Modal Sequent Calculi Labelled with Truth Values: Cut Elimination

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Abstract

Cut elimination is shown, in a constructive way, to hold in sequent calculi labelled with truth values for a wide class of normal modal logics, supporting global and local reasoning and allowing a general frame semantics. The complexity of cut elimination is studied in terms of the increase of logical depth of the derivations. A hyperexponential worst case bound is established. The subformula property and a similar property for the label terms are shown to be satisfied by that class of modal sequent calculi. Modal logics presented by these calculi are proven to be globally and locally consistent.

Keywords: Cut elimination, complexity of cut elimination, modal sequent calculus, labelled deduction

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1 Context

Cut elimination is a central issue of structural proof theory. It was first established by Gentzen [13] for the sequent systems LK and LJ, for classical and intuitionistic first order logic, respectively. There are a number of interesting properties in first order logic that capitalize on the existence of a cut free derivation for every derivation. For instance, the consistency of the calculus, Herbrand's theorem, the mid sequent theorem, Craig interpolation, and Beth definability. Cut elimination provides also the kind of analysis necessary to deal with plausible reasoning and other kinds of incomplete proofs as described by Polya [25], and can be used to construct elementary proofs from non-elementary ones [36].

Automated logical reasoning is also a motivation for studying cut elimination since inference rules like the cut rule make efficient proof search infeasible. Moreover, the extraction of programs from proofs requires cut elimination as a preparatory step. However cuts help making proofs structured and intelligible and have a strong effect on the length of proofs. So a consequence of cut elimination is that the length of cut free proofs may be much bigger than proofs with cuts.

Sequent calculus, since its introduction by Gentzen in 1934, has been one of the preferred deductive frameworks to represent and study logics [13, 5, 37]. Contrasting with propositional or first order logics whose representation in sequent calculus is perfect, modal

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logic has faced difficulties in finding a natural representation and simultaneously enjoying properties like symmetry, explicitness, and elimination of cut [40].

Curry seems to have been the first to seek sequent systems for modal logics [9] soon followed by Ohnishi and Matsumoto [23] and Kanger [15]. More recently, several modal sequent calculi answering some of the problems faced by the initial presentations have been proposed [20, 40, 4, 12, 21, 1, 39, 8, 17, 35, 29]. For example, in [39] a sequent style proof theory, based on Belnap's [3] display logic, is presented for any modal logic obtained by combination of D, T, K4, K5, B, Tr, V and Alt1. In [20], sequent calculi labelled with truth values are presented for modal logics obtained by combination, among others, of D, T, K4, K5, B, L, and of logics characterized either by irreflexive, or asymmetric, or antisymmetric frames. Additionally the calculi in [20] support global and local reasoning, and allow a general frame semantics. In [4] it is introduced a sequent calculus for S5 satisfying cut elimination as well as the subformula property. Moreover a useful review of modal sequent calculi for S5 is addressed. In [1] a cut free hypersequent calculus for S5 is presented. A different approach to devise sequential versions of the modal propositional logics S5, B, KB, DB, and K4B satisfying the subformula property was followed in [35] by requiring that the cut formula is a subformula of some formula in the lower sequent.

The complexity of cut elimination, in the context of modal sequent systems, has not been studied. This fact contrasts with the situation in propositional and first order logic which has being deserving close attention [34, 24, 26, 5, 6, 7, 22]. For instance the increase in depth when eliminating cuts in propositional sequent calculus, independently of the procedure, was shown to be elementary in the worst case, and with respect to classical first order logic the increase in depth when eliminating cuts, independently of the particular procedure, was shown to be non-elementary.

Herein we show that cuts can be eliminated in sequent calculi labelled with truth values, based on [20], for a wide class of modal logics. The great advantage of considering labelled deduction systems [10, 38, 28, 20, 32, 27] is that they allow to present in a natural, uniform and modular way families of logic systems. In order to accomplish that desiderata, the basic declarative unit in these presentations is a formula plus a label and there are rules expressing the properties of labels. So, the rules for the logical connectives can be immutable between the different logics in a family, and a particular logic can be obtained simply by adjoining to the core set specific rules involving only the labels. Another positive aspect of labelled deduction systems is the expressive power. For example in [20] it is presented a modal sequent calculi labelled with truth values characterizing the class of irreflexive frames although there is no modal axiom to define that class in the Hilbert context, in [38] cut free sequent systems labelled by worlds are presented for the class of modal logics whose accessibility relation is axiomatizable by Horn-clause, and in [12] cut free goal directed labelled deduction methods for the implicational fragments of several modal logics are provided and cut admissibility is proved. Note that in [12] labels represent worlds and the calculi are organized in a database composed by a finite set of labelled formulas and by a set of links between the labels. Approaches similar to labelled deduction have been devised. For instance in [21] it is developed an indexed sequent calculi covering all the combinations of the axiom schemata T, B and 4 above propositional K, and it is presented a cut-elimination proof for that calculi. Differently from labelled deduction systems, in this approach labels are not coupled with formulae but with sequents.

The labelled sequent calculi presented in this paper have three types of assertions: indivisibility assertions Ωt, truth value comparison assertions $t_1 \sqsubseteq t_2$, and labelled formulae $t \leq \varphi$; and have three types of rules: the left and the right rules for the logical connectives, structural rules, and modal accessibility rules. Modal accessibility rules express the properties of the accessibility relation of each specific modal logic. Cut rules are part of the structural rules of the calculi and there is one cut rule for each type of assertion.

Cut elimination, that is, the result that establishes that given a modal derivation there is a cut free derivation for the same end-sequent, is showed as a particular case of a more general statement. We develop a procedure that given any well behaved derivation returns a cut free derivation for the same end-sequent. This result is more general since from each modal derivation it is possible to extract, and we show how, a well behaved derivation for the same consequence without increasing the logical depth and the number and complexity of cuts.

Cuts over indivisibility assertions are immediately eliminated when constructing a well behaved derivation from a modal derivation. Cuts over labelled formulae are more complex to eliminate due to the number of rules that can introduce a logical connective and to the presence of labels. Based on the fact that the only labels needed in well behaved derivations are $\top$ and bound variables, we develop a procedure to eliminate cuts over formulae labelled by $\top$ and another procedure to eliminate cuts over formulae labelled by atomic terms. The order of application of the procedures is important since the elimination of cuts over formulae labelled by $\top$ may originate cuts over formulae labelled by atomic terms, and the elimination of cuts over formulae labelled by atomic terms may originate cuts over truth value comparison assertions. Cuts over truth value comparison assertions are shown to be eliminable in the context of any calculus with a compatible set of modal accessibility rules. The compatibility requirement is necessary since, in general, the rules that introduce truth value comparison assertions do not reduce the complexity of the terms that appear in the conclusion of the rule. Nevertheless those calculi present a large class of modal logics like T, B, K4, L, K5, logics characterized by irreflexive, asymmetric or antisymmetric frames, and most of the logics resulting from their combination.

The complexity of cut elimination was, at the same time, analyzed in terms of the increase of logical depth of the derivations after the application of the procedures. We establish that cut elimination increases hyperexponentially the logical depth of the derivation in the worst case. From the analysis of the bound obtained it is possible to see that the fact that the calculi presented in this paper allows to reason globally and locally, and the great expressive power of the calculi, both capitalizing in the presence of labels, causes in the worst case an exponential increase in the complexity of the cut elimination procedure, which is not significant compared to the hyperexponential worst case.

The subformula property and a related but weaker property for truth value terms are shown to be satisfied by the modal sequent calculi with a compatible set of modal accessibility rules. This happens as a corollary of the cut elimination theorem and due to the form of the rules. Another important consequence of the cut elimination theorem is the global and local consistency proof for modal logics presented by these calculi.

The treatment of labels by the cut elimination procedures, the way to eliminate cuts over truth value comparison assertions, and the move from modal derivations to well behaved derivations, which makes cuts simpler to eliminate cuts than in the original derivations, constitute novelties compared to [37], which was our reference.

The paper is organized in five sections. In Section 2 we present the modal sequent calculi labelled with truth values, starting by introducing the language of the systems.

Then the modal sequent calculi are described and its rules briefly explained. (The modal calculi share a common core set of rules consisting of structural rules, and the left and the right rules for the logical connectives. To obtain a calculus for a specific modal logic we add to the common set of rules, specific rules for the desired modal logic). In Section 3 we prove that cuts can be eliminated in the modal sequent calculi with a compatible set of modal accessibility rules. Moreover, we study the complexity of cut elimination and we provide a hyperexponential worst case bound for the procedure. As a corollary of the cut elimination theorem we show in Section 4 that the subformula property and a related but weaker property for terms are satisfied by the modal sequent calculi with a compatible set of modal accessibility rules. Additionally, the global and the local consistency of the modal logics presented by these calculi are proven. Finally, in Section 5 we draw some conclusions and sketch possible lines of future work.

2 Sequent calculi

The modal sequent calculi labelled with truth values presented in this paper is a reshaping of the calculi introduced in [20]. Basic definitions are shared between the two calculi. (Details not necessary for this work are hidden.) In the following we will briefly review some definitions.

Language

The sequent framework developed in [20] can be used to present a wide variety of logics and not only modal logic. The language of each specific calculus is defined by a *signature* where the connectives of the logic are declared, as well as its arity and type. In a more rigorous way a signature Σ is a tuple $\langle C, O, X, Y, Z \rangle$ where C and O are families of countable sets indexed by the natural numbers, $\top$ is in O_0, and X, Y and Z are countable sets. The elements of each C_k are *formula constructors* of arity k, the elements of O_k are *truth value* operators of arity k, the elements of X are *truth value unbound variables*, the elements of Y are *truth value bound variables* and the elements of Z are *formula unbound variables*. The truth value unbound variables in X and the formula unbound variables in Z are necessary when proving completeness, as is explained in [20]. The truth value bound variables in Y are needed in order to be able to universally quantify in the inference rules. In the following, when a signature Σ is not explicitly defined, we assume that their constituents are denoted by C, O, X, Y and Z.

The *modal signature* is the tuple $\langle C, O, X, Y, Z \rangle$ defined as follows:

- $C_0 = \{\mathrm{p}_i : i \in I\!N\}$, $C_1 = \{\neg, \Box, \Diamond\}$, $C_2 = \{\wedge, \vee, \Rightarrow\}$, and $C_k = \emptyset$ for $k \geq 3$;
- $O_0 = \{\top\}$, $O_1 = \{\mathbf{N}\}$, and $O_k = \emptyset$ for $k \geq 2$;
- $X = \{\mathbf{x}_i : i \in I\!N\}$, $Y = \{\mathbf{y}_i : i \in I\!N\}$, and $Z = \{\mathbf{z}_i : i \in I\!N\}$.

The connective $\mathbf{N}$ is a syntactic reference to the modal accessibility relation, as it will be clear when presenting the rules of a modal sequent calculus. The other connectives have the expected meaning.

It is assumed given three sets of meta-variables: the set of formula meta-variables $\{\xi_i : i \in I\!N\}$, the set of term meta-variables $\{\tau_i : i \in I\!N\}$ and the set of assertions multiset meta-variables $\{\Gamma_i : i \in I\!N\}$. The set of *(schema) formulae* is inductively defined in the usual way by (the meta-variables ξ_i and) the connectives in C and by the variables in Z. The set of *(schema) truth value terms*, or simply *(schema) terms*, is inductively defined in the usual way by (the meta-variables τ_i and) the connectives in O and by the variables in X and in Y.

The set of *(schema) assertions* over a signature is composed of expressions of six different forms: $t \sqsubseteq t'$ and $t \not\sqsubseteq t'$, denoted by *positive and negative truth value comparison assertions*, respectively, $t \leq \varphi$ and $t \not\leq \varphi$ denoted by *positive and negative labelled formulae*, respectively, and Ωt and $\mho t$ denoted by *positive and negative truth value indivisibility assertions*, respectively. Sometimes we use the term *atomic* instead of indivisibility.

The denotation of terms and formulae is defined in [20] over a natural two-sorted algebraic semantics. In this context, the intended meaning of $t \sqsubseteq t'$ is to assert that the truth value resulting from the denotation of t is less than or equal to the truth value resulting from the denotation of t'. The intended meaning of $t \leq \varphi$ is to assert that truth value obtained from the denotation of t is less than or equal to the denotation of the formula φ, and the intended meaning of Ωt is to express that the truth value resulting from the denotation of t is atomic, that is, there is no truth value strictly smaller than it besides falsum.

Deduction

A *sequent* over a signature is a pair $\langle \Psi, \Delta \rangle$, written $\Psi \to \Delta$ where Ψ and Δ are finite multisets composed of schema assertions and of assertions multiset meta-variables. A *(schema) substitution* σ is a map that associates each formula meta-variable ξ_i to a (schema) formula, each term meta-variable τ_i to a (schema) term, and each assertions multiset meta-variable Γ_i to a finite multiset of (schema) assertions. We extend the application of a (schema) substitution to a schema assertion, a multiset of schema assertions and to a sequent as is expected. Moreover we denote the application of a (schema) substitution σ to a sequent s by $s\sigma$. Similarly for the application to schema formulae, schema terms, schema assertions and multisets of schema assertions.

As is usual in most deduction calculi the application of a rule may be restricted to certain formulae, terms and assertions, satisfying some conditions. An application of a rule consists of the replacement of each meta-variable in the rule by the value attributed by a substitution to that meta-variable. So, associated to an application of a rule is always a substitution, and the way to constrain the application of a rule to certain formulae, terms and assertions, is to restrict the substitutions that can be used in the application of the rule.

A *proviso* π is map that associates to each substitution the value 0 if the substitution does not satisfy the proviso and the value 1 otherwise. A *rule* over a signature is a triple $\langle \{s_1, \ldots, s_p\}, s, \pi \rangle$ written

$$\frac{s_1 \ldots s_p}{s} \lhd \pi$$

where $s_1, \ldots, s_p, s$ are sequents and π is a proviso. A *sequent calculus* $\mathcal{C}$ is a pair $\langle \Sigma, \mathcal{R} \rangle$ where Σ is a signature and $\mathcal{R}$ is a finite set of rules over Σ. The *derivation* of a sequent s' from a set S of sequents in the context of a modal sequent calculus $\mathcal{C}$, written

$$S \vdash_{\mathcal{C}} s'$$

is defined in the usual way as a tree, see [37].

Modal sequent calculi

A *modal sequent calculus* $\mathcal{C}$ is a pair $\langle \Sigma, \mathcal{R} \rangle$ where Σ is the modal signature and $\mathcal{R}$ contains modal accessibility rules, the rules for introducing the logical connectives, and the structural rules. We now describe these types of rules. The *left and right rules for the logical connectives* are:

L¬ $\dfrac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1}{\Omega\tau_1,\tau_1\leq(\neg\xi_1),\Gamma_1\to\Gamma_2}$ R¬ $\dfrac{\Omega\tau_1,\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\neg\xi_1)}$

L∧ $\dfrac{\Omega\tau_1,\tau_1\leq\xi_1,\tau_1\leq\xi_2,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\xi_1\wedge\xi_2),\Gamma_1\to\Gamma_2}$ R∧ $\dfrac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1 \quad \Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\xi_1\wedge\xi_2)}$

L∨ $\dfrac{\Omega\tau_1,\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2 \quad \Omega\tau_1,\tau_1\leq\xi_2,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\xi_1\vee\xi_2),\Gamma_1\to\Gamma_2}$ R∨ $\dfrac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1,\tau_1\leq\xi_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\xi_1\vee\xi_2)}$

L⇒ $\dfrac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1 \quad \Omega\tau_1,\tau_1\leq\xi_2,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\xi_1\Rightarrow\xi_2),\Gamma_1\to\Gamma_2}$ R⇒ $\dfrac{\Omega\tau_1,\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\xi_1\Rightarrow\xi_2)}$

L□ $\dfrac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\Omega\tau_2 \quad \Omega\tau_2,\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\mathbf{N}(\tau_1) \quad \Omega\tau_2,\tau_2\leq\xi_1,\Omega\tau_1,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\Box\xi_1),\Gamma_1\to\Gamma_2}$

R□ $\dfrac{\Omega\tau_2,\tau_2\sqsubseteq\mathbf{N}(\tau_1),\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_2\leq\xi_1}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\Box\xi_1)} \lhd \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$

L◇ $\dfrac{\Omega\tau_2,\tau_2\sqsubseteq\mathbf{N}(\tau_1),\tau_2\leq\xi_1,\Omega\tau_1,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\Diamond\xi_1),\Gamma_1\to\Gamma_2} \lhd \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$

R◇ $\dfrac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\Omega\tau_2 \quad \Omega\tau_2,\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\mathbf{N}(\tau_1) \quad \Omega\tau_2,\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_2\leq\xi_1}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\Diamond\xi_1)}$

where the proviso $\tau_2 : \mathbf{y}$ gives the value 1 to a substitution σ if σ associates to τ_2 an element of Y, and the proviso $\tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$ gives the value 1 to a substitution σ if σ associates to τ_2 a term that is not present in $\tau_1\sigma$, $\Gamma_1\sigma$ and $\Gamma_2\sigma$. Together they express that, when applying the rule in a derivation, τ_2 is *fresh*. The intuitive meaning of an atomic assertion Ωt, in the context of Kripke semantics, is that t represents a truth value with only a world. So, in that context, the intuitive meaning of R□ is that if for any world accessible from τ_1 the formula ξ_1 holds then the formula $\Box\xi_1$ holds at τ_1. The *structural rules* of a modal sequent calculus are:

AxΩ $\dfrac{}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\Omega\tau_1}$ AxT $\dfrac{}{\tau_1\sqsubseteq\tau_2,\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_2}$

AxF $\dfrac{}{\tau_1\leq\xi,\Gamma_1\to\Gamma_2,\tau_1\leq\xi}$

LwΩ $\dfrac{\Gamma_1\to\Gamma_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2}$ RwΩ $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1\to\Gamma_2,\Omega\tau_1}$

LwT $\dfrac{\Gamma_1\to\Gamma_2}{\tau_1\sqsubseteq\tau_2,\Gamma_1\to\Gamma_2}$ RwT $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_2}$

LwF $\dfrac{\Gamma_1\to\Gamma_2}{\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2}$ RwF $\dfrac{\Gamma_1\to\Gamma_2}{\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1}$

LcΩ $\dfrac{\Omega\tau_1,\Omega\tau_1,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2}$ RcΩ $\dfrac{\Gamma_1\to\Gamma_2,\Omega\tau_1,\Omega\tau_1}{\Gamma_1\to\Gamma_2,\Omega\tau_1}$

LcT $\dfrac{\tau_1\sqsubseteq\tau_2,\tau_1\sqsubseteq\tau_2,\Gamma_1\to\Gamma_2}{\tau_1\sqsubseteq\tau_2,\Gamma_1\to\Gamma_2}$ RcT $\dfrac{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_2,\tau_1\sqsubseteq\tau_2}{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_2}$

LcF $\dfrac{\tau_1\leq\xi_1,\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2}{\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2}$ RcF $\dfrac{\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1,\tau_1\leq\xi_1}{\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1}$

$$\text{Lxi}\Omega \quad \frac{\Gamma_1 \to \Gamma_2, \Omega\tau_1}{\mho\tau_1, \Gamma_1 \to \Gamma_2} \qquad \text{Rxi}\Omega \quad \frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \mho\tau_1}$$

$$\text{LxiT} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2} \qquad \text{RxiT} \quad \frac{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}$$

$$\text{LxiF} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2} \qquad \text{RxiF} \quad \frac{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}$$

$$\text{Lxe}\Omega \quad \frac{\Gamma_1 \to \Gamma_2, \mho\tau_1}{\Omega\tau_1, \Gamma_1 \to \Gamma_2} \qquad \text{Rxe}\Omega \quad \frac{\mho\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$$

$$\text{LxeT} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2} \qquad \text{RxeT} \quad \frac{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$$

$$\text{LxeF} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2} \qquad \text{RxeF} \quad \frac{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$$

$$\text{cut}\Omega \quad \frac{\Gamma_1 \to \Gamma_2, \Omega\tau_1 \quad \Omega\tau_1, \Gamma_1' \to \Gamma_2'}{\Gamma_1 \Gamma_1' \to \Gamma_2 \Gamma_2'}$$

$$\text{mcutT} \quad \frac{\Gamma_1 \to \Gamma_2, (\tau_1 \sqsubseteq \tau_2)^n \quad (\tau_1 \sqsubseteq \tau_2)^m, \Gamma_1' \to \Gamma_2'}{\Gamma_1 \Gamma_1' \to \Gamma_2 \Gamma_2'} \qquad \text{mcutF} \quad \frac{\Gamma_1 \to \Gamma_2, (\tau_1 \leq \xi_1)^n \quad (\tau_1 \leq \xi_1)^m, \Gamma_1' \to \Gamma_2'}{\Gamma_1 \Gamma_1' \to \Gamma_2 \Gamma_2'}$$

where the cut rules over labelled formulae and truth value comparison assertions are similar to the multi-cut rules introduced by Gentzen. There are structural rules to deal with each type of assertions. The *label structural rules* of a modal sequent calculus are:

$$\text{ex}\Omega \quad \frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2} \lhd \tau_1 : \mathbf{y}, \tau_1 \notin \Gamma_1, \Gamma_2$$

$$\text{LgenF} \quad \frac{\Gamma_1 \to \Gamma_2, \Omega\tau_2 \quad \Omega\tau_2, \tau_2 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\top \leq \xi_1, \Gamma_1 \to \Gamma_2} \qquad \text{RgenF} \quad \frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \top \leq \xi_1} \lhd \tau_1 : \mathbf{y}, \tau_1 \notin \Gamma_1, \Gamma_2.$$

These rules express essential properties of the algebra of truth values for the calculi and its relation with formulae. For instance, rule exΩ is important in establishing that in an algebra of truth values for the calculi there is at least an atomic truth value. The intuition behind this rule is that if in a derivation it is possible to conclude a sequent having in its left side an assertion stating that an arbitrary bound variable, not appearing anywhere else in the sequent, is atomic, then it is possible to conclude that sequent without that assumption. This happens since there is at least one atomic truth value. (This rule has a similar flavor with the disjunction elimination rule in a natural deduction calculus for classical or intuitionistic logic.) Rules LgenF and RgenF allow to transform global reasoning in local reasoning and vice-versa. Specific modal logics are obtained by adding some *modal accessibility rules*:

$$4 \quad \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3 \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_3 \sqsubseteq \mathbf{N}(\tau_1) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1)}$$

$$\text{C} \quad \frac{\begin{array}{c}\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3 \qquad \Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \to \Gamma_2, \tau_3 \sqsubseteq \mathbf{N}(\tau_1) \\ \Omega\tau_4, \tau_4 \sqsubseteq \mathbf{N}(\tau_2), \tau_4 \sqsubseteq \mathbf{N}(\tau_3), \Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \tau_2 \sqsubseteq \mathbf{N}(\tau_1), \Gamma_1 \to \Gamma_2\end{array}}{\Omega\tau_1, \Omega\tau_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1), \Gamma_1 \to \Gamma_2} \lhd \tau_4 : \mathbf{y}, \tau_4 \notin \tau_1, \tau_2, \tau_3, \Gamma_1, \Gamma_2$$

$$\text{L} \quad \frac{\begin{array}{ccc}\Omega\tau_1, \Omega\tau_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_2), \Gamma_1 \to \Gamma_2 & \Omega\tau_1, \Omega\tau_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1), \Gamma_1 \to \Gamma_2 & \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3 \\ \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_3) & \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3) & \end{array}}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$$

$$5 \quad \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3 \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_2)}$$

$$\text{B} \quad \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_2)}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1)}$$

$$\text{T} \quad \frac{}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_1)}$$

X $\dfrac{}{\Omega\tau_1,\tau_1 \sqsubseteq \mathbf{N}(\tau_1),\Gamma_1 \to \Gamma_2}$

Z $\dfrac{\Omega\tau_1,\Omega\tau_2,\Gamma_1 \to \Gamma_2,\tau_1 \sqsubseteq \mathbf{N}(\tau_2)}{\Omega\tau_1,\Omega\tau_2,\tau_2 \sqsubseteq \mathbf{N}(\tau_1),\Gamma_1 \to \Gamma_2}$

Y $\dfrac{\Omega\tau_1,\Omega\tau_2,\Gamma_1 \to \Gamma_2,\tau_1 \sqsubseteq \mathbf{N}(\tau_2) \quad \Omega\tau_1,\Omega\tau_2,\Gamma_1 \to \Gamma_2,\tau_2 \sqsubseteq \mathbf{N}(\tau_1)}{\Omega\tau_1,\Omega\tau_2,\Gamma_1 \to \Gamma_2,\tau_1 \sqsubseteq \tau_2}$.

The correspondence between a modal accessibility rule and a property of the accessibility relation of Kripke frames is established in [20]. So, rule 4 expresses that the accessibility relation is transitive, rule C expresses that the relation is confluent, rule L that the relation is right-linear, rule 5 that the relation is Euclidean, rule B that the relation is symmetric, rule T that the relation is reflexive, rule X that the relation is irreflexive, rule Z that the relation is asymmetric, and the rule Y that the relation is antisymmetric. As showed in [20], modal calculi with a combination of these rules are complete over general Kripke structures whose accessibility relation satisfies the properties defined by the rules. In order to obtain a calculus complete with respect to the standard Kripke structures whose accessibility relation satisfy the properties established by the modal accessibility rules, it is necessary to add $@\mathbf{x}$ and $@\mathbf{y}$ to C_0 for all unbound variable $\mathbf{x}$ and bound variable $\mathbf{y}$, respectively, and to add the rule

$$@\Omega \quad \frac{}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq @\tau_1}.$$

Hence, to obtain a system for modal logic S4 it is necessary to add rules T and 4 and $@\Omega$, and to obtain a system for S5 it is necessary to add the rules T and 5 and $@\Omega$. A more detailed account of the modal accessibility rules and of their relation with the modal axioms are provided in [20].

Consequence relation

Given a set $\mathcal{P}$ of modal accessibility rules we denote by $\mathcal{C}_\mathcal{P}$ the modal sequent calculus with that rules and by $\mathcal{L}(\mathcal{P})$ the modal logic presented by $\mathcal{C}_\mathcal{P}$. As showed in [20], the *global* and the *local* consequence relations in $\mathcal{L}(\mathcal{P})$ can be defined by means of $\mathcal{C}_\mathcal{P}$ as follows:

$$\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{L}(\mathcal{P})} \varphi \quad \text{iff} \quad \vdash_{\mathcal{C}_\mathcal{P}} \top \leq \psi_1, \ldots, \top \leq \psi_k \to \top \leq \varphi$$

and

$$\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{L}(\mathcal{P})} \varphi \quad \text{iff} \quad \vdash_{\mathcal{C}_\mathcal{P}} \Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi.$$

The relations $\vdash^{\mathrm{g}}_{\mathcal{L}(\mathcal{P})}$ and $\vdash^{\ell}_{\mathcal{L}(\mathcal{P})}$ are denoted simply by $\vdash^{\mathrm{g}}_{\mathcal{P}}$ and $\vdash^{\ell}_{\mathcal{P}}$.

3 Cut elimination and complexity

We start by reviewing the technical notions necessary in the cut elimination proofs and by stating two auxiliary propositions. Then we show that given a global or a local modal derivation for a sequent it is possible to extract a well behaved derivation for that sequent without increasing the logical depth and the number and complexity of the cuts. Since one of the properties for a derivation to be well behaved is not to have cuts over indivisibility assertions, the procedure to extract a well behaved derivation eliminates that type of cut.

Elimination of the other types of cuts is then proved in an incremental manner for well behaved derivations. First, it is showed that cuts over formulae labelled by $\top$ can be eliminated at the expense, in the worst case, of an exponential increase in the logical depth, and possibly with an increase on the number of cuts over formulae labelled by atomic terms. After that, cuts over formulae labelled by atomic terms are shown to be removable, at the expense, in the worst case, of a hyperexponential increase in the logical depth, and possibly with an increase on the number of cuts over truth value comparison assertions. Finally, it is showed in the context of modal sequent calculi with a compatible set of modal accessibility rules, that it is possible to eliminate the only type of cuts not yet considered, the cuts over truth values comparison assertions, at the expense, in the worst case, of an exponential increase in the logical depth. So the elimination of all the cuts in a modal derivation increases the logical depth, in the worst case, hyperexponentially.

The cut elimination procedure for formulae labelled by atomic terms is illustrated in Example 3.12 by applying it to a simple derivation. The operation of the other procedures can also be inferred from the example. The illustration of the joint application of all the procedures is not feasible since the resulting derivation would soon become larger and bigger than the acceptable.

We now briefly recall some basic definitions needed throughout the paper. Our reference is [37]. We assume, given a derivation, that the sub-derivation leading to a sequent that is an axiom, is composed only of that sequent. The function *hyp* is defined by $\text{hyp}(x, 0, z) = z$ and $\text{hyp}(x, Sk, z) = x^{\text{hyp}(x,k,z)}$. The value $\text{hyp}(x, k, z)$ is denoted by x^z_k.

The *depth* of a formula φ, denoted by $|\varphi|$, is the maximum length of a branch in its construction tree minus 1. The *depth* of a derivation $\mathcal{D}$ is the maximum length of a branch in $\mathcal{D}$ minus 1. The *logical depth* of a derivation $\mathcal{D}$ denoted by $||\mathcal{D}||$ is the depth of $\mathcal{D}$ not counting the contractions and weakenings. The *level* of a cut is the sum of the depths of the derivations of the premises. The *rank* of a cut over a labelled formula $t \leq \varphi$ is $|\varphi| + 1$. The *cutrank* of a derivation $\mathcal{D}$, denoted by $\text{cr}(\mathcal{D})$, is the maximum of the ranks of cuts in $\mathcal{D}$ over labelled formulae. If there are no cuts in $\mathcal{D}$ over labelled formulae, the cutrank is 0.

We say that two derivation trees have the *same structure*, or are *structurally identical*, whenever they are equal modulo their labels, that is, modulo the sequents that label the tree, and the rules applied in both derivations are the same and at the same point.

We will use $\mathcal{C}_\mathcal{P}$ to denote a modal sequent calculus with an arbitrary collection of modal accessibility rules, unless otherwise explicitly stated. We now state two auxiliary propositions.

Proposition 3.1 Given a derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ and a map ι from Y to Y then there is a derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s\iota$ with the same structure as $\mathcal{D}$.

The proof of the proposition follows by complete induction on the depth of the derivation, taking care, when necessary, to choose an adequate renaming of the fresh bound variables appearing along the derivation.

Proposition 3.2 Given a derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_\mathcal{P}} \Psi \to \Delta$ and finite multisets Ψ' and Δ', there is a derivation for $\vdash_{\mathcal{C}_\mathcal{P}} \Psi'\Psi \to \Delta'\Delta$ structurally identical to $\mathcal{D}$.

The proof of the proposition follows by complete induction on the depth of the derivation. It may be necessary to invoke Proposition 3.1 and to consider an adequate renaming of the fresh bound variables in the derivation.

Well behaved derivations

Given a derivation for a global or a local consequence it is possible to find a well behaved derivation for the same end-sequent with at most the same logical depth as the original derivation and with the same cutrank. A derivation is well behaved if it satisfies a collection of properties that simplify cut elimination.

Definition 3.3 A derivation $\mathcal{D}$ is *well behaved* iff it has no applications of cutΩ and for any s in $\mathcal{D}$,

1. there are no negative assertions in s;
2. if Ωt is in s then t is a bound variable and Ωt is in the left side of s;
3. each formula in s is labelled either by an atomic term or by $\top$.

The fact that there are no applications of cutΩ is important since that rule is a cut, and since eliminating it reduces the ways that an indivisibility assertion can be introduced. Property 1 avoids to consider the rules $\_x\_$ in the proofs of cut elimination over well behaved derivations. Property 2 is important since in cut elimination proofs it is frequent to appear derivations whose end-sequent is justified by an application of a rule where one of the premises contains an indivisibility assertion Ωt in the right side, and in another premise, t is used to label a formula. Thanks to this property we know that t is a bound variable, and that the derivation for the first premise has depth 0 since it is an axiom. So in the other premise the formula is labelled by an atomic term. This would allow, for instance, the application of an induction hypothesis. The importance of Property 3 comes from the fact that it reduces significantly the complexity of labelled formulae. It permits, in the cut elimination procedures, to consider only formulae labelled either by $\top$ or by an atomic term.

The following lemma constitutes the first step to obtain a well behaved derivation from a given modal derivation. It shows that it is possible to eliminate the negative assertions and so the applications of the rules $\_x\_$.

Lemma 3.4 Given a derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_\mathcal{P}} \Psi \to \Delta$ there is a derivation $\mathcal{D}'$ for $\vdash_{\mathcal{C}_\mathcal{P}} \Psi' \to \Delta'$ where

- Ψ' is Ψ without the negative assertions and with the conjugate of the negative assertions in Δ;
- Δ' is Δ without the negative assertions and with the conjugate of the negative assertions in Ψ;

such that the cutrank in $\mathcal{D}'$ and in $\mathcal{D}$ is the same, $||\mathcal{D}'|| \leq ||\mathcal{D}||$ and any sequent s' in $\mathcal{D}'$ do not have negative assertions.

The proof follows by complete induction on the depth of the derivation. The intuition is that a negated assertion can be replaced by the corresponding positive assertion in the other side of the sequent. The rules $\_x\_$ are no longer needed since, in the resulting derivation, there are not negative assertions.

The following lemma is important to show that given a derivation for a global or a local modal consequence, there is a well behaved derivation for the same end-sequent.

Lemma 3.5 Given a derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ where each sequent do not have negative assertions, and where s denotes $\Psi \to \Delta$ and is such that

- if Ωt is in Ψ then t is a bound variable;
- the formulae in Δ are labelled either by an atomic bound variable or by $\top$;

then there is a derivation $\mathcal{D}'$ for $\vdash_{\mathcal{C}_\mathcal{P}} s'$ where

- there are no applications of cutΩ;
- s' is s without the formulae that are not labelled by an atomic term or by $\top$;
- if Ωt is in a sequent s'_i in $\mathcal{D}'$ then t is a bound variable and there is Ωt in Ψ'_i;
- all the formulae appearing in $\mathcal{D}'$ are labelled either by an atomic term or by $\top$;
- there are no negative assertions in $\mathcal{D}'$;
- $||\mathcal{D}'|| \leq ||\mathcal{D}||$;
- the cutrank of $\mathcal{D}'$ is equal to the cutrank of $\mathcal{D}$.

The proof of this lemma follows by complete induction on the depth of the derivation. The idea is that in a derivation satisfying the sufficient conditions of the lemma, the formulae labelled neither by an atomic term nor by $\top$, at the left side of a sequent in the derivation, and the indivisibility assertions in the right side of a sequent and not in the left side, can be eliminated without disturbing the derivation. This happens since that assertions are either not needed or if needed they and the rule which needs them can be eliminated from the derivation without preventing the derivation from concluding the expected end-sequent.

Hence, a well behaved derivation is obtained from a modal derivation by applying Lemma 3.4 to the modal derivation, followed by Lemma 3.5. Note that the application of these lemmas results in derivations with the same end-sequent as the original derivation due to the definition of global and local consequence. Moreover, note that the resulting well behaved derivation has the same cutrank as the original derivation and at most the same logical depth. So, it is possible to establish the following result.

Lemma 3.6 Given a derivation for $\varphi_1, \ldots, \varphi_n \vdash^{\mathrm{g}}_{\mathcal{P}} \psi$ then there is a well behaved derivation for it with the same cutrank as the original derivation and at most with the same logical depth. Similarly for $\vdash^{\ell}_{\mathcal{P}}$.

Elimination of cuts over formulae labelled by $\top$

We now prove that given a well behaved derivation $\mathcal{D}$, it is possible to obtain a derivation without cuts over formulae labelled by $\top$, with logical depth less than or equal to $2^{||\mathcal{D}||-1}$, with a cutrank not greater than $\mathrm{cr}(\mathcal{D})$, and well behaved. In order to prove it we need first to show that it is possible to eliminate a cut over formulae labelled by $\top$, in a well behaved derivation with only cuts of that type, and whose last rule is precisely that cut. Moreover if we denote by $\mathcal{D}_1$ and by $\mathcal{D}_2$ the premises of the cut then the logical depth of the resulting derivation is less than or equal to $||\mathcal{D}_1|| + ||\mathcal{D}_2||$. From the analysis of the proof it is straightforward to see that this worst case bound happens when the cut assertions are not needed in the derivation except in the cut. In this case the cut elimination procedure pushes the derivation of one of the premises into above the derivation of the other premise, and so the resulting derivation will have a logical depth that is the sum of the logical depths of the derivations of the premises. We present a more detailed proof of the lemma in order to have the possibility of omitting in other proofs the parts common to this one.

Lemma 3.7 Given a well behaved derivation $\mathcal{D}^\circ$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ where s is obtained by a cut over formulae labelled by $\top$ from derivations $\mathcal{D}$ and $\mathcal{D}'$ free of that cuts, then there is a well behaved derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with

$$||\mathcal{D}^\bullet|| \leq ||\mathcal{D}|| + ||\mathcal{D}'||$$

without cuts over formulae labelled by $\top$, and with a cutrank not greater than $\mathrm{cr}(\mathcal{D}^\circ)$.

Proof We prove the lemma by complete induction on the level of the cut. Consider two cases:

(The level is 0*).* Let

$$\dfrac{\dfrac{}{(\top \leq \varphi)^n, \Psi \rightarrow \Delta}\ r \qquad \dfrac{}{\Psi' \rightarrow \Delta', (\top \leq \varphi)^m}\ r'}{\Psi\Psi' \rightarrow \Delta\Delta'}\ \mathrm{mcutF}$$

be a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$. Then we can consider two situations:

a) the cut assertions are not needed either in r or in r'. Suppose without loss of generality that the cut assertions are not needed in r. Then

$$\dfrac{}{\Psi\Psi' \rightarrow \Delta\Delta'}\ r$$

is a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over formulae labelled by $\top$ and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$.

b) a cut assertion is needed in r and in r'. Then r and r' are AxF since AxF is the unique rule without premises that needs labelled formulae. So

$$\dfrac{}{\Psi\Psi' \rightarrow \Delta\Delta'}\ \mathrm{AxF}$$

is a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over formulae labelled by $\top$ and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$.

(The level is not 0*)* Let l be a natural number greater than or equal to 0 and $\mathcal{D}^\circ$ be

$$\dfrac{\dfrac{\mathcal{D}}{(\top \leq \varphi)^n, \Psi \rightarrow \Delta}\ r \qquad \dfrac{\mathcal{D}'}{\Psi' \rightarrow \Delta', (\top \leq \varphi)^m}\ r'}{\Psi\Psi' \rightarrow \Delta\Delta'}\ \mathrm{mcutF}$$

a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$ where there are no other cuts over formulae labelled by $\top$, and such that the level of the cut is $l + 1$. Note that n and m are not 0. Consider the cases:

a) the cut assertions are not needed in r or in r'. Suppose, without loss of generality, that the cut assertions are not needed in r. a.1) The case where r does not has premises is similar to the corresponding case when the level is 0, so we omit it. a.2) Consider the case where r has premises. Assume $\mathcal{D}$ is

$$\dfrac{\dfrac{\mathcal{D}_1}{(\top \leq \varphi)^n, \Psi_1 \to \Delta_1}\, r_1 \quad \ldots \quad \dfrac{\mathcal{D}_k}{(\top \leq \varphi)^n, \Psi_k \to \Delta_k}\, r_k}{(\top \leq \varphi)^n, \Psi \to \Delta}\, r$$

then, if r involves fresh variables we assume that the fresh variables of r are not used in Ψ' and in Δ' (if a fresh variable of r is used in Ψ', Δ' then we consider a derivation structurally identical to $\mathcal{D}$ where the fresh variables of r are renamed such that they do not appear in Ψ' and in Δ'). Note that in modal sequent calculi k is at most 3 and if k is 3, then r is L$\Box$ or R$\Diamond$ and so, since $\mathcal{D}$ is well behaved, there is a premise obtained by AxΩ. Suppose, without loss of generality that k is 3 (the most complicated case) and that $\mathcal{D}_1$ consists of the premise obtained by AxΩ. Consider the well behaved derivation

$$\dfrac{\dfrac{\mathcal{D}_j}{(\top \leq \varphi)^n, \Psi_j \to \Delta_j}\, r_j \quad \dfrac{\mathcal{D}'}{\Psi' \to \Delta', (\top \leq \varphi)^m}\, r'}{\Psi_j \Psi' \to \Delta_j \Delta'}\, \text{mcutF}$$

where j is equal to 2 and 3, which satisfies the induction hypothesis conditions. So there is a well behaved derivation

$$\dfrac{\mathcal{D}_j^\bullet}{\Psi_j \Psi' \to \Delta_j \Delta'}\, r_j^\bullet$$

with no cuts over formulae labelled by $\top$, with logical depth less than or equal to $||\mathcal{D}_j|| + ||\mathcal{D}'||$, and with a cutrank not greater than $\text{cr}(\mathcal{D}^\circ)$. So

$$\dfrac{\dfrac{}{\Psi_1 \Psi' \to \Delta_1 \Delta'}\, \text{Ax}\Omega \quad \dfrac{\mathcal{D}_2^\bullet}{\Psi_2 \Psi' \to \Delta_2 \Delta'}\, r_2^\bullet \quad \dfrac{\mathcal{D}_3^\bullet}{\Psi_3 \Psi' \to \Delta_3 \Delta'}\, r_3^\bullet}{\Psi \Psi' \to \Delta \Delta'}\, r$$

is a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over formulae labelled by $\top$, with a cutrank not greater than $\text{cr}(\mathcal{D}^\circ)$, and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$ since $\max(||\mathcal{D}_2|| + ||\mathcal{D}'||, ||\mathcal{D}_3|| + ||\mathcal{D}'||) + 1 = (||\mathcal{D}|| - 1) + ||\mathcal{D}'|| + 1$.

b) a cut assertion is needed in r and in r'. Note that r and r' can only be LwF, RwF, LcF, RcF, AxF, LgenF or RgenF. We now proceed by case analysis on r and r':

b.1) r or r' is LwF or RwF. Suppose r is LwF and $\mathcal{D}$ is

$$\dfrac{\dfrac{\mathcal{D}_1}{(\top \leq \varphi)^{n-1}, \Psi \to \Delta}\, r_1}{(\top \leq \varphi)^n, \Psi \to \Delta}\, \text{LwF}$$

and assume $n - 1$ is not 0 (when n is 1 the proof follows by weakening the premise of LwF). Then, the derivation

$$\dfrac{\dfrac{\mathcal{D}_1}{(\top \leq \varphi)^{n-1}, \Psi \to \Delta}\, r_1 \quad \dfrac{\mathcal{D}'}{\Psi' \to \Delta', (\top \leq \varphi)^m}\, r'}{\Psi \Psi' \to \Delta \Delta'}\, \text{mcutF}$$

is well behaved and satisfies the induction hypothesis conditions. So, there is a derivation

$$\dfrac{\mathcal{D}^\bullet}{\Psi \Psi' \to \Delta \Delta'}\, r^\bullet$$

for $\vdash_{\mathcal{C}_\mathcal{P}} s$, well behaved, with no cuts over formulae labelled by $\top$, with a cutrank not greater than $\text{cr}(\mathcal{D}^\circ)$, and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$.

b.2) r or r' is LcF or RcF. Suppose r is LcF and $\mathcal{D}$ is

$$\dfrac{\dfrac{\mathcal{D}_1}{(\top \leq \varphi)^{n+1}, \Psi \to \Delta}\, r_1}{(\top \leq \varphi)^{n}, \Psi \to \Delta}\,\text{LcF}$$

and consider the well behaved derivation

$$\dfrac{\dfrac{\mathcal{D}_1}{(\top \leq \varphi)^{n+1}, \Psi \to \Delta}\, r_1 \qquad \dfrac{\mathcal{D}'}{\Psi' \to \Delta', (\top \leq \varphi)^{m}}\, r'}{\Psi\Psi' \to \Delta\Delta'}\,\text{mcutF}$$

satisfying the induction hypothesis conditions. Then, there is a well behaved derivation

$$\dfrac{\mathcal{D}^{\bullet}}{\Psi\Psi' \to \Delta\Delta'}\, r^{\bullet}$$

for $\vdash_{\mathcal{C}_{\mathcal{P}}} s$ with no cuts over formulae labelled by $\top$, with a cutrank not greater than $\mathrm{cr}(\mathcal{D}^{\circ})$, and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$.

b.3) r or r' is AxF. Suppose r is AxF. Then the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}'}{\Psi' \to \Delta', (\top \leq \varphi)^{m}}\, r'}{\vdots}\,\text{\_w\_}}{\Psi\Psi' \to \Delta\Delta', (\top \leq \varphi)^{m}}\,\text{\_w\_}}{\dfrac{\vdots}{\Psi\Psi' \to \Delta\Delta'}\,\text{RcF}}\,\text{RcF}$$

is a well behaved derivation for $\vdash_{\mathcal{C}_{\mathcal{P}}} s$ with no cuts over formulae labelled by $\top$, with a cutrank not greater than $\mathrm{cr}(\mathcal{D}^{\circ})$, and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$.

b.4) r is LgenF and r' is RgenF. In this case we replace the cut by a cut over formulae labelled by an atomic term. Assume $\mathcal{D}$ is

$$\dfrac{\dfrac{\mathcal{D}_1}{(\top \leq \varphi)^{n-1}, \Psi \to \Delta, \Omega t_2}\, r_1 \qquad \dfrac{\mathcal{D}_2}{\Omega t_2, t_2 \leq \varphi, (\top \leq \varphi)^{n-1}, \Psi \to \Delta}\, r_2}{(\top \leq \varphi)^{n}, \Psi \to \Delta}\,\text{LgenF}$$

and $\mathcal{D}'$ is

$$\dfrac{\dfrac{\mathcal{D}'_1}{\Omega\mathbf{y}_1, \Psi' \to \Delta', (\top \leq \varphi)^{m-1}, \mathbf{y}_1 \leq \varphi}\, r'_1}{\Psi' \to \Delta', (\top \leq \varphi)^{m}}\,\text{RgenF}$$

and suppose $m-1$ and $n-1$ are distinct of 0 (the case where $m-1$ or $n-1$ are 0 is similar, the only difference is that instead of applying the induction hypothesis, the derivations ${\mathcal{D}'}^{\mathbf{y}_1}_{1t_2}$ or $\mathcal{D}_2$ are weakened, respectively). Note that Ωt_2 is in Ψ and t_2 is a bound variable, since $\mathcal{D}$ is well behaved. Consider the well behaved derivation

$$\dfrac{\dfrac{\mathcal{D}}{(\top \leq \varphi)^{m}, \Psi \to \Delta}\, r \qquad \dfrac{{\mathcal{D}'}^{\mathbf{y}_1}_{1t_2}}{\Omega t_2, \Psi' \to \Delta', (\top \leq \varphi)^{m-1}, t_2 \leq \varphi}\, r'_1}{\Omega t_2, \Psi\Psi' \to \Delta\Delta', t_2 \leq \varphi}\,\text{mcutF}$$

satisfying the induction hypothesis conditions, and where ${\mathcal{D}'}^{\mathbf{y}_1}_{1t_2}$ is identical to $\mathcal{D}'_1$ except that the fresh variable $\mathbf{y}_1$ is replaced by t_2 (see Proposition 3.1). So there is the well behaved derivation

$$\dfrac{\mathcal{D}_1'^{\bullet}}{\Omega t_2, \Psi\Psi' \to \Delta\Delta', t_2 \leq \varphi}\; r_1'^{\bullet}$$

and by a similar reasoning there is the well behaved derivation

$$\dfrac{\mathcal{D}_2^{\bullet}}{\Omega t_2, t_2 \leq \varphi, \Psi\Psi' \to \Delta\Delta'}\; r_2^{\bullet}$$

with no cuts over formulae labelled by $\top$, with a cutrank not greater than $\mathrm{cr}(\mathcal{D}^{\circ})$, and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}_1'||$ and $||\mathcal{D}_2|| + ||\mathcal{D}'||$, respectively. Then the well behaved derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_2^{\bullet}}{t_2 \leq \varphi, \Omega t_2, \Psi\Psi' \to \Delta\Delta'}\, r_2^{\bullet} \qquad \dfrac{\mathcal{D}_1'^{\bullet}}{\Omega t_2, \Psi\Psi' \to \Delta\Delta', t_2 \leq \varphi}\, r_1'^{\bullet}}{\Omega t_2, \Omega t_2, \Psi\Psi\Psi'\Psi' \to \Delta\Delta\Delta'\Delta'}\,\mathrm{mcutF}}{\vdots}\,\_c\_}{\Psi\Psi' \to \Delta\Delta'}\,\_c\_$$

for $\vdash_{\mathcal{C}_{\mathcal{P}}} s$ has no cuts over formulae labelled by $\top$, has a cutrank not greater than $\mathrm{cr}(\mathcal{D}^{\circ})$, and its logical depth is less than or equal to $\max(||\mathcal{D}_2|| + ||\mathcal{D}'||, ||\mathcal{D}|| + ||\mathcal{D}_1'||) + 1$ which is less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$. QED

The following lemma shows that the elimination of all cuts over formulae labelled by $\top$ increases the logical depth of the derivation exponentially.

Lemma 3.8 Given a well behaved derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_{\mathcal{P}}} s$ there is a well behaved derivation $\mathcal{D}^{\bullet}$ for $\vdash_{\mathcal{C}_{\mathcal{P}}} s$ with

$$||\mathcal{D}^{\bullet}|| \leq 2^{||\mathcal{D}||-1}$$

with no cuts over formulae labelled by $\top$, and with a cutrank not greater than $\mathrm{cr}(\mathcal{D})$.

Proof The proof follows by complete induction on the depth of the derivation. When the depth of a derivation $\mathcal{D}$ is 0, let $\mathcal{D}^{\bullet}$ be $\mathcal{D}$, since $\mathcal{D}$ satisfies the conditions of the lemma. Suppose that the depth of the well behaved derivation $\mathcal{D}$

$$\dfrac{\dfrac{\mathcal{D}_1}{\Psi_1 \to \Delta_1}\, r_1 \quad \ldots \quad \dfrac{\mathcal{D}_k}{\Psi_k \to \Delta_k}\, r_k}{\Psi \to \Delta}\, r$$

for $\vdash_{\mathcal{C}_{\mathcal{P}}} s$ is n where s is the conclusion of the derivation and $n-1$ is greater than or equal to 0. Suppose without loss of generality that k is 2. The proofs for other values of k are similar. Denote by $\mathcal{D}_1^{\bullet}$ the well behaved derivation with no cuts over formulae labelled by $\top$ obtained by the application of the induction hypothesis to $\mathcal{D}_1$. Note that $||\mathcal{D}_1^{\bullet}|| \leq 2^{||\mathcal{D}_1||-1}$. Similarly for $\mathcal{D}_2^{\bullet}$. Denote the well behaved derivation

$$\dfrac{\dfrac{\mathcal{D}_1^{\bullet}}{\Psi_1 \to \Delta_1}\, r_1^{\bullet} \quad \dfrac{\mathcal{D}_2^{\bullet}}{\Psi_2 \to \Delta_2}\, r_2^{\bullet}}{\Psi \to \Delta}\, r$$

by $\mathcal{D}'$ and consider two cases:

a) r is a cut over formulae labelled by $\top$. Then by Lemma 3.7 there is a well behaved derivation $\mathcal{D}^{\bullet}$ for $\vdash_{\mathcal{C}_{\mathcal{P}}} s$ without cuts over formulae labelled by $\top$ and with logical depth less than or equal to $||\mathcal{D}_1^{\bullet}|| + ||\mathcal{D}_2^{\bullet}||$. Note that $||\mathcal{D}_1^{\bullet}|| + ||\mathcal{D}_2^{\bullet}|| \leq 2 * 2^{||\mathcal{D}||-2}$ since $||\mathcal{D}_1||$ and

$||\mathcal{D}_2||$ are less than or equal to $||\mathcal{D}|| - 1$. So $||\mathcal{D}^\bullet|| \le 2^{||\mathcal{D}||-1}$.

b) r is not a cut over formulae labelled by $\top$. Then $\mathcal{D}'$ is a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$ without cuts over formulae labelled by $\top$. Consider two subcases:

b.i) r is not a weakening rule neither a contraction rule. Then $||\mathcal{D}^\bullet||$ is $\max(||\mathcal{D}_1^\bullet||, ||\mathcal{D}_2^\bullet||)+1$. If $||\mathcal{D}_1||$ and $||\mathcal{D}_2||$ are 0 then $||\mathcal{D}^\bullet||$ is 1 and $2^{||\mathcal{D}||-1}$ is 1. Otherwise $||\mathcal{D}|| - 1$ is greater than 0 and so $2^{||\mathcal{D}||-2} + 1 \le 2^{||\mathcal{D}||-1}$. Then $||\mathcal{D}^\bullet|| \le 2^{||\mathcal{D}||-1}$ since $\max(||\mathcal{D}_1^\bullet||, ||\mathcal{D}_2^\bullet||) + 1 \le 2^{||\mathcal{D}||-2} + 1$.

b.ii) r is a weakening rule or a contraction rule. Then $||\mathcal{D}^\bullet||$ is $\max(||\mathcal{D}_1^\bullet||, ||\mathcal{D}_2^\bullet||)$, and $||\mathcal{D}||$ is equal to $||\mathcal{D}_1||$ or $||\mathcal{D}_2||$. So $||\mathcal{D}^\bullet|| \le 2^{||\mathcal{D}||-1}$ since $\max(||\mathcal{D}_1^\bullet||, ||\mathcal{D}_2^\bullet||) \le 2^{||\mathcal{D}||-1}$. QED

Elimination of cuts over formulae labelled by atomic terms

Cuts over formulae labelled by atomic terms are eliminated (see Lemma 3.11) by repeatedly applying the cutrank reduction lemma (Lemma 3.10) until the cutrank be 0. The cutrank reduction lemma establishes the worst case bound of $4^{||\mathcal{D}||}$, where $\mathcal{D}$ is the original derivation. So, the elimination of all cuts over formulae labelled by atomic terms is, in the worst case, bounded by $4_k^{||\mathcal{D}||}$, where $\mathcal{D}$ is the original derivation and k is its cutrank.

To prove the cutrank reduction lemma it is necessary first to state a preliminary result establishing that given a derivation ending in a cut with rank k over formulae labelled by atomic terms, without cuts over formulae labelled by $\top$, and where all the other cuts over labelled formulae have lower rank, then there is a well behaved derivation for the same end-sequent where all cuts have rank lower than k. The main reason for the lemma to hold is that in a modal sequent calculus the left and the right rules for the logical connectives reduce the derivation of a compound formula to the derivation(s) of strict subformula(e) labelled by atomic terms. Moreover, if we denote by $\mathcal{D}$ and $\mathcal{D}'$ the premises of the cut, the resulting derivation has logical depth less than or equal to $2(||\mathcal{D}|| + ||\mathcal{D}'||)$. The bound being twice the sum of the logical depths of the premises, and not only the sum of the logical depths of the premises as in Lemma 3.7, is due to the elimination of cuts over labelled formulae where the main connective is $\Box$.

For illustration purposes we will apply the procedures to a well behaved derivation for $\vdash^\ell (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)$ with a cut over formulae labelled by atomic terms (see Example 3.12).

Lemma 3.9 *(Cut reduction lemma)* Given a well behaved derivation $\mathcal{D}^\circ$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over formulae labelled by $\top$, where s is obtained by a cut over formulae labelled by an atomic term from derivations $\mathcal{D}$ and $\mathcal{D}'$ with a lower cutrank than $\mathcal{D}^\circ$, then there is a well behaved derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with

$$||\mathcal{D}^\bullet|| \le 2(||\mathcal{D}|| + ||\mathcal{D}'||)$$

with no cuts over formulae labelled by $\top$, and with a lower cutrank than $\mathcal{D}^\circ$.

Proof The lemma is proved by complete induction on the level of the cut. We show only how to reduce the rank of the cut when the cut assertions are needed by the rules L$\Box$ and R$\Box$. The other cases are either similar to this case or are similar to the cases presented in Lemma 3.7. So, let l be a natural number and let $\mathcal{D}^\circ$ be the well behaved derivation

$$\dfrac{\dfrac{\mathcal{D}}{(t_1 \leq \Box\varphi_1)^n, \Psi \to \Delta}\ \text{L}\Box \qquad \dfrac{\mathcal{D}'}{\Psi' \to \Delta', (t_1 \leq \Box\varphi_1)^m}\ \text{R}\Box}{\Psi\Psi' \to \Delta\Delta'}\ \text{mcutF}$$

with no cuts over formulae labelled by $\top$ and where the level of the cut is $l+1$, and t_1 is a bound variable. Assume $\mathcal{D}$ is

$$\dfrac{\dfrac{\mathcal{D}_1}{(t_1 \leq \Box\varphi_1)^{n-1}, \Psi \to \Delta, \Omega t}\ r_1 \quad \dfrac{\mathcal{D}_2}{(t_1 \leq \Box\varphi_1)^{n-1}, \Omega t, \Psi \to \Delta, t \sqsubseteq \mathbf{N}(t_1)}\ r_2 \quad \dfrac{\mathcal{D}_3}{(t_1 \leq \Box\varphi_1)^{n-1}, \Omega t, t \leq \varphi_1, \Psi \to \Delta}\ r_3}{(t_1 \leq \Box\varphi_1)^n, \Psi \to \Delta}\ \text{L}\Box$$

and $\mathcal{D}'$ is

$$\dfrac{\dfrac{\mathcal{D}_1'}{\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \mathbf{N}(t_1), \Psi' \to \Delta', \mathbf{y}_1 \leq \varphi_1, (t_1 \leq \Box\varphi_1)^{m-1}}\ r_1'}{\Psi' \to \Delta', (t_1 \leq \Box\varphi_1)^m}\ \text{R}\Box$$

Consider the case where $n-1$ and $m-1$ are greater than 0 (the case where $m-1$ or $n-1$ are 0 is similar, the only difference is that instead of applying the induction hypothesis, the derivations ${\mathcal{D}'}_{1t}^{\mathbf{y}_1}$ or, $\mathcal{D}_2$ and $\mathcal{D}_3$ are weakened, respectively). Note that Ωt is in Ψ since $\mathcal{D}$ is well behaved. Consider the well behaved derivation:

$$\dfrac{\dfrac{\mathcal{D}}{(t_1 \leq \Box\varphi_1)^n, \Psi \to \Delta}\ r \qquad \dfrac{{\mathcal{D}'}_{1t}^{\mathbf{y}_1}}{\Omega t, t \sqsubseteq \mathbf{N}(t_1), \Psi' \to \Delta', t \leq \varphi_1, (t_1 \leq \Box\varphi_1)^{m-1}}\ r_1'}{\Omega t, t \sqsubseteq \mathbf{N}(t_1), \Psi\Psi' \to \Delta\Delta', t \leq \varphi_1}\ \text{mcutF}$$

where ${\mathcal{D}'}_{1t}^{\mathbf{y}_1}$ is identical to $\mathcal{D}_1'$ except that the fresh variable $\mathbf{y}_1$ is replaced by t (see Proposition 3.1). Then by the induction hypothesis there is a well behaved derivation

$$\dfrac{{\mathcal{D}'}_1^{\bullet}}{\Omega t, t \sqsubseteq \mathbf{N}(t_1), \Psi\Psi' \to \Delta\Delta', t \leq \varphi_1}\ {r'}_1^{\bullet}$$

with cutrank less than $\text{cr}(\mathcal{D}^\circ)$ and logical depth less than or equal to $2||\mathcal{D}|| + 2||\mathcal{D}'|| - 2$ and with no cuts over formulae labelled by $\top$. By a similar reasoning we obtain the well behaved derivations

$$\dfrac{\mathcal{D}_2^{\bullet}}{\Omega t, \Psi\Psi' \to \Delta\Delta', t \sqsubseteq \mathbf{N}(t_1)}\ r_2^{\bullet}$$

and

$$\dfrac{\mathcal{D}_3^{\bullet}}{\Omega t, t \leq \varphi_1, \Psi\Psi' \to \Delta\Delta'}\ r_3^{\bullet}$$

and so, the well behaved derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_3^{\bullet}}{\Omega t, t \leq \varphi_1, \Psi\Psi' \to \Delta\Delta'}\ r_3^{\bullet} \quad \dfrac{{\mathcal{D}'}_1^{\bullet}}{\Omega t, t \sqsubseteq \mathbf{N}(t_1), \Psi\Psi' \to \Delta\Delta', t \leq \varphi_1}\ {r'}_1^{\bullet}}{\Omega t, \Omega t, t \sqsubseteq \mathbf{N}(t_1), \Psi\Psi\Psi'\Psi' \to \Delta\Delta\Delta'\Delta'}\ \text{mcutF} \quad \dfrac{\mathcal{D}_2^{\bullet}}{\Omega t, \Psi\Psi' \to \Delta\Delta', t \sqsubseteq \mathbf{N}(t_1)}\ r_2^{\bullet}}{\Omega t, \Omega t, \Omega t, \Psi\Psi\Psi\Psi'\Psi'\Psi' \to \Delta\Delta\Delta\Delta'\Delta'\Delta'}\ \text{mcutT}}{\vdots}\ \_\text{c}\_}{\Psi\Psi' \to \Delta\Delta'}\ \_\text{c}\_$$

for $\vdash_{\mathcal{C}_\mathcal{P}} s$, has cutrank less than $\text{cr}(\mathcal{D}^\circ)$ and logical depth less than or equal to $2(||\mathcal{D}||+||\mathcal{D}'||)$ and has no cuts over formulae labelled by $\top$. QED

The following lemma shows that given a well behaved derivation $\mathcal{D}$ it is possible to find another derivation $\mathcal{D}^\bullet$ for the same end-sequent with a cutrank lower than the original derivation at the cost, in the worst case, of an exponential increase on the logical depth of the derivation. We omit the proof of the lemma since it is similar to the proof of Lemma 3.8.

Lemma 3.10 *(Cutrank reduction lemma)* Given a well behaved derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with non null cutrank and with no cuts over formulae labelled by $\top$, then there is a well behaved derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over formulae labelled by $\top$, with

$$||\mathcal{D}^\bullet|| \leq 4^{||\mathcal{D}||}$$

and with lower cutrank than $\mathcal{D}$.

Finally, capitalizing on the cutrank reduction lemma, it is possible to eliminate all the cuts over formulae labelled by atomic terms. The idea is to obtain successively well behaved derivations with lower cutrank until obtain a derivation with cutrank equal to 0. The iteration of the cutrank reduction lemma, which has an exponential worst case bound, would then cause that the logical depth of the derivation with no cuts over labelled formulae may, at the worst case, be hyperexponential on the cutrank of the original derivation.

Lemma 3.11 Given a well behaved derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over formulae labelled by $\top$, then there is a well behaved derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over labelled formulae and such that

$$||\mathcal{D}^\bullet|| \leq 4_k^{||\mathcal{D}||}$$

where k is the cutrank of $\mathcal{D}$.

Proof The proof follows by complete induction on the cutrank of the derivation. The case where the cutrank is 0 follows straightforwardly, since in this case, if $\mathcal{D}^\bullet$ is equal to $\mathcal{D}$, the conditions of the lemma are satisfied. Assume that the cutrank of the well behaved derivation $\mathcal{D}$, with no cuts over formulae labelled by $\top$, is greater than or equal to 1, and denote it by k. Then by Lemma 3.10 there is a derivation $\mathcal{D}'$ with no cuts over formulae labelled by $\top$, with cutrank k' less than k, and with logical depth less than or equal to $4^{||\mathcal{D}||}$. So, by induction hypothesis, there is a well behaved derivation $\mathcal{D}^\bullet$ with no cuts over labelled formulae and with logical depth less than or equal to $4_{k'}^{||\mathcal{D}'||}$. So the logical depth of $\mathcal{D}^\bullet$ is less than or equal to $4_{k-1}^{4^{||\mathcal{D}||}}$, i.e., $||\mathcal{D}^\bullet|| \leq 4_k^{||\mathcal{D}||}$. QED

We now illustrate the procedures described above, by applying them to a well behaved derivation for a modal consequence, with a cut over a formula labelled by an atomic term.

Example 3.12 Consider the well behaved derivation

$$\dfrac{\dfrac{\dfrac{}{\mathbf{y} \leq p_1 \vee \neg p_1, \Omega\mathbf{y}, \mathbf{y} \leq \Box p_0 \to \mathbf{y} \leq p_1 \vee \neg p_1}\,\text{AxF}}{\mathbf{y} \leq p_1 \vee \neg p_1, \Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}\,\text{R}\Rightarrow \quad \dfrac{\dfrac{\dfrac{}{\Omega\mathbf{y}, \mathbf{y} \leq p_1 \to \mathbf{y} \leq p_1}\,\text{AxF}}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1, \mathbf{y} \leq \neg p_1}\,\text{R}\neg}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\,\text{R}\vee}{\dfrac{\Omega\mathbf{y}, \Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}{\Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}\,\text{Lc}\Omega}\,\text{mcutF}$$

for $\vdash^{\ell} (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)$, in the context of modal logic K, denoted by $\mathcal{D}$. Note that the derivation

$$\dfrac{\dfrac{}{\mathbf{y} \leq p_1 \vee \neg p_1, \Omega\mathbf{y}, \mathbf{y} \leq \Box p_0 \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{AxF} \qquad \dfrac{\dfrac{\dfrac{}{\Omega\mathbf{y}, \mathbf{y} \leq p_1 \to \mathbf{y} \leq p_1}\text{AxF}}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1, \mathbf{y} \leq \neg p_1}\text{R}\neg}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{R}\vee}{\Omega\mathbf{y}, \Omega\mathbf{y}, \mathbf{y} \leq \Box p_0 \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{mcutF}$$

gives rise, by the procedure associated to the proof of the cut reduction lemma (Lemma 3.9) to the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Omega\mathbf{y}, \mathbf{y} \leq p_1 \to \mathbf{y} \leq p_1}\text{AxF}}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1, \mathbf{y} \leq \neg p_1}\text{R}\neg}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{R}\vee}{\Omega\mathbf{y}, \Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{Lw}\Omega}{\Omega\mathbf{y}, \Omega\mathbf{y}, \mathbf{y} \leq \Box p_0 \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{LwF}$$

So, the derivation

$$\dfrac{\dfrac{\dfrac{}{\mathbf{y} \leq p_1 \vee \neg p_1, \Omega\mathbf{y}, \mathbf{y} \leq (\Box p_0) \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{AxF}}{\mathbf{y} \leq p_1 \vee \neg p_1, \Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}\text{R}\Rightarrow \qquad \dfrac{\dfrac{\dfrac{}{\Omega\mathbf{y}, \mathbf{y} \leq p_1 \to \mathbf{y} \leq p_1}\text{AxF}}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1, \mathbf{y} \leq \neg p_1}\text{R}\neg}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{R}\vee}{\Omega\mathbf{y}, \Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}\text{mcutF}$$

by the same procedure, originates

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Omega\mathbf{y}, \mathbf{y} \leq p_1 \to \mathbf{y} \leq p_1}\text{AxF}}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1, \mathbf{y} \leq \neg p_1}\text{R}\neg}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{R}\vee}{\Omega\mathbf{y}, \Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{Lw}\Omega}{\Omega\mathbf{y}, \Omega\mathbf{y}, \mathbf{y} \leq \Box p_0 \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{LwF}}{\Omega\mathbf{y}, \Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}\text{R}\Rightarrow$$

Hence, derivation $\mathcal{D}$, by Lemma 3.11, gives rise to the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Omega\mathbf{y}, \mathbf{y} \leq p_1 \to \mathbf{y} \leq p_1}\text{AxF}}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1, \mathbf{y} \leq \neg p_1}\text{R}\neg}{\Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{R}\vee}{\Omega\mathbf{y}, \Omega\mathbf{y} \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{Lw}\Omega}{\Omega\mathbf{y}, \Omega\mathbf{y}, \mathbf{y} \leq \Box p_0 \to \mathbf{y} \leq p_1 \vee \neg p_1}\text{LwF}}{\Omega\mathbf{y}, \Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}\text{R}\Rightarrow}{\Omega\mathbf{y} \to \mathbf{y} \leq (\Box p_0) \Rightarrow (p_1 \vee \neg p_1)}\text{Lc}\Omega$$

with no cuts over formulae labelled by atomic terms and with logical depth equal to 3, less than or equal to $4^{||\mathcal{D}||}_{\mathrm{cr}(\mathcal{D})}$ which is 4^3_3.

Elimination of cuts over truth value comparison assertions

There are two main difficulties in eliminating cuts over truth value comparison assertions. One of the difficulties is that, in most of the cases, there are no left rule nor right rule for term operators. Another problem is that most of the rules that need truth value comparison assertions in the conclusion, do not reduce the complexity of the terms. So, depending of the modal accessibility rules in the calculus, when trying to eliminate cuts over truth value comparison assertions there may be situations where it is not possible to propagate the cut up and it is not possible to eliminate it.

We now define what is two modal accessibility rules to be compatible. We say that two *truth value comparison assertions have the same structure* when there is a renaming of the term meta-variables that make the two assertions equal.

Definition 3.13 In a modal sequent calculus two modal accessibility rules are *compatible* when

- either the truth value comparison assertion in the conclusion sequent of both rules appears on the same side,
- or the truth value comparison assertion appears in opposite sides of the conclusion sequent of the rules but
  - either they do not have the same structure,
  - or they have the same structure but one of the rules has only one premise, that has the same structure as the conclusion sequent.

We now define what is a set of modal accessibility rules to be compatible.

Definition 3.14 In a modal sequent calculus a set $\mathcal{P}$ of modal accessibility rules is *compatible* if any two rules in $\mathcal{P}$ are compatible.

Cuts over truth value comparison assertions are shown to be removable in modal sequent calculi with a compatible set of modal accessibility rules (see Lemma 3.16).

Note that a set $\mathcal{P}$ of the modal accessibility rules T (reflexivity), B (symmetry), 4 (transitivity), L (right-linearity), 5 (Euclideaness), C (confluence), X (irreflexivity), Y (antisymmetry) and Z (asymmetry) is compatible whenever T or 4 or L or 5, are not present simultaneously with, C or X or Z; and B is not present simultaneously with Z or C. The class of modal logics characterized by an accessibility relation satisfying the properties associated to a compatible set of modal accessibility rules is quite vast and encompasses the modal logics S4, S5, KBE, C, and logics characterized by frames with an irreflexive of asymmetric relation, among others.

We now show that, in the context of a modal sequent calculus with a compatible set of modal accessibility rules, it is possible to eliminate cuts over truth value comparison assertions with, in the worst case, an exponential increase of the logical depth. To prove this, we show first Lemma 3.15 where we present a procedure to eliminate a cut over truth value comparison assertions in a derivation ending with that cut.

Lemma 3.15 Let $\mathcal{P}$ be a compatible set of modal accessibility rules. Given a well behaved derivation $\mathcal{D}^\circ$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$, where s is obtained by a cut over truth value comparison assertions from cut free derivations $\mathcal{D}$ and $\mathcal{D}'$, there is a well behaved derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with

$$||\mathcal{D}^\bullet|| \leq ||\mathcal{D}|| + ||\mathcal{D}'||$$

and with no cuts.

Proof The proof follows by complete induction on the level of the cut:

The level is 0 Let

$$\dfrac{\dfrac{}{(t_1 \sqsubseteq t_2)^n, \Psi \to \Delta}\ r \qquad \dfrac{}{\Psi' \to \Delta', (t_1 \sqsubseteq t_2)^m}\ r'}{\Psi\Psi' \to \Delta\Delta'}\ \text{mcutT}$$

be a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$, where s is the conclusion of the derivation, and m and n are greater than 0. a) Consider first the case where the cut assertions are not needed either in r or in r'. Suppose without loss of generality that the cut assertions are not needed in r. Then

$$\dfrac{}{\Psi\Psi' \to \Delta\Delta'}\ r$$

is a well behaved derivation with no cuts. b) Consider now the case where a cut assertion is needed in r and in r'. Then r can be AxT or X, and r' can be AxT or T. We now proceed by case analysis: b.1) r or r' is AxT. Suppose that r is AxT. Then Δ contains $t_1 \sqsubseteq t_2$. Hence

$$\dfrac{}{\Psi\Psi' \to \Delta\Delta'}\ r'$$

is a well behaved derivation with no cuts. b.2) r is X. Then r' is AxT since X and T are not compatible. So this case is a particular case of b.1. Note that in any of the cases considered $||\mathcal{D}^\bullet|| \leq ||\mathcal{D}|| + ||\mathcal{D}'||$.

(The level is not 0*)* Let l be a natural number greater than or equal to 0, and

$$\dfrac{\dfrac{\mathcal{D}}{(t_1 \sqsubseteq t_2)^n, \Psi \to \Delta}\ r \qquad \dfrac{\mathcal{D}'}{\Psi' \to \Delta', (t_1 \sqsubseteq t_2)^m}\ r'}{\Psi\Psi' \to \Delta\Delta'}\ \text{mcutT}$$

be a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} s$ where s is the conclusion of the derivation, there are no other cuts, m and n are greater than 0, and the level of the cut is $l+1$. We consider only the cases where a cut assertion is needed in r and in r' and r is C, X, or Z, and r' is T, B, 4, L, 5, or Y. The case where the cut assertions are not needed in r or r', and the case where a cut assertion is needed in r and in r' and r is AxT, LwT or LcT and r' is AxT, RwT or RcT, are similar to the cases presented in the proof of Lemma 3.7 so we omit them. For the other cases we proceed by case analysis on r and r':

b.1) r is C and r' is T or B or 4 or 5, or r is X and r' is 4 or 5, or r is Z and r' is T or B or 4 or 5. These cases are not possible since $\mathcal{P}$ is a compatible set of modal accessibility rules (see Definition 3.14).

b.2) r is C or X or Z and r' is L or Y. These cases are not possible since by rules C or X or Z one of the labels is not a bound variable and, since $\mathcal{D}'$ is well behaved, by rules L or Y the two labels are bound variables.

b.3) r is X and r' is B. Then t_2 is $\mathbf{N}(t_1)$ and t_1 is a bound variable. Assume $\mathcal{D}'$ is

$$\dfrac{\dfrac{\mathcal{D}'_1}{\Psi' \to \Delta', (t_1 \sqsubseteq \mathbf{N}(t_1))^m}\ r'_1}{\Psi' \to \Delta', (t_1 \sqsubseteq \mathbf{N}(t_1))^m}\ \text{B}$$

and consider the well behaved derivation

$$\dfrac{\dfrac{}{(t_1 \sqsubseteq \mathbf{N}(t_1))^n, \Psi \to \Delta}\,\mathrm{X} \qquad \dfrac{\mathcal{D}'_1}{\Psi' \to \Delta', (t_1 \sqsubseteq \mathbf{N}(t_1))^m}\, r'_1}{\Psi\Psi' \to \Delta\Delta'}\,\mathrm{mcutT}$$

satisfying the induction hypothesis conditions. Then there is the well behaved derivation

$$\dfrac{\mathcal{D}'^{\bullet}_1}{\Psi\Psi' \to \Delta\Delta'}\, r'^{\bullet}_1$$

for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts and with logical depth less than or equal to $||\mathcal{D}|| + ||\mathcal{D}'||$. QED

The following lemma establishes that all cuts over truth value comparison assertions can be eliminated. Moreover, assuming $\mathcal{D}$ is a derivation satisfying the conditions of the lemma, it states that, in the worst case, that elimination would increase the logical depth to $2^{||\mathcal{D}||-1}$. We refer the reader interested in an intuitive justification of this bound to the paragraphs before Lemma 3.8 since the proofs are very similar. The proof of the lemma is omitted.

Lemma 3.16 Given a well behaved derivation $\mathcal{D}$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over labelled formulae, there is a well behaved derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with

$$||\mathcal{D}^\bullet|| \leq 2^{||\mathcal{D}||-1}$$

and with no cuts.

Elimination of all cuts

It is now possible to present the theorem establishing the elimination of all cuts in a derivation for a global or a local consequence and establishing the worst case bound of $2^{4_k^{2^{||\mathcal{D}||-1}}-1}$ for the logical depth of the cut free derivation, where $\mathcal{D}$ is the original derivation and k is its cutrank.

Theorem 3.17 For each compatible set $\mathcal{P}$ of modal accessibility rules, if $\mathcal{D}$ is a derivation for $\psi_1, \ldots, \psi_n \vdash^{\mathrm{g}}_{\mathcal{P}} \varphi$ then there is a well behaved derivation $\mathcal{D}^\bullet$ for it with no cuts and with

$$||\mathcal{D}^\bullet|| \leq 2^{4_{\mathrm{cr}(\mathcal{D})}^{2^{||\mathcal{D}||-1}}-1}.$$

Similarly for $\vdash^{\ell}_{\mathcal{P}}$.

Proof Let $\mathcal{D}$ be a derivation for $\psi_1, \ldots, \psi_n \vdash^{\mathrm{g}}_{\mathcal{P}} \varphi$, i.e., for $\vdash_{\mathcal{C}_\mathcal{P}} s$ where s is $\top \leq \psi_1, \ldots, \top \leq \psi_n \to \top \leq \varphi$. Denote by k the cutrank of $\mathcal{D}$. By Lemma 3.6 there is a well behaved derivation $\mathcal{D}^\circ$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with cutrank equal to k, and with $||\mathcal{D}^\circ|| \leq ||\mathcal{D}||$. Then we can apply Lemma 3.8 on derivation $\mathcal{D}^\circ$ to conclude that there is a well behaved derivation $\mathcal{D}^\circ_1$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over formulae labelled by $\top$, with cutrank k_1 not greater than k and with $||\mathcal{D}^\circ_1|| \leq 2^{||\mathcal{D}^\circ||-1}$, and therefore with $||\mathcal{D}^\circ_1|| \leq 2^{||\mathcal{D}||-1}$. Applying Lemma 3.11 on derivation $\mathcal{D}^\circ_1$ we can conclude that there is a well behaved derivation $\mathcal{D}^\circ_2$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts over labelled formulae and with $||\mathcal{D}^\circ_2|| \leq 4_{k_1}^{||\mathcal{D}^\circ_1||}$. Therefore $||\mathcal{D}^\circ_2|| \leq 4_k^{2^{||\mathcal{D}||-1}}$. Finally applying Lemma 3.16 to derivation $\mathcal{D}^\circ_2$ we obtain a well behaved derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathcal{C}_\mathcal{P}} s$ with no cuts and with $||\mathcal{D}^\bullet|| \leq 2^{||\mathcal{D}^\circ_2||-1}$, and so with $||\mathcal{D}^\bullet|| \leq 2^{4_k^{2^{||\mathcal{D}||-1}}-1}$. The proof is similar for local consequences in modal logic $\mathcal{L}(\mathcal{P})$. QED

There is however a question pending. Intuitively it was expected that the bound for local consequence be $||\mathcal{D}^\bullet|| \leq 2^{4^{||\mathcal{D}||-1}_{\mathrm{cr}(\mathcal{D})}}$ since it would be possible to find a derivation without formulae labelled by $\top$ for local consequence, and so there would be no need to eliminate cuts over formulae labelled by $\top$. The question is that in general given a derivation ending in a sequent with formulae labelled only by atomic terms we were not able to prove that there is a derivation ending in the same sequent and not using formulae labelled by $\top$. The cause of this situation is the presence of the rule LgenF which is important in [20] to prove the completeness of the modal sequent calculi. We believe it is possible to find another completeness theorem for local consequence that would not need that rule. So we establish the following conjecture: for each compatible set $\mathcal{P}$ of modal accessibility rules, if $\mathcal{D}$ is a derivation for $\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{P}} \varphi$ then there is a derivation $\mathcal{D}^\bullet$ for it with no cuts and with

$$||\mathcal{D}^\bullet|| \leq 2^{4^{||\mathcal{D}||}_{\mathrm{cr}(\mathcal{D})}-1}.$$

4 Subformula property and consistency

An important result strongly related with cut elimination is the subformula property. We show in this section that modal sequent calculi with a compatible set of modal accessibility rules satisfy the subformula property. Moreover we establish also a related property for truth value terms that we call local weak term property. A derivation $\mathcal{D}$ satisfy the *local weak term property* if for any sequent s in $\mathcal{D}$ obtained by rule r from the sequents $s_1, \ldots, s_k$ and a term t in s_j and not in s then either

- t is $\mathbf{N}(\mathbf{y})$ for some bound variable $\mathbf{y}$ with $\Omega\mathbf{y}$ in the left side of s, or
- t is a bound variable and Ωt is in the left side of s, or
- t is a fresh variable according to r.

It is now possible to state the following proposition.

Proposition 4.1 For each compatible set $\mathcal{P}$ of modal accessibility rules, if $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{P}} \varphi$ then there is a derivation for it satisfying the subformula property and the local weak term property. Similarly for $\vdash^{\ell}_{\mathcal{P}}$.

The proof of the proposition follows straightforwardly from the cut elimination theorem, Theorem 3.17, and by analysis of the rules present in the calculus.

We now prove that modal sequent calculi with a compatible set of modal accessibility rules are globally and locally consistent. Before the main theorem we show an auxiliary lemma. Observe the importance of the existence of a derivation without cut applications in the proof of the lemma.

Lemma 4.2 For any compatible set $\mathcal{P}$ of modal accessibility rules,

$$\not\vdash_{\mathcal{C}_{\mathcal{P}}} \Omega\mathbf{y}_1, \ldots, \Omega\mathbf{y}_n \rightarrow$$

for n in $\mathbb{N}_0$ and bound variables $\mathbf{y}_1, \ldots, \mathbf{y}_n$.

Proof Let $\mathcal{P}$ be a compatible set of modal accessibility rules. We show that for any n in $I\!N_0$ and bound variables $\mathbf{y}_1, \ldots, \mathbf{y}_n$ if $\mathcal{D}$ is a well behaved derivation, with no cuts, for $\vdash_{\mathcal{C}_\mathcal{P}} \Omega\mathbf{y}_1, \ldots, \Omega\mathbf{y}_n \rightarrow$ then there is an absurd. The proof follows by complete induction on the depth of the derivation. Let n be in $I\!N_0$ and $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be bound variables and let $\mathcal{D}$ be a well behaved derivation for $\vdash_{\mathcal{C}_\mathcal{P}} \Omega\mathbf{y}_1, \ldots, \Omega\mathbf{y}_n \rightarrow$ with no cuts and with depth m. Note that the rule r used in $\mathcal{D}$ to conclude $\Omega\mathbf{y}_1, \ldots, \Omega\mathbf{y}_n \rightarrow$ can only be (i) either LwΩ or LcΩ or exΩ if n is different of 0, or (ii) exΩ if n is 0. Denote by $\mathcal{D}_1$ the derivation $\mathcal{D}$ without the conclusion sequent. Then $\mathcal{D}_1$, independently of the value of n, satisfies the induction hypothesis conditions. So by the induction hypothesis we obtain an absurd. Hence for any n in $I\!N_0$ and bound variables $\mathbf{y}_1, \ldots, \mathbf{y}_n$ we have that $\not\vdash_{\mathcal{C}_\mathcal{P}} \Omega\mathbf{y}_1, \ldots, \Omega\mathbf{y}_n \rightarrow$ as we wanted to show. QED

We are now in conditions to prove the global and the local consistency of modal logics presented by labelled sequent calculus with a compatible set of modal accessibility rules.

Theorem 4.3 For each compatible set $\mathcal{P}$ of modal accessibility rules, the modal logic $\mathcal{L}(\mathcal{P})$ is globally and locally consistent.

Proof Let $\mathcal{P}$ be a compatible set of modal accessibility rules and φ a formula. Suppose by absurd that $\vdash^{\mathrm{g}}_{\mathcal{P}} \varphi$ and $\vdash^{\mathrm{g}}_{\mathcal{P}} \neg\varphi$ and denote by $\mathcal{D}$ and $\mathcal{D}'$ the derivations for $\vdash_{\mathcal{C}_\mathcal{P}} \rightarrow \top \leq \varphi$ and $\vdash_{\mathcal{C}_\mathcal{P}} \rightarrow \top \leq \neg\varphi$, respectively. So $\vdash_{\mathcal{C}_\mathcal{P}} \Omega\mathbf{y}_1 \rightarrow \mathbf{y}_1 \leq \varphi$ and $\vdash_{\mathcal{C}_\mathcal{P}} \Omega\mathbf{y}_1, \mathbf{y}_1 \leq \varphi \rightarrow$. Hence $\vdash_{\mathcal{C}_\mathcal{P}} \Omega\mathbf{y}_1, \Omega\mathbf{y}_1 \rightarrow$. But this contradicts Lemma 4.2. QED

5 Conclusions

We gave a procedure to obtain a cut free derivation for a local or a global modal derivation, in the context of a modal sequent calculus with a compatible set of modal accessibility rules. At the same time we showed that, denoting the original derivation by $\mathcal{D}$, the logical depth increases, in the worst case, $2^{4^{2^{||\mathcal{D}||-1}}_{\mathrm{cr}(\mathcal{D})}-1}$, where the exponent $2^{||\mathcal{D}||-1}$ results from the elimination on $||\mathcal{D}||$ of cuts over formulae labelled by $\top$, the hyperexponential term results from the elimination of cuts over formulae labelled by atomic terms in derivations resulting from the elimination of cuts over formulae labelled by $\top$, and the exterior power of 2 results from the elimination of cuts over truth value comparison assertions on derivations resulting from the elimination of cuts over labelled formulae.

The great expressive power of the calculi presented, and the possibility to reason globally and locally, essentially due to the presence of labels, causes only, in the worst case, an exponential increase of the logical depth. This is not significant since the elimination of cuts over formulae labelled by atomic terms causes, in the worst case, a hyperexponential increase. Our results compare with the results obtained for first order logic where cut elimination is possible but also at the expense, in the worst case, of a hyperexponential increase in the logical depth, see [5].

Contrasting with first order logic where the complexity of cut elimination has been deserving a continuous attention [30, 42, 2], and with some subsystems of linear logic where the complexity of the cut elimination problem has being studied [16], in modal logic we do not know of any work besides our own on the complexity of cut elimination in a modal sequent calculus. There are instead a widely variety of results on complexity of decision procedures [33, 14]. Various reasons contribute to this happening. One is certainly the fact that modal logics are usually presented by Hilbert systems and not by sequent systems. Another reason is that when the modal logics are presented by sequent systems either they are already cut free and so it is studied, at most, cut admissibility, or the cut elimination proof is done by non-constructive methods like by semantic methods or by translations to other cut free calculi.

The work presented in this paper raises several problems that we try to summarize in the next paragraphs.

The first is the development of specific methods of cut elimination, like Baaz and Leitsch did for first order logic [2], applying perhaps only to certain classes of end-sequents and of cuts, but with a better complexity than the general method presented in this paper. A different way was followed by Zhang in [42] who worked in the general method of cut elimination for first order logic and showed that it is possible to obtain improved bounds on cut elimination after refining notions like the level of the cut or taking into account the role of contractions and quantifications. Another direction of investigation is the analysis if the hyperexponential bound we established for the worst case is in fact unavoidable, in line with the work of Statman and Orevkov [34, 24] for first order logic.

Fibring is a technique of combination of logics that is deserving close attention [11, 41, 28]. One of the principal goals when studying fibring is to obtain preservation results that apply to a wide class of logics. So it would certainly be very interesting to investigate preservation by fibring of cut elimination and its complexity. A similar issue can be addressed for modulated fibring [31] which is a refinement of fibring preventing collapses. Then, in the application front, it would be very interesting to use those results to prove properties about probabilistic logics like [18] and also about quantum logics [19].

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