

# Secure $N$ -dimensional simultaneous dense coding and applications

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## Abstract

Simultaneous dense coding guarantees that Bob and Charlie simultaneously receive their respective information from Alice in their respective processes of dense coding. The idea is to use the so-called locking operation to “lock” the entanglement channels, thus requiring a joint unlocking operation by Bob and Charlie in order to simultaneously obtain the information sent by Alice. We present some new results on simultaneous dense coding: (1) We propose three simultaneous dense coding protocols, which use different  $N$ -dimensional entanglement (Bell state, W state and GHZ state). (2) Besides the quantum Fourier transform, two new locking operators are introduced (the double controlled-NOT operator and the SWAP operator). (3) In the case that spatially distant Bob and Charlie have to finalise the protocol by implementing the unlocking operation through communication, we improve our protocol’s fairness, with respect to Bob and Charlie, by implementing the unlocking operation in series of steps. (4) We improve the security of simultaneous dense coding against the intercept-resend attack. (5) We show that simultaneous dense coding can be used to implement a fair contract signing protocol. (6) We also show that the  $N$ -dimensional quantum Fourier transform can act as the locking operator in simultaneous teleportation of  $N$ -level quantum systems.

**Keywords:** Quantum communication, Teleportation, Dense coding

## 1 Introduction

Due to the Holevo bound[1], at most  $\log_2 N$  bits of information can be transmitted via a quNit ( $N$ -level quantum system). Dense coding, proposed by Bennett and Wiesner[2] in 1992, increases the classical capacity of a quantum communication channel with the help of prior entanglement[3]. If the sender and the receiver share a pair of entangled quNits,  $2\log_2 N$  bits can be transmitted via a quNit[2].

In the simplest case of dense coding two parties, Alice and Bob, share a pair of entangled qubits (2-level quantum systems) in a Bell state. Alice first performs one of the four local unitary operations  $I$ ,  $\sigma_x$ ,  $i\sigma_y$  and  $\sigma_z$  (where  $\sigma_j$  are the Pauli matrices) on her qubit to encode 2 bits of information, transforming the entangled pair into one of the four mutually orthogonal Bell states. Then Alice sends her qubit to Bob through a quantum channel. Bob is now able to measure both qubits in the Bell basis to obtain one of the four possible outcomes correlated with the operations performed by Alice. Thus, Alice can transmit 2 bits of information to Bob by manipulating and sending only one qubit.

Thus far, dense coding has been extensively studied in various ways. For example, dense coding that utilises high-dimensional entangled states has been studied in References [4, 5, 6],

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non-maximally entanglement channels in References [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], while multipartite entanglement channels have also been considered in References [20, 21, 22, 23, 24, 25, 26, 27]. Another generalisation is to perform the communication task under the control of a third party, so-called controlled dense coding[28, 29, 30, 31].

Inspired by the simultaneous teleportation scheme proposed by Wang *et al* [32], we have proposed a simultaneous dense coding (SDC) scheme[33], which guarantees that Bob and Charlie (the receivers) simultaneously receive their respective information from Alice (the sender) in their respective processes of dense coding. In this scheme, Alice first performs a locking operation to entangle the particles from two independent quantum entanglement channels, and therefore the receivers cannot obtain their respective information separately, before performing the unlocking operation together. The quantum Fourier transform and its inverse are used as the locking and unlocking operators, respectively.

*Simultaneous dense coding may be relevant and useful in various applications in which Bob and Charlie can be either close or separated. If Alice has two different secrets, one for Bob and another for Charlie, she can utilise simultaneous dense coding to guarantee that Bob and Charlie simultaneously reveal their respective secrets. Bob does not know Charlie's secret and vice versa. For example, boss wants two employees to simultaneously carry out two confidential commercial activities under the condition that the sensitive information of each activity is only revealed to whom is in charge of that activity.*

*There are other applications improving some models or tasks of quantum communication. In Sec. 5, we show that simultaneous dense coding can be used to implement a fair contract signing protocol [34] between spatially distant Bob and Charlie. In this case, Bob and Charlie are separated so that there is a problem of how to implement the unlocking operation fairly. We discuss this problem in Sec.3.*

In this paper, we give some new results on simultaneous dense coding and teleportation. The main improvements over earlier proposals are: (i) introduction of new locking operations (the double-controlled-NOT and the SWAP operators) for SDC schemes that were not used before, showing that the protocol can be achieved by using three different  $N$ -dimensional quantum states (Bell, W and GHZ states) that were not mentioned in that context in the literature before; (ii) designing a strategy which achieves the fairness of SDC, allowing for the protocol to be performed even when mutually mistrustful Bob and Charlie are situated on spatially distant locations, a feature which previous SDC proposals do not allow; (iii) designing a strategy which also achieves the security of SDC against intercept-resend attack, improving the security of SDC over the previous proposals; (iv) introducing simultaneous teleportation scheme for the transfer of arbitrary number of  $N$ -level quantum states. In addition to that, we proposed contract signing protocol based on a fair and secure SDC protocol. Having more options for achieving simultaneous transmission of (densely coded) information presents opportunities for designing SDC protocols with improved security features, and in addition provides wider range of possibly suitable experimental realisations. Indeed, achieving the fairness of SDC scheme, a feature not satisfied by previous SDC proposals, is only possible with the double-controlled-NOT and the SWAP locking operator. Also, the security of the protocol against intercept-resend attack is a novel feature of SDC schemes introduced in our paper.

The paper is organised as follows. In Sec. 2, we introduce the locking operators (i.e. the quantum Fourier transform, the double controlled-NOT operator and the SWAP operator) and propose three simultaneous dense coding protocols utilising different  $N$ -dimensional quantum states (i.e., Bell states, W states, and GHZ states). In Sec. 3, we improve the fairness of the protocol by implementing the unlocking operation between spatially distant Bob and Charlie. In Sec. 4, we improve the security of simultaneous dense coding against the intercept-resend attack. In Sec. 5, we show that simultaneous dense coding can be used to implement a fair contract signing protocol. In Sec. 6, we show that the  $N$ -dimensional quantum Fourier transform can act as the locking operator in simultaneous teleportation of quNits. A brief conclusion follows in Sec. 7.

## 2 $N$ -dimensional simultaneous dense coding

A quNit is an  $N$ -dimensional quantum system. States of  $N$ -dimensional quantum systems can be mapped onto  $n = \log_2(N)$  qubits, and throughout the paper we will assume that a quNit is realised as an ordered array of qubits. We also assume that the dimension  $N$  of quNits satisfies the requirement  $N = 2^n$ , with  $n \in \mathbb{N}$ . In other words, one quNit consists of (or can be mapped to)  $n$  qubits.

In the task of  $N$ -dimensional simultaneous dense coding (SDC), Alice intends to use dense coding to send two cNits (arrays of classical bits)  $b_1, b_2 \in \{0, 1, \dots, N-1\}$  to Bob and two cNits  $c_1, c_2 \in \{0, 1, \dots, N-1\}$  to Charlie, under the condition that Bob and Charlie must collaborate to simultaneously find out what she sends.

In the following subsections, we propose three protocols, one using  $N$ -dimensional Bell states, the other using W states, and the third using GHZ states, as the entanglement channels, respectively. The idea behind these protocols is to perform the locking operator on Alice's quNits before sending them to Bob and Charlie. After receiving Alice's quNits, the states of Bob's subsystem and Charlie's subsystem are independent of  $(b_1, b_2)$  and  $(c_1, c_2)$ , respectively, so that they know nothing about the encoded information. Only after performing the unlocking operator (the inverse of the locking operator) together, Bob and Charlie can obtain  $(b_1, b_2)$  and  $(c_1, c_2)$ , respectively.

In this section, we assume that Bob and Charlie are at the same site. In order to implement the unlocking operator, they can input their particles into a physical device that can unlock the particles, and then get their respective output particles. The problem of how distant Bob and Charlie can perform the unlocking operator fairly is discussed in Sec. 3.

### 2.1 DCNOT locking operator

Before introducing the protocols, let us first have a look at the DCNOT (double controlled-NOT) operator, which is used as the locking operator. The DCNOT operator is composed of two CNOT (controlled-NOT) operators. The CNOT operator has two input qubits. If the first qubit (the control qubit) is in state  $|1\rangle$ , CNOT flips the state of the second qubit (the target qubit). If the control qubit is in state  $|0\rangle$ , CNOT does nothing to the target qubit ( $|0\rangle$  and  $|1\rangle$  are the basis vectors). Its action can be described as ( $x, y = 0, 1$  are bit values):

$$|x\rangle|y\rangle \rightarrow |x\rangle|x \oplus y\rangle, \quad (1)$$

where  $\oplus$  denotes bitwise addition modulo 2.

The DCNOT operator is formed by performing the first CNOT with the first qubit as the control and the second being the target, and then a second CNOT with inverse roles of the qubits (the first qubit as the target and the second being the control). Its action can be described as:

$$|x\rangle|y\rangle \rightarrow |y\rangle|x \oplus y\rangle, \quad (2)$$

and the inverse DCNOT operator can be described as:

$$|x\rangle|y\rangle \rightarrow |x \oplus y\rangle|x\rangle. \quad (3)$$

Our quNits  $|x\rangle_{A_1}|y\rangle_{A_2}$ , with  $x, y \in \{0, 1, \dots, N-1\}$ , are arrays of  $n$  qubits, and can be written as:

$$|x_1, \dots, x_n\rangle_{A_{1,1} \dots A_{1,n}} |y_1, \dots, y_n\rangle_{A_{2,1} \dots A_{2,n}}, \quad (4)$$

where  $x_1 \dots x_n$  and  $y_1 \dots y_n$  are binary representations of  $x$  and  $y$ , respectively, and  $A_{i,j}$  denote different qubits ( $i = 1, 2$  and  $j = 1, 2, \dots, N$ ). We define the  $N$ -dimensional DCNOT operator by applying the DCNOT operator on each pair of  $A_{1,j}A_{2,j}$ :

$$\begin{aligned} &|x_1, \dots, x_n\rangle_{A_{1,1} \dots A_{1,n}} |y_1, \dots, y_n\rangle_{A_{2,1} \dots A_{2,n}} \\ &\rightarrow |y_1, \dots, y_n\rangle_{A_{1,1} \dots A_{1,n}} |x_1 \oplus y_1, \dots, x_n \oplus y_n\rangle_{A_{2,1} \dots A_{2,n}}, \end{aligned} \quad (5)$$

that is ( $A_1 = A_{1,1}A_{1,2} \dots A_{1,N}$  and analogously for  $A_2$ ),

$$|x\rangle_{A_1}|y\rangle_{A_2} \rightarrow |y\rangle_{A_1}|x \oplus y\rangle_{A_2}. \quad (6)$$

Analogously, by applying the inverse DCNOT operator on each pair of  $A_{1,j}A_{2,j}$ , we have the  $N$ -dimensional inverse DCNOT operator:

$$\begin{aligned} &|x_1, \dots, x_n\rangle_{A_{1,1} \dots A_{1,n}}|y_1, \dots, y_n\rangle_{A_{2,1} \dots A_{2,n}} \\ &\rightarrow |x_1 \oplus y_1, \dots, x_n \oplus y_n\rangle_{A_{1,1} \dots A_{1,n}}|x_1, \dots, x_n\rangle_{A_{2,1} \dots A_{2,n}}, \end{aligned} \quad (7)$$

that is,

$$|x\rangle_{A_1}|y\rangle_{A_2} \rightarrow |x \oplus y\rangle_{A_1}|x\rangle_{A_2}. \quad (8)$$

## 2.2 SDC Protocol 1: using $N$ -dimensional Bell state

Protocol 1 uses the following  $N$ -dimensional Bell basis [35], for composite systems of two quNits 1 and 2:

$$|\phi(xy)\rangle_{12} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{2\pi i}{N} jx} |j+y\rangle_1 |j\rangle_2, \quad (9)$$

where  $x, y \in \{0, 1, \dots, N-1\}$ . In the rest of the paper, addition and subtraction inside kets are done modulo  $N$ .

The local (acting onto system 1 only) unitary operators  $U(xy)$  transform  $|\phi(00)\rangle_{12}$  into  $|\phi(xy)\rangle_{12}$ :

$$\left( U(xy) \otimes I \right) |\phi(00)\rangle_{12} = |\phi(xy)\rangle_{12}, \quad (10)$$

where  $U(xy) = X^y Z^x$ ,  $X : |j\rangle \rightarrow |j+1\rangle$  is the shift operator, and  $Z : |j\rangle \rightarrow e^{\frac{2\pi i}{N} j} |j\rangle$  is the rotation operator.

The set  $\{|\phi(xy)\rangle_{12}\}$  forms an orthonormal basis of completely distinguishable states, and for that reason each of its elements can be used to carry information  $(xy)$ . The unitary operators  $U(xy)$  are then used to encode that information  $(xy)$  into the initial state  $|\phi(00)\rangle_{12}$ .

In the initialisation phase of Protocol 1, Alice, Bob and Charlie share two pairs of entangled quNits  $|\phi(00)\rangle_{A_1 B}$  and  $|\phi(00)\rangle_{A_2 C}$ , where subscripts  $A_1$  and  $A_2$  denote Alice's two quNits, subscript  $B$  denotes Bob's quNit and subscript  $C$  denotes Charlie's quNit. The initial quantum state of the composite system is (note that for reasons of simplicity, we drop the subscript  $A_1 B A_2 C$  for  $\Omega$ -states  $|\Omega(0)\rangle$  etc.)

$$|\Omega(0)\rangle = |\phi(00)\rangle_{A_1 B} \otimes |\phi(00)\rangle_{A_2 C}. \quad (11)$$

Protocol 1 consists of five steps:

(1) *Encoding*. Alice performs unitary operators  $U(b_1 b_2)$  on quNit  $A_1$  and  $U(c_1 c_2)$  on quNit  $A_2$  to encode  $(b_1, b_2)$  and  $(c_1, c_2)$ , respectively, like in the original dense coding scheme [2]. After that, the state of the composite system becomes

$$|\Omega(1)\rangle = U_{A_1}(b_1 b_2) \otimes U_{A_2}(c_1 c_2) |\Omega(0)\rangle = |\phi(b_1 b_2)\rangle_{A_1 B} \otimes |\phi(c_1 c_2)\rangle_{A_2 C}. \quad (12)$$

(2) *Locking*. Alice performs the DCNOT operator on quNits  $A_1 A_2$  to lock the entanglement channels. The state of the composite system becomes

$$|\Omega(2)\rangle = DCNOT_{A_1 A_2} \left( |\phi(b_1 b_2)\rangle_{A_1 B} \otimes |\phi(c_1 c_2)\rangle_{A_2 C} \right). \quad (13)$$

(3) *Communication*. Alice sends quNit  $A_1$  to Bob and quNit  $A_2$  to Charlie, like the original dense coding scheme [2].

(4) *Unlocking.* Bob and Charlie collaborate to perform the inverse *DCNOT* operator on quNits  $A_1A_2$ . The state of the composite system becomes

$$|\Omega(3)\rangle = DCNOT_{A_1A_2}^\dagger |\Omega(2)\rangle = |\phi(b_1b_2)\rangle_{A_1B} \otimes |\phi(c_1c_2)\rangle_{A_2C}. \quad (14)$$

(5) *Decoding.* Bob and Charlie measure quNits  $A_1B$  and quNits  $A_2C$  in the  $N$ -dimensional Bell basis respectively to obtain  $(b_1, b_2)$  and  $(c_1, c_2)$ , like the original dense coding scheme[2].

The following theorem demonstrates the validity of Protocol 1.

**Theorem 1.** *Neither Bob nor Charlie alone can learn the encoded information from the states of their subsystems before step 4 (Unlocking) of Protocol 1.*

*Proof* After step 2 (*Locking*), the state of the composite system becomes

$$\begin{aligned} |\Omega(2)\rangle &= DCNOT_{A_1A_2} \left( \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{2\pi i}{N} j b_1} |j + b_2\rangle_{A_1} |j\rangle_B \otimes \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} k c_1} |k + c_2\rangle_{A_2} |k\rangle_C \right) \\ &= \frac{1}{N} \sum_{j,k=0}^{N-1} e^{\frac{2\pi i}{N} (j b_1 + k c_1)} |k + c_2\rangle_{A_1} |(j + b_2) \oplus (k + c_2)\rangle_{A_2} |j\rangle_B |k\rangle_C, \end{aligned} \quad (15)$$

and the reduced density matrix in subsystem  $A_1B$  is

$$\begin{aligned} \rho_{A_1B} &= \sum_{j,k=0}^{N-1} (A_2 \langle j|_C \langle k|) |\Omega(2)\rangle \langle \Omega(2)| (|j\rangle_{A_2} |k\rangle_C) \\ &= \frac{1}{N^2} \sum_{j,k=0}^{N-1} |j\rangle_{A_1} |k\rangle_{BA_1} \langle j|_B \langle k| \\ &= I_{A_1B} / N^2. \end{aligned} \quad (16)$$

We can calculate the reduced density matrix in subsystem  $A_2C$  in the same way and get  $\rho_{A_2C} = I_{A_2C} / N^2$ . Because  $\rho_{A_1B}$  and  $\rho_{A_2C}$  are independent of  $(b_1, b_2)$  and  $(c_1, c_2)$ , Bob and Charlie know nothing about the encoded information before step 4 (*Unlocking*).  $\square$

### 2.3 SDC Protocol 2: using $N$ -dimensional W state

Li and Qiu [26] presented a sufficient and necessary condition for a  $N$ -dimensional W state to be suitable for perfect teleportation and dense coding, and then they generalised the states of W-class to multi-particle systems with  $N$ -dimension:

$$|W(00)\rangle_{123} = \frac{1}{\sqrt{N}} \left( \frac{1}{\sqrt{2}} \sum_{j=1}^{N-1} |j-1\rangle_1 (|0j\rangle + |j0\rangle)_{23} + |N-1\rangle_1 |00\rangle_{23} \right). \quad (17)$$

Alice uses unitary operators

$$U(xy) = \sum_{j=0}^{N-1} e^{\frac{2\pi i}{N} (j-y)x} |j-y\rangle \langle j| \quad (18)$$

to encode her information:

$$|W(xy)\rangle_{123} = \left( U(xy) \otimes I \otimes I \right) |W(00)\rangle_{123}, \quad (19)$$

where  $x, y \in \{0, 1, \dots, N-1\}$ .

In the initialisation phase of Protocol 2, Alice, Bob and Charlie share two pairs of entangled quNits  $|W(00)\rangle_{A_1B}$  and  $|W(00)\rangle_{A_2C}$ , where subscript  $A_1$  and  $A_2$  denote Alice's two quNits,

subscript  $B$  denotes Bob's two quNits and subscript  $C$  denotes Charlie's two quNits. Protocol 2 consists of five steps:

- (1) *Encoding.* Alice performs unitary operators  $U(b_1 b_2)$  on quNit  $A_1$  and  $U(c_1 c_2)$  on quNit  $A_2$  to encode  $(b_1, b_2)$  and  $(c_1, c_2)$ , respectively.
- (2) *Locking.* Alice performs the DCNOT operator on quNits  $A_1 A_2$ .
- (3) *Communication.* Alice sends quNit  $A_1$  to Bob and quNit  $A_2$  to Charlie.
- (4) *Unlocking.* Bob and Charlie collaborate to perform the inverse *DCNOT* operator on quNits  $A_1 A_2$ .
- (5) *Decoding.* Bob and Charlie make the von Neumann measurement using the orthogonal states  $\{|W(xy)\rangle\}$  on quNits  $A_1 B$  and quNits  $A_2 C$ , respectively, to obtain  $(b_1, b_2)$  and  $(c_1, c_2)$ .

The following theorem demonstrates the validity of Protocol 2.

**Theorem 2.** *Neither Bob nor Charlie alone can learn the encoded information from the states of their subsystems before step 4 (Unlocking) of Protocol 2.*

*Proof* The proof is similar to that of Protocol 1. After step 2 (*Locking*), the reduced density matrix

$$\begin{aligned} \rho_{A_1 B} &= \frac{1}{N^2} \sum_{j=0}^{N-1} |j\rangle_{A_1} \langle 00|_{B A_1} \langle j|_B \langle 00| + \frac{1}{2N^2} \sum_{k=1}^{N-1} \sum_{j=0}^{N-1} |j\rangle_{A_1} (|0k\rangle + |k0\rangle)_{B A_1} \langle j|_B (\langle 0k| + \langle k0|) \\ &= \frac{1}{N} \left[ \sum_{j=0}^{N-1} |j\rangle_{A_1} \langle j| \right] \otimes \frac{1}{N} \left[ |00\rangle_B \langle 00| + \frac{1}{2} \sum_{k=1}^{N-1} (|0k\rangle + |k0\rangle)_B (\langle 0k| + \langle k0|) \right], \end{aligned} \quad (20)$$

and analogously for  $\rho_{A_2 C}$ . Because  $\rho_{A_1 B}$  and  $\rho_{A_2 C}$  are independent of  $(b_1, b_2)$  and  $(c_1, c_2)$ , Bob and Charlie know nothing about the encoded information before step 4 (*Unlocking*).  $\square$

## 2.4 SDC Protocol 3: using $N$ -dimensional GHZ state

Protocol 3 uses the following  $N$ -dimensional GHZ states [4]:

$$|GHZ(xy)\rangle_{123} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{2\pi i}{N} jx} |j+y\rangle_1 |jj\rangle_{23}, \quad (21)$$

where  $x, y \in \{0, 1, \dots, N-1\}$ .

The unitary operators  $U(xy)$  transform  $|GHZ(00)\rangle_{123}$  into  $|GHZ(xy)\rangle_{123}$ :

$$\left( U(xy) \otimes I \otimes I \right) |GHZ(00)\rangle_{123} = |GHZ(xy)\rangle_{123}, \quad (22)$$

where  $U(xy) = X^y Z^x$ ,  $X : |j\rangle \rightarrow |j+1\rangle$  is the shift operator,  $Z : |j\rangle \rightarrow e^{\frac{2\pi i}{N} j} |j\rangle$  is the rotation operator.

Protocol 3 is similar to Protocols 1 and 2, and the proof of its validity is analogous to those of Protocols 1 and 2.

## 2.5 Other locking operators

In this subsection, we introduce another two locking operators: the quantum Fourier transform and the SWAP operator.

The two-quNit quantum Fourier transform is defined by

$$|x\rangle_{A_1} |y\rangle_{A_2} \rightarrow \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N^2} (xN+y)(jN+k)} |j\rangle_{A_1} |k\rangle_{A_2}. \quad (23)$$

The SWAP operator simply swaps two qubits (or quNits):

$$|\varphi\rangle_{A_1}|\varphi'\rangle_{A_2} \longrightarrow |\varphi'\rangle_{A_1}|\varphi\rangle_{A_2}. \quad (24)$$

When the  $N$ -dimensional quantum Fourier transform or the SWAP operator are substituted for the DCNOT operator in the above three protocols, the states of Bob’s subsystem and Charlie’s subsystem after step 2 (*Locking*) are also independent of  $(b_1, b_2)$  and  $(c_1, c_2)$ . Neither Bob nor Charlie alone can learn the encoded information from the states of their subsystems before step 4 (*Unlocking*). Only after performing the inverse quantum Fourier transform or the SWAP operator together, they can achieve  $(b_1, b_2)$  and  $(c_1, c_2)$  respectively.

When the SWAP operator is used as the locking operator, Protocols 1-3 become simpler. Step 2 (*Locking*) can be omitted. In step 3 (*Communication*), Alice sends quNit  $A_1$  to Charlie and quNit  $A_2$  to Bob. In step 4 (*Unlocking*), Bob and Charlie swap quNit  $A_1$  and quNit  $A_2$ . However, if one of the receivers, say Bob, is malicious, he can fool Charlie in step 4 (*Unlocking*) by detaining the quNit that carries  $(c_1, c_2)$  (i.e. quNit  $A_2$ ) and swapping a randomly prepared fake quNit for the quNit that carries  $(b_1, b_2)$  (i.e. quNit  $A_1$ ). Then, Bob would receive  $(b_1, b_2)$  from quNits  $A_1B$ , but Charlie would not receive  $(c_1, c_2)$ , because quNit  $A_2$  is still at Bob’s side.

### 3 Improving the fairness of SDC

The clients (Bob and Charlie) can decode the messages only by the joint unlocking quantum operation, which requires either a quantum channel, shared entanglement, or direct interaction between them. By “direct interaction”, we mean that the clients meet and input their particles into a physical device that can unlock the particles, and then get their respective output particles.

In this section, we discuss the problem of achieving a global unlocking operation between the distant parties (Bob and Charlie) by the means of local operations and classical communication (LOCC), some prior shared resources (such as entanglement, etc.) and/or quantum communication. Also, we discuss the problem of the fairness of the protocol, in case the clients do not trust each other. The solution is a probabilistic protocol based on sequential exchange of information (quantum, or classical with the help of prior shared entanglement) between the clients.

#### 3.1 The fairness problem

The problem of achieving a global operation by the actions of spatially distant clients (Bob and Charlie) lies in the fact that the unlocking operation (whether it is the inverse DCNOT, quantum Fourier transform or SWAP<sup>1</sup>) has a feature that it can entangle initially separable states. Since it is impossible to create entanglement by the means of LOCC only, in order to implement the unlocking operation, Bob and Charlie have to share prior entanglement, or use a quantum channel for one of the clients to send his quNit to the other, who would then perform the unlocking operation on quNits  $A_1A_2$  locally (i.e. at his site).

The problem with any such protocol is that it can be neither simultaneous, nor symmetric, with respect to the two parties involved, which clearly sets an unfair situation.

This is a general problem that arises within asynchronous distributed networks[36]. Namely, in order to achieve the joint operation of the quNits  $A_1A_2$ , clients can each perform local measurements, and then send, conditional to the measurement outcomes, (classical or quantum) information to the other. Since the clients are far apart, it is impossible to achieve simultaneous message exchange - the messages are always sent one at a time, from one client to the other (the network is “asynchronous”). This means that, whatever the protocol that achieves the unlocking

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<sup>1</sup>The case of the SWAP operation is particularly interesting. Namely, after Bob and Charlie swap the states of qubits  $A_1$  and  $A_2$ , the very qubits do not become entangled with each other. Nevertheless, the distant sites of Bob and Charlie do become entangled: qubits  $A_1$  are now entangled with qubits  $C$ , while qubits  $A_2$  are entangled with qubits  $B$ . The other way to look at this is through the no-cloning theorem: there is no way for Bob to learn the unknown state of the system  $A_1B$  and transfer (swap) its partial state of  $A_1$  together with its entanglement with  $B$  using only LOCC, without prior entanglement shared with Charlie.

operation is, there always exists the *last message*, say from Bob to Charlie. But this means that, prior to sending his last message to Charlie, Bob has all the right (quantum) information needed to obtain his message  $(b_1, b_2)$ , while Charlie does not. This is obviously unfair, with respect to Charlie.

### 3.2 The solution

The situation is similar to the one presented in the contract signing problem[34]. The solution, similar to the one proposed for quantum contract signing[37], is to perform the unlocking operation on quNits  $A_1$  and  $A_2$  (i.e., arrays of qubits  $A_{1,1} \dots A_{1,n}$  and  $A_{2,1} \dots A_{2,n}$ ) in series of steps, such that after each step, one pair of qubits are unlocked. In the course of the unlocking stage, the clients increase their probabilities to obtain the needed classical information ( $(b_1, b_2)$  and  $(c_1, c_2)$ , respectively), such that at each step one client has slightly higher probability of successful recovery than the other. Therefore, the protocol is probabilistic, and also fair in the sense that at each step, one client is only slightly privileged over the other.

The information can be transferred in two equivalent ways: either by sending quantum information (qubits) via a quantum channel, or by teleporting qubits' quantum states using LOCC and shared entanglement. Without the loss of generality, we will assume that the clients are exchanging the actual qubits, rather than teleporting their states. The only difference between the two cases is that in the latter case, the clients use the previously shared entanglement and exchange classical instead of quantum information.

The main problem in this approach is that a client, say Charlie, cannot be sure if Bob sent him the right information or not. Therefore, we introduce additional  $2s$  "control qubits" that are used by the clients to check each other's honesty during the unlocking stage:  $s$  control qubits would be joined with  $n$  "message qubits" that carry the message  $(b_1, b_2)$ , to form  $(n + s)$  qubits  $A_{1,1} \dots A_{1,n+s}$ , and given to Bob; other  $s$  control qubits joined with  $n$  message qubits that carry  $(c_1, c_2)$ , to form  $A_{2,1} \dots A_{2,n+s}$  and given to Charlie.

The position  $Cpos_1(j) \in \{1, 2, \dots, n + s\}$  of each control qubit  $j = 1, 2, \dots, s$  within the  $(n + s)$  qubits  $A_{1,1} \dots A_{1,n+s}$  is chosen randomly by Alice and this information ( $s$  integers  $Cpos_1(j)$ ) is given to Bob; analogously, random positions  $Cpos_2(j)$  of control qubits from  $A_{2,1} \dots A_{2,n+s}$  are given to Charlie.

The control qubits are prepared by Alice in pure states and are uncorrelated from the message qubits. The state of each control qubit is randomly chosen by Alice from a publicly known set  $\mathcal{S}$  of pure states.  $\mathcal{S}$  is the union of two mutually unbiased bases, computational  $Z$  basis  $\{|0\rangle, |1\rangle\}$  and rotated  $X$  basis  $\{|+\rangle, |-\rangle\}$ , with  $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ ; therefore,  $\mathcal{S} = \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ . The pure state of  $j$ -th control qubit in the position  $Cpos_1(j)$  among qubits  $A_{1,1} \dots A_{1,n+s}$  is  $Cstate_1(j)$ , with  $Cstate_1(j) \in \mathcal{S}$ . Analogously, the states of control qubits from  $A_{2,1} \dots A_{2,n+s}$  are encoded by  $Cstate_2(j) \in \mathcal{S}$ . Alice gives the information  $\{Cstate_1(j) | j = 1, 2, \dots, s\}$  about the states of control qubits from  $A_{1,1} \dots A_{1,n+s}$  to Bob, and the information  $\{Cstate_2(j) | j = 1, 2, \dots, s\}$  about the states of control qubits from  $A_{2,1} \dots A_{2,n+s}$  to Charlie.

Therefore, prior to Alice sending the message to clients (via sending the qubits  $A_{1,1} \dots A_{1,n+s}$  to Bob and  $A_{2,1} \dots A_{2,n+s}$  to Charlie), Bob has, apart from quNit  $B$ , the information  $\{Cpos_1(j) | j = 1, 2, \dots, s\}$  of the positions and  $\{Cstate_1(j) | j = 1, 2, \dots, s\}$  of the the states of the control qubits from  $A_{1,1} \dots A_{1,n+s}$ , and analogously for Charlie.

During the initialisation phase, Alice, Bob and Charlie share two pairs of entangled quNits, denoted as

$$|\phi(00)\rangle_{A_1 B} \otimes |\phi(00)\rangle_{A_2 C}. \quad (25)$$

Alice and Bob also share classical information of the control qubits:  $\{Cpos_1(j) | j = 1, 2, \dots, s\}$  and  $\{Cstate_1(j) | j = 1, 2, \dots, s\}$ . Alice and Charlie also share classical information  $\{Cpos_2(j) | j = 1, 2, \dots, s\}$  and  $\{Cstate_2(j) | j = 1, 2, \dots, s\}$ .

The protocol of simultaneous dense coding of classical messages  $(b_1, b_2)$  and  $(c_1, c_2)$  works as follows:

(1) *Encoding.* Alice performs unitary operators  $U(b_1b_2)$  on quNit  $A_1$  and  $U(c_1c_2)$  on quNit  $A_2$  to encode  $(b_1, b_2)$  and  $(c_1, c_2)$ , respectively.

(2) *Locking.* Alice joins the  $s$  control qubits with  $n$  message qubits  $A_{1,1} \dots A_{1,n}$ , to form the ordered set of  $(n + s)$  qubits  $A_{1,1} \dots A_{1,n+s}$ . The  $j$ -th control qubit is prepared in  $Cstate_1(j) \in \mathcal{S}$  and is in the position  $Cpos_1(j)$ , while the relative positions of the  $n$  message qubits are the same as before. Analogously, she forms the set  $A_{2,1} \dots A_{2,n+s}$ .

For  $j = 1$  to  $n + s$ , Alice applies the locking operator on  $A_{1,j}A_{2,j}$ .

(3) *Communication.* Alice sends  $(n + s)$  qubits  $A_{1,1} \dots A_{1,n+s}$  to Bob, and  $(n + s)$  qubits  $A_{2,1} \dots A_{2,n+s}$  to Charlie.

(4) *Unlocking.* For  $j = 1$  to  $n + s$ , Bob sends qubit  $A_{1,j}$  to Charlie, and then Charlie returns  $A_{1,j}$  to Bob after performing the unlocking operator on  $A_{1,j}A_{2,j}$  at his site.

If  $\exists k, Cpos_1(k) = j$  (i.e., if the  $j$ -th qubit given to Bob is a controlled one), Bob measures  $A_{1,j}$  in either  $X$  or  $Z$  basis, according to  $Cstate_1(k)$ . If the measurement result does not match  $Cstate_1(k)$ , he knows that Charlie did not return the real  $A_{1,j}$  and stops the unlocking stage.

Analogously, if  $\exists k, Cpos_2(k) = j$ , Charlie measures  $A_{2,j}$  in either  $X$  or  $Z$  basis, according to  $Cstate_2(k)$ . If the measurement result does not match  $Cstate_2(k)$ , he knows that Bob did not send the real  $A_{1,j}$  and stops the unlocking stage.

The remaining  $n$  qubits received at Bob's site form the ordered set  $A_{1,1} \dots A_{1,n}$ , and the remaining  $n$  qubits received at Charlie's site form the ordered set  $A_{2,1} \dots A_{2,n}$ .

(5) *Decoding.* Bob and Charlie measure quNits  $A_1B$  and quNits  $A_2C$  in the Bell basis respectively to achieve  $(b_1, b_2)$  and  $(c_1, c_2)$ .

The unlocking of qubits  $A_{1,1} \dots A_{1,n+s}$  and  $A_{2,1} \dots A_{2,n+s}$  is done in  $(n + s)$  steps, such that in each step one pair of qubits is unlocked. The order of qubits must be maintained. Without the loss of generality, we assume that the unlocking operation is done at Charlie's site. We will assume that Bob is an honest client who sends qubits  $A_{1,1} \dots A_{1,n+s}$  to Charlie, as long as he is convinced that Charlie is returning the exact resulting qubit after the local unlocking operation at Charlie's site. The way to check if Charlie is indeed doing so is the following: in each step  $Cpos_1(j)$  of the unlocking stage (i.e., when the qubit given by Alice to Bob is a controlled one), Bob measures the state of the qubit received back from Charlie during that step in one of the two mutually unbiased bases –  $Z$  if  $Cstate_1(j)$  is in  $\{|0\rangle, |1\rangle\}$ ,  $X$  otherwise. If the measurement result matches the classical information  $Cstate_1(j)$ , i.e. if Bob's outcome is consistent with Charlie returning the control qubit in  $Cstate_1(j)$ , Bob continues with the unlocking stage. Otherwise, it means that Charlie did not return the control qubit in  $Cstate_1(j)$  and Bob stops the unlocking stage. Since Charlie does not know the positions of control qubits from  $A_{1,1} \dots A_{1,n+s}$ , he has to return all of the resulting qubits to Bob. Otherwise, he will inevitably return some qubits as control ones in states different from those prepared by Alice, and Bob will, with high probability, be able to detect it.

Note that the whole analysis is done for the ideal case where no measurement errors or decoherence effects occur. The existence of measurement errors will set the threshold value  $\eta > 0$  for the allowed number of wrong results obtained on control qubits (results inconsistent with Charlie sending the control qubits in  $Cstate_1(j)$ ), which will increase the number  $s$  of the control qubits. After receiving, in the course of exchange,  $k \leq s$  controlled qubits, we require that not more than  $\eta k$  wrong results are obtained. The parameter  $\eta$  is determined by the experimental set-up, which sets the probability of obtaining the wrong result when the right qubit is sent.

The protocol is *optimistic*[38]: if both clients are honest, if they execute the protocol by unlocking the qubits Alice gave them, upon finishing the unlocking stage both parties will have the whole information sent by Alice.

Unfortunately, if we insist on the *perfect* fidelity of data transmission, situations when one client (say Charlie) knows that he obtained the whole information, while the other hasn't, clearly puts Bob in a disadvantaged situation. For example, if the unlocking operation is done at Charlie's site, Charlie can decide not to return the last resulting qubit to Bob, which would leave him without the whole state  $|\phi(b_1b_2)\rangle_{A_1B}$ , and thus  $(b_1, b_2)$ , in case  $Cpos_1(s) < n + s$ . The perfect fidelity was for the same reasons relaxed for quantum contract signing as well, by introducing the factor  $\alpha < 1$  of the required fraction of correct results for the total number  $N$  of qubits sent to a client[37].

Therefore, we will also introduce factor  $\alpha < 1$  and require that a client has correct values of  $\alpha(2n)$  message bits sent to him by Alice.

The above protocol is clearly *probabilistic*. At each step of the unlocking stage, Bob has a finite probability, which increases with the execution of the protocol, to obtain correct values of  $2\alpha n$  bits of the message  $(b_1, b_2)$  that Alice sent him, and analogously for Charlie.

Due to its probabilistic nature, during the execution of the protocol one client is always privileged over the other. By privileged, we mean that one client has higher probability of obtaining the required fraction  $\alpha$  of the message Alice encoded for him, than the other. The difference between the clients' probabilities to obtain the information sent by Alice is due to the random distribution of control qubits. On the other side, since neither of the clients know the distribution of both sets of control qubits, even though one of them might be privileged over the other at a certain step of the unlocking stage, he would not know that: the protocol is *a priori* symmetric with respect to the clients.

Yet, the protocol is *fair*[39]: throughout the execution, one client is only slightly privileged over the other, the difference being smaller with the increase of  $s$  and could be made arbitrarily small for big enough  $s$ . The argument here is exactly the same as the one presented for the fairness of the quantum contract signing protocol[37].

Namely, the role of the control qubits is to signal possible *cheating* of a client. By cheating, we mean not sending the qubits received by Alice. For example, if one of the clients, say Charlie, starts sending qubits each randomly in one of four states  $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ , he will send wrong message qubits  $A_1$  sent by Alice, but also wrong control qubits. Therefore, for each control qubit he will have a finite probability of  $1/2$  of being detected cheating (he did not send the right control qubit). Therefore, since Bob's probability  $p_w$  to detect a wrong qubit (Charlie's cheating) is approaching to one exponentially fast,  $p_w = 1 - (1/2)^m$ , where  $m$  is a number of controlled qubits that are sent as random, it is not difficult to estimate the expected difference between Bob and Charlie, depending on the total number  $n$  of the message qubits to be sent from one client to another and the number  $s$  of the controlled qubits given to each client.

One simply has to estimate the expected value for  $m$ , the expected number  $\langle m \rangle$  of wrongly sent control qubits after which Bob notices the cheating and stops communication. During that time, Bob received certain number  $w$  of random qubits as the message qubits, while Charlie was still receiving the proper ones. Therefore, he is in advantage by having about  $w/2$  more correct values for message bits (even when guessing bit values at random, one has  $1/2$  of the probability to be correct). The value  $w$  is a simple function of  $m$ ,  $w = w(m)$ , and the dependence on  $m$  can be straightforwardly determined by  $n$  and  $s$ . For big enough  $n$ ,  $w(\langle m \rangle) \ll n$  and therefore Charlie's advantage is negligible.

Of course, Charlie can try to send qubits in some other states (completely random, etc.), but as long as he's not sending the right qubits in the right states, there will be a finite probability  $p > 0$  that Bob detects cheating in a single-shot measurement, so that his probability to detect cheating will again exponentially fast approach to one in the number of wrongly sent control qubits, making Charlie's privilege arbitrarily small, for big enough  $n$ . Charlie can try to decrease the value of  $p$ , but can never make it zero. Otherwise, he would be able to perfectly distinguish between the non-orthogonal states and this would violate the security of the BB84 cryptographic protocol[40], for example.

In the case of quantum contract signing[37] the role of message and control qubits was given to the same  $N$  qubits sent by a trusted party (in our case Alice) to clients (in our case Bob and Charlie), while in the case of simultaneous dense coding the two roles are given to separate sets of qubits. This will introduce a slight change in the expressions for the probabilities involved in calculation, but this change is minor, conceptually straightforward to calculate and does not affect the main result of protocol's fairness. Nevertheless, knowing the exact expressions for the probabilities in case of different cheating strategies is of crucial importance, and may technically be quite non-trivial, which clearly presents interesting and challenging topics for future research.

The fairness condition can be even straightened, as was done in[37], by requiring the negligible *probability to cheat*: the probability that one client has  $2\alpha n$  right bits, while the other does not. This quantity may be quite relevant in various possible scenarios, like in the case of signing

contracts for buying and selling the goods on the market (see Reference [37]). While for a fixed value of  $\alpha$  the probability to cheat may be as high as  $1/4$ , if it is unknown to the clients and chosen randomly by Alice from a certain interval  $I_\alpha \subset (1/2, 1)$ , the probability to cheat can be made as small as needed, for big enough  $n$  (see Reference [37]).

The locking and unlocking operators performed on two quNits (i.e., two arrays of qubits) in this section are actually products of two-qubit locking and unlocking operators. The  $N$ -dimensional DCNOT operator and the SWAP operator are of this kind. But the  $N$ -dimensional quantum Fourier transform cannot be done qubit(pair)-by-qubit(pair). Therefore, the quantum Fourier transform cannot be used to implement the SDC protocol proposed in this section.

## 4 Improving the security of SDC

We now consider the security of simultaneous dense coding. We assume that in the initialisation phase, the entangled pairs have been securely distributed among Alice, Bob and Charlie. Because only part of the entangled pairs travel through the quantum channel, an outsider knows nothing about the encoded information. If the two receivers are honest and follow the protocols exactly, they must collaborate to achieve their respective information. However, if one of the receivers, say Bob, is dishonest and has the ability to intercept and resend the qubits going through the quantum channel between Alice and Charlie, he can obtain  $(b_1, b_2)$  without collaborating with Charlie by the following intercept-resend attack: (1) intercept quNit  $A_2$ ; (2) perform the unlocking operation on quNits  $A_1A_2$ ; (3) measure quNits  $A_1B$  to obtain  $(b_1, b_2)$ ; (4) perform the locking operation on quNits  $A_1A_2$ ; (5) send quNit  $A_2$  back to Charlie.

To detect such a cheating behaviour, we insert additional  $2r$  “detect qubits” into the array of  $2n$  message qubits during the communication phase:  $r$  detect qubits would be joined with  $n$  message qubits that carry the message  $(b_1, b_2)$ , to form  $(n+r)$  qubits  $A_{1,1} \dots A_{1,n+r}$ , and given to Bob; other  $r$  detect qubits joined with  $n$  message qubits that carry  $(c_1, c_2)$ , to form  $A_{2,1} \dots A_{2,n+r}$  and given to Charlie.

The position  $Dpos_1(j) \in \{1, 2, \dots, n+r\}$  of each detect qubit  $j = 1, 2, \dots, r$  within the  $(n+r)$  qubits  $A_{1,1} \dots A_{1,n+r}$  is chosen randomly by Alice, and analogously for  $Dpos_2(j)$  of each detect qubit within  $A_{2,1} \dots A_{2,n+r}$ .

The state of each detect qubit is randomly chosen by Alice from set

$$\mathcal{S} = Z \cup X = \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}.$$

The pure state of  $j$ -th detect qubit in the position  $Dpos_1(j)$  among qubits  $A_{1,1} \dots A_{1,n+r}$  is  $Dstate_1(j)$ , with  $Dstate_1(j) \in \mathcal{S}$ . Analogously, the states of detect qubits from  $A_{2,1} \dots A_{2,n+r}$  are encoded by  $Dstate_2(j) \in \mathcal{S}$ .

Only after the transmission of all the message qubits and detect qubits, Alice tells Bob and Charlie the positions and the bases of the detect qubits and requires them to return the results of the measurements performed on the detect qubits. Therefore Alice can check if after the transmission the states of the detect qubits have been altered by a dishonest client.

Because the position of the detect qubits in the array are chosen randomly by Alice, curious Bob does not know which qubits are the message qubits. If his intercept-resend attack involves a detect qubit, the state of the detect qubit may probably be changed, the probability of being detected grows exponentially with the increase of  $r$  and could be made arbitrarily large for big enough  $r$ .

During the initialisation phase, Alice, Bob and Charlie share two pairs of entangled quNits, denoted as

$$|\phi(00)\rangle_{A_1B} \otimes |\phi(00)\rangle_{A_2C}. \quad (26)$$

The protocol of simultaneous dense coding of classical messages  $(b_1, b_2)$  and  $(c_1, c_2)$  works similarly as in the previous case. For completeness, in A we present a detailed description of the combination of the two strategies that assure fair and secure simultaneous dense coding protocol.

We summarise the above protocols in Table 1 in order to demonstrate their qualitative and quantitative differences. The original SDC protocol in Ref. [33] and the  $N$ -dimensional SDC protocol in Sec. 2 are vulnerable to intercept-resend attacks and require the receivers to perform the unlocking operation at the same site, otherwise the fairness problem arises. The improved SDC protocol in Sec. 3 can guarantee fairness in the unlocking phase even if the receivers are separated spatially, by implementing the unlocking operation in series of steps of communication and introducing additional control qubits for cheat detection. On the other hand, the improved SDC protocol in Sec. 4 can guarantee security against the intercept-resend attack in the communication phase, by introducing additional detect qubits in the communication phase. The protocol in Appendix incorporates both fairness and security properties. We can also see from the table that the cost of the protocol increase with these improvements.

Protocol	Properties	Extra cost
protocol in Sec. 2 and Ref. [33]		
protocol in Sec. 3	fair	$s$ control qubits
protocol in Sec. 4	secure	$r$ detect qubits
protocol in Appendix	fair, secure	$s$ control qubits + $r$ detect qubits

Table 1: Comparison between different protocols

## 5 Applications to contract signing

Contract signing [34] is a security task involving two parties, Charlie and Bob, that do not trust each other and want to exchange a common contract signed with each other’s signature. At the end of the protocol Charlie should have the contract signed by Bob and vice-versa. The purpose of the signed contract is to bind the parties to the terms of the contract, which can be enforced by a judge (Alice). The real challenge to this problem is when both Charlie and Bob are physically apart and want to remotely sign the contract. This situation is becoming more and more common due to e-business, and may lead to fraud. For instance, if Bob gets the contract signed by Charlie without committing himself, Charlie and Bob are in an unfair situation. By having Charlie’s commitment, Bob is able to appeal to Alice to enforce the contract, while Charlie has no means to do the same, since he does not possess the contract signed by Bob. Note that even if Bob did not commit, but having Charlie’s commitment, puts him in a position to later in time choose whether to bind the contract or not, while Charlie has no power to do either of the two.

A simple solution to this unfair situation is to have a trusted third party (again Alice) mediating the transaction – Bob sends to Alice the contract signed by him and Charlie does the same; then Alice exchanges the contracts only after she has received both of the commitments. Note that this procedure increases significantly the cost of remote contract signing, as Alice’s time and resources are expensive. What is particularly costly is Alice being constantly online and alert waiting for the clients to contact her. Also, avoiding the communication with the trusted party at the very moment of determining a contract and committing to it removes the danger of overloading Alice and creating a bottleneck.

Unfortunately, it has been shown that the attendance of Alice is mandatory [34, 36], if the protocol is to fulfil the following two important properties:

- *fairness*: either both parties get each other’s commitment or none gets;
- *viability*: if both parties behave honestly, they will both get each other’s commitments.

One way to overcome this difficulty is to consider *optimistic protocols* that do not require communication with Alice unless something wrong comes up [38]. Another workaround is to relax the fairness condition, allowing one agent to have  $\epsilon$  more probability of binding the contract than the other agent (*probabilistic fairness*). In this case, for an arbitrary small  $\epsilon$  solutions have been found where the number of exchanged messages between the agents is minimised [39].

In this section we present a probabilistically fair quantum protocol based on a secure SDC for remote agents, such as the one described in A (a combination of protocols presented in Section 3.2 and Section 4). In such protocol, Alice, the trusted party, does not interact with the signing parties while they (Bob and Charlie) are determining the contract and then committing to it (the Exchange Phase). Note that in the below construction of a contract signing scheme, a SDC protocol is used iteratively many times as a *black box*. Thus, *any* protocol that achieves simultaneous transmission of (densly) coded information that satisfies the fairness and the security conditions, as presented in Section 3 and Section 4, respectively, allows for constructing probabilistically fair contract signing scheme.

In order to lower the use of resources, it is possible to map long contracts into messages of a small fixed size, say of  $k$  bits. Such short messages (digests) are obtained by so called hash functions (such as SHA1) and are well established in the field of cryptography [41]. Hash functions are not injective, that is, there exist pairs of different messages  $x, x'$  with the same digest  $d$ . Nevertheless, given a message  $x$  and its digest  $d$ , it is computationally hard to find a different message  $x' \neq x$  with the same digest  $d$ . For this reason, digests can be used to identify a message. From this point on, instead of contracts themselves we consider their digests, obtained by some hash function, with  $k$  bits, say  $(b_1 \dots b_k)$ , where  $b_i \in \{0, 1\}$ .

We also assume that Alice can sign messages using some public key signing scheme (such as DSS, for more detail see [41]). In short, a public key signature scheme for Alice is a pair of functions  $(sig, ver)$  together with a pair of keys  $(A, \hat{A})$  where  $A$  is the private key of Alice (known only by Alice) and  $\hat{A}$  is the corresponding public key (known by Alice, Bob and Charlie). For Alice to sign a message  $m$ , she uses her private key  $A$  and obtains the signature  $sig_A(m)$ . Bob (or Charlie) verifies if  $sig_A(m)$  is the message  $m$  signed by Alice by checking whether  $ver_{\hat{A}}(m, sig_A(m)) = 1$ . If  $ver_{\hat{A}}(m, sig_A(m)) \neq 1$  then  $sig_A(m)$  does not correspond to the signature of Alice over  $m$ . Signing can be seen as an encryption, unique to Alice: only she can do it using her private key  $A$ . But Bob and Charlie can, given the message  $m$  and a signature  $s$ , by using Alice's public key  $\hat{A}$  verify whether  $s$  is indeed the signature of  $m$  by Alice or not.

In our contract signing protocol, Alice does not know a priori to which particular contract/digest  $(b_1 \dots b_k)$  Bob and Charlie are going to agree upon. However, at the end of the protocol Alice (and also Bob and Charlie) needs some irrefutable proof of the particular contract that was agreed upon. Moreover, we do not want Alice to be contacted during the exchange phase. A solution for this problem is for Alice to produce  $4k$  triples  $\{(b, i, \text{Bob}), (b, i, \text{Charlie}) : b \in \{0, 1\}, i = 1 \dots k\}$  that can be used to represent any particular contract  $(b_1 \dots b_k)$  that Bob and Charlie will agree upon latter. Next, Alice signs those  $4k$  triples and prepares  $2k$  SDC protocols for Bob and Charlie, where in each of these SDC protocols the messages to be simultaneously received are  $sig_A(b, i, \text{Charlie})$  by Bob and  $sig_A(b, i, \text{Bob})$  by Charlie, with  $b \in \{0, 1\}$  and  $i \in \{1 \dots k\}$ .

Thus, Bob can enforce the contract  $(b_1 \dots b_k)$  against Charlie (and no other person) if he shows to Alice the signatures  $sig_A(b_1, 1, \text{Charlie}) \dots sig_A(b_k, k, \text{Charlie})$ . By using SDC, it is possible to force Bob to obtain this information (and no other) only with the collaboration of Charlie, and vice-versa. In detail the contract signing protocol based on SDC works as follows:

*Initialization Phase:*

1. Alice signs the following  $4k$  messages:  $(0, i, \text{Bob}), (1, i, \text{Bob}), (0, i, \text{Charlie}), (1, i, \text{Charlie})$  with  $i = 1, 2, \dots, k$ .
2. Alice arranges for  $2k$  different SDC's for the case of distant parties such that the message to be sent to Bob in one of such SDC's is  $sig_A(b, i, \text{Charlie})$  and to Charlie is  $sig_A(b, i, \text{Bob})$  for  $i = 1, 2, \dots, k$  and  $b = 0, 1$ .

*Exchange Phase:*

1. Bob and Charlie agree on contract  $(b_1 \dots b_k) \in \{0, 1\}^k$ .
2. For  $i = 1$  to  $k$ , Bob and Charlie collaborate to obtain from the entangled quNits of the  $2k$  SDC's the messages  $sig_A(b_i, i, \text{Charlie})$  and  $sig_A(b_i, i, \text{Bob})$ , respectively, and ignore the remaining

quantum data sent by Alice. Thus, at the end of the SDC's Bob has  $sig_A(b_i, i, \text{Charlie})$ , for  $i = 1 \dots k$ , and mutatis mutandis for Charlie.

*Binding Phase:*

1. Alice enforces contract  $(b_1 \dots b_k)$  when either Bob presents

$$sig_A(b_1, 1, \text{Charlie}) \dots sig_A(b_k, k, \text{Charlie})$$

or Charlie presents  $sig_A(b_1, 1, \text{Bob}) \dots sig_A(b_k, k, \text{Bob})$ .

The size of each signature, say  $sig_A(b_i, i, \text{Charlie})$ , is given by the length of a message a client receives in each SDC, which is  $2n$  bits (the factor 2 comes from the fact that the coding is dense). Since each SDC is probabilistic, a client only needs to present to Alice  $\alpha_i(2n)$  bits, with  $\frac{1}{2} < \alpha_i < 1$ , of the signature  $sig_A(b_i, i, \text{Charlie})$  for each  $i = 1 \dots k$ . Moreover, if each  $\alpha_i$  is random we achieve even stronger fairness condition, namely the *expected* probability to cheat on each SDC can be made arbitrarily small (the probability to cheat is the probability that an agent obtains at least  $\alpha_i(2n)$  bits of a signature and the other does not [37]).

The contract signing protocol described above, unlike the one presented in [37], allows for Bob and Charlie to determine the contract after the Initialisation Phase. This is due to the fact that public key signatures were introduced. However, this introduction leads to a poorer security assumption, as public key signatures are only computationally secure (as well as hash functions).

We can improve the security of the above protocol by removing both hash functions and public key signatures. Removing hash function accounts to consider a large enough  $k$  such that all potential contracts by Bob and Charlie would fit in  $k$  bits. To remove public key signatures, which are not perfectly secure, Alice can, during the initialisation phase, share a symmetric key  $k_{AB}$  with Bob and another,  $k_{AC}$ , with Charlie. These symmetric keys might be made perfectly secure by using one-time pad cryptosystems (see, for example [41]). Then, in each SDC the message that Bob receives is  $k_{AB}(b_i, i, r_i(b_i))$  and the message that Charlie receives is  $k_{AC}(b_i, i, r_i(b_i))$ . Here,  $k_{AB}(m)$  is the encryption of  $m$  with the symmetric key  $k_{AB}$  and  $r_i(b_i)$  is a random string for each  $i$  and  $b_i$ , sampled and known only by Alice, that is associated to a contract between Bob and Charlie. So, for Bob to enforce contract  $(b_1 \dots b_k)$  against Charlie, he has to present to Alice the random numbers  $r_1(b_1) \dots r_k(b_k)$ . Note that in this case Alice has to store all these random numbers  $r_i(b_i)$  in her private memory keeping in mind to whom they are associated. In this way the protocol's perfect security is obtained by the laws of physics, which is stronger than computational security used in classical protocols.

## 6 $N$ -dimensional simultaneous teleportation

Dense coding and quantum teleportation are dual protocols intimately linked to each other, both in their purpose as well as in the construction. The former is used to transmit (densely coded) classical information, while the latter transfers quantum states (i.e., quantum information). In Introduction, as a motivation for SDC the following situation was described: an agent, say a boss of a company, needs for two employees to simultaneously carry out two different confidential tasks, such that each employee is unaware of the other's activity. SDC allows that the two tasks, encoded by classical information, are confidentially communicated to the employees. Similarly, one could imagine a situation in which the employees are requested to each execute a predetermined quantum protocol (say, a quantum computation algorithm) for a given confidential initial state. The solution for such situation is achieved by a simultaneous teleportation protocol.

Simultaneous teleportation was proposed by Wang *et al* [32], in which all the receivers simultaneously obtain their respective quantum states from the sender. In their scheme, the sender first performs a locking operation to entangle the particles from two independent quantum entanglement channels, and therefore the receivers cannot restore their quantum states separately before

performing the unlocking operation together. The locking operator is composed of the Hadamard and the CNOT operators.

Ref. [33] showed that the quantum Fourier transform can alternatively be used as the locking operator in simultaneous teleportation. In this section, we further investigate simultaneous teleportation of quNits using the  $N$ -dimensional quantum Fourier transform.

In the task of  $N$ -dimensional simultaneous teleportation, Alice intends to teleport the unknown quNits  $|\varphi_t\rangle_{T_t} = \sum_{s=0}^{N-1} \alpha_{t,s} |s\rangle_{T_t}$  to Bob $_t$  ( $1 \leq t \leq M$ ) under the condition that all the receivers must collaborate to simultaneously obtain  $|\varphi_t\rangle_{T_t}$ .

In the initialisation phase, Alice shares with each Bob $_t$  a pair of entangled quNits  $|\phi(00)\rangle_{A_t B_t}$ , where subscript  $A_t$  denotes Alice's quNit, and subscript  $B_t$  denotes Bob $_t$ 's quNit. The initial quantum state of the composite system is

$$\begin{aligned} |\chi(0)\rangle &= \bigotimes_{t=1}^M |\phi(00)\rangle_{A_t B_t} \bigotimes_{t=1}^M |\varphi_t\rangle_{T_t} \\ &= \frac{1}{\sqrt{N^M}} \sum_{j=0}^{N^M-1} |j\rangle_{A_1 \dots A_M} |j\rangle_{B_1 \dots B_M} \bigotimes_{t=1}^M |\varphi_t\rangle_{T_t}, \end{aligned} \quad (27)$$

where  $j$  is a base- $N$ -number which can be written as  $j_1 j_2 \dots j_M$ ,  $j_t \in \{0, 1, \dots, N-1\}$ .

The protocol for simultaneous teleportation of quNits consists of five steps:

(1) *Locking*. Alice performs the  $N$ -dimensional quantum Fourier transform

$$|j\rangle_{A_1 \dots A_M} \longrightarrow \frac{1}{\sqrt{N^M}} \sum_{k=0}^{N^M-1} e^{\frac{2\pi i}{N^M} jk} |k\rangle_{A_1 \dots A_M} \quad (28)$$

on quNits  $A_1 \dots A_M$  to lock the entanglement channels. After that, the state of the composite system becomes

$$\begin{aligned} |\chi(1)\rangle &= \frac{1}{\sqrt{N^M}} \sum_{j=0}^{N^M-1} QFT_{A_1 \dots A_M} |j\rangle_{A_1 \dots A_M} |j\rangle_{B_1 \dots B_M} \bigotimes_{t=1}^M |\varphi_t\rangle_{T_t} \\ &= \frac{1}{\sqrt{N^M}} \sum_{j=0}^{N^M-1} \frac{1}{\sqrt{N^M}} \sum_{k=0}^{N^M-1} e^{\frac{2\pi i}{N^M} jk} |k\rangle_{A_1 \dots A_M} |j\rangle_{B_1 \dots B_M} \bigotimes_{t=1}^M |\varphi_t\rangle_{T_t}. \end{aligned} \quad (29)$$

(2) *Measuring.* Alice measures each pair of quNits  $A_t T_t$  in the  $N$ -dimensional Bell basis.

$$\begin{aligned}
& \bigotimes_{t=1}^M A_t T_t \langle \phi(x_t y_t) | \chi(1) \rangle \\
&= \frac{1}{N^M} \left( \bigotimes_{t=1}^M \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N} j x_t} A_t \langle j + y_t | T_t \langle j | \right) \\
& \quad \left[ \sum_{k=0}^{N^M-1} \left( \bigotimes_{t=1}^M |k_t\rangle_{A_t} |\varphi_t\rangle_{T_t} \right) \frac{1}{\sqrt{N^M}} \sum_{j=0}^{N^M-1} e^{\frac{2\pi i}{N^M} j k} |j\rangle_{B_1 \dots B_M} \right] \\
&= \frac{1}{N^M} \sum_{k=0}^{N^M-1} \left( \prod_{t=1}^M \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N} j x_t} \langle j + y_t | k_t \rangle \langle j | \sum_{s=0}^{N-1} \alpha_{t,s} |s\rangle \right) QFT_{B_1 \dots B_M} |k\rangle_{B_1 \dots B_M} \\
&= \frac{1}{N^M} QFT_{B_1 \dots B_M} \sum_{k=0}^{N^M-1} \bigotimes_{t=1}^M e^{-\frac{2\pi i}{N} x_t (k_t - y_t)} \alpha_{t, k_t - y_t} |k_t\rangle_{B_t} \\
&= \frac{1}{N^M} QFT_{B_1 \dots B_M} \sum_{k=0}^{N^M-1} \bigotimes_{t=1}^M e^{-\frac{2\pi i}{N} x_t k_t} \alpha_{t, k_t} |k_t + y_t\rangle_{B_t} \\
&= \frac{1}{N^M} QFT_{B_1 \dots B_M} \sum_{k=0}^{N^M-1} \bigotimes_{t=1}^M \alpha_{t, k_t} X^{y_t} (Z^\dagger)^{x_t} |k_t\rangle_{B_t} \\
&= \frac{1}{N^M} QFT_{B_1 \dots B_M} \bigotimes_{t=1}^M X^{y_t} (Z^\dagger)^{x_t} \sum_{s=0}^{N-1} \alpha_{t,s} |s\rangle_{B_t}, \tag{30}
\end{aligned}$$

where  $X : |j\rangle \rightarrow |j+1\rangle$  is the shift operator,  $Z : |j\rangle \rightarrow e^{\frac{2\pi i}{N} j} |j\rangle$  is the rotation operator.

If the measurement result of quNits  $A_t T_t$  is  $|\phi(x_t y_t)\rangle$ , the state of quNits  $B_1 \dots B_M$  collapses into

$$|\chi(2)\rangle = QFT_{B_1 \dots B_M} \bigotimes_{t=1}^M X^{y_t} (Z^\dagger)^{x_t} |\varphi_t\rangle_{B_t}. \tag{31}$$

(3) *Communication.* Alice sends the measurement result  $(x_t, y_t)$  to each Bob $_t$ .

(4) *Unlocking.* All the receivers collaborate to perform the inverse quantum Fourier transform on quNits  $B_1 \dots B_M$ , and the state of quNits  $B_1 \dots B_M$  becomes

$$|\chi(3)\rangle = QFT_{B_1 \dots B_M}^\dagger |\chi(2)\rangle = \bigotimes_{t=1}^M X^{y_t} (Z^\dagger)^{x_t} |\varphi_t\rangle_{B_t}. \tag{32}$$

(5) *Recovering.* Each Bob $_t$  performs  $Z^{x_t} (X^\dagger)^{y_t}$  on quNit  $B_t$  to obtain  $|\varphi_t\rangle$ .

## 7 Conclusions

Dense coding [2] and teleportation [35] are important quantum communication tasks. Simultaneous dense coding [33] and simultaneous teleportation [32], which guarantee that the receivers simultaneously achieve their respective information from one sender, may be relevant and useful for improvement of some models or tasks of quantum communication. In this paper, we have given a number of new results on simultaneous dense coding and teleportation. More specifically, we have given three protocols for simultaneous dense coding utilising different  $N$ -dimensional quantum states (i.e., Bell state, W state, and GHZ state). Besides the quantum Fourier transform,

we have introduced two new locking operators (i.e. the double controlled-NOT operator and the SWAP operator) for simultaneous dense coding. Then we have analysed the fairness and the security of the simultaneous dense coding protocol and proposed a protocol which guarantees both the fairness and the security, thus allowing for mutually distant receivers to execute the protocol. We have shown that any fair simultaneous dense coding scheme that is secure against intercept-resend attack can be used to implement a fair contract signing protocol. In addition, we have shown that the  $N$ -dimensional quantum Fourier transform can act as the locking operator in simultaneous teleportation of quNits.

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## A Combination of the two strategies

In this appendix, we combine in one protocol the two strategies described in the above two sections: (i) if Bob and Charlie have the ability to intercept and resend the qubits going through the quantum channel, Alice can detect such behaviour and interrupt the protocol; (ii) if Bob and Charlie are spatially separated, they can fairly decode their respective messages simultaneously through communication.

In the initialisation phase, Alice, Bob and Charlie share two pair of entangled quNits, denoted as

$$|\phi(00)\rangle_{A_1B} \otimes |\phi(00)\rangle_{A_2C}. \quad (33)$$

Alice and Bob also share classical information of the control qubits:  $\{Cpos_1(j) \mid j = 1, 2, \dots, s\}$  and  $\{Cstate_1(j) \mid j = 1, 2, \dots, s\}$ . Alice and Charlie also share classical information  $\{Cpos_2(j) \mid j = 1, 2, \dots, s\}$  and  $\{Cstae_2(j) \mid j = 1, 2, \dots, s\}$ .

The protocol of simultaneous dense coding of classical messages  $(b_1, b_2)$  and  $(c_1, c_2)$  works as follows:

(1) *Encoding*. Alice performs unitary operators  $U(b_1b_2)$  on quNit  $A_1$  and  $U(c_1c_2)$  on quNit  $A_2$  to encode  $(b_1, b_2)$  and  $(c_1, c_2)$ , respectively.

(2) *Locking*. Alice joins the  $s$  control qubits with  $n$  message qubits  $A_{1,1} \dots A_{1,n}$ , to form the ordered set of  $(n + s)$  qubits  $A_{1,1} \dots A_{1,n+s}$ . The  $j$ -th control qubit is prepared in  $Cstate_1(j) \in \mathcal{S}$

and is in the position  $Cpos_1(j)$ , while the relative positions of the  $n$  message qubits are the same as before. Analogously, she forms the set  $A_{2,1} \dots A_{2,n+s}$ .

For  $j = 1$  to  $n + s$ , Alice applies the locking operator on qubits  $A_{1,j}A_{2,j}$ .

(3) *Communication.* Alice joins the  $r$  detect qubits with  $(n + s)$  message and control qubits  $A_{1,1} \dots A_{1,n+s}$ , to form the ordered set of  $(n+s+r)$  qubits  $A_{1,1} \dots A_{1,n+s+r}$ . The  $j$ -th detect qubit is prepared in  $Dstate_1(j) \in \mathcal{S}$  and is in the position  $Dpos_1(j)$ , while the relative positions of the remaining  $(n + s)$  qubits are the same as before. Analogously, she forms the set  $A_{2,1} \dots A_{2,n+s+r}$ .

Alice sends  $(n + s + r)$  qubits  $A_{1,1} \dots A_{1,n+s+r}$  to Bob, and  $(n + s + r)$  qubits  $A_{2,1} \dots A_{2,n+s+r}$  to Charlie.

Alice waits for Bob and Charlie's acknowledgements of receiving all the  $2(n+s+r)$  qubits. After they have sent their acknowledgements through unjammable classical communication channel, Alice sends  $\{Dpos_1(j) \mid j = 1, 2, \dots, r\}$  and the bases of  $\{Dstate_1(j) \mid j = 1, 2, \dots, r\}$  to Bob. Analogously, Alice sends  $\{Dpos_2(j) \mid j = 1, 2, \dots, r\}$  and the bases of  $\{Dstate_2(j) \mid j = 1, 2, \dots, r\}$  to Charlie.

For each  $j = Dpos_1(k)$ , Bob measures qubit  $A_{1,j}$  in either  $X$  or  $Z$  basis, according to  $Dstate_1(k)$ , and returns the measurement result to Alice. If the measurement result is not equal to  $Dstate_1(k)$ , Alice announces that a cheating behaviour has been detected and stops the protocol.

Analogously, for each  $j = Dpos_2(k)$ , Charlie measures qubit  $A_{2,j}$  in either  $X$  or  $Z$  basis, according to  $Dstate_2(k)$ , and returns the measurement result to Alice. If the measurement result is not equal to  $Dstate_2(k)$ , Alice announces that a cheating behaviour has been detected and stops the protocol.

Now the remaining  $(n + s)$  received qubits at Bob's site form the ordered set  $A_{1,1} \dots A_{1,n+s}$ , and the remaining  $(n + s)$  received qubits at Charlie's site form the ordered set  $A_{2,1} \dots A_{2,n+s}$ .

(4) *Unlocking.* For  $j = 1$  to  $n + s$ , Bob sends qubit  $A_{1,j}$  to Charlie, and then Charlie returns  $A_{1,j}$  to Bob after performing the unlocking operator on qubits  $A_{1,j}A_{2,j}$  at his site.

If  $\exists k, Cpos_1(k) = j$ , Bob measures qubit  $A_{1,j}$  in either  $X$  or  $Z$  basis, according to  $Cstate_1(k)$ . If the measurement result does not match  $Cstate_1(k)$ , he knows that Charlie did not return the real  $A_{1,j}$  and stops the unlocking stage.

Analogously, if  $\exists k, Cpos_2(k) = j$ , Charlie measures qubit  $A_{2,j}$  in either  $X$  or  $Z$  basis, according to  $Cstate_2(k)$ . If the measurement result does not match  $Cstate_2(k)$ , he knows that Bob did not send the real  $A_{1,j}$  and stops the unlocking stage.

Now the remaining  $n$  received qubits at Bob's site form the ordered set  $A_{1,1} \dots A_{1,n}$ , and the remaining  $n$  received qubits at Charlie's site form the ordered set  $A_{2,1} \dots A_{2,n}$ .

(5) *Decoding.* Bob and Charlie measure qubits  $A_1B$  and qubits  $A_2C$  in the Bell basis respectively to achieve  $(b_1, b_2)$  and  $(c_1, c_2)$ .