Sufficient Conditions for Cut Elimination with Complexity Analysis

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Abstract

Sufficient conditions for first order based sequent calculi to admit cut elimination by a Schütte-Tait style cut elimination proof are established. The worst case complexity of the cut elimination is analysed. The obtained upper bound is parameterized by a quantity related with the calculus. The conditions are general enough to be satisfied by a wide class of sequent calculi encompassing, among others, some sequent calculi presentations for the first order and the propositional versions of classical and intuitionistic logic, classical and intuitionistic modal logic S4, and classical and intuitionistic linear logic and some of its fragments. Moreover the conditions are such that there is an algorithm for checking if they are satisfied by a sequent calculus.

Key words: Cut elimination, Complexity of cut elimination, Sequent calculus
MSC: primary 03F05, 03F20, secondary 03F07, 03F03

1 Introduction

Cut elimination is a central method of structural proof theory that has been thoroughly investigated for a wide variety of calculi since its introduction by Gentzen [1]. Despite the extensive research, in most cases, cut elimination has been studied on a case by case basis. Recently the focus started changing. There has been a growing interest toward the definition of sufficient conditions for sequent calculi to admit cut elimination as well as to the characterization

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of necessary conditions for cut elimination [2–9]. The complexity of cut elimination however has not been studied in this context, despite the interest it has deserved motivated in part by the relationship between the length of proofs and computational complexity [10–14].

Sufficient conditions for cut elimination for a wide class of calculi have been pioneered by Belnap in his work on display logic [8]. There, eight conditions are presented that when satisfied by a propositional display calculus guarantee that it enjoys cut elimination. Based on that work Restall in [7] showed that cut is redundant in any propositional consecution system satisfying three general conditions related with cut propagation, the behavior of principal formulas when eliminating a cut, and with the occurrence of congruent parameters and congruent classes of parameters. Later, Miller and Pimentel in [4] showed that any first order based sequent system, possibly without the weakening or contraction rules, whose encoding has only clauses of a certain type, admits cut elimination if all its left and right introduction rules are dual. In a different context Ciabattoni in [3] showed that for a class of (first order) single-conclusion sequent calculi characterized by its types of rules, a calculus admits cut elimination if the rules are substitutive, a condition important to guarantee that a cut can be up propagated when the cut formula is not needed, and the introduction rules are reductive, important to guarantee that a cut can be replaced by cuts over subformulas when the cut formula is introduced by those rules.

Cut elimination proofs, as Gentzen style proofs or Schütte-Tait style proofs typically consist of a great deal of case analysis [1,15,12]. When establishing sufficient conditions for cut elimination, those conditions determine and are determined by the level of detail considered in the case analysis of the cut elimination proof. Establishing generic sufficient conditions, as in [2,5,3,4,6–9], has several advantages. One is that it is in general simpler to define conditions that are satisfied by most of the calculi that actually admit cut elimination in the class that is being considered, and so are closer to being necessary conditions, than defining conditions with a lower level of detail. Another advantage is that the cut elimination proof becomes simpler. For instance, for sufficient general conditions there is no need to analyze which specific rules and provisos are used by the premises of a cut; it is enough to know for instance that the rules are introduction rules.

Herein we follow a different approach. We investigate detailed relationships between the rules and provisos that guarantee cut elimination for a wide class of first order based sequent calculi by a Schütte-Tait style cut elimination proof. The collection of relationships obtained shows deep inter-connections between the rules and provisos of a calculus and are such that there exists an algorithm for checking if they hold. The universe of calculi we are concerned with may have a great variety of rules and provisos and some of them have not been considered in [2,5,3,4,6–9]. The subformula property is also established as well as the consistency of these calculi. The sufficient conditions, although satisfied by a wide class of sequent calculi encompassing, among others, some sequent calculi presentations for the first order and the propositional versions of classical and intuitionistic logic, classical and intuitionistic modal logic S4, classical and intuitionistic linear logic, and many of the linear logic fragments, may not be necessary conditions.

A hyper-exponential bound for the worst case complexity of cut elimination is also proved, as well as an interesting relation, as far as we know not yet explicitly reported in the literature, between the obtained upper bound and the greatest length of any minimal cut sequence for a pair of left and right introduction rules in the calculus for a constructor. Intuitively a cut sequence is formed by the premises of those rules used when a cut over a formula is replaced by a cut over its subformulas. The non-elementary bound in the worst case complexity of the cut elimination procedure was expected since it is a lower bound of cut elimination in sequent calculi for classical first order logic [16,17] and some of these calculi satisfy the sufficient conditions for cut elimination established herein.

Outline of the paper. In Section 2 definitions and results needed throughout the paper are introduced. The section starts by settling the basic notions of sequent calculus, rule and proviso. Then, the cut rules considered in this work are introduced, basic cut elimination definitions and results are presented, and the notion of a (bounded) cutrank decremental operator is defined. In Section 3 general cut elimination results are proved in the context of calculi that allow the definition of a cutrank decremental operator. When the cutrank decremental operator considered is bounded the worst case complexity of the cut elimination procedure can be estimated as a function of its bound. Cut suitable calculi are introduced in Section 4. This class is interesting since each calculus in the class allows the definition of a cutrank decremental operator bounded by its cut length, as it is shown in Section 5. The subformula property and the consistency of cut suitable calculi are briefly analysed in Section 6, and in Section 7 several sequent calculi described in [18,15,1] are considered in order to check which satisfy the conditions stated herein and which do not. Finally concluding the paper a brief comparison with related work is done in Section 8 and future work is outlined in Section 9.

2 Basics

In order to define sufficient conditions for cut elimination for a wide class of sequent calculi, those calculi must be presented using a common meta-theoretic language. Only in this way can the conditions refer uniformly to common properties and aspects of different calculi.

A *sequent calculus* $\mathcal{C}$ is a pair composed by a signature and a finite set of rules. A *signature* is a tuple $\langle F, C, Q\rangle$ where F, C and Q are families $\{F_i\}_{i\in\mathbb{N}_0}$, $\{C_i\}_{i\in\mathbb{N}_0}$ and $\{Q_i\}_{i\in\mathbb{N}}$, respectively, of sets. Given n in $\mathbb{N}_0$ the elements of F_n are *function symbols* of arity n and the elements of C_n are *connectives* of arity n. The elements of Q_n are *quantifiers* of arity n for each n in $\mathbb{N}$. By a *constructor* we mean either a *quantifier* or a *connective*. We assume given once and for all the set $\{x_i : i \in \mathbb{N}\}$ of *quantification variables*. Given a signature and the set of quantification variables it is possible to define in the usual way the language of *formulas* and the language of *terms*. A *sequent* is a pair, written $\Psi \to \Delta$, where Ψ and Δ are finite multisets of formulas.

In order to present a definition of rule general enough to be used in the context of a wide class of calculi, we introduce meta-variables. Meta-variables will be only used in the context of rules, and their role is to indicate the places where a term, a formula or a multiset of formulas can appear when using that rule in a deduction. So we assume given once and for all three denumerable sets: the set $\{\theta_i : i \in \mathbb{N}\}$ of *term meta-variables*, the set $\{\xi_i : i \in \mathbb{N}\}$ of *formula meta-variables*, and the set $\{\Gamma_i : i \in \mathbb{N}\}$ of *multiset meta-variables*. *Meta-formulas* and *meta-terms* are defined as expected from a signature, the set of quantification variables and the sets of meta-variables. *Meta-sequents* are pairs of finite multisets of meta-formulas and of multiset meta-variables. A *rule* is a triple $\langle\{s_1, \dots, s_p\}, s, \pi\rangle$ written $\frac{s_1 \dots s_p}{s} \lhd \pi$ where s and $s_1, \dots, s_p$ are meta-sequents and π is a *proviso*, that is, a predicate on substitutions of meta-variables. A *deduction* of a sequent s from a set S of sequents, in the context of a sequent calculus $\mathcal{C}$, written $S \vdash_{\mathcal{C}} s$ is defined in the usual way, see [15], as a finite tree labelled with sequents such that the label of the root is s and the label of any node either is in S and the node is a leaf or it and its immediate successors constitute an instance of a rule in the calculus (when presenting the deduction tree the name of the rule is written near the node).

The following notations

$$\begin{array}{c}\mathcal{D}\\ \Psi \to \Delta\end{array} \qquad\qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \Psi \to \Delta\end{array}}{\Psi' \to \Delta'}\, r^*$$

mean, from left to right: the deduction $\mathcal{D}$ with conclusion $\Psi \to \Delta$, and a deduction obtained from the deduction $\mathcal{D}$ by successive applications of rule r

until the conclusion $\Psi' \rightarrow \Delta'$ is obtained. Note that it may be the case that it is not even necessary to apply r in order to obtain $\Psi' \rightarrow \Delta'$.

We denote by $c(\vec{\varphi}')$ formulas or meta-formulas whose main constructor is c. For instance $\forall(\vec{\varphi}')$ may denote the formula $\forall x \varphi_1$. We will use the symbols x and y in rules and in provisos to denote term meta-variables that can be instantiated only by quantification variables.

2.2 Cut rules and cardinality provisos

Since it is not feasible to consider in this work all the vast diversity of cut rules present in the literature, we decided to concentrate on a significant collection of such rules. So, herein, cut elimination is investigated for the following cut rules:

∘ the *cut rule*, named Cut, with the form $\dfrac{\Gamma_1 \rightarrow \Gamma_2, \xi_1 \quad \xi_1, \Gamma_1' \rightarrow \Gamma_2'}{\Gamma_1, \Gamma_1' \rightarrow \Gamma_2, \Gamma_2'} \triangleleft \pi_{cd}$;

∘ the *generic multicut rule*, in the sequel named Multicut, with the form $\dfrac{\Gamma_1 \rightarrow \Gamma_2, {\xi_1}^m \quad {\xi_1}^n, \Gamma_1' \rightarrow \Gamma_2'}{\Gamma_1, \Gamma_1' \rightarrow \Gamma_2, \Gamma_2'} \triangleleft \pi_{cd},\ m, n > 0$;

∘ the *left or right generic multicut rule* which are similar to the generic multicut rule but with the restriction that either m or n is equal to 1, respectively;

∘ the *left multicut rule for a u-ary quantifier Q*, named LMulticut Q, with the form $\dfrac{\Gamma_1 \rightarrow \Gamma_2, Qx(\xi_1, \ldots, \xi_u) \quad Qx(\xi_1, \ldots, \xi_u)^n, \Gamma_1' \rightarrow \Gamma_2'}{\Gamma_1, \Gamma_1' \rightarrow \Gamma_2, \Gamma_2'} \triangleleft \pi_{cd},\ n > 1$;

∘ the *right multicut rule for a quantifier* and the *left or right multicut rule for a connective* which are rules similar to the left multicut rule for a quantifier;

where

$$\pi_{cd}$$

denotes a collection of provisos containing only *cardinality provisos*, that is, provisos like

$$|\Gamma| \leq a \qquad \text{or} \qquad |\Gamma| = a$$

that imposes an upper bound or a value, a, on the number of formulas in the multiset assigned to Γ when instantiating the rule where the proviso is. The value a is a non-negative integer.

2.3 Cut elimination notions

We now briefly recall some basic notions needed throughout the paper. Our reference is [15]. The function *hyp* is defined by $\mathrm{hyp}(x, 0, z) = z$ and $\mathrm{hyp}(x, Sk, z) = x^{\mathrm{hyp}(x,k,z)}$. The value $\mathrm{hyp}(x, k, z)$ is denoted by x^z_k. The *depth* of a formula φ, denoted by $|\varphi|$, is defined as follows: if φ is atomic then $|\varphi|$ is 0, otherwise if φ is $c(\varphi_1, \ldots, \varphi_n)$ for some connective c then $|\varphi|$ is $\max(|\varphi_1|, \ldots, |\varphi_n|) + 1$ and similarly for quantifiers. The *depth* of a deduction $\mathcal{D}$ is the maximum length of a branch in $\mathcal{D}$ minus 1. The *logical depth* of a deduction $\mathcal{D}$, denoted by $||\mathcal{D}||$, is the depth of $\mathcal{D}$ not counting the contractions and weakenings. The *level* of a cut is the sum of the depths of the deductions of the premises. The *rank* of a cut over a formula φ is $|\varphi| + 1$. The *cutrank* of a deduction $\mathcal{D}$, denoted by $\mathrm{cr}(\mathcal{D})$, is the maximum of the ranks of cuts in $\mathcal{D}$. If there are no cuts in $\mathcal{D}$ then the cutrank of $\mathcal{D}$ is 0.

In the context of first order based sequent calculi, for the cut elimination procedure to be applied to a deduction it is necessary that the deduction satisfies some conditions with respect to the occurrence of fresh variables and of bound and free variables. A deduction satisfying these conditions is named a pure-variable deduction.

A *pure-variable deduction* is a deduction where i) any quantification variable used as a fresh variable in a premise by a rule application occurs free only in the sub-deduction of that premise, and ii) no quantification variable occurs both bound and free in the deduction.

Condition i) on fresh variables is important in the cut elimination process since there are cases where it is necessary to consider a deduction where a rule should be applied with a context coming from some other part of the original deduction. Condition ii) on the bound and free variables of a deduction is important in order to avoid situations where cuts can not be eliminated since it is not possible to rename in an appropriate way all occurrences of a free variable because some of them may become bound.

Proposition 1. In the context of a sequent calculus $\mathcal{C}$, given a deduction $\mathcal{D}$ for $\vdash_{\mathcal{C}} s$, there is a pure-variable deduction $\mathcal{D}'$ for $\vdash_{\mathcal{C}} s'$ where s' may differ from s only in the names of bound variables, and $\mathcal{D}'$ may differ from $\mathcal{D}$ only in the names of quantification variables.

2.4 Cutrank decremental operator

There are several types of cut elimination proof methods, ranging from local methods, which are based on local transformations of the deduction, like for instance the proofs based on the pioneer work of Gentzen [1,19], or based on the proof of Schütte-Tait [20,21,19,15], and the global methods of cut elimination as the one presented in [12]. There are also proofs obtained by generalizing cut elimination to the problem of redundancy elimination performed by a resolution method [22]. Note that most of the case analysis present in a local cut elimination proof method is common to other local methods.

In this work we consider a Schütte-Tait style cut elimination proof since it is well suited to the simultaneous analysis of complexity. We propose a formulation of this method with an operator for reducing the cutrank of certain deductions. Typically this reduction is proved as a lemma. Besides improving the readability of the cut elimination proof this approach has the advantage of reducing cut elimination to the existence of such an operator, that is, once we define, such an operator enjoys cut elimination in the context of a calculus, see Section 3.

Definition 2. A *cutrank decremental operator* $\mathcal{R}$ over a sequent calculus $\mathcal{C}$ is an operator on $\mathcal{C}$-deductions that given a pure-variable $\mathcal{C}$-deduction $\mathcal{D}^\circ$ for $\vdash_{\mathcal{C}} s$ where s is obtained by a cut from deductions $\mathcal{D}$ and $\mathcal{D}'$ with a lower cutrank than $\mathcal{D}^\circ$, gives as result a pure-variable $\mathcal{C}$-deduction $\mathcal{R}(\mathcal{D}^\circ)$ with the same endsequent and with lower cutrank than $\mathcal{D}^\circ$. A cutrank decremental operator is *b-bounded* whenever $||\mathcal{R}(\mathcal{D}^\circ)|| \leq b(||\mathcal{D}|| + ||\mathcal{D}'||)$.

3 Cut elimination with complexity analysis

We now show that a certain calculus with a given cutrank decremental operator enjoys cut elimination by a Schütte-Tait style cut elimination proof. Moreover if the cutrank decremental operator is bounded then it is possible to bound the complexity of the cut free deduction. Before proving the cut elimination theorem it is shown that it is possible to reduce the cutrank of any deduction with non-null cutrank.

Lemma 3. Given a pure-variable deduction $\mathcal{D}^\circ$ for $\vdash_{\mathcal{C}} s$ with non-null cutrank, where $\mathcal{C}$ is a sequent calculus with a cutrank decremental operator, there is a pure-variable deduction $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}} s$ with lower cutrank than $\mathcal{D}^\circ$. Moreover $||\mathcal{D}^\bullet|| \leq (2b)^{||\mathcal{D}^\circ||}$ if the cutrank decremental operator is b-bounded.

The proof of Lemma 3 is completely standard and so we omit it. Applying successively the previous lemma it is possible to obtain deductions with a decreasing cutrank until finally obtaining a deduction without cuts. The negative side is the worst case exponential increase of the logical depth each time Lemma 3 is applied. This is the cause of the hyper-exponential worst case complexity of this cut elimination process.

Theorem 4. Given a pure-variable deduction $\mathcal{D}^\circ$ for $\vdash_\mathcal{C} s$ where $\mathcal{C}$ is a sequent calculus with a cutrank decremental operator there is a pure-variable deduction $\mathcal{D}^\bullet$ for $\vdash_\mathcal{C} s$ without cuts. Moreover $||\mathcal{D}^\bullet|| \leq (2b)^{||\mathcal{D}^\circ||}_{\mathrm{cr}(\mathcal{D}^\circ)}$ if the cutrank decremental operator is b-bounded.

Theorem 4 is an obvious consequence of Lemma 3 and so its proof is omitted. A result similar to Theorem 4 can be established for deductions not necessarily pure-variable if the end-sequent is allowed to differ from the end-sequent of the original deduction by the name of bound variables, see Proposition 1.

4 Cut suitable calculi

The reduction from the cut elimination problem to the problem of defining a cutrank decremental operator, see Theorem 4, increases the importance of establishing classes of sequent calculi over which that operator can be defined.

Herein we propose a class of sequent calculi for which, in Section 5, a cutrank decremental operator is proposed. That class is composed of calculi satisfying some conditions and belonging to a certain universe of calculi. As in [5,6,2,4,3] the universe of sequent calculi is characterized by the rules and provisos that they may have. Although the types of rules considered in those papers are equal to the ones we consider: axiom rule, structural rules, cut rules and introduction rules, there are differences. For instance we allow structural and cut rules over constructors, and rules with provisos like the closure proviso or the cardinality proviso which were not considered in any of those papers. And, differently from [5,6,2,7,8] and as in [3,4] it is possible to consider introduction rules for first order quantifiers.

4.1 Universe of sequent calculi

The calculi in which we are interested in studying sufficient conditions for the definition of a cutrank decremental operation, and so for cut elimination, are characterized by the rules and provisos that they may have. We denote by $\mathcal{U}$ the collection of such calculi.

Provisos

Besides the cardinality provisos introduced in Subsection 2.2 the following provisos may be used:

- *ξ' is $\xi[x/\theta]$ where θ is free for x in ξ;*

- Γ *is c closed*;
- ξ *is in L*;
- ξ' *is* $\xi[x/y]$ *where y is free for x in* ξ, $y \notin FV(\Gamma_1, \Gamma_2)$, $y \equiv x$ *or* $y \notin FV(\xi)$;

called *substitution proviso*, *c closure proviso*, *restriction to L proviso* and *fresh proviso*, respectively, where c is a constructor in the underlying signature and L is a set of formulas in the underlying language of formulas. A closure proviso requires that all the formulas in the instance of a multiset meta-variable have a certain constructor as main constructor and a restriction to a set proviso requires that the instance of a formula meta-variable is in a certain set. In the context of a fresh or a substitution proviso, the formula meta-variable ξ' is said to be related or constrained by ξ, and in the context of a fresh proviso, the variable y is said to be the fresh variable.

Rules

In general a rule may have characteristics that can change with the calculi where the rule is. For instance the axiom rule may have or not contexts, depending on the calculus that is being considered. So, it is important to clearly define for each rule which are the characteristics that can change and how can they change. Most of these characteristics are established by the provisos the rule may have.

Besides cut rules, introduced in Subsection 2.2, the following rules may appear:

∘ the *left generic weakening rule*, named Lw, with the form $\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1, \xi_1 \to \Gamma_2} \lhd \pi_{cd}$;

∘ the *right generic weakening rule*, named Rw. Similar to Lw;

∘ the *left weakening rule for a u-ary connective c*, named Lw c, with the form $\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1, c(\xi_1, \ldots, \xi_u) \to \Gamma_2} \lhd \pi_{cd}$;

∘ the *right weakening rule for a connective* and the *left or right weakening rule for a quantifier* which are similar to the left weakening rule for a connective;

∘ the *axiom rule*, named Ax, with the form $\dfrac{}{\Gamma_1, \xi_1 \to \xi_1, \Gamma_2} \lhd \pi$ where π maybe composed by cardinality provisos over Γ_1 or Γ_2 and by a restriction to a set proviso over ξ_1;

∘ the *left generic contraction rule*, named Lc, with the form $\dfrac{\Gamma_1, \xi_1, \xi_1 \to \Gamma_2}{\Gamma_1, \xi_1 \to \Gamma_2} \lhd \pi_{cd}$;

∘ the *right generic contraction rule*, named Rc. Similar to Lc;

∘ the *left contraction rule for a u-ary quantifier* Q, named $\mathrm{Lc}\,Q$, with the form $\dfrac{\Gamma_1, Qx(\xi_1,\ldots,\xi_u), Qx(\xi_1,\ldots,\xi_u) \to \Gamma_2}{\Gamma_1, Qx(\xi_1,\ldots,\xi_u) \to \Gamma_2} \lhd \pi_{cd}$;

∘ the *right contraction rule for a quantifier* and the *left or right contraction rule for a connective* which are similar to the left contraction for a quantifier;

∘ *multiplicative or additive introduction rules for constructors*. Given an n-ary quantifier Q an additive left rule for Q has the form

$$\frac{\Gamma_1, \Psi_1 \to \Delta_1, \Gamma_2 \quad \ldots \quad \Gamma_1, \Psi_k \to \Delta_k, \Gamma_2}{\Gamma_1, Qx(\xi_1,\ldots,\xi_n) \to \Gamma_2} \lhd \pi$$

and a multiplicative left rule the form

$$\frac{\Gamma_{11}, \Psi_1 \to \Delta_1, \Gamma_{21} \quad \ldots \quad \Gamma_{1k}, \Psi_k \to \Delta_k, \Gamma_{2k}}{\Gamma_{11},\ldots,\Gamma_{1k}, Qx(\xi_1,\ldots,\xi_n) \to \Gamma_{21},\ldots,\Gamma_{2k}} \lhd \pi$$

and are named $\mathrm{L}\,Q$ and are such that k is greater than or equal to 0 and π does not have restriction to a set provisos. Similarly for right rules and for rules for connectives. These rules satisfy the conditions:

- if a formula meta-variable appears in more than one premise then it appears in the same side in each of the premises;
- Ψ_i and Δ_i for $i = 1,\ldots,k$ are multisets of either i. the formula whose connective was introduced by the rule, at the same side, or ii. formula meta-variables either in $\xi_1,\ldots,\xi_n$, or related to a meta-variable in $\xi_1,\ldots,\xi_n$ by a substitution or fresh proviso in π;
- if a substitution or fresh proviso for x is in π then the rule is for a quantifier and x is the bound variable of the quantifier;
- any formula meta-variable constrained by other variable in a substitution or fresh proviso is not in the conclusion of the rule;
- if a fresh proviso is in π then its multiset meta-variables and the constrained formula meta-variable are in the same premise;
- a formula meta-variable is not constrained by other variable in more than one substitution or fresh proviso, and if it is in a fresh proviso then it requires that the only place, in the premise(s) with the formula meta-variable, where the fresh variable can appear free is precisely in that formula meta-variable.

Examples

It is possible using the rules and provisos introduced above to present sequent calculi for a wide variety of logics. For illustration purposes we now consider a sequent calculus presentation for intuitionistic modal logic S4 and a sequent calculus presentation for the classical multiplicative fragment of linear logic [23,15].

Example 5. Consider the following sequent calculus for *classical multiplicative linear logic* constituted by a signature with a countable set of propositional symbols, the 0-ary connective **1**, the unary connectives $\sim, !, ?$, the binary connectives $\star, \multimap$ and the unary quantifier $\forall$, and with the following rules:

Ax $\dfrac{}{\Gamma_1,\xi_1\to\xi_1,\Gamma_2} \lhd |\Gamma_1| = 0$ and $|\Gamma_2| = 0$

RMulticut? $\dfrac{\Gamma_1\to\Gamma_2,(?\xi_1)^m \quad ?\xi_1,\Gamma_1'\to\Gamma_2'}{\Gamma_1,\Gamma_1'\to\Gamma_2,\Gamma_2'} \lhd m > 1$

LMulticut! $\dfrac{\Gamma_1\to\Gamma_2,!\xi_1 \quad (!\xi_1)^n,\Gamma_1'\to\Gamma_2'}{\Gamma_1,\Gamma_1'\to\Gamma_2,\Gamma_2'} \lhd n > 1$

Cut $\dfrac{\Gamma_1\to\Gamma_2,\xi_1 \quad \xi_1,\Gamma_1'\to\Gamma_2'}{\Gamma_1,\Gamma_1'\to\Gamma_2,\Gamma_2'}$

Lw! $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1,!\xi_1\to\Gamma_2}$

Lc! $\dfrac{\Gamma_1,!\xi_1,!\xi_1\to\Gamma_2}{\Gamma_1,!\xi_1\to\Gamma_2}$

L! $\dfrac{\Gamma_1,\xi_1\to\Gamma_2}{\Gamma_1,!\xi_1\to\Gamma_2}$

R! $\dfrac{\Gamma_1\to\xi_1,\Gamma_2}{\Gamma_1\to!\xi_1,\Gamma_2} \lhd \Gamma_1$ is ! closed, and Γ_2 is ? closed

Rw? $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1\to?\xi_1,\Gamma_2}$

Rc? $\dfrac{\Gamma_1\to?\xi_1,?\xi_1,\Gamma_2}{\Gamma_1\to?\xi_1,\Gamma_2}$

L? $\dfrac{\Gamma_1,\xi_1\to\Gamma_2}{\Gamma_1,?\xi_1\to\Gamma_2} \lhd \Gamma_1$ is ! closed, and Γ_2 is ? closed

R? $\dfrac{\Gamma_1\to\xi_1,\Gamma_2}{\Gamma_1\to?\xi_1,\Gamma_2}$

L**1** $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1,\mathbf{1}\to\Gamma_2}$

R**1** $\dfrac{}{\Gamma_1\to\mathbf{1},\Gamma_2} \lhd |\Gamma_1| = 0$ and $|\Gamma_2| = 0$

L$\sim$ $\dfrac{\Gamma_1\to\xi_1,\Gamma_2}{\Gamma_1,\sim\xi_1\to\Gamma_2}$

R$\sim$ $\dfrac{\Gamma_1,\xi_1\to\Gamma_2}{\Gamma_1\to\sim\xi_1,\Gamma_2}$

L$\star$ $\dfrac{\Gamma_1,\xi_1,\xi_2\to\Gamma_2}{\Gamma_1,\xi_1\star\xi_2\to\Gamma_2}$

R$\star$ $\dfrac{\Gamma_1\to\xi_1,\Gamma_2 \quad \Gamma_1'\to\xi_2,\Gamma_2'}{\Gamma_1,\Gamma_1'\to\xi_1\star\xi_2,\Gamma_2,\Gamma_2'}$

L$\multimap$ $\dfrac{\Gamma_1\to\xi_1,\Gamma_2 \quad \Gamma_1',\xi_2\to\Gamma_2'}{\Gamma_1,\Gamma_1',\xi_1\multimap\xi_2\to\Gamma_2,\Gamma_2'}$

R$\multimap$ $\dfrac{\Gamma_1,\xi_1\to\xi_2,\Gamma_2}{\Gamma_1\to\xi_1\multimap\xi_2,\Gamma_2}$

L$\forall$ $\dfrac{\Gamma_1,\xi_2\to\Gamma_2}{\Gamma_1,\forall x\xi_1\to\Gamma_2} \lhd \xi_2$ is $\xi_1[x/\theta]$ where θ is free for x in ξ_1

R$\forall$ $\dfrac{\Gamma_1\to\xi_2,\Gamma_2}{\Gamma_1\to\forall x\xi_1,\Gamma_2} \lhd \xi_2$ is $\xi_1[x/y]$ where y is free for x in $\xi_1, y \notin FV(\Gamma_1,\Gamma_2), y \equiv x$ or $y \notin FV(\xi_1)$.

Example 6. Consider the following sequent calculus for *intuitionistic modal logic* S4 constituted by a signature with a countable number of propositional symbols, the unary connectives $\neg, \Box, \Diamond$, the binary connectives $\wedge, \vee, \Rightarrow$ and the unary quantifiers $\forall, \exists$, and by the rules

Ax $\dfrac{}{\Gamma_1,\xi_1\to\xi_1,\Gamma_2} \lhd |\Gamma_1| = 0$ and $|\Gamma_2| = 0$

L$\bot$ $\dfrac{}{\Gamma_1,\bot\to\Gamma_2} \lhd |\Gamma_1| = 0$ and $|\Gamma_2| = 0$

LMulticut $\dfrac{\Gamma_1\to\Gamma_2,\xi_1 \quad {\xi_1}^n,\Gamma_1'\to\Gamma_2'}{\Gamma_1,\Gamma_1'\to\Gamma_2,\Gamma_2'} \lhd n > 0, |\Gamma_2| = 0$ and $|\Gamma_2'| \leq 1$

Lw $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1,\xi_1\to\Gamma_2} \lhd |\Gamma_2| \leq 1$

Rw $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1\to\xi_1,\Gamma_2} \lhd |\Gamma_2| = 0$

Lc $\dfrac{\Gamma_1,\xi_1,\xi_1\to\Gamma_2}{\Gamma_1,\xi_1\to\Gamma_2} \lhd |\Gamma_2| \leq 1$

L⇒ $\frac{\Gamma_1 \to \xi_1, \Gamma_2 \quad \Gamma_1', \xi_2 \to \Gamma_2'}{\Gamma_1, \Gamma_1', \xi_1 \Rightarrow \xi_2 \to \Gamma_2, \Gamma_2'} \lhd |\Gamma_2| = 0 \text{ and } |\Gamma_2'| \le 1$ R⇒ $\frac{\Gamma_1, \xi_1 \to \xi_2, \Gamma_2}{\Gamma_1 \to \xi_1 \Rightarrow \xi_2, \Gamma_2} \lhd |\Gamma_2| = 0$

L∧$_i$ $\frac{\Gamma_1, \xi_i \to \Gamma_2}{\Gamma_1, \xi_1 \wedge \xi_2 \to \Gamma_2} \lhd |\Gamma_2| \le 1 \ (i = 1, 2)$ R∧ $\frac{\Gamma_1 \to \xi_1, \Gamma_2 \quad \Gamma_1 \to \xi_2, \Gamma_2}{\Gamma_1 \to \xi_1 \wedge \xi_2, \Gamma_2} \lhd |\Gamma_2| = 0$

L∨ $\frac{\Gamma_1, \xi_1 \to \Gamma_2 \quad \Gamma_1, \xi_2 \to \Gamma_2}{\Gamma_1, \xi_1 \vee \xi_2 \to \Gamma_2} \lhd |\Gamma_2| \le 1$ R∨$_i$ $\frac{\Gamma_1 \to \xi_i, \Gamma_2}{\Gamma_1 \to \xi_1 \vee \xi_2, \Gamma_2} \lhd |\Gamma_2| = 0 \ (i = 1, 2)$

L∀ $\frac{\Gamma_1, \xi_2 \to \Gamma_2}{\Gamma_1, \forall x \xi_1 \to \Gamma_2} \lhd \xi_2$ is $\xi_1[x/\theta]$ where θ is free for x in ξ_1 and $|\Gamma_2| \le 1$

R∀ $\frac{\Gamma_1 \to \xi_2, \Gamma_2}{\Gamma_1 \to \forall x \xi_1, \Gamma_2} \lhd \xi_2$ is $\xi_1[x/y]$ where y is free for x in $\xi_1, y \notin FV(\Gamma_1, \Gamma_2), y \equiv x$ or $y \notin FV(\xi_1), |\Gamma_2| = 0$

L∃ $\frac{\Gamma_1, \xi_2 \to \Gamma_2}{\Gamma_1, \exists x \xi_1 \to \Gamma_2} \lhd \xi_2$ is $\xi_1[x/y]$ where y is free for x in $\xi_1, y \notin FV(\Gamma_1, \Gamma_2), y \equiv x$ or $y \notin FV(\xi_1), |\Gamma_2| \le 1$

R∃ $\frac{\Gamma_1 \to \xi_2, \Gamma_2}{\Gamma_1 \to \exists x \xi_1, \Gamma_2} \lhd \xi_2$ is $\xi_1[x/\theta]$ where θ is free for x in ξ_1, and $|\Gamma_2| = 0$

R□ $\frac{\Gamma_1 \to \xi_1, \Gamma_2}{\Gamma_1 \to \Box \xi_1, \Gamma_2} \lhd \Gamma_1$ is □ closed, Γ_2 is ◇ closed, and $|\Gamma_2| = 0$ L□ $\frac{\Gamma_1, \xi_1 \to \Gamma_2}{\Gamma_1, \Box \xi_1 \to \Gamma_2} \lhd |\Gamma_2| \le 1$

L◇ $\frac{\Gamma_1, \xi_1 \to \Gamma_2}{\Gamma_1, \Diamond \xi_1 \to \Gamma_2} \lhd \Gamma_1$ is □ closed, Γ_2 is ◇ closed, and $|\Gamma_2| \le 1$ R◇ $\frac{\Gamma_1 \to \xi_1, \Gamma_2}{\Gamma_1 \to \Diamond \xi_1, \Gamma_2} \lhd |\Gamma_2| = 0.$

4.2 Cut suitable calculi

Having settled the universe $\mathcal{U}$ of sequent calculi we now present sufficient conditions for a calculus in that universe to allow the definition of a bounded cutrank decremental operator, and so to enjoy cut elimination by a Schütte-Tait style cut elimination proof. In order to simplify the presentation, the conditions are written in compact form. So, when writing a condition like for instance, if a left (right) introduction rule r for a constructor c is in the calculus then all the left (right) introduction rules for c have the same provisos, we in fact mean two conditions: 1. if a *left* introduction rule r for a constructor c is in the calculus then all the *left* introduction rules for c have the same provisos, and 2. if a *right* introduction rule r for a constructor c is in the calculus then all the *right* introduction rules for c have the same provisos.

Cut suitable pair of rules

We consider first the specific case where the deduction ends in a cut inference with the main constructor of the cut formula introduced in both premises. In this case it is important to guarantee the existence of a deduction with the same end-sequent but with cuts over subformulas of the original cut formula. In [5,6] that deduction is guaranteed to exist by a condition requiring that the clauses associated to the premises of any pair of rules that introduce a same constructor in different sides are classically inconsistent (the empty sequent is deduced) using only the cut rule. Since the calculi considered in that work have all the standard structural rules that condition is enough to guarantee that the original end-sequent is derived. The calculi we are interested in may not have all the standard structural rules so to deduce the original end-sequent it is not enough to deduce the empty sequent. The same happens in [2,3] where the condition provided refers not to the deduction of the empty sequent but to the deduction of the original end-sequent by the use of the axiom rule and the structural rules in the calculus besides the cut rule. In [4] capitalizing on the characteristics of the encoding of the object-calculi and on the fixed number of combinations of structural rules allowed, it was sufficient to require only that the collection of premises of the object-level rules are inconsistent.

Herein we adopt an approach in between [5,6] and [2,3]. By one side we provide conditions to impose that the empty sequent is deduced, and by the other side we guarantee that it is possible to recover the contexts of the original end-sequent by means of additional specific conditions. This has the advantage that there is an algorithm to check these conditions, see Remark 8. The deduction of the empty sequent is required in the context of a calculus $\mathcal{C}^+$ obtained by enriching the original calculus $\mathcal{C}$ with a countable set of new propositional symbols, one for each formula meta-variable. In the context of $\mathcal{C}^+$ the *clause* associated with a meta-sequent p, denoted by p^+, is the instance of p obtained by instantiating the multiset meta-variables with the empty multiset and the formula meta-variables with the corresponding new propositional symbols.

Definition 7. A right and a left introduction rule for the same constructor constitute a *cut suitable pair* in the context of a sequent calculus $\mathcal{C}$ with either the cut rule or a generic multicut rule (referred to below as cut) whenever there is a sequence $(p_1; \ldots ; p_u)$, called a *cut sequence*, of premises of both rules, with no repetitions, for a natural u, such that

1. for i from 2 to u there are deductions $\mathcal{D}_{1,\ldots,i}$ in $\mathcal{C}^+$ obtained by applying cut to the end-sequent of $\mathcal{D}_{1,\ldots,i-1}$ and to p_i^+, where $\mathcal{D}_1$ is p_1^+ and such that the conclusion of $\mathcal{D}_{1,\ldots,u}$ is the empty sequent;

2. each formula meta-variable in $(p_1; \ldots ; p_u)$ constrained in a rule by a substitution proviso is constrained in the other rule by a fresh proviso, moreover if a formula meta-variable appears repeated in the same side of a premise in $(p_1; \ldots ; p_u)$ then $\mathcal{C}$ has a multicut rule either generic or generic for that side;

3. if the left (right) introduction rule, denoted by r, does neither have closure provisos nor cardinality provisos requiring that both contexts are empty then a.

$$\mathcal{C} \text{ has } \begin{cases} \text{Rw (Lw)} & \text{if np} < \text{cnp} \\ \text{Rc (Lc)} & \text{if np} > \text{cnp} \end{cases}$$

b. if $\mathcal{C}$ does not have a cardinality proviso over the left (right) side then

$$\mathcal{C} \text{ has } \begin{cases} \text{Lw (Rw)} & \text{if np} < \text{cnp} \\ \text{Lc (Rc)} & \text{if np} > \text{cnp} \end{cases}$$

where *cnp* is either the number of premises of r if r is multiplicative or is 1 if r is additive and *np* is the number of premises of r in $(p_1; \ldots; p_u)$ plus $a \times \text{cnp}$ where a is the number of premises in $(p_1; \ldots; p_u)$ of the other rule with the formula whose connective was introduced by r.

Intuitively, condition 1 of Definition 7 is important to guarantee that the cuts over the suformulas of the original cut formula do not leave a subformula in the end-sequent. For the sake of an example consider the sequent calculus presented in Example 5, for a fragment of linear logic. Rules R $\multimap$ and L $\multimap$ constitute a cut suitable pair, $(\Gamma_1, \xi_1 \to \xi_2, \Gamma_2; \Gamma_1', \xi_2 \to \Gamma_2'; \Gamma_1 \to \xi_1, \Gamma_2)$ constitute a cut sequence for that pair, $\mathcal{D}_1$ is the deduction composed by the sequent $p_1^+ \to p_2^+$, $\mathcal{D}_{1,2}$ is the deduction

$$\dfrac{p_1^+ \to p_2^+ \qquad p_2^+ \to}{p_1^+ \to} \text{Cut}$$

and

$$\dfrac{\to p_1^+ \qquad \begin{array}{c}\mathcal{D}_{1,2}\\ p_1^+ \to\end{array}}{\to} \text{Cut}$$

is the deduction $\mathcal{D}_{1,2,3}$. So when eliminating a cut over a formula introduced by R$\multimap$ and L$\multimap$, for instance, in the deduction

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi_1 \to \varphi_3, \varphi_2\end{array}}{\to \varphi_3, \varphi_1 \multimap \varphi_2}\text{R}\multimap \qquad \dfrac{\begin{array}{c}\mathcal{D}_1'\\ \varphi_4 \star \varphi_6 \to \varphi_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2'\\ \varphi_2 \to \varphi_5\end{array}}{\varphi_1 \multimap \varphi_2, \varphi_4 \star \varphi_6 \to \varphi_5}\text{L}\multimap}{\varphi_4 \star \varphi_6 \to \varphi_3, \varphi_5}\text{Cut}$$

it is possible to consider the deduction

$$\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \varphi_4 \star \varphi_6 \to \varphi_1\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi_1 \to \varphi_3, \varphi_2\end{array} \qquad \begin{array}{c}\mathcal{D}_2'\\ \varphi_2 \to \varphi_5\end{array}}{\varphi_1 \to \varphi_3, \varphi_5}\text{Cut}}{\varphi_4 \star \varphi_6 \to \varphi_3, \varphi_5}\text{Cut}$$

which has a meta-structure similar to $\mathcal{D}_{1,2,3}$ and so has no subformulas of the original cut formula in the end-sequent. Condition 2 is important to guarantee that cuts can be considered even over subformulas constrained by substitution provisos. Finally condition 3 ensures that the original contexts are obtained after performing the cuts over the subformulas. Conditions 2 and 3 do not appear explicitly in any of the works [2–9].

Remark 8. Given a calculus and a right and a left introduction rule for the same constructor note that i) the number of possible sequences of premises that can be a cut sequence is finite and for each such sequence there is an algorithm to test if the empty sequent is derived in the enriched calculus according to condition 1 of Definition 7; ii) checking if a formula meta-variable is constrained by certain specific provisos, if provisos are in rules and if rules are in the calculus is immediate. So there is an algorithm for testing if that pair of rules is a cut suitable pair.

Sufficient conditions

Cut suitable calculi are now introduced as the members of $\mathcal{U}$ that satisfy the collection of conditions described below. Those conditions are organized in terms of the rules and provisos to which they mainly apply. Note that the conditions could be obtained by requiring, without going into the detail of the relationship between the rules and provisos, that the calculus is such that the crucial cases of the definition of a cutrank decremental operation hold or that certain deductions adequate for those cases exist. This approach, which was followed in [2,5,3,4,6–9], has several advantages as was outlined in Section 1.

Herein we follow a different and hard way. We investigate relationships between the rules and provisos that guarantee that a calculus admits the definition of a cutrank decremental operator and so cut elimination by a Schütte-Tait style proof. Working at this level of detail and concentrating on that specific proof has several advantages and some disadvantages. One of the advantages is to bring to clarity deep relationships between rules and provisos sufficient for a calculus to admit the definition of such an operator. Another is the possibility of conceiving of algorithm for checking if a given calculus satisfy the conditions we propose. Another advantage is that it is possible to study the complexity of the cut elimination procedure and to establish a relation as far as we know not yet explicitly mentioned between this complexity and the parameter that bounds the cutrank decremental operation considered. Yet another advantage is that a better understanding of the characteristics and potentialities of that specific cut elimination proof method is achieved. One of the disadvantages is that the conditions are typically in a greater number than those other conditions. Another consequence is that there may exist calculi that admit cut elimination by other cut elimination proof methods and that do not satisfy the sufficient conditions obtained herein.

Although the definition of the cutrank decremental operator in Section 5 is the place where the conditions really show their importance, we decided to illustrate the significance of some of the conditions by an example immediately after its introduction. In that example a sequent calculus will be described that does not satisfy the condition, and thus we will present a deduction with a cut whose elimination illustrates the kind of problems the condition prevents. Although the proposed calculi seem in some cases artificially designed to show the significance of the conditions, they are nevertheless calculi in $\mathcal{U}$. In order to simplify the presentation only the aspects of the calculi and of the deductions relevant to the cut elimination situation are described. Only space limitations prevent us from presenting examples for all the conditions.

Definition 9. A sequent calculus $\mathcal{C}$ in $\mathcal{U}$ is *cut suitable* whenever the following conditions are satisfied:

C1. if the *axiom rule* is in $\mathcal{C}$, and
- the generic weakening rule over side l is not present,
- $\mathcal{C}$ does not impose a cardinality proviso over side l,
- there exists a rule with a closure or a cardinality proviso over side l

then the axiom rule has a proviso requiring that the l context is empty;

Example: condition C1 is concerned with the interplay between the axiom rule and rules with provisos. For the sake of an example note that C1 is not satisfied by a calculus in $\mathcal{U}$ including the cut rule, the rules

$$Ax\,\frac{}{\Gamma_1,\xi_1\to\xi_1,\Gamma_2} \qquad \textit{and} \qquad L\mathbf{0}\,\frac{}{\Gamma_1,\mathbf{0}\to\Gamma_2} \lhd |\Gamma_1|=0 \textit{ and } |\Gamma_2|=0$$

and whose weakening rules are all over constructors. Note further that the elimination of the cut in the deduction

$$\cfrac{\cfrac{}{\varphi_1,\varphi_2,\mathbf{0}\to\varphi_3,\mathbf{0}}\,Ax \quad \cfrac{}{\mathbf{0}\to}\,L\mathbf{0}}{\varphi_1,\varphi_2,\mathbf{0}\to\varphi_3}\,Cut$$

contributes to illustrate the importance of this condition since the deduction

$$\cfrac{\cfrac{}{\mathbf{0}\to}\,L\mathbf{0}}{\varphi_1,\varphi_2,\mathbf{0}\to\varphi_3}\,w^*$$

is not a correct deduction because Lw and Rw are not in the calculus.

C2. *weakening rules* are such that:
1. if the left (right) weakening rule for a constructor c is in the calculus then
   i) there is a right (left) introduction rule for c with closure provisos,
   ii) the right (left) weakening rule for c is not in the calculus,
   iii) the left (right) generic weakening rule is not in the calculus;
2. if a generic weakening rule is in $\mathcal{C}$ then the other generic weakening rule is in $\mathcal{C}$ except if $\mathcal{C}$ imposes that the number of formulas at the other side is 1;
3. the cardinality provisos in these rules are the ones imposed by the calculus;

Example: condition C2.2 is concerned with the interplay between the generic weakening rules. The significance of this condition is illustrated in the following example. Consider a calculus in $\mathcal{U}$ whose rules include the cut rule, the rule

$$R\vee \frac{\Gamma_1 \rightarrow \xi_1, \xi_2, \Gamma_2}{\Gamma_1 \rightarrow \xi_1 \vee \xi_2, \Gamma_2}$$

and the left generic weakening rule, and that do not include the right generic weakening rule. Then the elimination of the cut in the following deduction

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \varphi_3 \rightarrow \varphi_4, \varphi_1, \varphi_2\end{matrix}}{\varphi_3 \rightarrow \varphi_4, \varphi_1 \vee \varphi_2}\, R\vee \quad \dfrac{\begin{matrix}\mathcal{D}_1'\\ \rightarrow \varphi_5\end{matrix}}{\varphi_1 \vee \varphi_2 \rightarrow \varphi_5}\, Lw}{\varphi_3 \rightarrow \varphi_4, \varphi_5}\, Cut$$

contributes to illustrate the importance of this condition since the deduction

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \rightarrow \varphi_5\end{matrix}}{\varphi_3 \rightarrow \varphi_5}\, Lw}{\varphi_3 \rightarrow \varphi_4, \varphi_5}\, Rw$$

is not possible because rule Rw is not in the calculus.

C3. *contraction rules* are such that:

1. the left contraction rule and the right contraction rule for a same constructor are not simultaneously in $\mathcal{C}$;
2. if the left (right) contraction rule, either generic or for a constructor, is in $\mathcal{C}$ then $\mathcal{C}$ has no cardinality proviso over the left (right) side;
3. a contraction rule for a constructor over side l is in the calculus only if the generic contraction rule for that side is not;
4. the cardinality provisos in these rules are the ones imposed by the calculus;

C4. *cut and multicut rules* are such that:

1. if the left (right) multicut rule for a constructor c is in the calculus then i) introduction rules with closure provisos, the axiom rule and generic weakening rules, are the only rules in the calculus with the principal formula in the right (left) side of the conclusion with the possibility of having c as main constructor, ii) Cut is in $\mathcal{C}$, iii) the left (right) contraction rule for c is in $\mathcal{C}$, and iv) the right (left) generic contraction rule is not in $\mathcal{C}$;
2. if the generic multicut rule is in the calculus then the left and the right generic contraction rules are in the calculus;
3. if the left (right) generic multicut rule is in the calculus then i) the left (right) generic contraction rule is in $\mathcal{C}$ and the right (left) generic contraction rule is not, and ii) $\mathcal{C}$ has a right (left) cardinality proviso;
4. the cardinality provisos in these rules are the ones imposed by the calculus;

Example: we now illustrate the significance of condition C4 item 3. Consider a calculus in $\mathcal{U}$ including the rule RMulticut, the rules

$$L+\frac{\Gamma_1,\xi_1\to\Gamma_2 \quad \Gamma_1',\xi_2\to\Gamma_2'}{\Gamma_1,\Gamma_1',\xi_1+\xi_2\to\Gamma_2,\Gamma_2'} \qquad \textit{and} \qquad L\neg\frac{\Gamma_1\to\xi_1,\Gamma_2}{\Gamma_1,\neg\xi_1\to\Gamma_2}$$

the right generic contraction rule, and no other contraction rules. Note that C4 item 3 is not satisfied by this calculus. Consider the following deduction

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\varphi_1\to\neg\varphi_4} \quad \overset{\mathcal{D}_2}{\varphi_2\to\neg\varphi_4}}{\varphi_1+\varphi_2\to\neg\varphi_4{}^2}\,L+ \qquad \dfrac{\overset{\mathcal{D}_1'}{\varphi_3\to\varphi_4}}{\neg\varphi_4,\varphi_3\to}\,L\neg}{\varphi_1+\varphi_2,\varphi_3\to}\,\textit{RMulticut}$$

then, although the cut can be propagated to the premises of L+, it is not possible to consider the deduction

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1}{\varphi_1\to\neg\varphi_4} \quad \dfrac{\overset{\mathcal{D}_1'}{\varphi_3\to\varphi_4}}{\neg\varphi_4,\varphi_3\to}\,L\neg}{\varphi_1,\varphi_3\to}\,\textit{RMulticut} \qquad \dfrac{\overset{\mathcal{D}_2}{\varphi_2\to\neg\varphi_4} \quad \dfrac{\overset{\mathcal{D}_1'}{\varphi_3\to\varphi_4}}{\neg\varphi_4,\varphi_3\to}\,L\neg}{\varphi_2,\varphi_3\to}\,\textit{RMulticut}}{\varphi_1+\varphi_2,\varphi_3,\varphi_3\to}\,L+}{\varphi_1+\varphi_2,\varphi_3\to}\,Lc$$

C5. if an *introduction rule* r for a constructor c at side l is in $\mathcal{C}$ then:

1. one of the following conditions holds
   - i) r does not have any cardinality proviso,
   - ii) r has the cardinality proviso imposed by the calculus,
   - iii) r has a cardinality proviso requiring that all the contexts are empty,
   - iv) if r does not have premises and has a proviso requiring that the l' context is empty then $\mathcal{C}$ has a generic weakening rule over side l';
2. if r does not have premises and if the generic weakening rule over side l' is not in the calculus then r does not have closure provisos over side l';
3. all the l introduction rules for c have the same provisos;

C6. if *a rule r with a closure proviso* is in $\mathcal{C}$ then its number is at most 2 and

1. if it is 2 then the provisos are over opposite sides and, assuming that one is for c at left and the other for c' at right then
   - i) either r is a right introduction rule for c or a left rule introducing c',
   - ii) if r is a right (left) introduction rule then all the left (right) introduction rules for c' (c) in the calculus have the same provisos as r;
2. if it is 1 and the proviso is for a constructor c at left (right) then
   - i) r is a right (left) introduction rule for c,
   - ii) the calculus imposes a cardinality proviso over the right (left) side;
3. if r has a c closure proviso over side l then there are in the calculus
   - i) a contraction rule over side l generic or for the constructor c if $\mathcal{C}$ does not impose a cardinality proviso over side l,
   - ii) a weakening rule over side l generic or for the constructor c if $\mathcal{C}$ does not impose that the number of formulas at side l is 1;

C7. each *pair of rules with closure provisos* is such that either they have for each side the same closure provisos, or the closure provisos are all over different constructors;

Example: to illustrate condition C7 consider a calculus in $\mathcal{U}$ *with the cut rule, the generic multicut rule, the generic weakening rules, the rules*

$$R!\ \frac{\Gamma_1\to\xi_1,\Gamma_2}{\Gamma_1\to !\xi_1,\Gamma_2} \lhd \Gamma_1 \textit{ is } ! \textit{ closed, } \Gamma_2 \textit{ is } \Box \textit{ closed} \qquad R\Box\,\frac{\Gamma_1\to\xi_1,\Gamma_2}{\Gamma_1\to\Box\xi_1,\Gamma_2} \lhd \Gamma_1 \textit{ is } \Box \textit{ closed, } \Gamma_2 \textit{ is } \Diamond \textit{ closed}$$

and the generic contraction rules. Suppose the goal is to eliminate the cut in the following deduction:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\to\varphi_1,\Box\varphi_3}}{\to !\varphi_1,\Box\varphi_3}\,R! \qquad \dfrac{\overset{\mathcal{D}'_1}{\Box\varphi_3,\Box\varphi_4\to\varphi_2}}{\Box\varphi_3,\Box\varphi_4\to\Box\varphi_2}\,R\Box}{\Box\varphi_4\to !\varphi_1,\Box\varphi_2}\,\mathit{Cut}$$

so, consider

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1}{\to\varphi_1,\Box\varphi_3}}{\to !\varphi_1,\Box\varphi_3}\,R! \qquad \overset{\mathcal{D}'_1}{\Box\varphi_3,\Box\varphi_4\to\varphi_2}}{\Box\varphi_4\to !\varphi_1,\varphi_2}\,\mathit{Cut}}{\Box\varphi_4\to !\varphi_1,\Box\varphi_2}\,R\Box$$

but this is not a correct deduction since rule $R\Box$ *can only be applied when all the formulas in its right context are closed for* $\Diamond$*;*

C8. if $\mathcal{C}$ *has a cardinality proviso for side* l then either it requires that the number of formulas at side l of any sequent in a rule is at most 1 or it requires that the number of formulas at side l of any sequent in a rule is 1;

C9. *each pair of rules* formed by a right and a left introduction rule for the same constructor is *cut suitable.*

Example: we now illustrate the importance of condition C8. Consider a calculus in $\mathcal{U}$ *imposing that the maximum number of formulas in the right side of a sequent appearing in a deduction is* 2*, and including the rules*

$$L\multimap\ \frac{\Gamma_1\to\xi_1,\Gamma_2 \quad \Gamma'_1,\xi_2\to\Gamma'_2}{\Gamma_1,\Gamma'_1,\xi_1\multimap\xi_2\to\Gamma_2,\Gamma'_2} \lhd |\Gamma_2|\le 1 \textit{ and } |\Gamma_2\Gamma'_2|\le 2 \qquad R\multimap\ \frac{\Gamma_1,\xi_1\to\xi_2,\Gamma_2}{\Gamma_1\to\xi_1\multimap\xi_2,\Gamma_2} \lhd |\Gamma_2|\le 1$$

and the cut rule with the appropriate cardinality proviso. Then, the elimination of the cut in the deduction

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\varphi_1\to\varphi_3,\varphi_2}}{\to\varphi_3,\varphi_1\multimap\varphi_2}\,R\multimap \qquad \dfrac{\overset{\mathcal{D}'_1}{\varphi_1\multimap\varphi_2\to\varphi_4,\varphi_5} \quad \overset{\mathcal{D}'_2}{\varphi_6\to}}{\varphi_1\multimap\varphi_2,\varphi_4\multimap\varphi_6\to\varphi_5}\,L\multimap}{\varphi_4\multimap\varphi_6\to\varphi_3,\varphi_5}\,\mathit{Cut}$$

contributes to understand better the significance of this condition since

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\varphi_1 \to \varphi_3, \varphi_2}}{\to \varphi_3, \varphi_1 \multimap \varphi_2}\,R\multimap \qquad \overset{\mathcal{D}'_1}{\varphi_1 \multimap \varphi_2 \to \varphi_4, \varphi_5}}{\to \varphi_3, \varphi_4, \varphi_5}\,Cut$$

is not a correct deduction. It is not difficult to imagine similar deductions with the same problem in calculi that have a cardinality proviso imposing a fixed limit, greater than 2, to the number of formulas at a certain side.

4.3 Cut length of a calculus

The cut length of a calculus indicates how many cuts over subformulas have to be considered in the worst case in order to eliminate a cut in a deduction ending in it and where the cut formula is introduced in both premises. This quantity bounds the cutrank decremental operator for cut suitable calculi defined in Section 5 and so it appears as a parameter in the worst case complexity cost of eliminating cuts in these calculi, see Theorem 4.

Definition 10. The *cut length* of a calculus $\mathcal{C}$ is the greatest length minus 1 of any minimal cut sequence for any cut suitable pair of rules in $\mathcal{C}$.

Example 11. The cut length of both the sequent calculus for intuitionistic modal logic S4 presented in Example 6 and the sequent calculus for classical multiplicative linear logic presented in Example 5 is 2. Note that the sequence $(\Gamma_1, \xi_1, \xi_2 \to \Gamma_2;\ \Gamma_1 \to \xi_1, \Gamma_2;\ \Gamma'_1 \to \xi_2, \Gamma'_2)$ is a cut sequence for L$*$ and R$*$ in the calculus for classical multiplicative linear logic presented in Example 5.

5 Cutrank decremental operator

Once we define a cutrank decremental operator on a sequent calculus we can conclude by Theorem 4 that it enjoys cut elimination. Moreover when that operator is b-bounded it can be established that the logical depth of the cut-free deduction will be, in the worst case, hyper-exponentially greater, with base $2b$, than the logical depth of the original deduction.

We now propose a cutrank decremental operator $\mathcal{R}$ for each cut suitable sequent calculus $\mathcal{C}$. Moreover we show that the operator is bounded by the cut length of the calculus or by 1 in the case the cut length is less than 1. As far as we know, this fact was not already reported in the literature.

During the definition of the cutrank decremental operator it is explained for some important cases why the deduction returned by the operator is well defined. The conditions a cut suitable calculus has to satisfy are most of the times crucial. The analysis of why the operator proposed is bounded by the cut length of the calculus is presented at the end of the section. To simplify the presentation, since a rule may be present in some cut suitable calculi and not in others, when presenting a deduction by

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Psi \to \Delta\end{matrix}}{\Psi' \to \Delta'}\ r^*$$

we are requiring that the underlying calculus has rule r only if the conclusion of $\mathcal{D}$ is not $\Psi' \to \Delta'$. So, if the calculus does not have rule r then we are considering that $\mathcal{D}$ ends in $\Psi' \to \Delta'$. The name of the rule used in a single sequent deduction is sometimes omitted when more than one rule can instantiate to that sequent.

We define $\mathcal{R}$ inductively on the level of the cut that ends the deduction given. Let $\mathcal{D}^\circ$ be the pure-variable deduction

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Psi \to \Delta, \varphi^m\end{matrix} \qquad \begin{matrix}\mathcal{D}'\\ \varphi^n, \Psi' \to \Delta'\end{matrix}}{\Psi, \Psi' \to \Delta, \Delta'}\ \text{cut}$$

where $\mathcal{D}$ is

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Psi_1 \to \Delta_1, \varphi^{m_1}\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{D}_k\\ \Psi_k \to \Delta_k, \varphi^{m_k}\end{matrix}}{\Psi \to \Delta, \varphi^{m+a}}\ r}{\Psi \to \Delta, \varphi^m}\ ctr^a$$

and $\mathcal{D}'$ is

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}'_1\\ \varphi^{n_1}, \Psi'_1 \to \Delta'_1\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{D}'_{k'}\\ \varphi^{n_{k'}}, \Psi'_{k'} \to \Delta'_{k'}\end{matrix}}{\varphi^{n+a'}, \Psi' \to \Delta'}\ r'}{\varphi^n, \Psi' \to \Delta'}\ ctr^{a'}$$

both with lower cutrank than $\mathcal{D}^\circ$, and where k, k', a and a' are greater than or equal to 0, and r and r' are not contractions of the cut formula. Let

$$\mathcal{D}_i^* := \begin{cases} \mathcal{R}\left(\dfrac{\dfrac{\begin{matrix}\mathcal{D}_i\\ \Psi_i \to \Delta_i, \varphi^{m_i}\end{matrix}}{\Psi_i \to \Delta_i, \varphi^{m_i - a_i}}\ ctr^{a_i} \qquad \begin{matrix}\mathcal{D}'\\ \varphi^n, \Psi' \to \Delta'\end{matrix}}{\Psi_i, \Psi' \to \Delta_i, \Delta'}\ \text{cut}_i \right) & \text{if } m_i \neq 0 \\[2ex] \mathcal{D}_i & \text{if } m_i = 0 \end{cases}$$

and

$$\mathcal{D}'^*_j := \begin{cases} \mathcal{R}\left(\dfrac{\begin{array}{c}\mathcal{D}\\ \Psi \to \Delta, \varphi^m\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}'_j\\ \varphi^{n_j}, \Psi'_j \to \Delta'_j\end{array}}{\varphi^{n_i - a'_j}, \Psi'_j \to \Delta'_j}\, ctr^{a'_j}}{\Psi, \Psi'_j \to \Delta, \Delta'_j}\, \text{cut}'_j \right) & \text{if } n_j \neq 0 \\ \\ \mathcal{D}'_j & \text{if } n_j = 0 \end{cases}$$

where a_i is the minimum number of contractions such that $m_i - a_i$ is less than or equal to m, and similarly for a'_j, for $i = 1, \ldots, k$ and $j = 1, \ldots, k'$. Denote by $\Psi_{-\varphi}$ the multiset $\Psi \setminus \varphi$. Consider the following cases:

1. φ is not needed, neither by r nor by r'. Then

$$\mathcal{R}(\mathcal{D}^\circ) := \begin{cases} \dfrac{\overline{\Psi \to \Delta, \Delta'}\, r}{\Psi, \Psi' \to \Delta, \Delta'}\, \text{Lw}^* & \text{if } k = 0 \text{, Lw is in } \mathcal{C} \text{ and Rw is not in } \mathcal{C} \\ & \textit{(similarly if } k' = 0 \textit{ and for the other cases of Lw and Rw)} \\ \dfrac{\dfrac{\mathcal{D}'^*_1 \quad \ldots \quad \mathcal{D}'^*_{k'}}{\Psi^o, \Psi' \to \Delta^o, \Delta'}\, r'}{\Psi, \Psi' \to \Delta, \Delta'}\, c^* & \text{if } k \neq 0,\ k' \neq 0 \text{, and either cut is a right multicut for a constructor and } r \text{ and } r' \text{ do not have closure provisos or } r \text{ does and } r' \text{ does not have closure provisos} \\ \dfrac{\dfrac{\mathcal{D}^*_1 \quad \ldots \quad \mathcal{D}^*_k}{\Psi, \Psi'^o \to \Delta, \Delta'^o}\, r}{\Psi, \Psi' \to \Delta, \Delta'}\, c^* & \text{otherwise} \end{cases}$$

We now briefly explain why the deduction $\mathcal{R}(\mathcal{D}^\circ)$ is well defined. Namely we analyse the impact in $\mathcal{R}(\mathcal{D}^\circ)$ of the contexts in the r and r' inferences and of the presence or not presence of the contraction or the weakening rules in $\mathcal{C}$.

a. contexts. *a.1.* Suppose $\mathcal{C}$ has a cardinality proviso over the left side and $k \neq 0$ and $k' \neq 0$. The sequents in $\mathcal{R}(\mathcal{D}^\circ)$ that could violate the proviso are either the end-sequent of the r or r' inference or a premise with $m_j \neq 0$ or $n_j \neq 0$ respectively. But those premises satisfy the proviso since either Ψ'_j or Ψ' is empty by **C8**. The end-sequent does not violate the proviso when the rule applied is r' since $n = 1$ and so $o = 1$. If the rule applied is r the proviso is not violated by the end-sequent because Ψ' is empty. *a.2.* $k = 0$, Lw is in $\mathcal{C}$ and Rw is not. Suppose r is Ax. Then r does not have a cardinality proviso imposing that the right context is empty and $\mathcal{C}$ does not impose a cardinality proviso over the right side. So $\mathcal{C}$ does not have a rule with a cardinality proviso by **C1**. If r is a left introduction rule then r does not have a proviso imposing that the contexts are empty. Moreover if $\mathcal{C}$ imposes a cardinality proviso over the right side then Δ is empty. Note that r does not have a cardinality proviso requiring that the right context is empty by **C5.1**. *a.3.* fresh provisos are not an issue since deductions are pure-variable. *a.4.* Closure provisos are not a problem if $k = 0$ or $k' = 0$ due to **C5.2**. Otherwise if r and r' have closure provisos, by **C6** both rules have provisos for the main constructor c''' of the cut formula for different sides, so by **C7** both rules have two provisos for that constructor. Moreover r and r' are introduction rules for c'''. If r has closure provisos and r' does not, r' can be applied, see **C6**.

b. contraction rules. Suppose rule Lc is not in $\mathcal{C}$ and $k, k' \neq 0$. Then Multicut and LMulticut are not in $\mathcal{C}$ by **C4.2** and **C4.3** respectively. Denote the main constructor of the cut formula by c'''. *b.1.* rule r does have, and r' does not have, closure provisos. If r has one closure proviso then $n = 1$ and $o = 1$ by **C6.2**. If r has two closure provisos then consider **C6.3.i**. If $\mathcal{C}$ has a generic right contraction rule then it does not have a left multicut rule for c''' by **C4.1.iv**. Otherwise if $\mathcal{C}$ has a right contraction rule for c''' then Lc c''' is not in $\mathcal{C}$ by **C3.1** as well as LMulticut c''' by **C4.1.iii**. So in both cases $n = 1$ and $o = 1$. If $\mathcal{C}$ has a cardinality proviso over the right side then Δ is empty and r is a left introduction rule for c''' in a certain formula by **C6.1.i**. If n is greater than 1 than that formula can be contracted since the cut formula can be contracted. The other formulas in Ψ can be contracted by **C6.3.i**. *b.2.* both r and r' do not have closure provisos. If the cut shown in $\mathcal{D}^\circ$ is a right multicut for a constructor then $n = 1$ and $o = 1$. Otherwise if it is a generic right multicut than Rc is in $\mathcal{C}$ by **C4.3** and Ψ' is empty. *b.3.* both r and r' have closure provisos. By **C6** both rules have provisos for c''' for different sides, so by **C7** both have two provisos for c'''. Moreover r and r' introduce c'''.

2. φ is needed by r and not needed by r'. Then

$$\mathcal{R}(\mathcal{D}^\circ) := \begin{cases} \dfrac{\overline{\Psi' \to \Delta, \Delta'}\; r'}{\Psi, \Psi' \to \Delta, \Delta'}\,\mathrm{Lw}^* & \text{if } k' = 0\text{ , Lw is in } \mathcal{C} \text{ and Rw is not in } \mathcal{C} \\ & \textit{(similarly for the other cases of Lw and Rw)} \\ \dfrac{\dfrac{\begin{array}{c}\mathcal{D}'[\Psi_{-\varphi}|\emptyset] \\ \varphi^n, \Psi_{-\varphi}, \Psi' \to \Delta'\end{array}}{\Psi, \Psi' \to \Delta'}\,c^*}{\Psi, \Psi' \to \Delta, \Delta'}\,\mathrm{Rw}^* & \text{if } k' \neq 0,\ r \text{ is Ax, Lw is not in } \mathcal{C} \text{ and Rw is in } \mathcal{C} \\ & \textit{(similarly for the other cases of Lw and Rw)} \\ \dfrac{\mathcal{D}_1^*}{\Psi, \Psi' \to \Delta, \Delta'}\,w^* & \text{if } k' \neq 0 \text{ and either } r \text{ is Rw or } r' \text{ has closure} \\ & \text{provisos and } r \text{ is a weakening rule not generic} \\ \dfrac{\dfrac{\mathcal{D}_1'^* \quad \ldots \quad \mathcal{D}_{k'}'^*}{\Psi^o, \Psi' \to \Delta^o, \Delta'}\,r'}{\Psi, \Psi' \to \Delta, \Delta'}\,c^* & \text{otherwise} \end{cases}$$

c. contexts. *c.1* Lw is not in $\mathcal{C}$ and Rw is. The case when $k' = 0$ is omitted since a similar case was analysed in *a*. Suppose $k' \neq 0$, r is Ax and $\Psi_{-\varphi}$ is not empty. Then $\mathcal{C}$ does not impose a cardinality proviso over the left side, see **C8**, and does not have a rule with either a cardinality or a closure proviso by **C1**. *c.2.* $k' \neq 0$ and r is an introduction rule. *c.2.1.* cardinality provisos. Suppose $\mathcal{C}$

has a cardinality proviso over the right side. Note that both the end-sequent of the resulting deduction and the premises for which $n_j \neq 0$ satisfy the proviso since Δ is empty. *c.2.2.* fresh provisos are not an issue since the deductions are pure-variable. *c.2.3.* rule r' has closure provisos. Then r' has a closure proviso for the left side for the main constructor c''' of the cut formula by **C6.2**. If r' has one closure proviso then it is a right introduction rule for c''' by **C6.2.i**, and so r has the same provisos as r' by **C5.3**. Hence r' can be applied with the contexts of the r inference. If r' has two closure provisos and is a left introduction rule then r has the same provisos as r' by **C6.1.ii**.

d. contraction and weakening rules. *d.1.* Lw is not in $\mathcal{C}$. *d.1.1.* $k' \neq 0$ and r is Rw. Then $\mathcal{C}$ imposes a cardinality proviso over the left side by **C2.2**. So Ψ' is empty by **C8**. *d.1.2.* $k' \neq 0$, r is a right weakening rule for a constructor c, r' is a rule with closure proviso(s) and $m_1 = 0$. If r' has one closure proviso then by **C6.2** it introduces c at the right side, $\mathcal{C}$ has a cardinality proviso for that side, the proviso is for c at the left side, and by **C6.3.ii** either Lw or Lw c are in $\mathcal{C}$, which is impossible by **C2.1.ii**, or $\mathcal{C}$ has a cardinality proviso for the left side which implies that Ψ' is empty by **C8**. The case when r' has two closure provisos is analogous since r' is a right introduction rule. *d.2.* generic contraction rules are not in $\mathcal{C}$. *d.2.1.* $k' \neq 0$ and r is Ax. Cut formulas can be contracted as can be seen by case analysis on multicut rules in $\mathcal{C}$, see **C4.1**, **C4.2** and **C4.3**. *d.2.2.* $k' \neq 0$ and r is an introduction rule. *d.2.2.1.* cut shown in $\mathcal{D}^\circ$ is LMulticut. Then $\mathcal{C}$ has a cardinality proviso for the right side by **C4.3**, so Δ is empty, and Lc is in $\mathcal{C}$ by **C4.2**. *d.2.2.2.* cut shown in $\mathcal{D}^\circ$ is LMulticut c. Hence r has closure provisos by **C4.1.i** and so for each side l either $\mathcal{C}$ has an appropriate contraction rule for the l context of r or it is empty, see **C6**.

3. φ is needed by r' and not needed by r. Omitted (it is similar to case **2**).

4. φ is needed by r and by r'. Then

$$\mathcal{R}(\mathcal{D}^\circ) := \begin{cases} \dfrac{\dfrac{\begin{array}{c}\mathcal{D}[\Psi'|\emptyset]\\ \Psi,\Psi'\to\Delta,\varphi^m\end{array}}{\Psi,\Psi'\to\Delta,\varphi}\,c^*}{\Psi,\Psi'\to\Delta,\Delta'}\,\mathrm{Rw}^* & \text{if } r' \text{ is Ax, Lw is not in } \mathcal{C} \text{ and Rw is in } \mathcal{C} \\ & \textit{(similarly if } r \textit{ is } \mathrm{Ax} \textit{ and for the other} \\ & \textit{cases of Lw and Rw)} \\[2ex] \dfrac{\mathcal{D}_1'^{*}}{\Psi,\Psi'\to\Delta,\Delta'}\,w^* & \text{if } r' \text{ is a weakening and } r \text{ is not Ax} \\ & \textit{(similarly if } r \textit{ is a weakening)} \\[2ex] \dfrac{\begin{array}{c}\hat{\mathcal{D}}\\ \Psi^o,\Psi'^{o'}\to\Delta^o,\Delta'^{o'}\end{array}}{\Psi,\Psi'\to\Delta,\Delta'}\,c^*+w^* & \text{if } r' \text{ and } r \text{ are introduction rules for the} \\ & \text{same constructor} \end{cases}$$

where, when r and r' are a pair of introduction rules for the same constructor and $(p_1; \ldots; p_u)$ is a cut sequence with minimal length for them, $\hat{\mathcal{D}}$ is the pure-variable deduction corresponding to the deduction whose end-sequent is empty in Definition 7, defined using the deductions

$$\hat{\mathcal{D}}_i^* := \begin{cases} \mathcal{D}_l^*[t'_1/y_1, \dots, t'_v/y_v] & \text{if } p_i \text{ is the } l\text{-th premise of } r \text{ and } y_1, \dots, y_v \text{ are the instantiation of the fresh variables of } p_i, \text{ if any, in } \mathcal{D} \text{ and } t'_1, \dots, t'_v \text{ the instantiation of those variables in } \mathcal{D}' \\ \mathcal{D}_l'^*[t_1/y'_1, \dots, t_v/y'_v] & \text{if } p_i \text{ is the } l\text{-th premise of } r' \text{ and } y'_1, \dots, y'_v \text{ are the instantiation of the fresh variables of } p_i, \text{ if any, in } \mathcal{D}' \text{ and } t_1, \dots, t_v \text{ the instantiation of those variables in } \mathcal{D} \end{cases}$$

for $i = 1, \dots, u$. For example, the deduction $\hat{\mathcal{D}}$ may be

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\hat{\mathcal{D}}_3^*\\ \hat{s}_3^*\end{array} \quad \dfrac{\begin{array}{c}\hat{\mathcal{D}}_1^*\\ \hat{s}_1^*\end{array} \quad \begin{array}{c}\hat{\mathcal{D}}_2^*\\ \hat{s}_2^*\end{array}}{s_{1,2}}\,\mathrm{cut}_1}{s_{1,2,3}}\,\mathrm{cut}_2 \\ \ddots \end{array} \qquad \begin{array}{c}\hat{\mathcal{D}}_u^*\\ \hat{s}_u^*\end{array}}{s_{1,\dots,u}}\,\mathrm{cut}_{u-1}$$

e. contexts. *e.1.* Cut rules with additional contexts when r and r' are introduction rules for the same constructor. The case where $\mathcal{C}$ imposes a cardinality proviso over side l and after applying a cut rule the corresponding end-sequent violates it, see **C4.4**, is not possible since by **C8** both premises have at most one formula at side l and one of them is the cut formula. *e.2.* r' is Ax, Lw is not in $\mathcal{C}$ and Rw is. Suppose Ψ' is not empty. Then $\mathcal{C}$ does not impose a left cardinality proviso. So by **C1** there is not a rule in $\mathcal{C}$ with either a cardinality or a closure proviso.

f. contraction and weakening rules. In the case that r and r' are introduction rules for the same constructor, **C9** and the definition of cut suitable pair of rules, Definition 7, ensures that $\mathcal{C}$ has the appropriate contraction and weakening. Suppose r' is Lw, r is not Ax and Rw is not in $\mathcal{C}$. Note that r is an introduction rule. Then $\mathcal{C}$ has a cardinality proviso over the right side by **C2.2**. So Δ is empty. If r' is Lw c then r has closure provisos by **C2.1.i**. Moreover the cut formulas can be contracted if m or n is greater than zero by **C4.1**, **C4.2** and **C4.3**.

We now briefly explain why $\mathcal{R}$ is a bounded operator, see Definition 2, and satisfies the cutrank and the bound conditions with respect to the cut length of $\mathcal{C}$. For that we do a case analysis on $\mathcal{R}(\mathcal{D}^\circ)$. We distinguish three major forms of $\mathcal{R}(\mathcal{D}^\circ)$: **1**. $\mathcal{R}(\mathcal{D}^\circ)$ is $\mathcal{D}$ or $\mathcal{D}'$ possibly enriched with additional contexts and with weakenings or contractions at the end. In this case $\mathrm{cr}(\mathcal{R}(\mathcal{D}^\circ)) < \mathrm{cr}(\mathcal{D}^\circ)$ since $\mathrm{cr}(\mathcal{D}), \mathrm{cr}(\mathcal{D}') < \mathrm{cr}(\mathcal{D}^\circ)$ by hypothesis. Moreover $||\mathcal{R}(\mathcal{D}^\circ)|| \leq b(||\mathcal{D}|| + ||\mathcal{D}'||)$; **2**. $\mathcal{R}(\mathcal{D}^\circ)$ either is $\mathcal{D}_j^*$ or $\mathcal{D}_{j'}'^*$, or is obtained from those deductions by application of r or r', possibly with weakenings and contractions at the end. Note that $\mathrm{cr}(\mathcal{D}_j^*), \mathrm{cr}(\mathcal{D}_{j'}'^*) < \mathrm{cr}(\mathcal{D}^\circ)$, and if r or r' is a cut it has cutrank lower than $\mathrm{cr}(\mathcal{D}^\circ)$. So $\mathrm{cr}(\mathcal{R}(\mathcal{D}^\circ)) < \mathrm{cr}(\mathcal{D}^\circ)$; *2.1* $\mathcal{R}(\mathcal{D}^\circ)$ is obtained from r and $||\mathcal{D}_j|| = ||\mathcal{D}||$ for some j. Then r is a weakening or contraction and so $||\mathcal{R}(\mathcal{D}^\circ)|| \leq b(||\mathcal{D}|| + ||\mathcal{D}'||)$. Similarly for r'; *2.2* $\mathcal{R}(\mathcal{D}^\circ)$ is obtained from r and $||\mathcal{D}_j|| < ||\mathcal{D}||$. Then $||\mathcal{D}^*_j|| \leq b(||\mathcal{D}|| + ||\mathcal{D}'||) - b$ for all j and so the result follows since $||\mathcal{R}(\mathcal{D}^\circ)|| \leq b(||\mathcal{D}|| + ||\mathcal{D}'||) - b + 1$ and b is greater than 1. Similarly for r'; **3**. $\mathcal{R}(\mathcal{D}^\circ)$ is $\hat{\mathcal{D}}$ possibly with weakenings and contractions. Then $\mathrm{cr}(\mathcal{R}(\mathcal{D}^\circ)) < \mathrm{cr}(\mathcal{D}^\circ)$ since $\mathrm{cr}(\mathcal{D}^*_j), \mathrm{cr}(\mathcal{D}'^*_j) < \mathrm{cr}(\mathcal{D}^\circ)$ and the remaining cuts are over subformulae of the original cut formula. Note that $||\mathcal{D}_j|| < ||\mathcal{D}||$. So $||\mathcal{D}^*_j|| \leq b(||\mathcal{D}|| + ||\mathcal{D}'||) - b$. Hence $||\mathcal{R}(\mathcal{D}^\circ)|| \leq b(||\mathcal{D}|| + ||\mathcal{D}'||)$ since the cuts over subformulae of the original cut formula are at most $\mathrm{cl}(\mathcal{C})$.

6 Subformula property and consistency

By induction on the depth of a deduction one easily sees that every cutfree deduction in a cut suitable calculus satisfies the subformula property. Together with Theorem 4 this yields that any cut suitable calculus is consistent.

7 Sequent calculi in the literature

The results established in this paper can also be seen as providing conditions for a sequent calculus to enjoy cut admissibility. Using the results of this paper, a cut is admissible in a calculus if there is an enrichment, with cut rules and possibly with admissible weakening or contraction rules, of the original calculus, in order to make it cut suitable. The cut elimination proof can then be replaced by a decidable check of whether the enriched calculus satisfies the cut suitable conditions. For the sake of an illustration we now consider the calculi proposed by Gentzen in his pioneering work [1] and some of the calculi proposed in [18] and in [15].

The Gentzen system **G1c** was shown to admit cut elimination in [15] by proving the elimination of the rule Multicut in the calculus obtained by enriching **G1c** with that rule. Capitalizing on the results presented herein it would not be necessary to explicitly prove cut elimination in **G1c**+Multicut since this calculus is cut suitable. Moreover the upper bound obtained in [15] for the complexity of cut elimination in **G1c**+Multicut coincides with the bound we establish herein using the general concept of cut length of the calculus. Analogously, the Gentzen systems **G2c**, **G[12][mi]**, **G3[c]**, **G3[mi]**, **G[ic]l**, **G3s** and **G[12]s** admit cut elimination since the calculi

- **G2c**+Multicut
- **G[12][mi]**+LMulticut

- **G3[c]**+Lc+Rc+Cut
- **G3[mi]**+Lc+Cut
- **G[ic]l**+LMulticut!+RMulticut?
- **G3s**+Lc+Rc+Cut
- **G[12]s**+Cut

are cut suitable and the contraction rules are admissible in **G3[mic]** and in **G3s**. Note that the calculus presented in Example 6 is an intuitionistic version of the calculus **G1s** enriched with the rule LMulticut and is cut suitable, as well as the calculus presented in Example 5 which is a multiplicative fragment of the calculus **Gcl** enriched with LMulticut! and RMulticut?.

Cut admissibility is studied in [18] for several sequent calculi. The Cut rule was shown to be admissible in **G0[i]**, **G0[c]**, as well as in **G3[ic][p]** after having proved admissibility of contraction. The results presented herein can be seen as applying to these calculi since **G0[i]**+Cut, **G0[c]**+Cut, **G3[i][p]**+Lc+Cut, **G3[c][p]**+Lc+Rc+Cut are cut suitable. The calculi **GN**, **GM** and **G3im** and its enrichments do not belong to the universe of sequent calculi studied herein since some introduction rules in that calculi have a form not considered in this work.

The pioneer calculi **LJ** and **LK** of Gentzen [1] are cut suitable modulo the Interchange rule which can be seen as being hidden in the multiset data structure.

8 Comparison with related work

It is a common aspect of the papers [2–4,6,9] and of our paper to start by defining the universe of calculi under study and only after that to establish sufficient conditions or characterization conditions for cut elimination in the context of that calculi. In all those papers, that universe is defined by the types of rules and provisos that can be present in the calculi. The papers [7,8] are concentrated on display and on consecution systems.

As in [3,4] we consider sequent calculi that can have rules for first order quantifiers. Propositional based systems are considered in [2,5–9]. Structural rules and cut rules that apply only to formulas with a specific main constructor can be in the calculi considered herein. The papers [2,3,6,9] do not consider calculi with such kinds of rules. With respect to provisos we consider calculi with rules that may have cardinality and closure provisos. Rules with these types of provisos are not allowed in [5,3,6–9]. In [2] due to conditions (log2) and (log3) in the definition of simple calculus we conclude that those calculi can not have introduction rules with cardinality or closure provisos.

There are a number of differences between the previous work [2,5,3,4,6–9] and the work of this paper, with respect to level of detail, names, and presentation of sufficient conditions; however, there are several common aspects. This was expected since there are crucial cases common to most of the cut elimination proofs. One such crucial case is the propagation of cut to the subformulas of the original cut formula if it is introduced by a left or a right introduction rule. All the works have a condition for guaranteeing that there is such a deduction. That condition changes accordingly with the universe of calculi that is being considered. For instance Avron and Lev in [5,6] in the context of a canonical propositional calculus require that for every left and right logical rules for the same connective, the set of clauses obtained from its premises is classically inconsistent, that is, the empty clause can be derived from them using cuts. A similar notion had previously appeared in the work of Baaz and Leitsch on CERES, a cut elimination by resolution method for classical first order logic [22]. In the work of Ciabattoni and Terui in [2] the derivation of the empty sequent is not enough to guarantee the deduction of the original end-sequent since the calculi considered may not have all the structural rules. So they require that there is a deduction for the end-sequent using the identity axiom, the structural rules and the cut rule. The condition we propose herein can be seen as staying between those two kinds of conditions. We require that it is possible to derive the empty sequent by using only cut rules, as in [5,6], but also require additional conditions related with the presence of the structural rules so that it is possible to derive the contexts of the end-sequent by using them.

Another crucial case present in almost all the cut elimination proofs is the propagation of the cut over the cut formula to the premises of a rule. This case appears when the cut formula is not needed by the rule used in one of the premises. In [2] the authors propose the weak substitutive condition to take care of it. In our work we guarantee that that case is not problematic by requiring a condition for each specific subcase of that crucial case. Note however that there is a difference in the level of detail in our condition and the similar condition proposed for instance by [2]. In general the conditions we propose are more specific than in the other related works.

The study of the complexity of cut elimination in a class of calculi, as well as the possibility of checking the conditions by a computer program, was not considered in the works [2,5,3,4,6–9].

9 Concluding remarks

Besides the results about the sufficient conditions for cut elimination and its complexity analysis, this work also provides another point of view concerning the intimate relationships between the rules and the provisos in a calculus that guarantee that it admits cut elimination by a Schütte-Tait style cut elimination proof. A complete characterization of cut elimination for the class of calculi described in this paper is not straightforward but is one of the most interesting directions of future work that we intend to pursue. For this we would like to start by characterizing the semantics of the logics with calculi in this class or in a large subclass of it. Another interesting direction of future work is the refinement of the complexity analysis of cut elimination done herein by differentiating between propositional connectives and quantifiers as studied in [24–28].

General modularity results like the ones obtained in [9] is another very interesting goal to pursue in the future. The idea is to study under which conditions cut elimination is preserved when a certain calculus enjoying cut elimination is enriched with rules of a certain type. Another possible direction of future work is to generalize the type of calculi considered, e.g. to infinitary systems, and the universe of rules and provisos [29–33]. Taking into account that the conditions presented are proved based on a specific type of cut elimination proof it would also be interesting to identify sufficient conditions for cut elimination when the cut elimination proof method is different from the one considered in this work.

Finally note that the approach followed in this paper can be applied to other properties like for instance the Beth definability property and the Craig interpolation property.

Acknowledgements

The author wishes to express his gratitude to the anonymous referee(s) for the valuable comments and suggestions given in earlier versions of the work. Moreover the author thanks all the colleagues that contributed with useful comments and suggestions. The author gratefully acknowledges the partial support of FCT and EU FEDER through POCTI, POCI, PTDC and PPCDT namely via CLC, SQIG-IT, the QuantLog MAT/55796/2004 Project of CLC and the QSec PTDC/EIA/67661/2006 Project of SQIG-IT.

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