# Interpolation via translations

Walter Carnielli[2,3] João Rasga[1,3] Cristina Sernadas[1,3]

[1] DM, IST, TU Lisbon, Portugal
[2] CLE and IFCH, UNICAMP, Brazil
[3] SQIG - Instituto de Telecomunicações, Portugal

**Abstract**

A new technique is presented for proving that a consequence system enjoys Craig interpolation or Maehara interpolation based on the fact that these properties hold in another consequence system. This technique is based on the existence of a back and forth translation satisfying some properties between the consequence systems. Some examples of translations satisfying those properties are described. Namely a translation between the global/local consequence systems induced by fragments of linear logic, a Kolmogorov-Gentzen-Gödel style translation, and a new translation between the global consequence systems induced by full Lambek calculus and linear logic, mixing features of a Kiriyama-Ono style translation with features of a Kolmogorov-Gentzen-Gödel style translation. These translations establish a strong relationship between the logics involved and are used to obtain new results about whether Craig interpolation and Maehara interpolation hold in that logics.

AMS Classification: 03C40, 03F03, 03B22
Keywords: Craig interpolation, Maehara interpolation, preservation of interpolation, negative translation, Kiriyama-Ono translation

## 1 Introduction

The well-known family of negative translations (Kolmogorov-Gentzen-Gödel) from classical logic into intuitionistic logic, were introduced as a way of proving the consistency of classical logic by reducing it to the consistency of intuitionistic logic (see the works of Kolmogorov [18] and Glivenko [11] for the propositional case and the works of Gentzen [10] and Gödel [12] for the predicate case).

Subsequently, special designed translations were used to show that other properties besides consistency, like for instance cut-elimination [16] and undecidability [17], hold on a deductive system relative to another, that is, hold on a deductive system whenever they hold on another deductive system. Similar results can be found in other scientific areas, like for instance in complexity theory, where frequently the answer to a problem is shown to be related, usually through a kind of translation, to the answer of a different problem.

These kind of preservation results are, despite their importance, usually shown in a case by case basis, although recently there can be seen a widely and growing effort to generalize preservation results so that they can be applied once to a whole class of entities [4, 5, 26, 28].

Herein we propose a new method for checking whether Craig interpolation and Maehara interpolation hold in a logic. This method is based on the existence of a special kind of translation between an abstract consequence system in which we want to investigate those properties and another abstract consequence system. If it is the case that such a translation exists, then, if the destination system enjoys those properties than the source consequence system would also enjoys them. Otherwise, even if the destination system does not enjoy the properties, the fact that there exists such a translation allows to reduce that question to the same question but over that destination system (a system in which it may be simpler to investigate the interpolation properties).

The use of abstract consequence systems make the preservation results very general and applicable in a wide range of situations. In fact the results presented in this work can be applied either to consequence systems defined model-theoretically or to consequence systems induced by deductive systems (for instance consequence systems based on the local or the global consequence relation induced by a deductive system). Moreover they can be applied to propositional based consequence systems or to first-order based consequence systems.

The generalized translation schema investigated herein is general enough to be satisfied by several interesting translations (see Section 5). Some of the translations developed in this work are variants of well studied translations, adapted to the specific logic systems being considered in order to satisfy the conditions of the generalized translation schema. For instance, a new kind of translation mixing features of a Kolmogorov-Gentzen-Gödel style translation and features of a Kiriyama-Ono style translation is developed between the global consequence systems induced by full Lambek calculus and linear logic.

New results about whether Craig interpolation and Maehara interpolation over local consequence and global consequence hold in some logics are proved. We show that full Lambek calculus with exchange and weakening enriched with multiplicative rules of linear logic enjoys Craig interpolation and Maehara interpolation over global consequence. We reduce the problem of checking whether Craig interpolation and Maehara interpolation hold in a linear logic with classical and intuitionistic rules, to its intuitionistic component. Finally we show that the logic resulting from the fibring of intuitionistic and classical logic enjoys Craig interpolation and Maehara interpolation both over its local consequence relation and over its global consequence relation .

Although not explored in this work, we observe also that another use of the results obtained herein is proving that Craig interpolation and Maehara interpolation does not hold in a logic. In fact, in order to show that a consequence system does not enjoy those properties it is sufficient to establish a translation to that system from a consequence system which does not enjoy those interpolation properties.

*Organization.* The basic notions of consequence system, consequence relation, Craig interpolation and Maehara interpolation are defined in Section 2. In Section 3, Maehara generalized translation schemas and Craig generalized translation schemas, over generic abstract consequence systems, are introduced, and the theorems regarding the preservation of Craig interpolation and Maehara interpolation are proved. Examples illustrating the preservation results and generalized translation schemas are presented in Section 5 for the global and the local consequence relations induced by a deductive system. These deductive consequence relations were defined in Section 4 after introducing the basic notions of sequent, sequent calculus and data structure associated to a sequent. Moreover interesting relationships between those consequence relations are also established in Section 4. Finally, in Section 6, we draw the conclusions.

For the benefit of the reader there is an Appendix [2] to this work where we provide the details of the proofs that are just sketched or not proved at all herein (proofs that follow in an expected way are just sketched or not proved at all in order to simplify the presentation).

## 2 Basics

Translation schemas are defined quite generally through abstract consequence systems so that examples of such translations can be found in several different areas. By an *abstract consequence system*, or simply, a *consequence system*, we consider a pair $(L, \vdash)$ where $L$ is a set and $\vdash \subseteq (\wp L) \times L$ is such that:

- (reflexivity) $\{\varphi\} \vdash \varphi$;
- (monotonicity) $\Gamma' \vdash \varphi$ whenever $\Gamma \vdash \varphi$ and $\Gamma$ is contained in $\Gamma'$;
- (transitivity) $\Gamma, \Psi, \Delta \vdash \varphi$ whenever $\Psi \vdash \psi$ and $\Gamma, \psi, \Delta \vdash \varphi$;

where $\Gamma$, $\Gamma'$, $\Psi$ and $\Delta$ are contained in $L$ and $\psi$ and $\varphi$ are elements of $L$. The set $L$ is called the *language* of the system and its elements are called *formulas*. The relation $\vdash$ is called the *consequence relation*. To simplify the presentation, in the rest of the paper, when writing $\Gamma \vdash \Psi$, we mean $\Gamma \vdash \psi$, for every $\psi$ in $\Psi$. Moreover, given a set $L$, a function from $L$ to the power set of another set, is called an *$L$-function*, and given an $L$-function $f$ to the power set of a set $S$ and a subset $S'$ of $S$, we denote by $L_{f,S'}$ the set $\{\varphi \in L : f(\varphi) \subseteq S'\}$. We may simply write $L_{S'}$ for $L_{f,S'}$ when there is no ambiguity about which $L$-function is being considered.

We now propose a very general definition of Craig interpolation and Maehara interpolation over abstract consequence systems. A consequence system $(L, \vdash)$ enjoys *Craig interpolation with respect to* an $L$-function var, whenever, if

$$\Gamma \vdash \varphi \quad \text{and} \quad \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$$

then

$$\text{there is} \quad \text{non-empty } \Psi \subseteq L_{\mathrm{var}(\Gamma)\cap\mathrm{var}(\varphi)} \quad \text{such that} \quad \Gamma \vdash \Psi \text{ and } \Psi \vdash \varphi$$

for every $\Gamma \subseteq L$ and $\varphi \in L$, and it enjoys *Maehara interpolation with respect to* an $L$-function var, whenever, if

$$\Gamma, \Delta \vdash \varphi \quad \text{and} \quad \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Delta) \cup \mathrm{var}(\varphi)) \neq \emptyset$$

then

$$\text{there is} \quad \text{non-empty } \Psi \subseteq L_{\mathrm{var}(\Gamma)\cap(\mathrm{var}(\Delta)\cup\mathrm{var}(\varphi))} \quad \text{such that} \quad \Gamma \vdash \Psi \text{ and } \Psi, \Delta \vdash \varphi$$

for every $\Gamma, \Delta \subseteq L$ and $\varphi \in L$. Given a consequence system, in order for the definitions we propose to coincide with the usual definitions of Craig interpolation and Maehara interpolation for that system, it is sufficient to define the function var so that it maps a formula to the set of symbols relevant for interpolation according to that usual convention for that system. For the sake of illustration consider the case of classical first-order logic. Some authors when considering Craig interpolation in the context of that logic require that the set of predicate symbols in the interpolant is contained in the common predicate symbols of the hypothesis and the conclusion, while others, besides predicate symbols, require also a similar condition over function symbols and free variables. The Craig interpolation definition proposed herein is appropriate in both situations since the set of symbols relevant for interpolation is a parameter of the definition, and so, it can be adjusted depending of the case. Note however that the logical aspect of the definition remains unchanged. Hence, when in the first situation, it is considered an $L$-function that maps a first-order formula to the set of predicate symbols appearing in it, and when in the other situation, the $L$-function considered would map a first-order formula to the set of predicate symbols, free variables and function symbols appearing in the formula.

We observe that Maehara interpolation is also known in the literature as Craig-Robinson interpolation property or Strong interpolation property (see for instance [8, 27, 6]). Interesting relationships can be established between these interpolation properties when the underlying consequence system enjoys some additional characteristics. For instance, when the consequence system enjoys compactness and has implication those two interpolation properties are equivalent, and when it enjoys Craig interpolation, metatheorem of modus ponens and metatheorem of deduction then Maehara interpolation holds (see for instance [4] for more information on this subject). Note that the metatheorem of deduction and the metatheorem of modus ponens can be defined over consequence systems that may not have an implication, and that there exist consequence systems with implication enjoying only a constrained form of metatheorem of deduction as is the case for instance of classical first-order logic.

# 3 Generalized translation schema

In this section we introduce Maehara generalized translation schemas and show that they can be used to prove the preservation of Maehara interpolation. Similarly we define a weaker notion of translation, called Craig generalized translation schema, that can be used to prove the preservation of Craig interpolation. The translation schemas are defined as general as possible in order to preserve those properties, and so, as we show in Section 5, several interesting translations can be shown to satisfy their conditions.

A *Maehara generalized translation schema from* $(L,\vdash)$ *to* $(L',\vdash')$ *via* $(L^\circ,\vdash^\circ)$ *with respect to* an $L$-function var and an $L'$-function var$'$, where $L^\circ \supseteq L \cup L'$ and $\vdash^\circ \supseteq \vdash \cup \vdash'$, is a tuple $(h_1, h_2, h)$ where $h_1, h_2 : L \to L'$ and $h : L'_{\mathrm{var}'(h_1(L))} \to L$ such that

1. $h_1(\Psi) \vdash' h_2(\varphi)$ whenever $\Psi \vdash \varphi$;
2. $\varphi \vdash^\circ h_1(\varphi)$ and $h_2(\varphi) \vdash^\circ \varphi$;
3. $\Psi \vdash h(\varphi')$ whenever $\Psi \vdash^\circ \varphi'$;
4. $h(\Psi'), \Delta \vdash \varphi$ whenever $\Psi', \Delta \vdash^\circ \varphi$;
5. $\mathrm{var}'(h_1(\varphi)) \cap \mathrm{var}'(h_2(\psi)) \neq \emptyset$ whenever $\mathrm{var}(\varphi) \cap \mathrm{var}(\psi) \neq \emptyset$;
6. $\mathrm{var}(h(\Psi')) \subseteq \mathrm{var}(\Psi) \cap (\mathrm{var}(\Delta) \cup \mathrm{var}(\varphi))$ whenever $\mathrm{var}'(\Psi') \subseteq \mathrm{var}'(h_1(\Psi)) \cap (\mathrm{var}'(h_1(\Delta)) \cup \mathrm{var}'(h_2(\varphi)))$;

where $\Psi \cup \Delta \cup \{\varphi, \psi\}$ is contained in $L$ and $\Psi' \cup \{\varphi'\}$ is contained in $L'_{\mathrm{var}'(h_1(L))}$. A *Craig generalized translation schema* is a tuple $\langle h_1, h_2, h\rangle$ satisfying conditions similar to the ones of a Maehara generalized translation schema with only the following differences: i. $h : L'_{\mathrm{var}'(h_1(L)) \cap \mathrm{var}'(h_2(L))} \to L$; ii. the formulas in $\Psi' \cup \{\varphi'\}$ are in $L'_{\mathrm{var}'(h_1(L)) \cap \mathrm{var}'(h_2(L))}$; and iii. $\Delta = \emptyset$.

In Section 5 we present several interesting instances of the Maehara generalized translation schema: in Subsection 5.1 we describe an interesting translation between a consequence system for linear logic and a consequence system for its intuitionistic fragment, in Subsection 5.2 we present a new translation with Kiriyama-Ono style features and Kolmogorov-Gentzen-Gödel style features between a consequence system for full Lambek calculus enriched with some linear logic rules and a consequence system for a more weaker fragment of full Lambek calculus, and finally in Subsection 5.3 we describe a Kolmogorov-Gentzen-Gödel style translation between a consequence system for the logic resulting from the combination of intuitionistic and classical logic and a consequence system for intuitionistic logic.

The deep relationship established by a Maehara generalized translation schema between two consequence systems can be used to transfer Maehara interpolation between them, as we show in the next theorem.

**Theorem 3.1** A consequence system $(L, \vdash)$ enjoys Maehara interpolation with respect to an $L$-function var, if (i) there is a Maehara generalized translation schema from $(L, \vdash)$ to another consequence system $(L', \vdash')$ with respect to var and an $L'$-function var$'$; (ii) $(L', \vdash')$ enjoys Maehara interpolation with respect to var$'$.

**Proof:** Let $(h_1, h_2, h)$ be a Maehara generalized translation schema from $(L, \vdash)$ to $(L', \vdash')$ via a consequence system $(L^\circ, \vdash^\circ)$, with respect to an $L$-function var and an $L'$-function var$'$. Assume that $(L', \vdash')$ enjoys Maehara interpolation over var$'$. Suppose that $\Gamma, \Delta \vdash \varphi$ and $\text{var}(\Gamma) \cap (\text{var}(\Delta) \cup \text{var}(\varphi)) \neq \emptyset$. Then

$$h_1(\Gamma), h_1(\Delta) \vdash' h_2(\varphi) \quad \text{and} \quad \text{var}'(h_1(\Gamma)) \cap (\text{var}'(h_1(\Delta)) \cup \text{var}'(h_2(\varphi))) \neq \emptyset$$

using condition 1 and condition 5 of the definition of Maehara generalized translation schema. Since $(L', \vdash')$ has Maehara interpolation there is a non-empty finite set $\Psi'$ of formulas of $L'$ such that

$$h_1(\Gamma) \vdash' \Psi' \quad \text{and} \quad \Psi', h_1(\Delta) \vdash' h_2(\varphi)$$

and $\text{var}'(\Psi') \subseteq \text{var}'(h_1(\Gamma)) \cap (\text{var}'(h_1(\Delta)) \cup \text{var}'(h_2(\varphi)))$. So $h_1(\Gamma) \vdash^\circ \Psi'$ and $\Psi', h_1(\Delta) \vdash^\circ h_2(\varphi)$. Using condition 2 of the definition of Maehara generalized translation schema and the transitivity of the consequence relation the following holds

$$\Gamma \vdash^\circ \Psi', \quad \Psi', \Delta \vdash^\circ \varphi \quad \text{with} \quad \text{var}'(\Psi') \subseteq \text{var}'(h_1(\Gamma)) \cap (\text{var}'(h_1(\Delta)) \cup \text{var}'(h_2(\varphi))).$$

Finally, using condition 3, condition 4 and condition 6, $\Gamma \vdash h(\Psi')$ and $h(\Psi'), \Delta \vdash \varphi$ and $\text{var}(h(\Psi')) \subseteq \text{var}(\Gamma) \cap (\text{var}(\Delta) \cup \text{var}(\varphi))$. Hence $h(\Psi')$ is a Maehara interpolant in $(L, \vdash)$ for $\Gamma, \Delta \vdash \varphi$. ◇

A result similar to Theorem 3.1 can be proved for Craig interpolation when there is a Craig generalized translation schema between the consequence systems. The proof of Theorem 3.2 is provided in the Appendix [2] and omitted here since it is very similar to the proof of Theorem 3.1.

**Theorem 3.2** A consequence system $(L, \vdash)$ enjoys Craig interpolation with respect to an $L$-function var, if (i) there is a Craig generalized translation schema from $(L, \vdash)$ to another consequence system $(L', \vdash')$ with respect to var and an $L'$-function var$'$; (ii) $(L', \vdash')$ enjoys Craig interpolation with respect to var$'$.

We now prove that it is possible to extract a Craig generalized translation schema from a Maehara generalized translation schema without imposing any conditions. This useful result allows that, in Section 5, dedicated to examples, we only concentrate on Maehara generalized translation schemas. We provide its proof in the Appendix [2] and omit it here, since it follows straightforwardly.

**Proposition 3.3** Given a Maehara generalized translation schema $\langle h_1, h_2, h \rangle$ from the consequence system $(L, \vdash)$ to the consequence system $(L', \vdash')$ via the consequence system $(L^\circ, \vdash^\circ)$ with respect to an $L$-function var and an $L'$-function var$'$, the tuple

$$\langle h_1, h_2, h^- \rangle$$

where $h^-$ is the restriction of $h$ to the set $L'_{\text{var}'(h_1(L)) \cap \text{var}'(h_2(L))}$, is a Craig generalized translation schema from $(L, \vdash)$ to $(L', \vdash')$ via $(L^\circ, \vdash^\circ)$ with respect to var and var$'$.

The previous result about how from a Maehara generalized translation schema is possible to extract a Craig generalized translation schema, Proposition 3.3, allows to prove a theorem for preservation of Craig interpolation and Maehara interpolation relying only on the existence of a Maehara generalized translation schema.

**Corollary 3.4** A consequence system $(L, \vdash)$ enjoys Craig/Maehara interpolation with respect to an $L$-function var, if (i) there is a Maehara generalized translation schema from $(L, \vdash)$ to another consequence system $(L', \vdash')$ with respect to var and an $L'$-function var$'$; (ii) $(L', \vdash')$ enjoys Maehara interpolation with respect to var$'$.

The proof of the corollary is omitted since it follows immediately by using Proposition 3.3, Theorem 3.1 and Theorem 3.2.

# 4 Deductive consequence systems

In this section we introduce consequence systems generated by deductive systems. This type of consequence systems is going to be used in Section 5 when illustrating the definitions and results of Section 3. We use finitary sequent calculi as deductive systems. That is, in this work, a *deductive system* $\mathcal{D}$ is a pair $(L, R)$ where $L$ is the language of the system and $R$ is a set of rules presented using sequents. *Sequents* are pairs $\langle \Psi, \Delta \rangle$ of finite collections of formulas represented as $\Psi \to \Delta$. *Rules* are triples $(\{s_1, \ldots, s_n\}, s, \pi)$ represented as $\frac{s_1 \; \ldots \; s_n}{s} \lhd \pi$ , where $s_1, \ldots, s_n, s$ are sequents and $\pi$ is a proviso, that is, a condition that should be satisfied when using the rule in a derivation. There is an intimate relationship between the data structure used for the collections of formulas in sequents and the presence of some rules in the deductive system. For example, for some deductive systems, $\Psi$ and $\Delta$ are sequences, for others they are multisets and the permutation rule is not present int the system, and for others they are sets and no structural rules are present. In the sequel, given a deductive system $\mathcal{D}$, we refer to the type of structure (sequence, multiset or set) used for collections of formulas in sequents as $t_{\mathcal{D}}$. By a $t_{\mathcal{D}}$-collection of formulas we mean either a sequence, or multiset or set of formulas whenever $t_{\mathcal{D}}$ is sequence, multiset, or set, respectively. In the sequel, given a $t_{\mathcal{D}}$ collection of formulas $\Psi$, we denote by $\Psi^s$ the set with the formulas in $\Psi$. Derivations in a sequent calculus are defined in the usual way using sequents.

We consider two types of consequence relations over a sequent calculus $\mathcal{D}$, the *global consequence relation*, or simply the *global relation*, and the *local consequence relation*, or simply the *local relation*. The global relation $\vdash^g_{\mathcal{D}}$ is such that $\Psi \vdash^g_{\mathcal{D}} \varphi$ whenever $\to \varphi$ is derivable in $\mathcal{D}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$. The local relation $\vdash^l_{\mathcal{D}}$ is only considered on calculi where the Axiom rule, that is, a rule of the type $\frac{}{\varphi, \Gamma \to \Delta, \varphi}$ , and the Cut rule, that is, a rule of the type $\frac{\Gamma_1 \to \Delta_1, \varphi \quad \varphi, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}$ , are admissible, and is such that $\Psi \vdash^l_{\mathcal{D}} \varphi$ whenever $\Psi_c \to \varphi$ is derivable in $\mathcal{D}$, for some finite $t_{\mathcal{D}}$-collection $\Psi_c$ with ${\Psi_c}^s \subseteq \Psi$. So when using $\vdash^l_{\mathcal{D}}$ we are assuming that both the Axiom rule and the Cut rule are admissible in $\mathcal{D}$. Given a deductive system $\mathcal{D} = (L, R)$ it is not difficult to show that the pairs $(L, \vdash^l_{\mathcal{D}})$ and $(L, \vdash^g_{\mathcal{D}})$ constitute consequence systems. We may call $\vdash^l_{\mathcal{D}}$ as the local consequence relation induced by $\mathcal{D}$ or simply the local consequence when there is no ambiguity, and $\vdash^g_{\mathcal{D}}$ as the global consequence relation induced by $\mathcal{D}$ or simply the global consequence. Note that, in the context of deductive systems, Craig interpolation and Maehara interpolation over local consequence are not so well studied in the literature, as far as we know, as Craig interpolation and Maehara interpolation over global consequence (see [14, 7, 1, 21] mainly about Craig interpolation and Maehara interpolation over global consequence).

Interesting relationships can be established between the local consequence and the global consequence induced by a deductive system. The following proposition states that it is always the case that local consequence implies global consequence (by *local consequence implies global consequence* in the context of a deductive system $\mathcal{D}$, we mean that if $\Psi \vdash^l_{\mathcal{D}} \varphi$ then $\Psi \vdash^g_{\mathcal{D}} \varphi$, and by *global consequence implies local consequence* we mean that if $\Psi \vdash^g_{\mathcal{D}} \varphi$ then $\Psi \vdash^l_{\mathcal{D}} \varphi$).

**Proposition 4.1** Let $\mathcal{D}$ be a deductive system and $\Psi \cup \{\varphi\}$ a set of formulas of $\mathcal{D}$, then if $\Psi \vdash^l_{\mathcal{D}} \varphi$ then $\Psi \vdash^g_{\mathcal{D}} \varphi$.

**Proof:** The proof follows by showing that if $\Psi_c \to \Delta$ is derivable in $\mathcal{D}$ from a set of hypothesis $H$ then $\to \Delta$ is derivable in $\mathcal{D}$ from the set $\{\to \psi : \psi \text{ in } {\Psi_c}^s\} \cup H$, by induction on the number of formula occurrences in the $t_{\mathcal{D}}$-collection of formulas $\Psi_c$. The *base* follows straightforwardly. *Step* Assume that for all $t_{\mathcal{D}}$-collections of formulas $\Psi_c$ whose number of formula occurrences is $n$, if $\Psi_c \to \Delta$ is derivable in $\mathcal{D}$ from a set of hypothesis $H$ then $\to \Delta$ is derivable in $\mathcal{D}$ from the set $\{\to \psi : \psi \text{ in } {\Psi_c}^s\} \cup H$. Let $\Psi_c$ be a $t_{\mathcal{D}}$-collection of formulas whose number of formula occurrences is $n + 1$ and assume that $\Psi_c \to \Delta$ is derivable in $\mathcal{D}$ from a set of hypothesis $H$. Assume $\Psi_c$ is the $t_{\mathcal{D}}$-collection of formulas $\Psi_{cl}, \psi, \Psi_{cr}$. Then, there is a derivation in $\mathcal{D}$ for $\Psi_{cl}, \Psi_{cr} \to \Delta$ from $H \cup \{\to \psi\}$ since Cut is admissible. So, by induction hypothesis, $\to \Delta$ is derivable in $\mathcal{D}$ from the set $\{\to \psi : \psi \text{ in } (\Psi_{cl}, \Psi_{cr})^s\} \cup H \cup \{\to \psi\}$, that is, from the set $\{\to \psi : \psi \text{ in } {\Psi_c}^s\} \cup H$, as we wanted to show. $\diamond$

Although global consequence does not, in general, implies local consequence, it is however possible to prove that this is the case on deductive systems with rules and provisos satisfying some conditions, as is shown in the next proposition. For those deductive systems, global consequence and local consequence coincide and so results about preservation of Craig interpolation and Maehara interpolation over the local relation hold immediately for the global and vice-versa. The conditions of the next proposition hold in some of the deductive systems considered in the examples of Section 5.

**Proposition 4.2** Global consequence implies local consequence in a deductive system whose rules are of the following types:

- axiom, with possibly a proviso imposing that it is only over atomic formulas, that is, a rule of the following type $\frac{}{\varphi,\Gamma \to \Delta,\varphi}$;
- weakening rule, that is, a rule of the following type $\frac{\Gamma \to \Delta}{\varphi,\Gamma \to \Delta}$;
- contraction rule, that is, a rule of the following type $\frac{\varphi,\varphi,\Gamma \to \Delta}{\varphi,\Gamma \to \Delta}$;
- cut rule, that is, a rule of the following type $\frac{\Gamma_1 \to \Delta_1,\varphi \quad \varphi,\Gamma_2 \to \Delta_2}{\Gamma_1,\Gamma_2 \to \Delta_1,\Delta_2}$;
- additive introduction rules, that is, rules introducing a formula at the left or the right hand side of the conclusion sequent where the context formulas are shared between the premises and the conclusion;
- multiplicative introduction rules, that is, rules introducing a formula at the left or the right hand side of the conclusion sequent where the context in the conclusion sequent are obtained by putting together the contexts in the premises;
- other rules without premises;

and such that i. the rules with premises, in the deductive system, do not have provisos restricting their left contexts, and ii. when the deductive system has additive rules with more than one premise, the left weakening rule is admissible (that is, is derivable).

**Proof:** Let $\mathcal{D}$ be a deductive system with rules of one of the types described above, where rules with premises do not have provisos over their left contexts, and where the left weakening rule is admissible if the deductive system has additive rules with more than one premise. We show by complete induction on the depth of a derivation that if $\Psi \to \Delta$ is derivable from a set of hypothesis $\{\to \varphi : \varphi \in \Gamma\}$ then there exists a $t_{\mathcal{D}}$-collection $\Gamma'$ of formulas in $\Gamma$ such that $\Gamma', \Psi \to \Delta$ is derivable from the empty set of hypothesis. Note that $\Gamma'$ may be empty. Let

$$\frac{\begin{matrix} D_1 & & D_k \\ \Psi_1 \to \Delta_1 & \ldots & \Psi_k \to \Delta_k \end{matrix}}{\Psi \to \Delta}\ r$$

where $k$ is greater than or equal to 0, be a derivation in $\mathcal{D}$ of $\Psi \to \Delta$ from a set of hypothesis $\{\to \varphi : \varphi \in \Gamma\}$, where $r$ denotes the justification of the last inference, which may be hypothesis or a rule. Consider the following cases:

$r$ is a rule without premises. Then there is a derivation for $\Psi \to \Delta$ from the empty set of hypothesis;

$r$ is an hypothesis. Then $\Psi \to \Delta$ is $\to \psi$ where $\psi$ is the formula used in the hypothesis. Since Axiom is admissible in $\mathcal{D}$, there is a derivation for $\psi, \Psi \to \Delta$ from the empty set of hypothesis;

$r$ is a weakening or a contraction rule. Then the following derivation is a derivation for $\Gamma', \Psi \to \Delta$ from the empty set of hypothesis

$$\frac{\begin{matrix} D_1^\circ \\ \Gamma', \Psi_1 \to \Delta_1 \end{matrix}}{\Gamma', \Psi \to \Delta}\ r$$

where $\Gamma'$ is a $t_{\mathcal{D}}$-collection of formulas in $\Gamma$ and $D_1^\circ$ is a derivation for $\Gamma', \Psi_1 \to \Delta_1$ from the empty set of hypothesis, that exist by induction hypothesis. Note that $r$ can be applied since by assumption there is no restriction over formulas in the left contexts;

$r$ is the cut rule. We omit the proof of this case since it is very similar to the case when $r$ is weakening or a contraction rule;

$r$ is an introduction rule with only one premise or a multiplicative introduction rule with more than one premise. The thesis follows since it is possible to consider the following derivation

$$\frac{\begin{matrix} D_1^\circ & & D_k^\circ \\ \Gamma_1', \Psi_1 \to \Delta_1 & \ldots & \Gamma_k', \Psi_k \to \Delta_k \end{matrix}}{\Gamma_1', \ldots, \Gamma_k', \Psi \to \Delta}\ r$$

where, for $i = 1, \ldots, k$, $\Gamma_i'$ is a $t_{\mathcal{D}}$-collection of formulas in $\Gamma$ and $D_i^\circ$, that exist by induction hypothesis, is a derivation for $\Gamma_i', \Psi_i \to \Delta_i$ from the empty set of hypothesis. Note that $r$ can be applied since by assumption there is no restriction in $r$ over formulas in the left contexts;

$r$ is an additive introduction rule with more than one premise. For each derivation

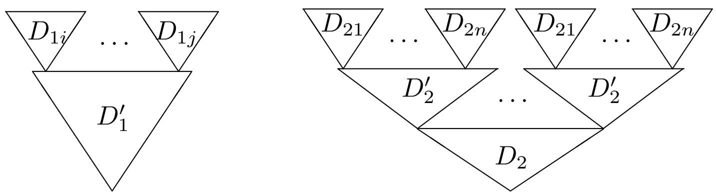
$$\begin{matrix} D_i^\circ \\ \Gamma_i', \Psi_i \to \Delta_i \end{matrix}$$

for $i = 1, \dots, k$, that exists by induction hypothesis, denote by $D^{\circ}_{i\,w}$ a derivation for $\Gamma', \Psi_i \to \Delta_i$ that exists since left weakening is admissible. Then

$$\dfrac{\begin{array}{ccc} D^{\circ}_{1\,w} & & D^{\circ}_{k\,w} \\ \Gamma', \Psi_1 \to \Delta_1 & \dots & \Gamma', \Psi_k \to \Delta_k \end{array}}{\Gamma', \Psi \to \Delta}\; r$$

is a derivation for $\Gamma', \Psi \to \Delta$ from the empty set of hypothesis. Observe that $r$ can be applied since by assumption there is no restriction in $r$ on formulas in the left contexts. $\diamond$

Consider the proof of Theorem 3.1 and assume that the consequence systems involved in the theorem are deductive consequence systems, for instance, $(L, \vdash^l_{\mathcal{D}})$, $(L', \vdash^l_{\mathcal{D}'})$ and $(L^{\circ}, \vdash^l_{\mathcal{D}^{\circ}})$. Then, the deductions for $\Gamma \vdash^l_{\mathcal{D}^{\circ}} \Psi'$ and $\Psi', \Delta \vdash^l_{\mathcal{D}^{\circ}} \varphi$, which are consequences referred to in that proof, have the structure depicted by the following diagrams




Observe that the central part of the deduction for $\Gamma \vdash^l_{\mathcal{D}^{\circ}} \Psi'$ is the deduction in $\mathcal{D}'$ for $h_1(\Gamma) \vdash^l_{\mathcal{D}'} \Psi'$ called $D'_1$. The remaining parts consist of the deductions $D_{1i}$ for $\Gamma \vdash^l_{\mathcal{D}^{\circ}} h_1(\gamma_i)$ for each $\gamma_i \in \Gamma$. Similarly for $\Psi', \Delta \vdash^l_{\mathcal{D}^{\circ}} \varphi$, where the significant part is the deduction $D'_2$ for $\Psi', h_1(\Delta) \vdash^l_{\mathcal{D}'} h_2(\varphi)$, and the remaining parts consist of the deductions $D_{2i}$ for $\Delta \vdash^l_{\mathcal{D}^{\circ}} h_1(\delta_i)$ for each $\delta_i \in \Delta$, and the deduction $D_2$ for $h_2(\varphi) \vdash^l_{\mathcal{D}^{\circ}} \varphi$.

# 5 Examples

In this section we present several interesting examples of Maehara generalized translations and use them to take conclusions about Craig interpolation and Maehara interpolation either in the local consequence systems or in the global consequence systems induced by the deductive systems described in each example.

For each example we start by fully describing the deductive systems involved, then for each deductive system we describe the function specifying the set of symbols relevant for interpolation, finally we define the maps of the translation and after that we prove the results about Craig interpolation and Maehara interpolation.

We stress that, in each example, the set of symbols considered relevant for interpolation (given by the $L$-function) is also used in the literature when investigating interpolation in that system.

The examples presented in this section are for propositional based logics. Moreover, in each example, we assume that each language is generated by a propositional based signature. A *propositional based signature* $C$, or simply, a *signature*, is a family

$$\{C_k\}_{k \in N}$$

of sets of connectives of arity $k$. The *language* generated by the signature $C$, denoted by $L_C$, or simply $L$ when there is no ambiguity, is the least set inductively defined as follows: $c(\varphi_1, \dots, \varphi_k) \in L$ for every $\varphi_1, \dots, \varphi_k \in L$, $c \in C_k$ and $k \in N$. A deductive system $(L, R)$ whose language $L$ is generated by a signature $C$ is presented as a pair $(C, R)$.

## 5.1 Interpolation in NLL⊕MALL

Consider the question of whether the linear logic associated with the deductive system having as rules the rules of MALL (additive and multiplicative rules) and the rules of NLL (intuitionistic rules), see [22, 15], enjoys Craig interpolation and Maehara interpolation over the global and the local consequence relation. (Interpolation for this logic was not studied in [25].) Using the results of this work we reduce this problem to the same problem over its intuitionistic (NLL) fragment. We denote this linear logic by NLL⊕MALL. The only difference of NLL with respect to the logic appearing in [22] is that we do not impose a restriction on the language of the logic. Moreover the translation we propose is based on the translation described in that paper. The motivation for this example is to illustrate how it is possible, using the results of this work, to reduce interpolation questions for a logic to interpolation questions over another logic.

### Deductive systems

The deductive system $\mathcal{D}_{n+ma}$ for NLL⊕MALL is such that, its language, $L_{n+ma}$, is freely generated by the following signature $C_{n+ma}$:

- $C_{n+ma_0}$ contains $\perp$, $\mathbf{1}$ and a denumerable set of propositional symbols $P$;
- $C_{n+ma_1} = \{\sim\}$;
- $C_{n+ma_2} = \{\oplus_n, \otimes_n, \multimap, \oplus_m, \otimes_m, \&, \nabla\}$;
- $C_{n+ma_i} = \emptyset$, for $i \geq 3$;

the $L_{n+ma}$-function

$$\mathrm{var}_{n+ma}$$

considered in this example associates to each formula $\varphi$ of $L_{n+ma}$ the symbols of $P$ present in $\varphi$. Sequents are pairs of finite multisets of formulas, and its sequent calculus is defined by the following rules:

Init $\frac{}{\xi_1 \to \xi_1}$

L$\otimes_n$ $\frac{\Gamma_1, \xi_1, \xi_2 \to \xi_3}{\Gamma_1, \xi_1 \otimes_n \xi_2 \to \xi_3}$ R$\otimes_n$ $\frac{\Gamma_1 \to \xi_1 \quad \Gamma_1' \to \xi_2}{\Gamma_1, \Gamma_1' \to \xi_1 \otimes_n \xi_2}$

L$\multimap$ $\frac{\Gamma_1 \to \xi_1 \quad \Gamma_1', \xi_2 \to \xi_3}{\Gamma_1, \Gamma_1', \xi_1 \multimap \xi_2 \to \xi_3}$ R$\multimap$ $\frac{\Gamma_1, \xi_1 \to \xi_2}{\Gamma_1 \to \xi_1 \multimap \xi_2}$

L$\oplus_n$ $\frac{\Gamma_1, \xi_1 \to \xi_3 \quad \Gamma_1, \xi_2 \to \xi_3}{\Gamma_1, \xi_1 \oplus_n \xi_2 \to \xi_3}$ R$_i\oplus_n$ $\frac{\Gamma_1 \to \xi_i}{\Gamma_1 \to \xi_1 \oplus_n \xi_2}$, $i = 1, 2.$

Cut $\frac{\Gamma_1 \to \Gamma_2, \xi_1 \quad \xi_1, \Gamma_1' \to \Gamma_2'}{\Gamma_1, \Gamma_1' \to \Gamma_2, \Gamma_2'}$

L$\mathbf{1}$ $\frac{\Gamma_1 \to \Gamma_2}{\Gamma_1, \mathbf{1} \to \Gamma_2}$ R$\mathbf{1}$ $\frac{}{\to \mathbf{1}}$

L$\perp$ $\frac{}{\perp \to}$ R$\perp$ $\frac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \perp, \Gamma_2}$

$$\text{L}\sim \ \frac{\Gamma_1 \to \xi_1, \Gamma_2}{\Gamma_1, \sim\xi_1 \to \Gamma_2} \qquad \text{R}\sim \ \frac{\Gamma_1, \xi_1 \to \Gamma_2}{\Gamma_1 \to \sim\xi_1, \Gamma_2}$$

$$\text{L}\otimes_m \ \frac{\Gamma_1, \xi_1, \xi_2 \to \Gamma_2}{\Gamma_1, \xi_1 \otimes_m \xi_2 \to \Gamma_2} \qquad \text{R}\otimes_m \ \frac{\Gamma_1 \to \xi_1, \Gamma_2 \quad \Gamma_1' \to \xi_2, \Gamma_2'}{\Gamma_1, \Gamma_1' \to \xi_1 \otimes_m \xi_2, \Gamma_2, \Gamma_2'}$$

$$\text{L}\nabla \ \frac{\Gamma_1, \xi_1 \to \Gamma_2 \quad \Gamma_1', \xi_2 \to \Gamma_2'}{\Gamma_1, \Gamma_1', \xi_1 \nabla \xi_2 \to \Gamma_2, \Gamma_2'} \qquad \text{R}\nabla \ \frac{\Gamma_1 \to \xi_1, \xi_2, \Gamma_2}{\Gamma_1 \to \xi_1 \nabla \xi_2, \Gamma_2}$$

$$\text{L}_i\& \ \frac{\Gamma_1, \xi_i \to \Gamma_2}{\Gamma_1, \xi_1 \& \xi_2 \to \Gamma_2}, \quad i = 1,2 \qquad \text{R}\& \ \frac{\Gamma_1 \to \xi_1, \Gamma_2 \quad \Gamma_1 \to \xi_2, \Gamma_2}{\Gamma_1 \to \xi_1 \& \xi_2, \Gamma_2}$$

$$\text{L}\oplus_m \ \frac{\Gamma_1, \xi_1 \to \Gamma_2 \quad \Gamma_1, \xi_2 \to \Gamma_2}{\Gamma_1, \xi_1 \oplus_m \xi_2 \to \Gamma_2} \qquad \text{R}_i\oplus_m \ \frac{\Gamma_1 \to \xi_i, \Gamma_2}{\Gamma_1 \to \xi_1 \oplus_m \xi_2, \Gamma_2}, \quad i = 1,2.$$

The deductive system $\mathcal{D}_n$ for NLL, is such that its language, $L_n$, is freely generated by the signature $C_n$:

- $C_{n0}$ is the set $P \cup \{p_\$\}$ where $p_\$$ is a symbol not in $P$.
- $C_{n2} = \{\oplus_n, \otimes_n, \multimap\}$;
- $C_{ni} = \emptyset$, for $i = 1$ or $i \geq 3$;

and, the $L_n$-function

$$\text{var}_n$$

considered in this example associates to each formula $\varphi$ of $L_n$ the symbols of $P \cup \{p_\$\}$ present in $\varphi$. Sequents are pairs of finite multisets of formulas, and its sequent calculus is defined by the following rules:

$$\text{Init} \ \frac{}{\xi_1 \to \xi_1} \qquad \text{Cut} \ \frac{\Gamma_1 \to \xi_1 \quad \xi_1, \Gamma_1' \to \xi_3}{\Gamma_1, \Gamma_1' \to \xi_3}$$

$$\text{L}\otimes_n \ \frac{\Gamma_1, \xi_1, \xi_2 \to \xi_3}{\Gamma_1, \xi_1 \otimes_n \xi_2 \to \xi_3} \qquad \text{R}\otimes_n \ \frac{\Gamma_1 \to \xi_1 \quad \Gamma_1' \to \xi_2}{\Gamma_1, \Gamma_1' \to \xi_1 \otimes_n \xi_2}$$

$$\text{L}\multimap \ \frac{\Gamma_1 \to \xi_1 \quad \Gamma_1', \xi_2 \to \xi_3}{\Gamma_1, \Gamma_1', \xi_1 \multimap \xi_2 \to \xi_3} \qquad \text{R}\multimap \ \frac{\Gamma_1, \xi_1 \to \xi_2}{\Gamma_1 \to \xi_1 \multimap \xi_2}$$

$$\text{L}\oplus_n \ \frac{\Gamma_1, \xi_1 \to \xi_3 \quad \Gamma_1, \xi_2 \to \xi_3}{\Gamma_1, \xi_1 \oplus_n \xi_2 \to \xi_3} \qquad \text{R}_i\oplus_n \ \frac{\Gamma_1 \to \xi_i}{\Gamma_1 \to \xi_1 \oplus_n \xi_2}, \quad i = 1,2.$$

The deductive system $\mathcal{D}_{n+_\$ma}$ is such that its signature is the union of the signatures of $\mathcal{D}_{n+ma}$ and $\mathcal{D}_n$, that is, is the signature of $\mathcal{D}_{n+ma}$, enriched with the additional symbol $p_\$$. We denote its language by $L_{n+_\$ma}$. The rules of this deductive system are the union of the rules of $\mathcal{D}_{n+ma}$ and $\mathcal{D}_n$ plus two additional rules expressing that $p_\$$ behaves as $\perp$:

$$\text{L}p_\$ \ \frac{}{p_\$ \to} \qquad \text{R}p_\$ \ \frac{\Gamma_1 \to}{\Gamma_1 \to p_\$}$$

together with the following derived rules

$$\text{L}\neg_\$ \ \frac{\Gamma_1 \to \xi_1}{\Gamma_1, \neg_\$ \xi_1 \to} \qquad \text{R}\neg_\$ \ \frac{\Gamma_1, \xi_1 \to}{\Gamma_1 \to \neg_\$ \xi_1}$$

where $\neg_\$ \varphi$ denotes the formula $\varphi \multimap p_\$$.

**Translation**

Consider the following maps $h_1, h_2 : L_{n+ma} \to L_n$ and $h : L_n \to L_{n+ma}$ inductively defined as follows:

- $h_1(\bot) = p_\$$;
- $h_1(\mathbf{1}) = p_\$ \multimap p_\$$;
- $h_1(p) = p$ for each $p$ in $P$;
- $h_1(\sim \varphi) = \neg_\$ h_1(\varphi)$;
- $h_1(\varphi_1 \nabla \varphi_2) = \neg_\$((\neg_\$ h_1(\varphi_1)) \otimes_n (\neg_\$ h_1(\varphi_2)))$;
- $h_1(\varphi_1 \& \varphi_2) = \neg_\$((\neg_\$ h_1(\varphi_1)) \oplus_n (\neg_\$ h_1(\varphi_2)))$;
- $h_1(\varphi_1 \circ_m \varphi_2) = (\neg_\$\neg_\$ h_1(\varphi_1)) \circ_n (\neg_\$\neg_\$ h_1(\varphi_2))$, for $\circ \in \{\otimes, \oplus\}$;
- $h_1(\varphi_1 \circ \varphi_2) = (\neg_\$\neg_\$ h_1(\varphi_1)) \circ (\neg_\$\neg_\$ h_1(\varphi_2))$, for $\circ \in \{\otimes_n, \oplus_n, \multimap\}$;

- $h_2(\varphi) = \neg_\$\neg_\$ h_1(\varphi)$;

- $h(p_\$) = \bot$;
- $h(p) = p$ for each $p$ in $P$;
- $h(\varphi_1 \circ \varphi_2) = h(\varphi_1) \circ h(\varphi_2)$ for $\circ \in \{\otimes_n, \multimap, \oplus_n\}$.

We now prove that $\langle h_1, h_2, h\rangle$ satisfies the conditions of a Maehara generalized translation schema from $(L_{n+ma}, \vdash^l_{\mathcal{D}_{n+ma}})$ to $(L_n, \vdash^l_{\mathcal{D}_n})$ via $(L_{n+_\$ma}, \vdash^l_{\mathcal{D}_{n+_\$ma}})$ with respect to $\text{var}_{n+ma}$ and $\text{var}_n$. Due to space constraints and in order to improve the readability of the paper we sketch the proofs and omit its details. We start by showing that it satisfies condition 1 of the definition of a Maehara generalized translation schema.

**Lemma 5.1** The pair of maps $h_1$ and $h_2$ is such that $h_1(\Psi) \vdash^l_{\mathcal{D}_n} h_2(\varphi)$ whenever $\Psi \vdash^l_{\mathcal{D}_{n+ma}} \varphi$ and $\Psi$ and $\{\varphi\}$ are contained in $L_{n+ma}$.

*Proof sketch:* The proof follows by showing by complete induction on the depth of sequent derivations that if $\Psi \to \Delta$ is a theorem in $\mathcal{D}_{n+ma}$ then $h_1(\Psi), \neg_\$ h_1(\Delta) \to p_\$$ is a theorem in $\mathcal{D}_n$. For full details consult the Appendix [2]. $\diamond$

In the next lemma it is established that $h$ satisfies condition 3 and condition 4 of the definition of Maehara generalized translation schema.

**Lemma 5.2** The map $h$ is such that

- $\Gamma \vdash^l_{\mathcal{D}_{n+ma}} h(\psi)$ whenever $\Gamma \vdash^l_{\mathcal{D}_{n+_\$ma}} \psi$
- $h(\Psi), \Delta \vdash^l_{\mathcal{D}_{n+ma}} \varphi$ whenever $\Psi, \Delta \vdash^l_{\mathcal{D}_{n+_\$ma}} \varphi$

where $\Gamma \cup \Delta \cup \{\varphi\}$ is contained in $L_{n+ma}$ and $\Psi \cup \{\psi\}$ is contained in $L_n$.

*Proof sketch:* The proof follows by showing by complete induction on the depth of a sequent derivation that if $\Psi$ and $\Delta$ are sets contained in $L_{n+_\$ma}$ and $\Psi \to \Delta$ is a theorem in $\mathcal{D}_{n+_\$ma}$ then $\overline{h}(\Psi) \to \overline{h}(\Delta)$ is a theorem in $\mathcal{D}_{n+ma}$ where $\overline{h}$ is a map from $L_{n+_\$ma}$ to $L_{n+ma}$ extending $h$ by establishing also an identity on the connectives of MALL. For full details consult the Appendix [2]. $\diamond$

In order to show that the proposed translation satisfies condition 2 of the definition of Maehara generalized translation schema, that is, $\varphi \vdash^l_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$ and $h_2(\varphi) \vdash^l_{\mathcal{D}_{n+_\$ma}} \varphi$ for $\varphi$ in $L_{n+ma}$, we use the following auxiliary map $h_a$ from $L_{n+ma}$ to $L_{n+ma}$ inductively defined as follows:

- $h_a(\bot) = \bot$;
- $h_a(p) = \sim\sim p$ for $p$ in $P$;
- $h_a(\mathbf{1}) = \sim\sim \mathbf{1}$;
- $h_a(\sim \varphi) = \sim h_a(\varphi)$;
- $h_a(\varphi_1 \circ \varphi_2) = \sim\sim ((\sim\sim h_a(\varphi_1)) \circ (\sim\sim h_a(\varphi_2)))$ for $\circ \in \{\otimes_m, \oplus_m, \otimes_n, \oplus_n\}$;
- $h_a(\varphi_1 \circ \varphi_2) = (\sim\sim h_a(\varphi_1)) \circ (\sim\sim h_a(\varphi_2))$ for $\circ \in \{\nabla, \&, \multimap\}$.

**Lemma 5.3** The maps $h_1$, $h_2$ and $h_a$ are such that

1. $\varphi \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_a(\varphi)$
2. $h_a(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$
3. $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{n+_\$ma}} h_a(\varphi)$

for $\varphi$ in $L_{n+ma}$.

*Proof sketch:* The proofs follow by complete induction on the complexity of a formula. For full details consult the Appendix [2]. $\diamond$

**Proposition 5.4** The tuple $\langle h_1, h_2, h\rangle$ is a Maehara generalized translation schema from $(L_{n+ma}, \vdash^l_{\mathcal{D}_{n+ma}})$ to $(L_n, \vdash^l_{\mathcal{D}_n})$ via $(L_{n+_\$ma}, \vdash^l_{\mathcal{D}_{n+_\$ma}})$ with respect to $\mathrm{var}_{n+ma}$ and $\mathrm{var}_n$.

**Proof:** The domain and co-domain of $h_1$, $h_2$ and $h$ are as indicated in the definition of Maehara generalized translation schema. We now show that they satisfy all the conditions of that definition. Condition 1. follows immediately by Lemma 5.1; condition 2. follows by Lemma 5.3; condition 3. and condition 4. follow by Lemma 5.2; condition 5. follows immediately by induction on the structure of a formula taking into account the way $h_1$ is defined inductively on the structure of a formula, and since $h_1(p) = p$ and $h_2(p) = \neg_\$\neg_\$ p$; condition 6. follows since $\mathrm{var}_n(h(h_i(\varphi))) = \mathrm{var}_n(\varphi)$ for $i = 1, 2$ and taking into account the way $h$ is defined inductively on the structure of a formula. $\diamond$

Using the fact that the proposed translation satisfies the conditions of the Maehara generalized translation schema from $(L_{n+ma}, \vdash^l_{\mathcal{D}_{n+ma}})$ to $(L_n, \vdash^l_{\mathcal{D}_n})$ via $(L_{n+_\$ma}, \vdash^l_{\mathcal{D}_{n+_\$ma}})$ with respect to $\mathrm{var}_{n+ma}$ and $\mathrm{var}_n$, see Proposition 5.4, it is possible to conclude by Corollary 3.4 the following result.

**Corollary 5.5** The logic NLL⊕MALL has Craig/Maehara interpolation over its local consequence relation with respect to $\mathrm{var}_{n+ma}$ whenever NLL has Craig/Maehara interpolation over its local consequence relation with respect to $\mathrm{var}_n$.

It is not possible to conclude, by using Corollary 3.4, preservation of Craig interpolation and Maehara interpolation over global consequence, since $\mathcal{D}_{n+ma}$ does not satisfy the conditions of Proposition 4.2. So, preservation of that properties has to be proved by showing that $\langle h_1, h_2, h\rangle$ satisfies the conditions of the Maehara generalized translation schema from $(L_{n+ma}, \vdash^g_{\mathcal{D}_{n+ma}})$ to $(L_n, \vdash^g_{\mathcal{D}_n})$ via $(L_{n+_\$ma}, \vdash^g_{\mathcal{D}_{n+_\$ma}})$ with respect to $\mathrm{var}_{n+ma}$ and $\mathrm{var}_n$, and then use Corollary 3.4.

**Lemma 5.6** The pair of maps $h_1$ and $h_2$ is such that $h_1(\Psi) \vdash^g_{\mathcal{D}_n} h_2(\varphi)$ whenever $\Psi \vdash^g_{n+ma} \varphi$ and $\Psi$ and $\{\varphi\}$ are contained in $L_{n+ma}$.

*Proof sketch:* The proof follows by showing by complete induction on the depth of a sequent derivation that if $\Gamma \to \Delta$ is derivable in $\mathcal{D}_{n+ma}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ then $h_1(\Gamma), \neg_\$ h_1(\Delta) \to p_\$$ is derivable in $\mathcal{D}_n$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$. For full details consult the Appendix [2]. $\diamond$

We now show that $h$ satisfies also condition 3 and condition 4 of the definition of Maehara generalized translation schema over the global consequence systems.

**Lemma 5.7** The map $h$ is such that

- $\Gamma \vdash^g_{\mathcal{D}_{n+ma}} h(\psi)$ whenever $\Gamma \vdash^g_{\mathcal{D}_{n+_\$ma}} \psi$
- $h(\Psi), \Delta \vdash^g_{\mathcal{D}_{n+ma}} \varphi$ whenever $\Psi, \Delta \vdash^g_{\mathcal{D}_{n+_\$ma}} \varphi$

where $\Gamma \cup \Delta \cup \{\varphi\}$ is contained in $L_{n+ma}$ and $\Psi \cup \{\psi\}$ is contained in $L_n$.

*Proof sketch:* We show by complete induction on the depth of a sequent derivation that if $\Gamma$ and $\Delta$ are sets contained in $L_{n+_\$ma}$ and $\Gamma \to \Delta$ is derivable in $\mathcal{D}_{n+_\$ma}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ then $\overline{h}(\Gamma) \to \overline{h}(\Delta)$ is derivable in $\mathcal{D}_{n+ma}$ from the set of hypothesis $\{\to \overline{h}(\psi) : \psi \in \Psi\}$ where $\overline{h}$ is a map from $L_{n+_\$ma}$ to $L_{n+ma}$ extending $h$ by establishing an identity on the connectives not in $\mathcal{D}_n$. For full details consult the Appendix [2]. $\diamond$

Finally we prove that $\langle h_1, h_2, h\rangle$ satisfies also condition 2 of Maehara generalized translation schema over the global consequence systems. The proof of the lemma follows immediately by Proposition 4.1 since similar deductions hold for the local consequence relation, see Lemma 5.3.

**Lemma 5.8** The maps $h_a$, $h_1$ and $h_2$ are such that

1. $\varphi \dashv\vdash^g_{\mathcal{D}_{n+_\$ma}} h_a(\varphi)$
2. $h_a(\varphi) \dashv\vdash^g_{\mathcal{D}_{n+_\$ma}} h_1(\varphi)$
3. $h_2(\varphi) \dashv\vdash^g_{\mathcal{D}_{n+_\$ma}} h_a(\varphi)$

for $\varphi$ in $L_{n+ma}$.

Henceforth, using the previous results, Lemma 5.6, Lemma 5.7 and Lemma 5.8, we conclude that the proposed translation is also a Maehara generalized translation schema over the global consequence systems with respect to $\text{var}_{n+ma}$ and $\text{var}_n$.

**Proposition 5.9** The tuple $\langle h_1, h_2, h\rangle$ is a Maehara generalized translation schema from $(L_{n+ma}, \vdash^g_{\mathcal{D}_{n+ma}})$ to $(L_n, \vdash^g_{\mathcal{D}_n})$ via $(L_{n+_\$ma}, \vdash^g_{\mathcal{D}_{n+_\$ma}})$ with respect to $\text{var}_{n+ma}$ and $\text{var}_n$.

Therefore it is now possible to conclude that NLL$\oplus$MALL enjoys also Craig interpolation and Maehara interpolation over its global consequence relation with respect to $\text{var}_{n+ma}$ if NLL enjoys them over with respect to $\text{var}_n$.

**Corollary 5.10** Logic NLL$\oplus$MALL enjoys Craig/Maehara interpolation over its global consequence relation with respect to $\text{var}_{n+ma}$ whenever NLL enjoys Craig/Maehara interpolation over its global consequence relation with respect to $\text{var}_n$.

## 5.2 Interpolation in FLew⊕MLL

In this example we prove that the full Lambek calculus with exchange and weakening enriched with multiplicative classical linear logic rules enjoys Craig interpolation and Maehara interpolation over its global consequence relation. These properties are proved by translating this logic to the fragment of full Lambek calculus with exchange, FLe, which enjoys those properties. As far as we know it was not known before whether full Lambek calculus with exchange and weakening enriched with multiplicative classical linear logic rules enjoyed Craig interpolation and Maehara interpolation over its global consequence relation. The novelty of the translation we propose resides on the combination of Kiriyama-Ono style translation features, see [17], with Kolmogorov-Gentzen-Gödel style translation features.

### Deductive systems

We start by presenting the deductive system $\mathcal{D}_{w+m}$ for the full Lambek calculus with exchange and weakening enriched with multiplicative classical linear logic rules. We note that this system can be seen as the combination of full Lambek calculus with exchange and weakening (an extended BCK logic), FLew, and multiplicative linear logic, MLL, and so is represented as FLew⊕MLL, see [17, 28, 3, 16]. The deductive system $\mathcal{D}_{w+m}$ is such that, its language, $L_{w+m}$, is generated by the signature $C_{w+m}$ with:

- $C_{w+m0}$ contains $\bot$, $\bot_m$, $\mathbf{0}$, $\mathbf{1}$, $\mathbf{1}_m$ and a denumerable set of propositional symbols $P$,
- $C_{w+m1} = \{\sim\}$,
- $C_{w+m2} = \{*, \supset, \wedge, \vee, \otimes, \nabla\}$, and
- $C_{w+mk} = \emptyset$, for $k \geq 3$,

where we may also use $\neg$ as an abbreviation for the Lambek negation, that is, $\neg\varphi$ is an abbreviation of $\varphi \supset \bot$. The $L_{w+m}$-function

$$\mathrm{var}_{w+m}$$

considered in this example associates to each formula $\varphi$ of $L_{w+m}$ the symbols of $P$ present in $\varphi$. Sequents in $\mathcal{D}_{w+m}$ are pairs of finite multisets of formulas, and its sequent calculus is defined by the following rules:

Init $\dfrac{}{\xi_1 \to \xi_1}$

Lw $\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1, \xi_1 \to \Gamma_2} \ \triangleleft\ |\Gamma_2| \leq 1$ Rw $\dfrac{\Gamma_1 \to}{\Gamma_1 \to \xi_1}$

L$\bot$ $\dfrac{}{\bot \to}$ R$\mathbf{1}$ $\dfrac{}{\to \mathbf{1}}$

L$*$ $\dfrac{\Gamma_1, \xi_1, \xi_2 \to \Gamma_2}{\Gamma_1, \xi_1 * \xi_2 \to \Gamma_2} \ \triangleleft\ |\Gamma_2| \leq 1$ R$*$ $\dfrac{\Gamma_1 \to \xi_1 \quad \Gamma_1' \to \xi_2}{\Gamma_1, \Gamma_1' \to \xi_1 * \xi_2}$

L$\supset$ $\dfrac{\Gamma_1 \to \xi_1 \quad \Gamma_1', \xi_2 \to \Gamma_2'}{\Gamma_1, \Gamma_1', \xi_1 \supset \xi_2 \to \Gamma_2'} \ \triangleleft\ |\Gamma_2'| \leq 1$ R$\supset$ $\dfrac{\Gamma_1, \xi_1 \to \xi_2}{\Gamma_1 \to \xi_1 \supset \xi_2}$

L$_k\wedge$ $\dfrac{\Gamma_1, \xi_k \to \Gamma_2}{\Gamma_1, \xi_1 \wedge \xi_2 \to \Gamma_2} \ \triangleleft\ |\Gamma_2| \leq 1, \quad k = 1, 2$ R$\wedge$ $\dfrac{\Gamma_1 \to \xi_1 \quad \Gamma_1 \to \xi_2}{\Gamma_1 \to \xi_1 \wedge \xi_2}$

L$\vee$ $\dfrac{\Gamma_1, \xi_1 \to \Gamma_2 \quad \Gamma_1, \xi_2 \to \Gamma_2}{\Gamma_1, \xi_1 \vee \xi_2 \to \Gamma_2} \ \triangleleft\ |\Gamma_2| \leq 1$ R$_k\vee$ $\dfrac{\Gamma_1 \to \xi_k}{\Gamma_1 \to \xi_1 \vee \xi_2}, \quad k = 1, 2$

L**0** $\dfrac{}{\Gamma_1,\mathbf{0}\rightarrow\Gamma_2} \lhd |\Gamma_2| \leq 1,$

L$\mathbf{1}_m$ $\dfrac{\Gamma_1\rightarrow\Gamma_2}{\Gamma_1,\mathbf{1}_m\rightarrow\Gamma_2}$

R$\mathbf{1}_m$ $\dfrac{}{\rightarrow\mathbf{1}_m}$

L$\bot_m$ $\dfrac{}{\bot_m\rightarrow}$

R$\bot_m$ $\dfrac{\Gamma_1\rightarrow\Gamma_2}{\Gamma_1\rightarrow\bot_m,\Gamma_2}$

L$\sim$ $\dfrac{\Gamma_1\rightarrow\xi_1,\Gamma_2}{\Gamma_1,\sim\xi_1\rightarrow\Gamma_2}$

R$\sim$ $\dfrac{\Gamma_1,\xi_1\rightarrow\Gamma_2}{\Gamma_1\rightarrow\sim\xi_1,\Gamma_2}$

L$\otimes$ $\dfrac{\Gamma_1,\xi_1,\xi_2\rightarrow\Gamma_2}{\Gamma_1,\xi_1\otimes\xi_2\rightarrow\Gamma_2}$

R$\otimes$ $\dfrac{\Gamma_1\rightarrow\xi_1,\Gamma_2 \quad \Gamma_1'\rightarrow\xi_2,\Gamma_2'}{\Gamma_1,\Gamma_1'\rightarrow\xi_1\otimes\xi_2,\Gamma_2,\Gamma_2'}$

L$\nabla$ $\dfrac{\Gamma_1,\xi_1\rightarrow\Gamma_2 \quad \Gamma_1',\xi_2\rightarrow\Gamma_2'}{\Gamma_1,\Gamma_1',\xi_1\nabla\xi_2\rightarrow\Gamma_2,\Gamma_2'}$

R$\nabla$ $\dfrac{\Gamma_1\rightarrow\xi_1,\xi_2,\Gamma_2}{\Gamma_1\rightarrow\xi_1\nabla\xi_2,\Gamma_2}$

The deductive system $\mathcal{D}_e$ for full Lambek calculus with exchange, FLe, is such that, its language, $L_e$, is generated by the signature $C_e$ with:

- $C_{e0}$ contains $\bot$, $\mathbf{0}$, $\mathbf{1}$ and $P$,
- $C_{e2} = \{*, \wedge, \supset, \vee\}$, and
- $C_{ek} = \emptyset$ for $k \neq 0, 2$,

where we may also use $\neg$ as the usual abbreviation. The $L_e$-function

$$\mathrm{var}_e$$

considered in this example associates to each formula $\varphi$ of $L_e$ the symbols of $P$ present in $\varphi$. Sequents in $\mathcal{D}_e$ are pairs of finite multisets of formulas, and its sequent calculus is defined by the following rules:

Init $\dfrac{}{\xi_1\rightarrow\xi_1}$

L**0** $\dfrac{}{\Gamma_1,\mathbf{0}\rightarrow\Gamma_2} \lhd |\Gamma_2| \leq 1$

L**1** $\dfrac{\Gamma_1\rightarrow\Gamma_2}{\Gamma_1,\mathbf{1}\rightarrow\Gamma_2} \lhd |\Gamma_2| \leq 1$

R**1** $\dfrac{}{\rightarrow\mathbf{1}}$

L$\bot$ $\dfrac{}{\bot\rightarrow}$

R$\bot$ $\dfrac{\Gamma_1\rightarrow}{\Gamma_1\rightarrow\bot}$

L$*$ $\dfrac{\Gamma_1,\xi_1,\xi_2\rightarrow\Gamma_2}{\Gamma_1,\xi_1*\xi_2\rightarrow\Gamma_2} \lhd |\Gamma_2| \leq 1$

R$*$ $\dfrac{\Gamma_1\rightarrow\xi_1 \quad \Gamma_1'\rightarrow\xi_2}{\Gamma_1,\Gamma_1'\rightarrow\xi_1*\xi_2}$

L$\supset$ $\dfrac{\Gamma_1\rightarrow\xi_1 \quad \Gamma_1',\xi_2\rightarrow\Gamma_2'}{\Gamma_1,\Gamma_1',\xi_1\supset\xi_2\rightarrow\Gamma_2'} \lhd |\Gamma_2'| \leq 1$

R$\supset$ $\dfrac{\Gamma_1,\xi_1\rightarrow\xi_2}{\Gamma_1\rightarrow\xi_1\supset\xi_2}$

L$_k\wedge$ $\dfrac{\Gamma_1,\xi_k\rightarrow\Gamma_2}{\Gamma_1,\xi_1\wedge\xi_2\rightarrow\Gamma_2} \lhd |\Gamma_2| \leq 1, \quad k = 1,2$

R$\wedge$ $\dfrac{\Gamma_1\rightarrow\xi_1 \quad \Gamma_1\rightarrow\xi_2}{\Gamma_1\rightarrow\xi_1\wedge\xi_2}$

L$\vee$ $\dfrac{\Gamma_1,\xi_1\rightarrow\Gamma_2 \quad \Gamma_1,\xi_2\rightarrow\Gamma_2}{\Gamma_1,\xi_1\vee\xi_2\rightarrow\Gamma_2} \lhd |\Gamma_2| \leq 1$

R$_k\vee$ $\dfrac{\Gamma_1\rightarrow\xi_k}{\Gamma_1\rightarrow\xi_1\vee\xi_2}, \quad k = 1,2.$

moreover, we may use the following derived rules

L$\neg$ $\dfrac{\Gamma_1\rightarrow\xi_1}{\Gamma_1,\neg\xi_1\rightarrow}$

R$\neg$ $\dfrac{\Gamma_1,\xi_1\rightarrow}{\Gamma_1\rightarrow\neg\xi_1}$

in order to simplify the presentation of derivations.

**Translation**

Consider the following maps $h_1, h_2 : L_{w+m} \to L_e$ and $h : L_e \to L_{w+m}$ inductively defined as follows:

- $h_2(\varphi) = \neg\neg h_1(\varphi)$;
- $h_1(\varphi) = (h_1^-(\varphi) \vee \bot)$;
- $h_1^-(p) = p$ for $p$ in $P$;
- $h_1^-(a) = a$ for $a$ in $\{\bot, \mathbf{0}, \mathbf{1}\}$;
- $h_1^-(\varphi_1 \supset \varphi_2) = (h_1^l(\varphi_1) \supset (\neg\neg h_1(\varphi_2)))$;
- $h_1^-(\varphi_1 \circ \varphi_2) = ((\neg\neg h_1(\varphi_1)) \circ (\neg\neg h_1(\varphi_2)))$ for $\circ$ in $\{\wedge, \vee\}$;
- $h_1^-(\bot_m) = \bot$;
- $h_1^-(\mathbf{1}_m) = \mathbf{1}$;
- $h_1^-(\sim \varphi_1) = \neg h_1^l(\varphi_1)$;
- $h_1^-(\varphi_1 \circ \varphi_2) = (\neg\neg h_1(\varphi_1) * \neg\neg h_1(\varphi_2))$ for $\circ$ in $\{\otimes, *\}$;
- $h_1^-(\varphi_1 \nabla \varphi_2) = (\neg h_2(\varphi_1)) \supset (\neg\neg h_1(\varphi_2))$;

- $h_1^l(\varphi) = (h_1^{l\,-}(\varphi) \wedge \mathbf{1})$;
- $h_1^{l\,-}(p) = p$ for $p$ in $P$;
- $h_1^{l\,-}(a) = a$ for $a$ in $\{\bot, \mathbf{0}, \mathbf{1}\}$;
- $h_1^{l\,-}(\varphi_1 \supset \varphi_2) = ((\neg\neg h_1(\varphi_1)) \supset h_1^l(\varphi_2))$;
- $h_1^{l\,-}(\varphi_1 \circ \varphi_2) = (h_1^l(\varphi_1) \circ h_1^l(\varphi_2))$ for $\circ$ in $\{\wedge, \vee\}$;
- $h_1^{l\,-}(\bot_m) = \bot$;
- $h_1^{l\,-}(\mathbf{1}_m) = \mathbf{1}$;
- $h_1^{l\,-}(\sim \varphi_1) = \neg h_2(\varphi_1)$;
- $h_1^{l\,-}(\varphi_1 \circ \varphi_2) = (h_1^l(\varphi_1) * h_1^l(\varphi_2))$ for $\circ$ in $\{\otimes, *\}$;
- $h_1^{l\,-}(\varphi_1 \nabla \varphi_2) = ((\neg h_1^l(\varphi_1)) \supset h_1^l(\varphi_2))$;

- $h(\varphi) = \varphi$.

We now show that $\langle h_1, h_2, h \rangle$ constitute a Maehara generalized translation schema from $(L_{w+m}, \vdash^g_{\mathcal{D}_{w+m}})$ to $(L_e, \vdash^g_{\mathcal{D}_e})$ via $(L_{w+m}, \vdash^g_{\mathcal{D}_{w+m}})$ with respect to $\mathrm{var}_{w+m}$ and $\mathrm{var}_e$. We start by observing that $h$ satisfies condition 3 and condition 4 of the definition of Maehara generalized translation schema, that is,

- $\Gamma \vdash^g_{\mathcal{D}_{w+m}} h(\psi)$ whenever $\Gamma \vdash^g_{\mathcal{D}_{w+m}} \psi$
- $h(\Psi) \vdash^g_{\mathcal{D}_{w+m}} \varphi$ whenever $\Psi \vdash^g_{\mathcal{D}_{w+m}} \varphi$

where $\Gamma \cup \{\varphi\}$ is contained in $L_{w+m}$ and $\Psi \cup \{\psi\}$ is contained in $L_e$. Condition 1 of the definition of Maehara generalized translation schema is also satisfied by the proposed translation, as shown in the following lemma.

**Lemma 5.11** The pair of maps $h_1$ and $h_2$ is such that $h_1(\Psi) \vdash^g_{\mathcal{D}_e} h_2(\varphi)$ whenever $\Psi \vdash^g_{\mathcal{D}_{w+m}} \varphi$ and $\Psi$ and $\{\varphi\}$ are contained in $L_{w+m}$.

*Proof sketch:* The proof follows by complete induction on the depth of a sequent derivation by showing that if $\Gamma \to \Delta$ is derivable in $\mathcal{D}_{w+m}$ from the set of hypothesis $\{\to \psi : \psi \in \Psi\}$ then $h^l_1(\Gamma), \neg h_1(\Delta) \to$ is derivable in $\mathcal{D}_e$ from the set of hypothesis $\{\to h_1(\psi) : \psi \in \Psi\}$. For full details consult the Appendix [2]. $\diamond$

In order to prove that the tuple $\langle h_1, h_2, h\rangle$ satisfies condition 2, we consider an auxiliary map $h_a$ that transforms a formula to an equivalent one that can circumvent the difficulties imposed by reasoning with rules with cardinality restrictions on its right side. The map $h_a$ is inductively defined as follows:

- $h_a(p) = \sim\sim p$ for $p$ in $P$;
- $h_a(a) = a$ for $a$ in $\{\bot, \mathbf{0}, \mathbf{1}\}$;
- $h_a(\bot_m) = \bot_m$;
- $h_a(\mathbf{1}_m) = \mathbf{1}_m$;
- $h_a(\varphi_1 \circ \varphi_2) = \sim\sim (h_a(\varphi_1) \circ h_a(\varphi_2))$ for $\circ$ in $\{\otimes, *, \supset, \wedge, \vee\}$;
- $h_a(\varphi_1 \nabla \varphi_2) = \sim\sim ((\sim\sim h_a(\varphi_1))\nabla(\sim\sim h_a(\varphi_2)))$;
- $h_a(\sim \varphi_1) = \sim h_a(\varphi_1)$.

**Lemma 5.12** The maps $h_1$, $h_2$, $h^l_1$ and $h_a$ are such that

1. $h_1(\varphi) \dashv\vdash^g_{\mathcal{D}_{w+m}} h_a(\varphi)$, $h_2(\varphi) \dashv\vdash^g_{\mathcal{D}_{w+m}} h_a(\varphi)$ and $h_a(\varphi) \dashv\vdash^g_{\mathcal{D}_{w+m}} h^l_1(\varphi)$
2. $\varphi \dashv\vdash^g_{\mathcal{D}_{w+m}} h_a(\varphi)$

for $\varphi$ in $L_{w+m}$.

*Proof sketch:* The proof follows by complete induction on the complexity of the formula. For full details consult the Appendix [2]. $\diamond$

Putting together the previous lemmas we can now prove the following proposition.

**Proposition 5.13** The tuple $\langle h_1, h_2, h\rangle$ is a Maehara generalized translation schema from $(L_{w+m}, \vdash^g_{\mathcal{D}_{w+m}})$ to $(L_e, \vdash^g_{\mathcal{D}_e})$ via $(L_{w+m}, \vdash^g_{\mathcal{D}_{w+m}})$ with respect to $\text{var}_{w+m}$ and $\text{var}_e$.

**Proof:** The proof follows straightforwardly taking into account the previous lemmas, Lemma 5.11 and Lemma 5.12, the observation that $h$ satisfies condition 3 and condition 4 of the definition of Maehara generalized translation schema (observe that the deductive systems involved are the same and $h$ is the identity), the fact that $\mathbf{1}$ and $\bot$ are not symbols relevant for interpolation, the fact that $\text{var}_{w+m}(h(h_i(\varphi))) = \text{var}_{w+m}(\varphi)$, and by taking into account the way $h$ is defined inductively on the structure of a formula. $\diamond$

Moreover, by Proposition 3.3, there is a Craig generalized translation schema from $(L_{w+m}, \vdash^g_{\mathcal{D}_{w+m}})$ to $(L_e, \vdash^g_{\mathcal{D}_e})$ via $(L_{w+m}, \vdash^g_{\mathcal{D}_{w+m}})$ with respect to $\mathrm{var}_{w+m}$ and $\mathrm{var}_e$. So, by Theorem 3.1 and Theorem 3.2 we can now establish the following corollary, capitalizing on the fact that FLe enjoys Craig interpolation and Maehara interpolation over its global consequence relation with respect to $\mathrm{var}_e$, see [8, 23].

**Corollary 5.14** The logic FLew⊕MLL enjoys Craig interpolation and Maehara interpolation with respect to $\mathrm{var}_{w+m}$ over its global consequence relation.

## 5.3 Interpolation in IL⊕CL

In this subsection we investigate whether the logic resulting from the combination by fibring of intuitionistic logic and classical logic enjoys Craig interpolation and Maehara interpolation either over its local consequence or over its global consequence. We prove that this is indeed the case by reducing this question to the same question over intuitionistic logic. The reduction is made by defining a Kolmogorov-Gentzen-Gödel style translation between those logics. The motivation of this example is to show how the results obtained in this work can be used to answer this interesting question, and not in the answer itself, which was already investigated by the authors in [4], in the context of fibring, using preliminary tools and methods that inspired the ones developed herein.

The language of the logic resulting from the combination by fibring of intuitionistic logic and classical logic is generated from the signature with the connectives of both logics. The set of rules of a deductive system for the combined logic is simply the union of the set of rules of intuitionistic logic and the set of rules of classical logic.

#### Deductive systems

We start by presenting the deductive system $\mathcal{D}_{i+c}$ for the logic resulting from the fibring of intuitionistic logic and classical logic. The language of this system, $L_{i+c}$, is freely generated by the signature $C_{i+c}$ defined as follows:

- $C_{i+c0}$ contains $\bot_i$, $\bot_c$ and a denumerable set of propositional symbols $P$;
- $C_{i+c2} = \{\wedge_i, \vee_i, \Rightarrow_i, \Rightarrow_c\}$;
- $C_{i+ck} = \emptyset$ for $k \neq 0, 2$;

where we may use $\wedge_c$, $\vee_c$ and $\neg_c$ as the usual classical abbreviations, and $\neg_i$ as the abbreviation for intuitionistic negation. The $L_{i+c}$-function

$$\mathrm{var}_{i+c}$$

considered in this example associates to each formula $\varphi$ of $L_{i+c}$ the symbols of $P$ present in $\varphi$. Sequents in $\mathcal{D}_{i+c}$ are pairs of multisets of formulas and its sequent calculus is defined by the following rules:

$$\mathrm{Lw}\ \frac{\Gamma_1 \to \Gamma_2}{\Gamma_1, \xi_1 \to \Gamma_2} \qquad \mathrm{Rw}\ \frac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \xi_1}$$

$$\mathrm{Lc}\ \frac{\Gamma_1, \xi_1, \xi_1 \to \Gamma_2}{\Gamma_1, \xi_1 \to \Gamma_2} \qquad \mathrm{Rc}\ \frac{\Gamma_1 \to \Gamma_2, \xi_1, \xi_1}{\Gamma_1 \to \Gamma_2, \xi_1}$$

$$\mathrm{Ax}\ \frac{}{\xi_1 \to \xi_1}$$

$$\mathrm{L}{\Rightarrow_i}\ \frac{\Gamma_1 \to \xi_1 \quad \Gamma_1', \xi_2 \to \Gamma_2'}{\Gamma_1, \Gamma_1', \xi_1 \Rightarrow_i \xi_2 \to \Gamma_2'} \lhd |\Gamma_2'| \leq 1 \qquad \mathrm{R}{\Rightarrow_i}\ \frac{\Gamma_1, \xi_1 \to \xi_2}{\Gamma_1 \to \xi_1 \Rightarrow_i \xi_2}$$

$$\mathrm{L}{\wedge_{ij}}\ \frac{\Gamma_1, \xi_j \to \Gamma_2}{\Gamma_1, \xi_1 \wedge_i \xi_2 \to \Gamma_2} \lhd |\Gamma_2| \leq 1 \ (j = 1, 2) \qquad \mathrm{R}{\wedge_i}\ \frac{\Gamma_1 \to \xi_1 \quad \Gamma_1 \to \xi_2}{\Gamma_1 \to \xi_1 \wedge_i \xi_2}$$

L$\vee_i$ $\frac{\Gamma_1,\xi_1\to\Gamma_2 \quad \Gamma_1,\xi_2\to\Gamma_2}{\Gamma_1,\xi_1\vee_i\xi_2\to\Gamma_2}$ $\lhd$ $|\Gamma_2| \leq 1$

R$\vee_{ij}$ $\frac{\Gamma_1\to\xi_j}{\Gamma_1\to\xi_1\vee_i\xi_2}$ $(j = 1, 2)$

$\bot_i$ $\frac{}{\bot_i\to}$

$\bot_c$ $\frac{}{\bot_c\to}$

L$\Rightarrow_c$ $\frac{\Gamma_1\to\xi_1,\Gamma_2 \quad \Gamma_1,\xi_2\to\Gamma_2}{\Gamma_1,\xi_1\Rightarrow_c\xi_2\to\Gamma_2}$

R$\Rightarrow_c$ $\frac{\Gamma_1,\xi_1\to\xi_2,\Gamma_2}{\Gamma_1\to\xi_1\Rightarrow_c\xi_2,\Gamma_2}$

plus the following derived rules:

L$\neg_i$ $\frac{\Gamma_1\to\xi_1}{\Gamma_1,\neg_i\,\xi_1\to}$

R$\neg_i$ $\frac{\Gamma_1,\xi_1\to}{\Gamma_1\to\neg_i\,\xi_1}$

L$\neg_c$ $\frac{\Gamma_1\to\xi_1,\Gamma_2}{\Gamma_1,\neg_c\,\xi_1\to\Gamma_2}$

R$\neg_c$ $\frac{\Gamma_1,\xi_1\to\Gamma_2}{\Gamma_1\to\neg_c\,\xi_1,\Gamma_2}$

The language $L_i$ of the deductive system $\mathcal{D}_i$ for intuitionistic logic, is freely generated by the signature $C_i$ such that:

- $C_{i0}$ contains $\bot_i$ and a denumerable set of propositional symbols $P_i$;
- $C_{i2} = \{\wedge_i, \vee_i, \Rightarrow_i\}$;
- $C_{ik} = \emptyset$ for $k \neq 0, 2$;

where we may use $\neg_i$ as the usual abbreviation, and assume that $P_i \cap P = \emptyset$. The $L_i$-function

$$\text{var}_i$$

considered in this example associates to each formula $\varphi$ of $L_i$ the symbols of $P_i$ present in $\varphi$. Sequents in $\mathcal{D}_i$ are pairs of multisets of formulas and its sequent calculus is defined by the following rules:

Ax $\frac{}{\xi_1\to\xi_1}$

Lc $\frac{\Gamma_1,\xi_1,\xi_1\to\Gamma_2}{\Gamma_1,\xi_1\to\Gamma_2}$ $\lhd$ $|\Gamma_2| \leq 1$

Lw $\frac{\Gamma_1\to\Gamma_2}{\Gamma_1,\xi_1\to\Gamma_2}$ $\lhd$ $|\Gamma_2| \leq 1$

Rw $\frac{\Gamma_1\to}{\Gamma_1\to\xi_1}$

$\bot_i$ $\frac{}{\bot_i\to}$

L$\Rightarrow_i$ $\frac{\Gamma_1\to\xi_1 \quad \Gamma_1',\xi_2\to\Gamma_2'}{\Gamma_1,\Gamma_1',\xi_1\Rightarrow_i\xi_2\to\Gamma_2'}$ $\lhd$ $|\Gamma_2'| \leq 1$

R$\Rightarrow_i$ $\frac{\Gamma_1,\xi_1\to\xi_2}{\Gamma_1\to\xi_1\Rightarrow_i\xi_2}$

L$\wedge_{ij}$ $\frac{\Gamma_1,\xi_j\to\Gamma_2}{\Gamma_1,\xi_1\wedge_i\xi_2\to\Gamma_2}$ $\lhd$ $|\Gamma_2| \leq 1$ $(j = 1, 2)$

R$\wedge_i$ $\frac{\Gamma_1\to\xi_1 \quad \Gamma_1\to\xi_2}{\Gamma_1\to\xi_1\wedge_i\xi_2}$

L$\vee_i$ $\frac{\Gamma_1,\xi_1\to\Gamma_2 \quad \Gamma_1,\xi_2\to\Gamma_2}{\Gamma_1,\xi_1\vee_i\xi_2\to\Gamma_2}$ $\lhd$ $|\Gamma_2| \leq 1$

R$\vee_{ij}$ $\frac{\Gamma_1\to\xi_j}{\Gamma_1\to\xi_1\vee_i\xi_2}$ $(j = 1, 2)$

plus the following derived rules

L$\neg_i$ $\frac{\Gamma_1\to\xi_1}{\Gamma_1,\neg_i\,\xi_1\to}$

R$\neg_i$ $\frac{\Gamma_1,\xi_1\to}{\Gamma_1\to\neg_i\,\xi_1}$.

We now define the deductive system $\mathcal{D}_{i+_\iota c}$. Fix a bijective map $\iota$ from $P$ to $P_i$. The language of the deductive system is freely generated by the signature $C_{i+c} \cup C_i$, that is, $C_{i+c}$ with the intuitionistic propositional symbols. Its consequence relation is generated by the rules of $\mathcal{D}_{i+c}$ enriched with two additional rules expressing the relationship between the propositional symbols in the fibring and their intuitionistic counterparts:

L$\iota$ $\frac{}{\iota(\xi_1)\to\xi_1}$ $\lhd$ $\xi_1 \in P$

R$\iota$ $\frac{}{\xi_1\to\neg_i\neg_i\iota(\xi_1)}$ $\lhd$ $\xi_1 \in P$

moreover we denote the language of this system by $L_{i+_\iota c}$. Observe also that imposing that $\iota$ is a bijection is not a restriction since both $P$ and $P_i$ are denumerable sets.

**Translation**

A Kolmogorov-Gentzen-Gödel style translation from $\mathcal{D}_{i+c}$ to $\mathcal{D}_i$ is now established. Consider the maps $h_1, h_2 : L_{i+c} \to L_i$ and $h : L_i \to L_{i+c}$ where $h_1$ is inductively defined as follows:

- $h_1(p) = \neg_i \neg_i \iota(p)$ for $p \in P$;
- $h_1(c_i(\varphi_1, \dots, \varphi_n)) = c_i(\neg_i \neg_i h_1(\varphi_1), \dots, \neg_i \neg_i h_1(\varphi_n))$ for $c_i$ in $C_{i+c_n}$, $n \geq 0$;
- $h_1(\bot_c) = \bot_i$;
- $h_1(\varphi_1 \Rightarrow_c \varphi_2) = (\neg_i \neg_i h_1(\varphi_1)) \Rightarrow_i (\neg_i \neg_i h_1(\varphi_2))$;

and $h_2(\varphi) = \neg_i \neg_i h_1(\varphi)$, and $h$ is such that

- $h(\iota(p)) = p$ for $p \in P$;
- $h(c(\varphi_1, \dots, \varphi_n)) = c(h(\varphi_1), \dots, h(\varphi_n))$ for $c \in C_{in}$.

This translation is similar to the Kolmogorov-Gentzen-Gödel translation for instance with respect to the prefixing of some subformulas with a double negation. However it generalizes that type of translation by not imposing that both deductive systems have the same propositional symbols. We now prove some lemmas useful to establish that $\langle h_1, h_2, h \rangle$ is a Maehara generalized translation schema from $(L_{i+c}, \vdash^l_{\mathcal{D}_{i+c}})$ to $(L_i, \vdash^l_{\mathcal{D}_i})$ via $(L_{i+_\iota c}, \vdash^l_{\mathcal{D}_{i+_\iota c}})$ with respect to $\text{var}_{i+c}$ and $\text{var}_i$.

**Lemma 5.15** The pair of maps $h_1$ and $h_2$ is such that $h_1(\Psi) \vdash^l_{\mathcal{D}_i} h_2(\varphi)$ whenever $\Psi \vdash^l_{\mathcal{D}_{i+c}} \varphi$, and $\Psi$ and $\{\varphi\}$ are contained in $L_{i+c}$.

*Proof sketch:* The proof follows by complete induction on the depth of a sequent derivation by showing that if $\Psi \to \Delta$ is a theorem in $\mathcal{D}_{i+c}$ then $h_1(\Psi), \neg_i h_1(\Delta) \to$ is a theorem in $\mathcal{D}_i$. For full details consult the Appendix [2]. $\diamond$

In order to show that $h_1$ and $h_2$ satisfy condition 2 of the definition of Maehara generalized translation schema we introduce an auxiliary map $h_c$ from $L_{i+c}$ to $L_{i+c}$ inductively defined as follows:

- $h_c(\varphi) = \varphi$ whenever $\varphi$ is either $\bot_i$ or $\bot_c$;
- $h_c(\varphi) = \neg_c \neg_c \varphi$ whenever $\varphi$ is in $P$;
- $h_c(\varphi_1 \Rightarrow_c \varphi_2) = (\neg_c \neg_c h_c(\varphi_1)) \Rightarrow_c (\neg_c \neg_c h_c(\varphi_2))$;
- $h_c(\varphi_1 \vee_i \varphi_2) = \neg_c \neg_c (\neg_c \neg_c h_c(\varphi_1)) \vee_i (\neg_c \neg_c h_c(\varphi_2))$;
- $h_c(c_i(\varphi_1, \varphi_2)) = c_i(\neg_c \neg_c h_c(\varphi_1), \neg_c \neg_c h_c(\varphi_2))$ whenever $c_i \in \{\wedge_i, \Rightarrow_i\}$.

which capitalizes on the existence of a classical negation in order to show the equivalence between a formula and its translation.

**Lemma 5.16** The maps $h_c$, $h_1$ and $h_2$ are such that

- $\varphi \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$;
- $h_c(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_1(\varphi)$;
- $h_2(\varphi) \dashv\vdash^l_{\mathcal{D}_{i+_\iota c}} h_c(\varphi)$;

for $\varphi$ in $L_{i+c}$.

*Proof sketch:* The proofs follow by complete induction on the complexity of the formula. For full details consult the Appendix [2]. ◇

Finally we show that $h$ satisfies also condition 3 and condition 4 of the definition of Maehara generalized translation schema.

**Lemma 5.17** The map $h$ is such that

- $\Gamma \vdash^l_{\mathcal{D}_{i+c}} h(\psi)$ whenever $\Gamma \vdash^l_{\mathcal{D}_{i+_\iota c}} \psi$
- $h(\Psi), \Delta \vdash^l_{\mathcal{D}_{i+c}} \varphi$ whenever $\Psi, \Delta \vdash^l_{\mathcal{D}_{i+_\iota c}} \varphi$

where $\Gamma \cup \Delta \cup \{\varphi\}$ is contained in $L_{i+c}$ and $\Psi \cup \{\psi\}$ is contained in $L_i$.

*Proof sketch:* We show by complete induction on the depth of a sequent derivation that if $\Gamma$ and $\Delta$ are sets contained in $L_{i+_\iota c}$ and $\Gamma \to \Delta$ is a theorem of $\mathcal{D}_{i+_\iota c}$ then $\overline{h}(\Gamma) \to \overline{h}(\Delta)$ is a theorem of $\mathcal{D}_{i+c}$ where $\overline{h}$ is a map from $L_{i+_\iota c}$ to $L_{i+c}$ extending $h$ by establishing an identity on the connectives not in $\mathcal{D}_i$. For full details consult the Appendix [2]. ◇

Using the previous lemmas we can now prove that the proposed translation satisfies all the conditions of the Maehara generalized translation schema.

**Proposition 5.18** The triple $\langle h_1, h_2, h\rangle$ is a Maehara generalized translation schema from $(L_{i+c}, \vdash^l_{\mathcal{D}_{i+c}})$ to $(L_i, \vdash^l_{\mathcal{D}_i})$ via $(L_{i+_\iota c}, \vdash^l_{\mathcal{D}_{i+_\iota c}})$ with respect to $\text{var}_{i+c}$ and $\text{var}_i$.

**Proof:** The result follows straightforwardly by taking into account Lemma 5.15, Lemma 5.16 and Lemma 5.17. Moreover $\text{var}_{i+c}(h(h_i(\varphi))) = \text{var}_{i+c}(\varphi)$ for $\varphi$ in $L_{c+i}$ and $\text{var}_i(h_1(\varphi)) = \text{var}_i(h_2(\varphi))$ as can be shown by induction on the structure of $\varphi$. ◇

Our initial question about whether the fibring of classical and intuitionistic logic enjoys Craig interpolation or Maehara interpolation, either over its local consequence relation or over its global consequence relation, can now be positively answered.

**Theorem 5.19** The fibring of classical and intuitionistic logic enjoys Craig interpolation and Maehara interpolation with respect to $\text{var}_{i+c}$ over its local consequence relation and its global consequence relation.

**Proof:** Craig interpolation and Maehara interpolation over local consequence hold due to Corollary 3.4 since intuitionistic logic enjoys those properties with respect to $\text{var}_i$ and since there is a Maehara generalized translation schema from $(L_{i+c}, \vdash^l_{\mathcal{D}_{i+c}})$ to $(L_i, \vdash^l_{\mathcal{D}_i})$ via $(L_{i+_\iota c}, \vdash^l_{\mathcal{D}_{i+_\iota c}})$ with respect to $\text{var}_{i+c}$ and $\text{var}_i$, see Proposition 5.18. The fibring of classical and intuitionistic logic enjoys Craig interpolation and Maehara interpolation over global consequence since, by Theorem 3.4, in the deductive system considered herein for that logic, global consequence and local consequence coincide. ◇

# 6 Conclusions

Preservation results for showing that a logic enjoys Craig interpolation or Maehara interpolation, or for showing that a logic does not enjoy those properties are proved. In particular, the results distinguish between Craig interpolation and Maehara interpolation over local consequence and over global consequence. We present some interesting examples illustrating those preservation results. Contrarily to the traditional methods for proving Craig interpolation, this new alternative allows to capitalize on results about Craig interpolation and Maehara interpolation on other logics, more precisely on other logics for which there is a specific translation between the consequence systems involved satisfying some properties.


## Acknowledgments

The authors wish to thank an anonymous referee for the very appropriate comments and observations on a previous version of the paper. This work was partially supported by Fundação para a Ciência e a Tecnologia (FCT) and EU FEDER via SQIG (Security and Quantum Information Group) at Instituto de Telecomunicações and the projects QuantLog POCTI/MAT/55796/2004, QSec PTDC/EIA/67661/2006 and KLog PTDC/MAT/68723/2006. W. Carnielli acknowledges support from The State of São Paulo Research Foundation-FAPESP, Brazil, Thematic Project number 2004/14107-2 - ConsRel, and by an individual research grant from The National Council for Scientific and Technological Development-CNPq, Brazil.


## References


[1] A. Carbone. The Craig interpolation theorem for schematic systems. In *Collegium logicum*, volume 2, pages 87–100. Springer, 1996.

[2] W. Carnielli, J. Rasga, and C. Sernadas. Appendix on Interpolation via translations: proofs as expected. Technical report, SQIG-IT, DM, IST, 2008. The appendix contains the full details of most of the proofs that are just sketched or not proved at all in the paper since they follow in an expected way. Available at `http://wslc.math.ist.utl.pt/ftp/pub/RasgaJ/08-CRS-it-appendix.pdf`.

[3] W. A. Carnielli, M. E. Coniglio, D. Gabbay, P. Gouveia, and C. Sernadas. *Analysis and Synthesis of Logics - How To Cut And Paste Reasoning Systems*, volume 35 of *Applied Logic*. Springer, 2008.

[4] W. A. Carnielli, J. Rasga, and C. Sernadas. Preservation of interpolation features by fibring. *Journal of Logic and Computation*, 18(1):123–151, 2008.

[5] M. E. Coniglio and W. A. Carnielli. Transfers between logics and their applications. *Studia Logica*, 72(3):367–400, 2002.

[6] J. Czelakowski and D. Piggozi. Amalgamation and interpolation in abstract algebraic logic. In *Lecture Notes on Pure and Applied Mathematics*, volume 203, pages 187–265. Dekker, 1999.

[7] D. Gabbay and L. Maksimova. *Interpolation and Definability: Modal and Intuitionistic Logics*. Oxford University Press, Oxford, 2005.

[8] N. Galatos and H. Ono. Algebraization, parametrized local deduction theorem and interpolation for substructural logics over FL. *Studia Logica*, 83(1-3):279–308, 2006.

[9] G. Gentzen. Über das Verhältnis zwischen intuitionistischer und klassischer arithmetik. In M. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 53–67. North Holland, 1969. English translation of [10].

[10] G. Gentzen. Über das Verhältnis zwischen intuitionistischer und klassischer arithmetik. *Archiv für Mathematische Logik und Grundlagenforschung*, 16:119–132, 1974. Submitted and accepted by Mathematische Annalen in 1933 and then withdrawan. English translation: [9].

[11] V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Academie Royale de Belgique, Bulletins de la Classe des Sciences*, 15(5):183–188, 1929.

[12] K. Gödel. Zur intuitionistischen arithmetik und zahlentheorie. In *Ergebnisse eines mathematischen Kolloquiums*, volume 4, pages 34–38, 1933. English translation: [13].

[13] K. Gödel. On intuitionistic arithmetic and number theory. In *Collected Works, Vol I*, pages 287–295. Oxford University Press, 1986. English translation of [12].

[14] E. Hoogland. *Definability and interpolation: Model-theoretic investigations.* PhD thesis, ILLC, University of Amsterdam, 2001.

[15] H. Ishihara. A note on the Gödel-Gentzen translation. *Mathematical Logic Quarterly*, 46(1):135–137, 2000.

[16] N. Kamide. Synthesized substructural logics. *Mathematical Logic Quarterly*, 53(3):219–225, 2007.

[17] E. Kiriyama and H. Ono. The contraction rule and the decision problems for logics without structural rules. *Studia Logica*, 50(2):299–319, 1991.

[18] A. N. Kolmogorov. On the principle "tertium non datur". *Mathematicheskiĭ Sbornic*, 32:646–667, 1924/25. English translation: [19].

[19] A. N. Kolmogorov. On the principle of excluded middle. In J. van Heijenoort, editor, *From Frege to Gödel: a Source Book in Mathematical Logic, 1879-1931*, pages 416–437. Harvard University Press, 1967. English translation of [18].

[20] S. Kuroda. Intuitionistische Untersuchungen der formalistischen Logik. *Nagoya Mathematical Journal*, 2:35–47, 1951.

[21] S. Maehara. On the interpolation theorem of Craig. *Sûgaku*, 12, 1960-1961.

[22] M. Okada and K. Terui. Completeness proofs for linear logic based on the proof search method (preliminary report). In J. Garriguej, editor, *Proceedings of the Symposium on Type Theory and Its Applications*, pages 146–168. University of Kyoto, 1998.

[23] H. Ono. Proof-theoretic methods in nonclassical logic—an introduction. In *Theories of types and proofs (Tokyo, 1997)*, volume 2 of *MSJ Mem.*, pages 207–254. Math. Soc. Japan, 1998.

[24] J. Rasga. Sufficient conditions for cut elimination with complexity analysis. *Annals of Pure and Applied Logic*, 149(1-3):81–99, 2007.

[25] D. Roorda. Interpolation in fragments of classical linear logic. *Journal of Symbolic Logic*, 59(2):419–444, 1994.

[26] C. Sernadas, J. Rasga, and W. A. Carnielli. Modulated fibring and the collapsing problem. *Journal of Symbolic Logic*, 67(4):1541–1569, 2002.

[27] J. Shoenfield. *Mathematical Logic*. Addison-Wesley, Reading, Massachussets, 1967.

[28] A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *Journal of Symbolic Logic*, 66(1):414–439, 2001.