Rightarrow_c$. Denote by $\varphi_1 \Rightarrow_c \varphi_2$ the formula used by r and by Δ' the multiset Δ without that formula. Then consider

$$\dfrac{\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1(\varphi_1), h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi_2) \to \end{matrix}}{\neg_i\neg_i h_1(\varphi_1), h_1(\Psi), \neg_i h_1(\Delta') \to \neg_i\neg_i h_1(\varphi_2)}\ \neg_i{}^3}{h_1(\Psi), \neg_i h_1(\Delta') \to h_1(\varphi)}\ \text{R}\Rightarrow_i}{h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi) \to}\ \text{L}\neg_i$$

where D_1° exists by the induction hypothesis.

r is L$\Rightarrow_c$. Denote by φ the formula $\varphi_1 \Rightarrow_c \varphi_2$ introduced by r and by Ψ' the multiset Ψ without that formula. Then consider

$$\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1(\Psi'), \neg_i h_1(\Delta), \neg_i h_1(\varphi_1) \to \end{matrix}}{h_1(\Psi'), \neg_i h_1(\Delta) \to \neg_i\neg_i h_1(\varphi_1)}\ \text{R}\neg_i \qquad \dfrac{\begin{matrix} D_2^\circ \\ h_1(\varphi_2), h_1(\Psi'), \neg_i h_1(\Delta) \to \end{matrix}}{\neg_i\neg_i h_1(\varphi_2), h_1(\Psi'), \neg_i h_1(\Delta) \to}\ \neg_i{}^2}{h_1(\Psi'), h_1(\varphi), \neg_i h_1(\Delta) \to}\ \text{L}\Rightarrow_i$$

where D_1° and D_2° exist by induction hypothesis.

r is R$\wedge_i$. Denote by φ the formula $\varphi_1 \wedge_i \varphi_2$ introduced by r and by Δ' the multiset Δ without that formula. Then consider

$$\dfrac{\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi_1) \to \end{matrix}}{h_1(\Psi), \neg_i h_1(\Delta') \to \neg_i\neg_i h_1(\varphi_1)}\ \text{R}\neg_i \qquad \dfrac{\begin{matrix} D_2^\circ \\ h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi_2) \to \end{matrix}}{h_1(\Psi), \neg_i h_1(\Delta') \to \neg_i\neg_i h_1(\varphi_2)}\ \text{R}\neg_i}{h_1(\Psi), \neg_i h_1(\Delta') \to h_1(\varphi)}\ \text{R}\wedge_i}{h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi) \to}\ \text{L}\neg_i$$

where D_1° and D_2° exist by the induction hypothesis.

r is L$(\wedge_i)_j$ with $j = 1, 2$. Denote by φ the formula $\varphi_1 \wedge_i \varphi_2$ introduced by r and by Ψ' the multiset Ψ without that formula. Then consider

$$\dfrac{\dfrac{\begin{matrix} D_1^\circ \\ h_1(\varphi_j), h_1(\Psi'), \neg_i h_1(\Delta) \to \end{matrix}}{\neg_i\neg_i h_1(\varphi_j), h_1(\Psi'), \neg_i h_1(\Delta) \to}\ \neg_i{}^2}{h_1(\Psi'), h_1(\varphi), \neg_i h_1(\Delta) \to}\ \text{L}(\wedge_i)_j$$

where D_1° exists by the induction hypothesis.

r is R$(\vee_i)_j$ with $j = 1, 2$. Denote by φ the formula $\varphi_1 \vee_i \varphi_2$ introduced by r and by Δ' the multiset Δ without that formula. Then consider

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi_j) \rightarrow\end{array}}{h_1(\Psi), \neg_i h_1(\Delta') \rightarrow \neg_i\neg_i h_1(\varphi_j)}\,\mathrm{R}\neg_i}{h_1(\Psi), \neg_i h_1(\Delta') \rightarrow h_1(\varphi)}\,\mathrm{R}(\vee_i)_j}{h_1(\Psi), \neg_i h_1(\Delta'), \neg_i\, h_1(\varphi) \rightarrow}\,\mathrm{L}\neg_i
$$

where D_1° exists by the induction hypothesis.

r is $\mathrm{L}\vee_i$. Denote by φ the formula $\varphi_1 \vee_i \varphi_2$ introduced by r and by Ψ' the multiset Ψ without that formula. Then taking into account the induction hypothesis

$$
\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\varphi_1), h_1(\Psi'), \neg_i h_1(\Delta) \rightarrow\end{array}}{\neg_i\neg_i h_1(\varphi_1), h_1(\Psi'), \neg_i h_1(\Delta) \rightarrow}\,\neg_i{}^2 \qquad \dfrac{\begin{array}{c}D_2^\circ\\ h_1(\varphi_2), h_1(\Psi'), \neg_i h_1(\Delta) \rightarrow\end{array}}{\neg_i\neg_i h_1(\varphi_2), h_1(\Psi'), \neg_i h_1(\Delta) \rightarrow}\,\neg_i{}^2}{h_1(\Psi'), h_1(\varphi), \neg_i h_1(\Delta) \rightarrow}\,\mathrm{L}\vee_i
$$

where D_1° and D_2° exist by the induction hypothesis.

r is $\mathrm{R}{\Rightarrow_i}$. Denote by φ the formula $\varphi_1 \Rightarrow_i \varphi_2$ introduced by r and by Δ' the multiset Δ without that formula. Then consider

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\varphi_1), h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi_2) \rightarrow\end{array}}{\neg_i\neg_i h_1(\varphi_1), h_1(\Psi), \neg_i h_1(\Delta') \rightarrow \neg_i\neg_i h_1(\varphi_2)}\,\neg_i{}^3}{h_1(\Psi), \neg_i h_1(\Delta') \rightarrow h_1(\varphi)}\,\mathrm{R}{\Rightarrow_i}}{h_1(\Psi), \neg_i h_1(\Delta'), \neg_i h_1(\varphi) \rightarrow}\,\mathrm{L}\neg_i
$$

where D_1° exists by the induction hypothesis.

r is $\mathrm{L}{\Rightarrow_i}$. Denote by φ the formula $\varphi_1 \Rightarrow_i \varphi_2$ introduced by r and by Ψ' the multiset Ψ without that formula. Then consider

$$
\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_1(\Psi'), \neg_i h_1(\Delta), \neg_i h_1(\varphi_1) \rightarrow\end{array}}{h_1(\Psi'), \neg_i h_1(\Delta) \rightarrow \neg_i\neg_i h_1(\varphi_1)}\,\mathrm{R}\neg_i \qquad \dfrac{\begin{array}{c}D_2^\circ\\ h_1(\varphi_2), h_1(\Psi'), \neg_i h_1(\Delta) \rightarrow\end{array}}{\neg_i\neg_i h_1(\varphi_2), h_1(\Psi'), \neg_i h_1(\Delta) \rightarrow}\,\neg_i{}^2}{h_1(\Psi'), h_1(\varphi), \neg_i h_1(\Delta) \rightarrow}\,\mathrm{L}{\Rightarrow_i}
$$

where D_1° and D_2° exist by the induction hypothesis. ◇

Lemma 5.16 The maps h_c, h_1 and h_2 are such that

- $\varphi \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$;
- $h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi)$;
- $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$;

for φ in L_{i+c}.

Proof:
1. $(\varphi \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi))$

The proof follows by complete induction on the complexity of the formula:

φ is either $\bot_c$ or $\bot_i$. Note that $h_c(\varphi)$ is φ. Straightforward.

φ is in P. Note that $h_c(\varphi)$ is $\neg_c\neg_c\varphi$. Then

$$\dfrac{\dfrac{}{\varphi \to \varphi}\;\text{Ax}}{\neg_c\neg_c\varphi \to \varphi}\;{\neg_c}^2$$

is a derivation for $h_c(\varphi) \vdash^l_{\mathcal{D}_{i+_\iota c}} \varphi$. The derivation for the other direction follows analogously;

φ is $\varphi_1 \Rightarrow_c \varphi_2$. Note that $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ \varphi_1 \to h_c(\varphi_1)\end{array}}{\varphi_1 \to \neg_c\neg_c h_c(\varphi_1)}\;{\neg_c}^2}{\varphi_1 \to \neg_c\neg_c h_c(\varphi_1), \varphi_2}\;\text{Rw} \qquad \dfrac{\dfrac{\begin{array}{c}D_2^\circ\\ h_c(\varphi_2) \to \varphi_2\end{array}}{\neg_c\neg_c h_c(\varphi_2) \to \varphi_2}\;{\neg_c}^2}{\varphi_1, \neg_c\neg_c h_c(\varphi_2) \to \varphi_2}\;\text{Lw}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c \neg_c\neg_c h_c(\varphi_2), \varphi_1 \to \varphi_2}\;\text{L}{\Rightarrow_c}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2)) \to \varphi_1 \Rightarrow_c \varphi_2}\;\text{R}{\Rightarrow_c}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_3^\circ\\ h_c(\varphi_1) \to \varphi_1\end{array}}{h_c(\varphi_1) \to \varphi_1, h_c(\varphi_2)}\;\text{Rw} \qquad \dfrac{\begin{array}{c}D_4^\circ\\ \varphi_2 \to h_c(\varphi_2)\end{array}}{\varphi_2, h_c(\varphi_1) \to h_c(\varphi_2)}\;\text{Lw}}{\varphi_1 \Rightarrow_c \varphi_2, h_c(\varphi_1) \to h_c(\varphi_2)}\;\text{L}{\Rightarrow_c}}{\varphi_1 \Rightarrow_c \varphi_2, \neg_c\neg_c h_c(\varphi_1) \to \neg_c\neg_c h_c(\varphi_2)}\;{\neg_c}^4}{\varphi_1 \Rightarrow_c \varphi_2 \to (\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2))}\;\text{R}{\Rightarrow_c}$$

are derivations for $\varphi \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \wedge_i \varphi_2$. Note that $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_c(\varphi_1) \to \varphi_1\end{array}}{\neg_c\neg_c h_c(\varphi_1) \to \varphi_1}\;{\neg_c}^2}{(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2)) \to \varphi_1}\;\text{L}(\wedge_i)_1 \qquad \dfrac{\dfrac{\begin{array}{c}D_2^\circ\\ h_c(\varphi_2) \to \varphi_2\end{array}}{\neg_c\neg_c h_c(\varphi_2) \to \varphi_2}\;{\neg_c}^2}{(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2)) \to \varphi_2}\;\text{L}(\wedge_i)_2}{(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2)) \to \varphi_1 \wedge_i \varphi_2}\;\text{R}\wedge_i$$

and

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}D_3^\circ\\ \varphi_1 \to h_c(\varphi_1)\end{array}}{\varphi_1 \wedge_1 \varphi_2 \to h_c(\varphi_1)}\;\text{L}(\wedge_i)_1}{\varphi_1 \wedge_1 \varphi_2 \to \neg_c\neg_c h_c(\varphi_1)}\;{\neg_c}^2 \qquad \dfrac{\dfrac{\begin{array}{c}D_4^\circ\\ \varphi_2 \to h_c(\varphi_2)\end{array}}{\varphi_1 \wedge_1 \varphi_2 \to h_c(\varphi_2)}\;\text{L}(\wedge_i)_2}{\varphi_1 \wedge_1 \varphi_2 \to \neg_c\neg_c h_c(\varphi_2)}\;{\neg_c}^2}{\varphi_1 \wedge_1 \varphi_2 \to (\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2))}\;\text{R}\wedge_i$$

are derivations for $\varphi \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \vee_i \varphi_2$. Note that $h_c(\varphi)$ is $\neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1^\circ\\ h_c(\varphi_1) \to \varphi_1\end{array}}{\neg_c\neg_c h_c(\varphi_1) \to \varphi_1}\;{\neg_c}^2}{\neg_c\neg_c h_c(\varphi_1) \to \varphi_1 \vee_i \varphi_2}\;\text{R}(\vee_i)_1 \qquad \dfrac{\dfrac{\begin{array}{c}D_2^\circ\\ h_c(\varphi_2) \to \varphi_2\end{array}}{\neg_c\neg_c h_c(\varphi_2) \to \varphi_2}\;{\neg_c}^2}{\neg_c\neg_c h_c(\varphi_2) \to \varphi_1 \vee_i \varphi_2}\;\text{R}(\vee_i)_2}{(\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)) \to \varphi_1 \vee_i \varphi_2}\;\text{L}\vee_i}{\neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))) \to \varphi_1 \vee_i \varphi_2}\;{\neg_c}^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{\varphi_1 \to h_c(\varphi_1)}}{\varphi_1 \to \neg_c\neg_c h_c(\varphi_1)}\;{\neg_c}^2}{\varphi_1 \to (\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))}\;\mathrm{R}(\vee_i)_1 \qquad \dfrac{\dfrac{\dfrac{D_4^\circ}{\varphi_2 \to h_c(\varphi_2)}}{\varphi_2 \to \neg_c\neg_c h_c(\varphi_2)}\;{\neg_c}^2}{\varphi_2 \to (\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))}\;\mathrm{R}(\vee_i)_2}{\dfrac{\varphi_1 \vee_i \varphi_2 \to (\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))}{\varphi_1 \vee_i \varphi_2 \to \neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)))}\;{\neg_c}^2}\;\mathrm{L}\vee_i$$

are derivations for $\varphi \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \Rightarrow_i \varphi_2$. Note that $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\varphi_1 \to h_c(\varphi_1)}}{\varphi_1 \to \neg_c\neg_c h_c(\varphi_1)}\;{\neg_c}^2 \qquad \dfrac{\dfrac{\dfrac{D_2^\circ}{h_c(\varphi_2) \to \varphi_2}}{\neg_c\neg_c h_c(\varphi_2) \to \varphi_2}\;{\neg_c}^2}{\varphi_1, \neg_c\neg_c h_c(\varphi_2) \to \varphi_2}\;\mathrm{Lw}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)), \varphi_1 \to \varphi_2}\;\mathrm{L}{\Rightarrow_i}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)) \to \varphi_1 \Rightarrow_i \varphi_2}\;\mathrm{R}{\Rightarrow_i}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_c(\varphi_1) \to \varphi_1} \qquad \dfrac{D_4^\circ}{\varphi_2 \to h_c(\varphi_2)}}{\varphi_1 \Rightarrow_i \varphi_2, h_c(\varphi_1) \to h_c(\varphi_2)}\;\mathrm{L}{\Rightarrow_i}}{\varphi_1 \Rightarrow_i \varphi_2, \neg_c\neg_c h_c(\varphi_1) \to \neg_c\neg_c h_c(\varphi_2)}\;{\neg_c}^4}{\varphi_1 \Rightarrow_i \varphi_2 \to (\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2))}\;\mathrm{R}{\Rightarrow_i}$$

are derivations for $\varphi \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

2. $(h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi))$

The proof follows by complete induction on the complexity of the formula:

φ is $\bot_c$ or $\bot_i$. Note that $h_c(\varphi)$ is φ and $h_1(\varphi)$ is $\bot_i$, so the proof of this case follows straightforwardly.

φ is in P. Note that $h_1(\varphi)$ is $\neg_i\neg_i\iota(\varphi)$ and $h_c(\varphi)$ is $\neg_c\neg_c\varphi$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\iota(\varphi) \to \varphi}\;\mathrm{L}\iota}{\iota(\varphi), \neg_c\varphi \to}\;\mathrm{L}\neg_c}{\neg_i\neg_i\iota(\varphi), \neg_c\varphi \to}\;{\neg_i}^2}{\neg_i\neg_i\iota(\varphi) \to \neg_c\neg_c\varphi}\;\mathrm{R}\neg_c \qquad\qquad \dfrac{\dfrac{}{\varphi \to \neg_i\neg_i\iota(\varphi)}\;\mathrm{R}\iota}{\neg_c\neg_c\varphi \to \neg_i\neg_i\iota(\varphi)}\;{\neg_c}^2$$

are derivations for $h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi)$;

φ is $\varphi_1 \Rightarrow_c \varphi_2$. Observe that $h_1(\varphi)$ is $(\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2))$ and $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_c(\varphi_1) \to h_1(\varphi_1)}}{\neg_c\neg_c h_c(\varphi_1) \to \neg_i\neg_i h_1(\varphi_1)}\;{\neg_i}^2 + {\neg_c}^2}{\neg_c\neg_c h_c(\varphi_1), \neg_c h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_1)}\;\mathrm{Lw} \qquad \dfrac{\dfrac{\dfrac{D_2^\circ}{h_1(\varphi_2) \to h_c(\varphi_2)}}{\neg_i\neg_i h_1(\varphi_2), \neg_c h_c(\varphi_2) \to}\;{\neg_i}^2 + \mathrm{L}\neg_c}{\neg_i\neg_i h_1(\varphi_2), \neg_c\neg_c h_c(\varphi_1), \neg_c h_c(\varphi_2) \to}\;\mathrm{Lw}}{(\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2)), \neg_c\neg_c h_c(\varphi_1), \neg_c h_c(\varphi_2) \to}\;\mathrm{L}{\Rightarrow_i}}{(\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2)), \neg_c\neg_c h_c(\varphi_1) \to \neg_c\neg_c h_c(\varphi_2)}\;\mathrm{R}\neg_c}{(\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2)) \to (\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2))}\;\mathrm{R}{\Rightarrow_c}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_1(\varphi_1) \to h_c(\varphi_1)}}{\neg_i\neg_i h_1(\varphi_1) \to \neg_c\neg_c h_c(\varphi_1)}\ {\neg_i}^2+{\neg_c}^2}{\neg_i\neg_i h_1(\varphi_1) \to \neg_i\neg_i h_1(\varphi_2), \neg_c\neg_c h_c(\varphi_1)}\ \text{Rw} \qquad \dfrac{\dfrac{\dfrac{D_4^\circ}{h_c(\varphi_2) \to h_1(\varphi_2)}}{\neg_c\neg_c h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_2)}\ {\neg_i}^2+{\neg_c}^2}{\neg_c\neg_c h_c(\varphi_2), \neg_i\neg_i h_1(\varphi_1) \to \neg_i\neg_i h_1(\varphi_2)}\ \text{Lw}}{\dfrac{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2)), \neg_i\neg_i h_1(\varphi_1) \to \neg_i\neg_i h_1(\varphi_2)}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2)) \to (\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2))}\ \text{R}{\Rightarrow_i}}\ \text{L}{\Rightarrow_c}$$

are derivations for $h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \wedge_i \varphi_2$. Note that $h_1(\varphi)$ is $(\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))$ and $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_1(\varphi_1) \to h_c(\varphi_1)}}{\neg_i\neg_i h_1(\varphi_1) \to \neg_c\neg_c h_c(\varphi_1)}\ {\neg_c}^2+{\neg_i}^2}{\neg_i\neg_i h_1(\varphi_1), \neg_i\neg_i h_1(\varphi_2) \to \neg_c\neg_c h_c(\varphi_1)}\ \text{Lw} \qquad \vdots}{\neg_i\neg_i h_1(\varphi_1), \neg_i\neg_i h_1(\varphi_2) \to (\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2))}\ \text{R}{\wedge_i}}{(\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2)) \to (\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2))}\ \text{L}{\wedge_i}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_c(\varphi_1) \to h_1(\varphi_1)}}{h_c(\varphi_1) \to \neg_i\neg_i h_1(\varphi_1)}\ {\neg_i}^2}{h_c(\varphi_1), h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_1)}\ \text{Lw} \qquad \vdots}{h_c(\varphi_1), h_c(\varphi_2) \to (\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))}\ \text{R}{\wedge_i}}{\neg_c\neg_c h_c(\varphi_1), \neg_c\neg_c h_c(\varphi_2) \to (\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))}\ {\neg_c}^4}{(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2)) \to (\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))}\ \text{L}{\wedge_i}$$

are derivations for $h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \vee_i \varphi_2$. Observe that $h_1(\varphi)$ is $(\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2))$ and the image of φ by h_c is $\neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_c(\varphi_1) \to h_1(\varphi_1)}}{\neg_c\neg_c h_c(\varphi_1) \to \neg_i\neg_i h_1(\varphi_1)}\ {\neg_c}^2+{\neg_i}^2}{\neg_c\neg_c h_c(\varphi_1) \to (\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2))}\ \text{R}(\vee_i)_2 \qquad \vdots}{(\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)) \to (\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2))}\ \text{L}{\vee_i}}{\neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))) \to (\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2))}\ \neg_c^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_1(\varphi_1) \to h_c(\varphi_1)}}{\neg_i\neg_i h_1(\varphi_1) \to \neg_c\neg_c h_c(\varphi_1)}\ {\neg_c}^2+{\neg_i}^2}{\neg_i\neg_i h_1(\varphi_1) \to (\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))}\ \text{L}(\vee_i)_2 \qquad \vdots}{(\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2)) \to (\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))}\ \text{L}{\vee_i}}{(\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2)) \to \neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)))}\ {\neg_c}^2$$

are derivations for $h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \Rightarrow_i \varphi_2$. Observe that $h_1(\varphi)$ is $(\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2))$ and $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\vdots}{(\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2)), \neg_c\neg_c h_c(\varphi_1) \to \neg_c\neg_c h_c(\varphi_2)}\ \mathrm{R}\neg_c}{(\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2)) \to (\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2))}\ \mathrm{R}{\Rightarrow_i}$$

(the rest of the derivation is similar to the case when φ is $\varphi_1 \Rightarrow_c \varphi_2$) and

$$\dfrac{\dfrac{\dfrac{\begin{matrix}D_1^\circ\\ h_1(\varphi_1) \to h_c(\varphi_1)\end{matrix}}{\neg_i\neg_i h_1(\varphi_1) \to \neg_c\neg_c h_c(\varphi_1)}\ {\neg_c}^2 + {\neg_i}^2 \qquad \dfrac{\dfrac{\begin{matrix}D_2^\circ\\ h_c(\varphi_2) \to h_1(\varphi_2)\end{matrix}}{\neg_c\neg_c h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_2)}\ {\neg_c}^2 + {\neg_i}^2}{\neg_c\neg_c h_c(\varphi_2), \neg_i\neg_i h_1(\varphi_1) \to \neg_i\neg_i h_1(\varphi_2)}\ \mathrm{Lw}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)), \neg_i\neg_i h_1(\varphi_1) \to \neg_i\neg_i h_1(\varphi_2)}\ \mathrm{L}{\Rightarrow_i}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)) \to (\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2))}\ \mathrm{R}{\Rightarrow_i}$$

are derivations for $h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi)$ where D_1° and D_2° exist by induction hypothesis;

3. $(h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi))$

The proof follows by complete induction on the complexity of the formula:

φ is $\bot_c$ or $\bot_i$. Note that $h_c(\varphi)$ is φ and $h_2(\varphi)$ is $\neg_i\neg_i\bot_i$. Then

$$\dfrac{\dfrac{\dfrac{}{\bot_i \to}\ \mathrm{L}\bot_i}{\neg_i\neg_i\bot_i \to}\ {\neg_i}^2}{\neg_i\neg_i\bot_i \to \varphi}\ \mathrm{Rw} \qquad\qquad \dfrac{\dfrac{}{\varphi \to}\ \mathrm{L}\varphi}{\varphi \to \neg_i\neg_i\bot_i}\ \mathrm{Rw}$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$;

φ is in P. Note that $h_2(\varphi)$ is $\neg_i\neg_i\neg_i\neg_i\iota(\varphi)$ and $h_c(\varphi)$ is $\neg_c\neg_c\varphi$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\iota(\varphi) \to \varphi}\ \mathrm{L}\iota}{\iota(\varphi), \neg_c\varphi \to}\ \mathrm{L}\neg_c}{\neg_i\neg_i\neg_i\neg_i\iota(\varphi), \neg_c\varphi \to}\ {\neg_i}^4}{\neg_i\neg_i\neg_i\neg_i\iota(\varphi) \to \neg_c\neg_c\varphi}\ \mathrm{R}\neg_c \qquad\qquad \dfrac{\dfrac{}{\varphi \to \neg_i\neg_i\iota(\varphi)}\ \mathrm{R}\iota}{\neg_c\neg_c\varphi \to \neg_i\neg_i\neg_i\neg_i\iota(\varphi)}\ {\neg_i}^2 + {\neg_c}^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$;

φ is $\varphi_1 \Rightarrow_c \varphi_2$. Note that $h_2(\varphi)$ is $\neg_i\neg_i(h_2(\varphi_1) \Rightarrow_i h_2(\varphi_2))$ and $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}D_1^\circ\\ h_c(\varphi_1) \to h_2(\varphi_1)\end{matrix}}{\neg_c\neg_c h_c(\varphi_1) \to h_2(\varphi_1)}\ {\neg_c}^2}{\neg_c\neg_c h_c(\varphi_1), \neg_c h_c(\varphi_2) \to h_2(\varphi_1)}\ \mathrm{Lw} \qquad \dfrac{\dfrac{\begin{matrix}D_2^\circ\\ h_2(\varphi_2) \to h_c(\varphi_2)\end{matrix}}{h_2(\varphi_2), \neg_c h_c(\varphi_2) \to}\ \mathrm{L}\neg_c}{h_2(\varphi_2), \neg_c\neg_c h_c(\varphi_1), \neg_c h_c(\varphi_2) \to}\ \mathrm{Lw}}{(h_2(\varphi_1)) \Rightarrow_i (h_2(\varphi_2)), \neg_c\neg_c h_c(\varphi_1), \neg_c h_c(\varphi_2) \to}\ \mathrm{L}{\Rightarrow_i}}{\neg_i\neg_i((h_2(\varphi_1)) \Rightarrow_i (h_2(\varphi_2))), \neg_c\neg_c h_c(\varphi_1), \neg_c h_c(\varphi_2) \to}\ \neg_i^2}{\neg_i\neg_i((h_2(\varphi_1)) \Rightarrow_i (h_2(\varphi_2))), \neg_c\neg_c h_c(\varphi_1) \to \neg_c\neg_c h_c(\varphi_2)}\ \mathrm{R}\neg_c}{\neg_i\neg_i((h_2(\varphi_1)) \Rightarrow_i (h_2(\varphi_2))) \to (\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2))}\ \mathrm{R}{\Rightarrow_c}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_2(\varphi_1) \to h_c(\varphi_1)}}{h_2(\varphi_1) \to \neg_c\neg_c h_c(\varphi_1)}\,\neg_c{}^2}{h_2(\varphi_1) \to h_2(\varphi_2), \neg_c\neg_c h_c(\varphi_1)}\,\text{Rw} \qquad \dfrac{\dfrac{\dfrac{D_4^\circ}{h_c(\varphi_2) \to h_2(\varphi_2)}}{\neg_c\neg_c h_c(\varphi_2) \to h_2(\varphi_2)}\,\neg_c{}^2}{\neg_c\neg_c h_c(\varphi_2), h_2(\varphi_1) \to h_2(\varphi_2)}\,\text{Lw}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2)), h_2(\varphi_1) \to h_2(\varphi_2)}\,\text{L}{\Rightarrow_c}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2)) \to (h_2(\varphi_1)) \Rightarrow_i (h_2(\varphi_2))}\,\text{R}{\Rightarrow_i}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c\neg_c h_c(\varphi_2)) \to \neg_i\neg_i((h_2(\varphi_1)) \Rightarrow_i (h_2(\varphi_2)))}\,\neg_i^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \wedge_i \varphi_2$. Note that $h_2(\varphi)$ is $\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2)))$ and $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\neg_i\neg_i h_1(\varphi_1) \to h_c(\varphi_1)}}{\neg_i\neg_i h_1(\varphi_1), \neg_c h_c(\varphi_1) \to}\,\text{L}\neg_c}{(\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2)), \neg_c h_c(\varphi_1) \to}\,\text{L}(\wedge_i)_2}{\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))) \to \neg_c\neg_c h_c(\varphi_1)}\,\text{R}\neg_c + \neg_i{}^2 \qquad \vdots}{\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))) \to ((\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2)))}\,\text{R}\wedge_i$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{h_c(\varphi_1) \to \neg_i\neg_i h_1(\varphi_1)}}{h_c(\varphi_1), h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_1)}\,\text{Lw} \qquad \dfrac{\dfrac{D_4^\circ}{h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_2)}}{h_c(\varphi_1), h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_2)}\,\text{Lw}}{h_c(\varphi_1), h_c(\varphi_2) \to (\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))}\,\text{R}\wedge_i}{\neg_c\neg_c h_c(\varphi_1), \neg_c\neg_c h_c(\varphi_2) \to (\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))}\,\neg_c{}^4}{(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2)) \to (\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2))}\,\text{L}\wedge_i}{(\neg_c\neg_c h_c(\varphi_1)) \wedge_i (\neg_c\neg_c h_c(\varphi_2)) \to \neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \wedge_i (\neg_i\neg_i h_1(\varphi_2)))}\,\neg_i^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \vee_i \varphi_2$. Note that $h_2(\varphi)$ is $\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2)))$ and $h_c(\varphi)$ is $\neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)))$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{h_c(\varphi_1) \to \neg_i\neg_i h_1(\varphi_1)}}{\neg_c\neg_c h_c(\varphi_1) \to \neg_i\neg_i h_1(\varphi_1)}\,\neg_c{}^2}{\neg_c\neg_c h_c(\varphi_1) \to (\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2))}\,\text{R}(\vee_i)_2 \qquad \vdots}{(\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)) \to (\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2))}\,\text{L}\vee_i}{\neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))) \to \neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2)))}\,\neg_i^2 + \neg_c^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_3^\circ}{\neg_i\neg_i h_1(\varphi_1) \to h_c(\varphi_1)}}{\neg_i\neg_i h_1(\varphi_1) \to \neg_c\neg_c h_c(\varphi_1)}\,\neg_c{}^2}{\neg_i\neg_i h_1(\varphi_1) \to (\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2))}\,\text{L}(\vee_i)_2}{\neg_i\neg_i h_1(\varphi_1), \neg_c(\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)) \to}\,\text{L}\neg_c \qquad \vdots}{(\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2)), \neg_c(\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)) \to}\,\text{L}\vee_i}{\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \vee_i (\neg_i\neg_i h_1(\varphi_2))) \to \neg_c\neg_c((\neg_c\neg_c h_c(\varphi_1)) \vee_i (\neg_c\neg_c h_c(\varphi_2)))}\,\text{R}\neg_c + \neg_i{}^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis;

φ is $\varphi_1 \Rightarrow_i \varphi_2$. Note that $h_2(\varphi)$ is $\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2)))$ and $h_c(\varphi)$ is $(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2))$. Then

$$\dfrac{\dfrac{\vdots}{\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2))), \neg_c\neg_c h_c(\varphi_1) \to \neg_c\neg_c h_c(\varphi_2)}\ \mathrm{R}\neg_c}{\neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2))) \to ((\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)))}\ \mathrm{R}{\Rightarrow_i}$$

(the rest of this derivation is similar to the case when φ is $\varphi_1 \Rightarrow_c \varphi_2$) and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1^\circ}{\neg_i\neg_i h_1(\varphi_1) \to h_c(\varphi_1)}}{\neg_i\neg_i h_1(\varphi_1) \to \neg_c\neg_c h_c(\varphi_1)}\ {\neg_c}^2 \quad \dfrac{\dfrac{\dfrac{D_2^\circ}{h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_2)}}{\neg_c\neg_c h_c(\varphi_2) \to \neg_i\neg_i h_1(\varphi_2)}\ {\neg_c}^2}{\neg_c\neg_c h_c(\varphi_2), \neg_i\neg_i h_1(\varphi_1) \to \neg_i\neg_i h_1(\varphi_2)}\ \mathrm{Lw}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)), \neg_i\neg_i h_1(\varphi_1) \to \neg_i\neg_i h_1(\varphi_2)}\ \mathrm{L}{\Rightarrow_i}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)) \to (\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2))}\ \mathrm{R}{\Rightarrow_i}}{(\neg_c\neg_c h_c(\varphi_1)) \Rightarrow_i (\neg_c\neg_c h_c(\varphi_2)) \to \neg_i\neg_i((\neg_i\neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i\neg_i h_1(\varphi_2)))}\ \neg_i^2$$

are derivations for $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$ where D_1°, D_2°, D_3° and D_4° exist by induction hypothesis. $\diamond$

Lemma 5.17 The map h is such that

- $\Gamma \vdash^l_{\mathcal{D}_{i+c}} h(\psi)$ whenever $\Gamma \vdash^l_{\mathcal{D}_{i+_\iota c}} \psi$
- $h(\Psi), \Delta \vdash^l_{\mathcal{D}_{i+c}} \varphi$ whenever $\Psi, \Delta \vdash^l_{\mathcal{D}_{i+_\iota c}} \varphi$

where $\Gamma \cup \Delta \cup \{\varphi\}$ is contained in L_{i+c} and $\Psi \cup \{\psi\}$ is contained in L_i.

Proof: We show by complete induction on the depth of a sequent derivation that if Ψ and Δ are sets contained in $L_{i+_\iota c}$ and $\Psi \to \Delta$ is a theorem of $\mathcal{D}_{i+_\iota c}$ then

$$\overline{h}(\Psi) \to \overline{h}(\Delta)$$

is a theorem in $\mathcal{D}_{i+c}$ where $\overline{h}$ is a map from $L_{i+_\iota c}$ to L_{i+c} extending h by establishing an identity on the connectives not in $\mathcal{D}_i$. Let

$$\dfrac{\begin{matrix} D_1 & & D_k \\ \Psi_1 \to \Delta_1 & \ldots & \Psi_k \to \Delta_k \end{matrix}}{\Psi \to \Delta}\ r$$

be a derivation in $\mathcal{D}_{i+_\iota c}$ where k is greater than or equal to 0. We only present the proof for the rules in $\mathcal{D}_{i+_\iota c}$ and not in $\mathcal{D}_{i+c}$ since the proof of the other rules follow straightforwardly by induction hypothesis and applying the rule. So, consider the following cases:

r is Lι. Note that $\overline{h}(\Psi)$ is $h(\iota(p))$, which is p, and $\overline{h}(\Delta)$ is p. So, the thesis follows straightforwardly using Ax;

r is Rι. Note that $\overline{h}(\Psi)$ is p, and $\overline{h}(\Delta)$ is $\neg_i\neg_i h(\iota(p))$, which is $\neg_i\neg_i p$. Then

$$\dfrac{\dfrac{}{p \to p}\ \mathrm{Ax}}{p \to \neg_i\neg_i p}\ {\neg_i}^2$$

is a derivation for $\overline{h}(\Psi) \to \overline{h}(\Delta)$ in $\mathcal{D}_{i+c}$. $\diamond$