Paracategories II: Adjunctions, fibrations and examples from probabilistic automata theory

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Abstract

In this sequel to [HM02], we explore some of the global aspects of the category of paracategories. We establish its (co)completeness and cartesian closure. From the closed structure we derive the relevant notion of *transformation* for paracategories. We set-up the relevant notion of *adjunction* between paracategories and apply it to define (co)completeness and cartesian closure, exemplified by the paracategory of bivariant functors and dinatural transformations. We introduce *partial multicategories* to account for partial tensor products. We also consider fibrations for paracategories and their indexed-paracategory version. Finally, we instantiate all these concepts in the context of probabilistic automata.

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1 Introduction

Paracategories were originally proposed by Freyd [Fre96]. A paracategory can be understood as a category in which the composition of morphisms is a *partial* operation. Freyd produced an *enveloping category* construction, which allows us to understand paracategories as categories with a distinguished collection of morphisms. A motivating example is that of dinatural transformations [Mac98, §IX.4], whose pointwise composition is not necessarily dinatural.

In [HM02] we reexamined Freyd's notion (which is presented as a Horn theory) and produced a different (but equivalent) axiomatisation. First, using the monoid classifier Δ, we recasted paramonoids as *strong monoidal saturated lax functors* $\mathsf{M} : \Delta \to \mathsf{Ptl}_{\mathcal{M}}(\mathbb{B})$ (with non-empty support) into a bicategory of partial maps. This notion gave us a suitable internal version of paramonoid/paracategory. With further assumptions on the ambient category $\mathbb{B}$, we reelaborated paramonoids as *saturated partial algebras* $x : \mathsf{M}x \rightharpoonup x$ for the free-monoid monad M. We also established the basic result of paramonoids at the level of partial algebras, namely, the existence of the enveloping algebra construction and the fact that saturated partial algebras can be recovered from their enveloping ones.

In this paper we continue our study of paracategories by exploring the global structure of $\mathcal{P}ar\mathcal{C}at$, the category of paracategories and functors. For this, we rely on the reflectiviy of $\mathcal{P}ar\mathcal{C}at$ in $\mathcal{C}at_P$, the category of categories-with-distinguished-subcollection-of-morphisms. This latter is seen to be (co)complete and cartesian closed (as it is fibred over $\mathcal{C}at$). An important property of the enveloping category construction is that it preserves products (Proposition 3.2), so that the reflection from $\mathcal{C}at_P$ into $\mathcal{P}ar\mathcal{C}at$ implies that the latter is cartesian closed as well. This closed structure yields a natural notion of *transformation* for paracategories.

Although $\mathcal{P}ar\mathcal{C}at$ with this 2-dimensional structure does not form a 2-category (its Hom's are only paracategories), we can nevertheless use it to obtain a meaningful notion of *adjunction*. The traditional definition of adjunction between categories [Mac98, §IV.1] is that of a natural bijection

$$\theta_{X,Y} \,:\, \mathbb{C}(FX, Y) \cong \mathbb{B}(X, GY)$$

for functors $F : \mathbb{B} \to \mathbb{C}$ and $G : \mathbb{C} \to \mathbb{B}$. After [Law66], where comma-categories were introduced, we can equally understand the above natural bijection as an isomorphism of comma-catgories $\theta \,:\, F \downarrow \mathbb{C} \cong \mathbb{B} \downarrow G$ compatible with the projections to $\mathbb{B}$ and $\mathbb{C}$. This approach is applicable in our present context: we can sensibly define *comma-paracategories* and demand an isomorphism as above. This definition yields the usual unit/counit data and the triangular identities, but it is stronger in that it guarantees that enough composites are defined so as to perform adjoint-transposition.

Similarly to the comma-paracategory construction, we also have at our disposal *cotensoring* of paracategories by categories (inherited via the reflection from $\mathcal{C}at_P$). Thus we can define the notion of *limit/colimit* for paracategories. In particular, we can make sense of *products* in a paracategory. Assuming these, we can further demand the relevant adjoints so as to have *cartesian closed* paracategories (§5.2). In fact, with these definitions we can accommodate Freyd's intended example of the cartesian closed paracategory of bivariant functors $T : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$ (with $\mathbb{D}$ a cartesian closed category) and dinatural transformations between these (Example 5.6).

In [Mat00], the combination of probabilistic automata involves a certain 'product without diagonals'. Following [Her00], we introduce *partial multicategories* (which is an instance of the abstract notion of paramonoid or saturated partial algebra from [HM02]) and define the appropriately restricted notion of *representability*. This framework allows us to exhibit the above 'product without diagonals' as the *tensor product* corresponding to the representability of a suitable partial multicategory.

We then examine the relevant notion of *fibration* of paracategories. We show that the traditional Grothendieck correspondence between fibrations and contravariant functors into *Cat* applies in this context (Proposition 7.6), once again appealing to the enveloping category construction.

We finally consider, as an extended example of the notions so far introduced, their application in the context of *probabilistic automata*. The study of this 'category with partial composites' of probabilistic automata was the motivation behind Mateus' *et al.* ([MSS99, Mat00]) preliminary study of the notion of *precategory*. A precategory is defined similarly to a paracategory but only with a *binary* partial composition. Assuming the relevant associativity, such binary partial composition gives rise to a paracategory (see [HM02, Example 2.2.(4)]), which is a more general (and natural) notion to explore. Thus we have concentrated here on the latter, incorporating the relevant results of *ibid.*, and presenting further developments.

Although all the concepts developed for paracategories here make sense at the internal level (as in [HM02]), at present our examples correspond to the ordinary, *Set*-based scenario. Therefore we have chosen to present matters concretely.

2 Freyd's paracategories

We recall from [Fre96] the elementary definitions of paramonoid and paracategory and their corresponding morphisms.

- A **paramonoid** consists of a set M and n-ary partial operations $\otimes_n : M^n \rightharpoonup M$, which we write indistinctly as $[\_]$ for any arity. These operations are subject to the following axioms:
    1. $\otimes_0 : M^0 (= 1) \rightharpoonup M$ is total
    2. $\otimes_1 = (id, id) : M \rightharpoonup M$
    3. If $[\vec{y}]$ is defined then $[\vec{x}[\vec{y}]\vec{z}] = [\vec{x}\vec{y}\vec{z}]$

  where the equality in the last axiom above is Kleene equality (if either side is defined so is the other and then they are equal).

- A **paracategory** $\mathbb{C}$ consists of a directed graph $C_0 \overset{d}{\leftarrow} C_1 \overset{c}{\rightarrow} C_0$ and partial n-ary operations $\circ_n : C_n \rightharpoonup C_1$, where

  $$C_n = C_1 \times_{C_0} \cdots \times_{C_0} C_1 = \{(f_1, \ldots, f_n) | cf_i = df_{i+1}, 1 \leq i \leq n-1\}$$

  is the set of composable n-tuples of morphisms. They are subject to the following axioms

  1. $\circ_0 = \iota : C_0 \rightharpoonup C_1$ is total. This yields **identity morphisms** $id_A : A \to A$ for every object $A \in C_0$.
  2. $\circ_1 = (id, id) : C_1 \rightharpoonup C_1$
  3. If $\circ_n \vec{y}$ is defined, then

     $$\circ_{m+1+n}(\vec{x}, \circ_k(\vec{y}), \vec{z}) = \circ_{m+k+n}(\vec{x}, \vec{y}, \vec{z})$$

     where $\vec{x} \in C_m$, $\vec{y} \in C_k$ and $\vec{z} \in C_n$.

- A **functor** between paracategories $\mathbb{C}$ and $\mathbb{D}$ is a morphism of graphs

  $$(f_0 : C_0 \to D_0, f_1 : C_1 \to D_1)$$

  such that if $[\vec{x}]$ is defined, then $f_1[\vec{x}] = [\vec{f_1 x}]$ (notice that this entails preservation of identities). The functor is called a **Kleene functor** if $[\vec{f_1 x}] = f_1 y$ implies $[\vec{x}] = y$.
- A **subparacategory** of a paracategory $\mathbb{C}$ is a subgraph such that the inclusion is a Kleene functor.

A category $\mathbb{C}$ and a subset of morphisms $P \subseteq C_1$ (including the identity morphisms) determines a subparacategory, to wit, that where $[\vec{x}]$ is defined if the composite of the tuple $\vec{x}$ in $\mathbb{C}$ belongs to P. Similarly a functor between categories $F : \mathbb{C} \to \mathbb{D}$ with distinguished subsets of morphisms $P \subseteq C_1$ and $Q \subseteq D_1$ such that $f_1(P) \subseteq Q$ determines a functor between the induced paracategories (see [HM02] for a more general version of this construction in the setting of partial algebras for a monad). Let $\mathcal{P}ar\mathcal{C}at$ denote the category of paracategories and functors and $\mathcal{C}at_P$ the category of categories with distinguished subsets of morphisms (containing the identities) and functors compatible with such subsets. The construction above yields a functor $U : \mathcal{C}at_P \to \mathcal{P}ar\mathcal{C}at$.

2.1. Proposition (Enveloping Category [Fre96]).
The functor $U : \mathcal{C}at_P \to \mathcal{P}ar\mathcal{C}at$ admits a (fully faithful) left adjoint EC.

Proof. Given a paracategory $\mathbb{C}$ define a category $\mathsf{EC}(\mathbb{C})$ as follows: let $\mathsf{M}(\mathbb{C})$ be the free category on the underlying graph of $\mathbb{C}$ and let $\mathsf{EC}(\mathbb{C})$ be the quotient of $\mathsf{M}(\mathbb{C})$ by the relation

$$\langle x_1, \ldots, x_n \rangle \sim \langle [x_1, \ldots, x_n] \rangle$$

whenever $[x_1, \ldots, x_n]$ is defined. This construction extends in an obvious manner to functors of paracategories to yield the desired left adjoint.

□

One important consequence of the above enveloping category construction is that it explains precisely how every paracategory arises, namely by specifying a collection of morphisms in a given category (the left adjoint EC being fully faithful is equivalent to the unit $\eta : \mathbb{C} \to U\mathsf{EC}(\mathbb{C})$ being an isomorphism [Mac98, §IV.3,Th.1]).

3 The global structure of the category of paracategories

Given the enveloping construction of Proposition 2.1, we can deduce properties of $\mathcal{P}ar\mathcal{C}at$ from the corresponding ones in $\mathcal{C}at_P$. So we analise this latter category first.

First of all, let us be more precise about $\mathcal{C}at_P$. Its objects are small categories, with graph $C_0 \overset{d}{\leftarrow} C_1 \overset{c}{\rightarrow} C_0$ and a subset $m : P \hookrightarrow C_1$ such that the identities $\iota : C_0 \to C_1$ belong to it:

$$\begin{tikzcd} C_0 \arrow[r, dashed] \arrow[rr, bend left, "\iota"] & P \arrow[r, hook, "m"'] & C_1 \end{tikzcd}$$

Notice that this includes the empty category with empty distinguished subobject of morphisms. There is an evident forgetful functor $\underline{U} : \mathcal{C}at_P \to \mathcal{C}at$ (throw away P), which is quite evidently a fibration. We refer to [Jac99, Bor94] for fibred categorical matters.

This fibration inherits a good many properties from the related fibration (considering only the object of morphisms of the categories) $\underline{U} : (\bullet/\mathcal{S}et)_P \to \bullet/\mathcal{S}et$, where $\bullet/\mathcal{S}et$ is the category of pointed sets and point-preserving morphisms. The fibres of this fibration are the collections of non-empty subsets of non-empty sets, and hence complete lattices and Heyting algebras. Since $\bullet/\mathcal{S}et$ is (co)complete and cartesian closed, it follows from [Her99, Cor.4.9,Cor.4.10,Cor.4.12] that $(\bullet/\mathcal{S}et)_P$ is (co)complete and cartesian closed as well. Thus, we get:

3.1. Proposition. *The category $\mathcal{C}at_P$ is (co)complete and cartesian closed. The functor $\underline{U} : \mathcal{C}at_P \to \mathcal{C}at$ preserves this structure.*

Proof. We apply [Her99, Cor.4.9,Cor.4.10,Cor.4.12] as per the pointed-sets situation:

- The fibres are (co)complete and cartesian closed and the reindexing preserves such structure.
- The base $\mathcal{C}at$ is (co)complete and cartesian closed.

□

Let us spell out the cartesian closed structure of $\mathcal{C}at_P$: given categories $\mathbb{C}$ (with a subset of morphisms C) and $\mathbb{D}$ (with a subset of morphisms D) we have:

product $\mathbb{C} \times \mathbb{D}$ with subset of morphisms $C \times D$

internal-hom $[\mathbb{C}, \mathbb{D}]$ with subset of morphisms (natural transformations)

$$C \Rightarrow D = \{\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D} \mid \forall (f : x \to y) \in C \implies (\alpha_f : Fx \to Gy) \in D\}$$

where α_f is the diagonal of the naturality square:

$$\begin{tikzcd} Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] \arrow[dr, "\alpha_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\alpha_y"'] & Gy \end{tikzcd}$$

Notice that, since we assume that the subsets of morphisms include the identities, the components $\alpha_x = \alpha_{id_x}$ of the transformations in the subset $C \Rightarrow D$ lie in D.

We now would like to transfer the above structure to $\mathcal{P}ar\mathcal{C}at$. The forgetful functor $U : \mathcal{C}at_P \to \mathcal{P}ar\mathcal{C}at$, being a right adjoint, preserves products. Since $\mathcal{C}at_P$ is a cartesian closed category, we would seek a similar structure on $\mathcal{P}ar\mathcal{C}at$ in such a way that U preserves it, thereby relating the enrichment of both categories. Thus, just like we obtain the partial composites in a paracategory by restriction of the composition in the ambient category, we would proceed in the same way for whatever additional structure this latter might carry.

Given a coreflection $J \dashv U : \mathbb{D} \to \mathbb{C}$ where both categories admit finite products and $\mathbb{D}$ is cartesian closed, $\mathbb{C}$ is cartesian closed whenever the embedding J preserves products [Str80, Day72]. This is indeed the case for the embedding $\mathsf{EC} : \mathcal{P}ar\mathcal{C}at \to \mathcal{C}at_P$. We show this for paramonoids to simplify the presentation, without loss of generality.

3.2. Proposition. *The enveloping monoid functor* $\mathsf{E} : \mathcal{P}ar\mathcal{M}on \to \mathsf{Mon}_P$ *preserves products.*

Proof. Recall from the proof of Proposition 2.1 above that the enveloping monoid $\mathsf{E}(M)$ of a paramonoid M is obtained as a quotient of the free monoid on M, $\mathsf{M}(M)$.

Given paramonoids M_1, M_2, their product is given by the product of their carriers and the corresponding tupling of partial compositions. The free-monoid monad $\mathsf{M} : \mathcal{S}et \to \mathcal{S}et$ preserves pullbacks (in fact, this is true in any elementary topos with a natural-numbers-object [Bén90]). Thus we have a pullback

$$\begin{tikzcd} \mathsf{M}(M_1 \times M_2) \arrow[r, "\mathsf{M}\pi'"] \arrow[d, "\mathsf{M}\pi"'] & \mathsf{M}M_2 \arrow[d, "\mathsf{M}!"] \\ \mathsf{M}M_1 \arrow[r, "\mathsf{M}!"'] & \mathsf{M}\mathbf{1} \end{tikzcd}$$

where $\mathsf{M}\mathbf{1} = \mathsf{N}$ is the natural-numbers-object of $\mathcal{S}et$. Since $\mathsf{M}(M_1 \times M_2)$ consists of *lists of pairs*, the above square identifies

lists of pairs = *pairs of lists of the same length* via the canonical comparison morphism $\langle \mathsf{M}\pi, \mathsf{M}\pi' \rangle : \mathsf{M}(M_1 \times M_2) \to \mathsf{M}M_1 \times \mathsf{M}M_2$.

Using the units of the monoids, we can transform any pair of lists into an equivalent pair whose lists have the same length: define $\phi : \mathsf{M}M_1 \times \mathsf{M}M_2 \to \mathsf{M}(M_1 \times M_2)$ by

$$\phi(\vec{x}, \vec{y}) = (\vec{x}.[e_1]^{n-|\vec{x}|}, \vec{y}.[e_2]^{n-|\vec{y}|})$$

where $n = \mathsf{max}(|\vec{x}|, |\vec{y}|)$ and e_1 and e_2 are the units of M_1 and M_2 respectively. Since upon passage to the quotient $(\vec{x}, \vec{y}) \sim \phi(\vec{x}, \vec{y})$, we conclude $\mathsf{E}(M_1 \times M_2) \cong \mathsf{E}(M_1) \times \mathsf{E}(M_2)$.

□

3.3. Corollary.

1. *The category $\mathcal{P}ar\mathcal{C}at$ is (co)complete and cartesian closed.*
2. *The functors $U : \mathcal{C}at_P \to \mathcal{P}ar\mathcal{C}at$ and $\mathsf{EC} : \mathcal{P}ar\mathcal{C}at \to \mathcal{C}at_{\mathsf{P}}$ preserve the closed structure.*
3. *The embeddings $J : \mathcal{C}at \to \mathcal{P}ar\mathcal{C}at$ and $J' : \mathcal{C}at \to \mathcal{C}at_P$ (which sends a category to itself with all the morphisms as distinguished subset) preserve the cartesian closed structure and*

$$\begin{tikzcd}
\mathcal{P}ar\mathcal{C}at \arrow[rr, bend left, "\mathsf{EC}"] \arrow[rr, phantom, "\perp"] & & \mathcal{C}at_P \arrow[ll, bend left, "U"] \\
& \mathcal{C}at \arrow[ul, "J"] \arrow[ur, "J'"'] &
\end{tikzcd}$$

The cartesian closed nature of $\mathcal{P}ar\mathcal{C}at$ means that we obtain a meaningful notion of transformation for paracategories, by restricting the transformations given in the internal-hom of $\mathcal{C}at_P$ to the corresponding functors induced by the envelope functor EC:

3.4. Definition. Given functors $F, G : \mathbb{C} \to \mathbb{D}$, a **transformation** $\alpha : F \Rightarrow G$ consists of a collection of morphisms $\{\alpha_x : Fx \to Gx\}_{x \in \mathbb{C}}$ of morphisms of $\mathbb{D}$ (indexed by the objects of $\mathbb{C}$) such that both composites in the square

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\
Fx \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

are defined and equal for every morphism $f : x \to y$ in $\mathbb{C}$. Thus $\mathcal{P}ar\mathcal{C}at(\mathbb{C}, \mathbb{D})$ denotes the paracategory of functors and transformations between them, with partial compositions defined pointwise in $\mathbb{D}$.

Notice that if $\mathbb{D}$ is a category, so is $\mathcal{P}ar\mathcal{C}at(\mathbb{C}, \mathbb{D})$. Just like in $\mathcal{C}at$, there are horizontal composites of transformations, obtained by vertical composites and (pre)composition of functors with transformations.

3.5. Remarks.

- The embedding $J' : \mathcal{C}at \to \mathcal{C}at_P$ is right adjoint to the evident forgetful functor $\underline{U} : \mathcal{C}at_P \to \mathcal{C}at$, which also preserves closed structure.
- The definition of transformations in $\mathcal{P}ar\mathcal{C}at$ is forced upon us by regarding $\mathcal{P}ar\mathcal{C}at$ as a $\mathcal{P}ar\mathcal{C}at$-category.
- Although we have formulated the above definitions and statements for ordinary paracategories, the whole matter internalises straightforwardly, assuming the ambient category $\mathbb{B}$ is cartesian closed, in addition to the requirements in [HM02].

4 Comma-paracategories

In order to produce an appropriate notion of adjunction for paracategories, we show that the usual comma-category construction applies in the context of paracategories

4.1. Definition. Given functors $F : \mathbb{A} \to \mathbb{C}$ and $G : \mathbb{B} \to \mathbb{C}$, its **comma paracategory** F/G is the parategory whose objects are triples $(x, a : Fx \to Gy, y)$ and whose morphisms $(f, g) : (x, a : Fx \to Gy, y) \to (x', a' : Fx' \to Gy', y')$ are pairs of morphims $f : x \to x'$ in $\mathbb{A}$ and $g : y \to y'$ in $\mathbb{B}$ such that $a' \circ Ff = Gg \circ a$ with both sides defined. The partial composites are inherited componentwise from $\mathbb{A}$ and $\mathbb{B}$. We have thus a 'universal transformation'

$$\begin{tikzcd} F/G \arrow[r, "\pi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\alpha \nearrow"] & \mathbb{B} \arrow[d, "G"] \\ \mathbb{A} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd}$$

where α has components $\alpha_{(x,a\,:\,Fx\,\to\,Gy,y)} = a$. The comma paracategory is characterised by the following universal property

$$\mathcal{P}ar\mathcal{C}at(\mathbb{X}, F/G) \cong \mathcal{P}ar\mathcal{C}at(\mathbb{X}, F)/\mathcal{P}ar\mathcal{C}at(\mathbb{X}, G)$$

naturally in (the paracategory) $\mathbb{X}$, which means that the *lax square* above is the universal one with respect to F and G.

When $G = id_{\mathbb{C}}$, we write $F/\mathbb{C}$ for F/G and similarly when $F = id_{\mathbb{C}}$.

An analogous construction of the *comma-object* (in the spirit of 2-category theory [Str73]) can be carried out in $\mathcal{C}at_P$: given functors $F : (\mathbb{A}, A) \to (\mathbb{C}, C)$ and $G : (\mathbb{B}, B) \to (\mathbb{C}, C)$ their comma-object F/G in $\mathcal{C}at_P$ is the category $C \cap (F/G)$ with

objects triples $(x, a : Fx \to Gy, y)$ with $a \in C$

morphisms $(f, g) : (x, a : Fx \to Gy, y) \to (x', a' : Fx' \to Gy', y')$ given by $f : x \to x'$ in $\mathbb{A}$ and $g : y \to y'$ in $\mathbb{B}$ such that the outer square

$$\begin{tikzcd} Fx \arrow[r, "a"] \arrow[d, "Ff"'] \arrow[dr, "\alpha_{f,g}"] & Gy \arrow[d, "Gg"] \\ Fx' \arrow[r, "a'"'] & Gy' \end{tikzcd}$$

commutes and the diagonal $\alpha_{f,g} \in C$ whenever $f \in A$ and $g \in B$. The distinguished collection of morphisms in $C \cap (F/G)$ is the set

$$\{(f,g) : (x, a : Fx \to Gy, y) \to (x', a' : Fx' \to Gy', y') \mid f \in A \wedge g \in B\}$$

5 Adjunctions of paracategories

In [SM99] the authors introduce a notion of transformation between functors of 'precategories' which amounts to the usual data for a natural transformation, subject to the requirement that the relevant composites in the naturality squares are defined, in accordance to our *derived* definition of transformation of paracategories above. The authors further analyse several elementary definitions of adjointness, and observe the important role of comma-categories in this setting.

As we explained in the introduction, we adopt the point of view of [Law66] and formulate the notion of adjunction in terms of isomorphism of comma-paracategories:

5.1. Definition. A functor $L : \mathbb{C} \to \mathbb{D}$ is **left adjoint** to $R : \mathbb{D} \to \mathbb{C}$ if there exists an isomorphism of comma-paracategories $\theta : L/\mathbb{D} \xrightarrow{\sim} \mathbb{C}/R$ such that

$$\begin{tikzcd}
& L/\mathbb{D} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \arrow[dd, "\theta"] & \\
\mathbb{C} & & \mathbb{D} \\
& \mathbb{C}/R \arrow[ul, "\pi"] \arrow[ur, "\pi'"'] &
\end{tikzcd}$$

Given an adjunction, which we write in the customary way $F \dashv G$, we obtain transformations $\eta : id \Rightarrow RL$ (the *unit*, $\eta = \theta(id_{L\_})$) and $\epsilon : LR \Rightarrow id$ (the *counit*, $\epsilon = \theta^{-1}(id_{R\_})$) verifying the *triangle identities*

$$\epsilon R \circ R\eta = id \qquad L\epsilon \circ \eta L = id$$

It is important to emphasise that the existence of the isomorphism θ is a *stronger* requirement than that of the existence of η and ϵ satisfying the above equations. Namely, the fact that η is a transformation of paracategories ensures that for any $f : X \to Y$ in $\mathbb{C}$, the composite $RL(f) \circ \eta_X$ is defined. The existence of θ demands more: that all composites of the form $R(g) \circ \eta_X$ (the *adjoint transpose* of g) exist, for given $g : LX \to Y$ in $\mathbb{D}$, the previous condition being the special case $g = Lf$. These issues are illustrated in Example 5.6.

5.2. Remark. One important consequence of the way we defined adjunctions above is that they compose: the composite of two left-adjoint functors is another such. This would not be the case if we adopted the *unit/counit/triangular-identities* notion, where composition would be partial.

The same definition of adjunction applies in $\mathcal{C}at_P$ (isomorphism of comma-objects $\theta : L/(\mathbb{D}, D) \xrightarrow{\sim} (\mathbb{C}, C)/R$) so that, as expected we have a precise correspondence between adjunctions in $\mathcal{P}ar\mathcal{C}at$ and $\mathcal{C}at_P$

5.3. Corollary (Correspondence of adjunctions).

1. *$F \dashv G : \mathbb{C} \to \mathbb{D}$ in $\mathcal{P}ar\mathcal{C}at$ iff $\mathsf{EC}F \dashv \mathsf{EC}G : \mathsf{EC}\mathbb{C} \to \mathsf{EC}\mathbb{D}$ in $\mathcal{C}at_P$*
2. *$F \dashv G : (\mathbb{C}, C) \to (\mathbb{D}, D)$ in $\mathcal{C}at_P$ iff $UF \dashv UG : U(\mathbb{C}, C) \to U(\mathbb{D}, D)$ in $\mathcal{P}ar\mathcal{C}at$*
3. *A functor $F : \mathbb{C} \to \mathbb{D}$ in $\mathcal{P}ar\mathcal{C}at$ admits a right(/left) adjoint iff the functor $\mathsf{EC}F : \mathsf{EC}\mathbb{C} \to \mathsf{EC}\mathbb{D}$ admits a right(/left) adjoint in $\mathcal{C}at_P$*

5.1 Limits and colimits

Using the embedding of $\mathcal{C}at$ in $\mathcal{P}ar\mathcal{C}at$ and the closed structure of the latter, we may speak of *cotensors* in $\mathcal{P}ar\mathcal{C}at$ paraphrasing the enriched-category notion [Kel82]. Thus for a paracategory $\mathbb{C}$ and a category X, the **cotensor** $\{X, \mathbb{C}\}$ is given by $\mathcal{P}ar\mathcal{C}at(J(X), \mathbb{C})$ which has the following universal property

$$\mathcal{P}ar\mathcal{C}at(\mathbb{A}, \{X, \mathbb{C}\}) \cong \mathcal{P}ar\mathcal{C}at(J(X), \mathcal{P}ar\mathcal{C}at(\mathbb{A}, \mathbb{C}))$$

naturally on the paracategory $\mathbb{A}$. We have the corresponding diagonal functor $\delta : \mathbb{C} \to \{X, \mathbb{C}\}$, induced by the functor $x \mapsto id$ in $\mathcal{C}at(X, \underline{U}\mathsf{EC}(\mathcal{P}ar\mathcal{C}at(\mathbb{C}, \mathbb{C})))$. We can thus adopt the adjoint characterisation of (co)limits [Mac98, §IV.2] as our definition in the paracategorical context:

5.4. Definition. Let X be a small category. A paracategory $\mathbb{C}$ admits X-limits (colimits) if the diagonal $\delta : \mathbb{C} \to \{X, \mathbb{C}\}$ has a right (left) adjoint.

Applying Corollary 5.3.(3) we can characterise (co)limit structure for a paracategory either in $\mathcal{P}ar\mathcal{C}at$ or $\mathcal{C}at_P$.

5.5. Corollary.

- *Given a category $\mathbb{D}$ with a subset of morphisms D, $(\mathbb{D}, D)$ has X-(co)limits (δ has an adjoint in $\mathcal{C}at_P$) iff $U(\mathbb{D}, D)$ has X-(co)limits.*
- *Given a paracategory $\mathbb{C}$, $\mathbb{C}$ has X-(co)limits iff $\mathsf{EC}(\mathbb{C})$ has X-(co)limits (in $\mathcal{C}at_P$).*

In elementary terms, the above means that if we consider a paracategory $\mathbb{C}$ arising from a category $\mathbb{D}$ and a subset of morphisms D, $\mathbb{C}$ will have X-limits if $\mathbb{D}$ has a cone (the counit of the adjunction) and diagonals (the unit of the adjunction) with components in D. Moreover given another cone with components in D, there must be a unique mediating morphism (in D) to the given (counit) cone. Notice that this property does not entail the existence of ordinary limits in $\mathbb{D}$. See §8.4 where the analogous situation with tensor products appears.

5.2 Cartesian closed paracategories

Given a paracategory $\mathbb{C}$ with finite products, every object $X \in \mathbb{C}$ determines a functor $X \times \_ : \mathbb{C} \to \mathbb{C}$. The object X is **exponentiable** if this functor admits a right-adjoint $X \Rightarrow \_ : \mathbb{C} \to \mathbb{C}$. The paracategory $\mathbb{C}$ is **cartesian-closed** if every object is exponentiable. We have thus a transformation $\epsilon^X : X \times (X \Rightarrow \_) \Rightarrow id$ satisfying the following universal property: given $f : X \times Y \to Z$, there is a unique morphism $\hat{f} : Y \to X \Rightarrow Z$ such that

$$\begin{tikzcd} X \times Y \arrow[r, "X \times \hat{f}"] \arrow[dr, "f"'] & X \times (X \Rightarrow Z) \arrow[d, "\epsilon^X_Z"] \\ & Z \end{tikzcd}$$

Of course, as for any adjunction, we also have the unit transformation $\eta^X : id \Rightarrow X \Rightarrow (X \times \_)$ which satisfies the triangular identities with ϵ:

$$\epsilon^X_{X \Rightarrow \_} \circ (X \times \eta^X) = id \qquad (X \Rightarrow \epsilon^X) \circ \eta^X_{X \Rightarrow \_} = id$$

Just like in the case of categories, the collection of $X \Rightarrow \_$ organize themselves into a bivariant functor $\_ \Rightarrow \_ : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$. Bearing in mind the connection between cartesian closed categories and simply typed lambda calculus [LS86, Jac99], we see that bivariant functors arise naturally in the categorical semantics of such calculi.

5.6. Example. Consider a (small) cartesian closed category $\mathbb{D}$ and the collection of bivariant functors $T : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$ from a given small category $\mathbb{C}$. Except for rather uninteresting $\mathbb{C}$, the category $\mathcal{C}at(\mathbb{C}^{op} \times \mathbb{C}, \mathbb{D})$ will *not* be cartesian closed with the pointwise structure in $\mathbb{D}$. However, the situation becomes more interesting if we consider *dinatural transformations* rather than only natural ones. Recall (from [Mac98, §IX.4]) that given bivariant functors $S, T : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$ a **dinatural transformation** $\alpha : S \Rightarrow T$ is given by a collection of morphisms $\{\alpha_C : S(C, C) \to T(C, C)\}_{C \in \mathbb{C}}$ in $\mathbb{D}$ such that for every morphism $f : B \to C$ in $\mathbb{C}$ we have the following commutative hexagon in $\mathbb{D}$:

$$\begin{tikzcd} & S(B,B) \arrow[r, "\alpha_B"] & T(B,B) \arrow[dr, "T(B,f)"] & \\ S(C,B) \arrow[ur, "S(f,B)"] \arrow[dr, "S(C,f)"'] & & & T(B,C) \\ & S(C,C) \arrow[r, "\alpha_C"'] & T(C,C) \arrow[ur, "T(f,C)"'] & \end{tikzcd}$$

A problem arises: the componentwise composition of two dinatural transformations is not necessarily dinatural. Hence with this composition we get only a paracategory $Dinat(\mathbb{C}^{op} \times \mathbb{C}, \mathbb{D})$ of bivariant functors and dinatural transformations.

The relationship between natural and dinatural transformations is the following:

- Given a natural transformation $\theta : S \Rightarrow T$, the collection of morphisms $\{\theta_{C,C} : S(C,C) \to T(C,C)\}_{C \in \mathbb{C}}$ is a dinatural transformation:

$$\begin{tikzcd}
 & S(B,B) \arrow[r, "\theta_{B,B}"] & T(B,B) \arrow[dr, "T(B,f)"] & \\
S(C,B) \arrow[ur, "S(f,B)"] \arrow[dr, "S(C,f)"'] \arrow[r, dashed, "\theta_{C,B}"] & T(C,B) \arrow[ur, "T(f,B)"] \arrow[dr, "T(C,f)"'] \arrow[rr, dashed, "T(f,f)"] & & T(B,C) \\
 & S(C,C) \arrow[r, "\theta_{C,C}"'] & T(C,C) \arrow[ur, "T(f,C)"'] &
\end{tikzcd}$$

- Natural transformations act by composition on dinatural ones: given dinatural transformations $\alpha : S' \Rightarrow S$ and $\beta : T \Rightarrow T'$, the collections

$$\{\theta_{C,C} \circ \alpha_C : S'(C,C) \to T(C,C)\}_{C \in \mathbb{C}}$$

and

$$\{\beta_C \circ \theta_{C,C} : S(C,C) \to T'(C,C)\}_{C \in \mathbb{C}}$$

are dinatural.

Given $S, T : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$ the bivariant functor $(S \times T)(B,C) = S(B,C) \times T(B,C)$ together with the dinatural transformations $\pi_{C,C} : (S \times T)(C,C) \to S(C,C)$ and $\pi'_{C,C} : (S \times T)(C,C) \to T(C,C)$ form a product in $\mathcal{D} = Dinat(\mathbb{C}^{op} \times \mathbb{C}, \mathbb{D})$. More generally, $\mathcal{D}$ inherits cocompleteness from $\mathbb{D}$, pointwise, since the relevant natural (co)limit cones are also dinatural and suitable closed under (pre)composition with dinaturals as indicated above.

For exponentials in $\mathcal{D}$, define

$$(S \Rightarrow T)(B,C) = S(C,B) \Rightarrow T(B,C)$$

We have an associated dinatural transformation with components

$$\epsilon^{S(C,C)}_{T(C,C)} : S(C,C) \times (S(C,C) \Rightarrow T(C,C)) \to T(C,C)$$

This yields a transformation $\epsilon : S \times (S \Rightarrow \pi') \Rightarrow \pi'$ in the paracategory sense: given a dinatural transformation $\theta : U \Rightarrow V$ we must show that both composites $\theta \circ \epsilon$ and $\epsilon \circ S \times (S \Rightarrow \theta)$ are defined and equal. Since $\mathbb{D}$ is cartesian closed, we work with the simply typed lambda-calculus which is its internal language [LS86] (or if the reader prefers, pretend $\mathbb{D}$ is $\mathcal{S}et$ and work with elements). So, $\epsilon^{S(C,C)}_{T(C,C)}(s, \phi) = ev \circ \langle \phi, s \rangle$ (evaluate function ϕ at argument s). Given a morphism $f : B \to C$ in $\mathbb{C}$, define $\epsilon^S_{f,T} : S(C,B) \times (S(B,C) \Rightarrow T(C,B)) \to T(C,B)$ as

$$\epsilon^S_{f,T}(s, \phi) = ev \circ \langle \phi, S(f,f)s \rangle$$

so that $\epsilon^{S(C,C)}_{T(C,C)} = \epsilon^S_{id,T}$. We have the following commutative diagram

$$
\begin{tikzcd}
& S(B,B)\times(S(B,B)\Rightarrow U(B,B)) \arrow[r, "\epsilon^{S(B,B)}_{U(B,B)}"] & U(B,B) \arrow[r, "\theta_B"] & V(B,B) \arrow[dr, "V(B,f)"] & \\
S(C,B)\times(S(B,C)\Rightarrow U(C,B)) \arrow[ur, "S(f,B)\times(S(B,f)\Rightarrow U(f,B))"] \arrow[r, "\epsilon^S_{f,U}"] \arrow[dr, "S(C,f)\times(S(f,C)\Rightarrow U(C,f))"'] & U(C,B) \arrow[ur, "U(f,B)"] \arrow[dr, "U(C,f)"'] & & & V(B,C) \\
& S(C,C)\times(S(C,C)\Rightarrow U(C,C)) \arrow[r, "\epsilon^{S(C,C)}_{U(C,C)}"'] & U(C,C) \arrow[r, "\theta_C"'] & V(C,C) \arrow[ur, "V(f,C)"'] &
\end{tikzcd}
$$

so that $\theta \circ \epsilon$ is a dinatural transformation. The equality $\epsilon \circ S\times(S \Rightarrow \theta) = \theta \circ \epsilon$ is easily verified, so that the left-hand side is a dinatural transformation as well. Similarly, the unit of the exponential adjunction in $\mathbb{D}$ becomes a dinatural transformation with components

$$\eta^{S(C,C)}_{T(C,C)} : T(C,C)) \to S(C,C) \Rightarrow (S(C,C)\times T(C,C))$$

In order to verify that we get a well-defined transformation $\eta^S : \pi' \Rightarrow S \Rightarrow (S \times \pi')$, define $\eta^S_{f,T} : T(B,C) \to S(C,B) \Rightarrow (S(B,C)\times T(B,C))$ (for $f : B \to C$ in $\mathbb{C}$) as

$$\eta^S_{f,T} t = \lambda s.\langle S(f,f)s, t\rangle$$

so that $\eta^{S(C,C)}_{T(C,C)} = \eta^S_{id,T}$. A similar diagram chase like the one above for $\epsilon^S_{f,T}$ shows that $\eta \circ \theta$ is a dinatural transformation whenever θ is. With all these data we can finally assert that every bivariant functor $S : \mathbb{C}^{op}\times\mathbb{C} \to \mathbb{D}$ is exponentiable:

$$\mathcal{D}(S\times T, U) \cong \mathcal{D}(T, S \Rightarrow U)$$

Indeed we are only left to verify that given a dinatural transformation $\theta : S\times U \Rightarrow T$, its adjoint transpose $\hat{\theta} = (S \Rightarrow \theta) \circ \eta$ is a dinatural transformation $\hat{\theta} : U \Rightarrow (S \Rightarrow T)$, which follows easily 'chasing elements' in λ-notaion: given an element $u \in U(C,B)$ the composite

$$(S(f,B) \Rightarrow T(B,f)) \circ \hat{\theta}_B \circ U(f,B)$$

corresponds to the term

$$u : U(C,B) \mid \lambda s : S(C,B).\, T(B,f)(\theta_B\langle S(f,B)s, U(f,B)u\rangle)$$

while the composite

$$(S(C,f) \Rightarrow T(f,C)) \circ \hat{\theta}_C \circ U(C,f)$$

corresponds to the term

$$u : U(C,B) \mid \lambda s : S(C,B).\, T(f,B)(\theta_C\langle S(C,f)s, U(C,f)u\rangle)$$

and the bodies of these λ-abstractions are identified by dinaturality of θ. $\square$

6 Partial multicategories

In [HM02] we have shown that the notions of paramonoid and paracategory are instances of that of *saturated partial algebra*. Here we present another useful instance of this latter notion, namely *partial multicategories*[1]. Such structures allow us to capture certain 'tensor products' which arise in the context of probabilistic automata (§8).

Our reference for the theory of multicategories is [Her00]. As indicated there, an internal multicategory with object-of-objects C_0 is an *internal monoid* in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})(\mathbf{C_0}, \mathbf{C_0})$. Hence a **partial multicategory** is an internal paramonoid in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})(\mathbf{C_0}, \mathbf{C_0})$. We will not indulge here in this internal version of partial multicategories and give instead a concrete description below. We do mention this abstract version though as it yields a neat application of the theory of partial algebras for a cartesian monad developed in [HM02]. Since the iterated composites of (multi)morphisms involve a rather heavy syntax, it would be easier to work with $T\text{-}\mathcal{A}lg_P$ in the terminology of *ibid.*, that is, regarding partial multicategories as determined by a multicategory together with a specified subset of morphisms (or multiarrows) to give a full-fledged explicit definition.

Let us recall the basic definition of a **multicategory** $\mathbb{M}$ in $\mathcal{S}et$: it consists of a set of objects, ranged over by $x, y, \ldots$, and a set of morphisms, ranged over by $f, g, \ldots$. Every morphism is endowed with *source-target* information, displayed as $f : \vec{x} \to y$, where $\vec{x}$ is a list $\langle x_1, \ldots, x_n\rangle$ of objects. These data consitute a *multigraph* $\mathsf{M}(C_0) \overset{d}{\leftarrow} C_1 \overset{c}{\to} C_0$. In addition, there are *identity* morphisms $id_x : \langle x\rangle \to x$ (for every object x) and an associative *composition*, which takes a morphism $f : \langle x_1, \ldots, x_n\rangle \to y$ and a sequence of morphisms

$$f_1 : \langle X_{11}, \ldots, X_{1m_1}\rangle \to X_1, \ldots, f_n : \langle X_{n1}, \ldots, X_{nm_n}\rangle \to X_n$$

and produces their composite $f\langle f_1, \ldots, f_n\rangle$ with source-target as displayed:

$$f\langle f_1, \ldots, f_n\rangle : \langle X_{11}, \ldots, X_{1m_1}, \ldots, X_{n1}, \ldots, X_{nm_n}\rangle \to Y$$

The point of view of [Her00] is that multicategories arise as auxiliary structures to axiomatise *monoidality* or *tensor products* as universal constructions. Here too, we adopt partial multicategories to axiomatise the corresponding notion of *tensor products*.

6.1. Definition. A **partial multicategory** consists of a multigraph $\mathsf{M}(C_0) \overset{d}{\leftarrow} C_1 \overset{c}{\to} C_0$ and partial n-ary composition operations $\circ_n : C_1^n \rightharpoonup C_1$ where C_1^n is the n-th tensor power of the multigraph as object in the monoidal category $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})(\mathbf{C_0}, \mathbf{C_0})$. More explicitly, C_1^n consists of 'trees of composable morphisms of height n', so that C_1^2 consists of the data above for composition in a multicategory. These partial operations are subject to the axioms that rule the partial compositions in a paracategory, suitably modifying the third one to accommodate multicomposition.

[1] The term *paramulticategory* is doubly incovenient for being a mouthful and causing havoc with line breaks.

More simply we can construe a partial multicategory as specified by a multicategory with a distinguished subset of morphisms $D \subseteq C_1$ (including the identities). The corresponding partial composites are those induced by the multicategory composites whenever the result lies in D. We write $(\mathbb{M}, D)$ for such data.

The evident notion of functor bewteen partial multicategories is equally obtained from that of functor between multicategories (morphism of multigraphs preserving composites) which is compatible with the partiality information: whenever the composite of the source multicategory is defined, so is the composite of its image in the target.

Using the second version of partial multicategory, we can easily describe the corresponding notion of *representability*. Recall that a multicategory $\mathbb{M}$ is representable whenever for every list of objects $\vec{x}$ there is a *universal* morphism $\pi_{\vec{x}} : \vec{x} \to \otimes\, \vec{x}$, so that every morphism $f : \vec{y} \to z$ with $\vec{x}$ a sublist of $\vec{y}$, factors uniquely through $\pi_{\vec{x}}$. Equivalently, we require the existence of v-universal morphisms $\pi_{\vec{x}} : \vec{x} \to \otimes\, \vec{x}$, such that every morphism $f : \vec{x} \to z$ factors uniquely through $\pi_{\vec{x}}$, and v-universals are closed under (multicategory) composition. In the case of partial multicategories, we restrict such universality condition to morphisms in D:

6.2. Definition. A partial multicategory $(\mathbb{M}, D)$ is **representable** whenever for every list of objects $\vec{x}$ there is a **universal morphism** $\pi_{\vec{x}} : \vec{x} \to \otimes\, \vec{x} \in D$, so that every morphism $f : \vec{x} \to z \in D$ factors uniquely through $\pi_{\vec{x}}$, and the factors belong to D. Such universal morphisms must be closed under composition. We write $\mathcal{RM}$ for the category of representable partial multicategories and functors between them which preserve universal morphisms.

Recall [Her00, Def. 6.7] that a multicategory $\mathbb{M}$ has an underlying category $\overline{\mathbb{M}}$ of 'linear morphisms' and that representability of $\mathbb{M}$ endows $\overline{\mathbb{M}}$ with a monoidal structure. The same correspondence between representability and monoidality applies to the partial case, as we state next.

6.3. Definition. Given a partial multicategory $(\mathbb{M}, D)$, its **underlying paracategory** $\overline{\mathbb{M}}$ has the same objects of $\mathbb{M}$ while its morphisms are those with a singleton source $f : \langle x \rangle \to y$. The partial composites of $\mathbb{M}$ restrict to give the corresponding ones in $\overline{\mathbb{M}}$.

Next we should specify what we mean by monoidal structure on a paracategory. The simplest way is to consider the strict version and regard a monoidal category as an internal monoid in $\mathcal{C}at$ (see [Mac98, §VII]). Since $\mathcal{P}ar\mathcal{C}at$ has finite products, we can equally well consider monoids in it. Furthermore, the coherence theorem for monoidal categories (see *e.g.*[JS93]) says that every monoidal category is equivalent to a strict one, and thus we can construe a monoidal category $\mathbb{V}$ as a category with a given equivalence to a strict monoidal category $\underline{\mathbb{V}}$, and use this procedure to define monoidal paracategories. The notion of *equivalence* in the paracategory context is the same as for categories: an adjunction whose unit and counit transformations are isomorphisms.

6.4. Definition. A **strict monoidal paracategory** is a monoid in $\mathcal{P}ar\mathcal{C}at$, that is, a paracategory $\mathbb{C}$ endowed with a unit $I \in \mathbb{C}$ and a multiplication $\otimes$: $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, associative and unitary. A **monoidal paracategory** is a paracategory $\mathbb{C}$ together with an equivalence $e : \mathbb{C} \to \underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ is a strict monoidal paracategory. A **strong monoidal functor** between monoidal paracategories is thus given by a pair of functors ($F : \mathbb{C} \to \mathbb{D}$,$\underline{F} : \underline{\mathbb{C}} \to \underline{\mathbb{D}}$) and a natural isomorphism $\phi : \underline{F}e \Rightarrow e'F$, such that $\underline{F}$ preserves the strict monoidal structure (on the nose). We have thus the category $\mathcal{M}on(\mathcal{P}ar\mathcal{C}at)$ of monoidal paracategories and strong monoidal functors.

6.5. Remark. In the context of $\mathcal{C}at_P$, it is clear that monoidal structure is the same as for ordinary categories, with the additional proviso that the associativity and unit isomorphisms should lie in the distinguished subset of arrows of the category. Translating this situation to paracategories, a monoidal paracategory in the sense above amounts to a paracategory together with the usual tensor and unit functors and coherent isomorphisms such that all the relevant composites (required to state that 'all diagrams commute' in the sense of [Mac98, §VII.2,Th.1]) are defined.

Applying the correspondence between monoidal categories and representable multicategories of [Her00, Th.9.8], we obtain one at the partial level:

$$\boxed{\mathcal{RM} \equiv \mathcal{M}on(\mathcal{P}ar\mathcal{C}at)}$$

Either side of this 'equivalence' is a suitable axiomatisation for the notion of tensor product for paracategories. Despite the restriction on the morphisms, we do get the usual properties of tensor products in this context, namely associativity: $\otimes \langle x, \otimes \langle y, z\rangle\rangle \cong \otimes \langle \otimes \langle x, y\rangle, z\rangle$ canonically.

6.6. Remark. The precise statement of the above correspondence would involve making explicit the 2-dimensional structure implicit on both sides, as we only have a 'biequivalence'. We have refrained from indulging in such technicalities in this presentation. Likewise, the construction of the free representable multicategory of [Her00, Th. 7.2] translates literally to the partial context, keeping track of the distinguished morphisms. Hence, every partial multicategory is fully and faithfully embedded into its free representable one. It is possible to carry out the adjoint characterisation of representability as in *ibid.* in this context, but that would require further technical machinery which goes beyond the scope of this paper.

In §8.4 we illustrate the above notions setting up representable partial multicategories $(\mathcal{B}or^{\eth}_{\blacktriangleright}, D)$ and $(\mathcal{MP}aut^{\eth}_{\blacktriangleright}, \underline{D})$; the induced tensor on the latter case accounts for the (free) aggregation of probabilistic automata.

7 Fibrations of paracategories

[Mat00] considers fibrations for *precategories*. We remind the reader that precategories correspond to the special case of paracategories where only a partial *binary* operation and identities are taken as primitives, subject to an associativity requirement which allows the consistent definition of (partial) composites of higher arities. The relevant notion of fibration translates literally to the context of paracategories:

7.1. Definition. Given a functor $p : \mathbb{E} \to \mathbb{B}$:

1. A morphism $f : x \to y$ in $\mathbb{E}$ is (strong) p-cartesian if for any other morphism $h : z \to y$ with a (well-defined) factorisation $ph = pf \circ v$ in $\mathbb{B}$, there exists a unique factorisation $h = f \circ \tilde{u}$ with $p\tilde{u} = u$

2. The functor p is a **fibration of paracategories** if for every object x in $\mathbb{E}$ and every morphism $v : i \to px$ in $\mathbb{B}$, there exists a (strong) p-cartesian morphism $\bar{v} : v^*(x) \to x$ such that $p\bar{v} = v$.

7.2. Remark. It is equivalent to require for p to be a fibration the weaker property of existence of p-cartesian morphisms, which are universal only with respect to morphisms $h : z \to y$ with $ph = pf$ provided that composable p-cartesian morphisms do compose whenever their images in the base category do, and their composite is p-cartesian as well.

For a functor of paracategories $p : \mathbb{E} \to \mathbb{B}$, with $\mathbb{B}$ a category, there is a simple adjoint characterisation of the property of being a fibration, analogous to the one for fibrations of categories [Str73].

7.3. Proposition. *Given a functor $p : \mathbb{E} \to \mathbb{B}$ with $\mathbb{B}$ a category, the following are equivalent:*

1. *p is a fibration of paracategories*

2. *The functor $\eta_p : \mathbb{E} \to \mathbb{B}/p$ induced as depicted below*

$$\begin{tikzcd}
\mathbb{E} \arrow[dr, dashed, "\eta_p"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "p"'] & & \\
& \mathbb{B}/p \arrow[r, "\pi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\alpha \Nearrow"] & \mathbb{E} \arrow[d, "p"] \\
& \mathbb{B} \arrow[r, "id"'] & \mathbb{B}
\end{tikzcd}$$

$\eta_p : x \mapsto (px, id : px \to px, x)$ has a right adjoint right inverse, i.e. with identity counit.

7.4. Remark. The reason we must restrict ourselves to $\mathbb{B}$ being a category is the following: in the ordinary categorical setting the functor $\pi : \mathbb{B}/p \to \mathbb{B}$ is a fibration. In fact, it is the *free fibration over p* [Str73]. But if $\mathbb{B}$ is only a paracategory, the relevant cartesian liftings:

$$f : x \to y \ , \ a : y \to pz \ \longmapsto a \circ f : x \to pz$$

may not be defined. This is also the reason why the isomorphism θ in Definition 5.1 cannot be said to be an isomorphism of *bimodules*, as it is the case for categories.

These considerations suggest we might adopt a more relaxed notion of fibration, with *partial* actions, so as to accomodate π. At present though, the situation in which we apply fibrations in §8 fits our restricted formulation above.

The indexed version poses no problems. We consider the split (strict) version only for simplicity.

7.5. Definition. Given a paracategory $\mathbb{C}$ and a functor $\mathcal{F} : \mathbb{C}^{op} \to \mathcal{P}ar\mathcal{C}at$, define its *Grothendieck paracategory* $\int \mathcal{F}$ as follows:

objects pairs (X, x) with $X \in \mathbb{C}$ and $x \in \mathcal{F}X$

morphisms $(b, f) : (X, x) \to (Y, y)$ is a pair $b : X \to Y$ in $\mathbb{C}$ and $f : x \to (\mathcal{F}b)y$ in the fibre $\mathcal{F}X$

identities $(id, id) : (X, x) \to (X, x)$

composition

$$[(b_1, f_1) \dots (b_n, f_n)] = ([b_1 \dots b_n], [f_1 (b_1)^* f_2 \dots (b_1 \dots b_{n-1})^* f_n])$$

where we have written $b^* f = (\mathcal{F}b) f$ and the second composition on the right takes place in the fibre $\mathcal{F}X_1$, with $b_1 : X_1 \to X_2$

There is an evident functor $\pi_{\mathcal{F}} : \int \mathcal{F} \to \mathbb{C}$ taking (X, x) to X.

We want to relate this construction to the usual Grothendieck construction for contravariant functors into $\mathcal{C}at$. Given a functor $\mathcal{F} : \mathbb{C}^{op} \to \mathcal{P}ar\mathcal{C}at$, since $\mathcal{P}ar\mathcal{C}at$ is a category, we get a functor $\widehat{\mathcal{F}} : \mathsf{EC}(\mathbb{C})^{op} \to \mathcal{P}ar\mathcal{C}at$, which we postcompose with $\underline{U}\mathsf{EC} : \mathcal{P}ar\mathcal{C}at \to \mathcal{C}at$ thus obtaining a contravariant $\mathcal{C}at$-valued functor $\overline{\mathcal{F}} \equiv \underline{U}\mathsf{EC}\widehat{\mathcal{F}}$, and consider its associated split fibration $\pi_{\overline{\mathcal{F}}} : \int \overline{\mathcal{F}} \to \mathsf{EC}(\mathbb{C})$.

7.6. Proposition. *Given a functor $\mathcal{F} : \mathbb{C}^{op} \to \mathcal{P}ar\mathcal{C}at$, the following hold:*

- $\pi_{\mathcal{F}} : \int \mathcal{F} \to \mathbb{C}$ *is a fibration of paracategories, and every fibration so arises.*
- $\mathsf{EC}(\pi_{\mathcal{F}}) \simeq \pi_{\overline{\mathcal{F}}}$

7.7. Remark. The above characterisation of fibration of paracategories in terms of the Grothendieck construction shows that a fibration of paracategories is a Kleene functor iff its fibres are categories.

Finally, to establish the relationship between fibrations of paracategories and that of categories, we must specify these latter in the context of $\mathcal{C}at_P$.

7.8. Definition. Given objects $(\mathbb{E}, E)$ and $(\mathbb{B}, B)$ of $\mathcal{C}at_P$, a functor $p : \mathbb{E} \to \mathbb{B}$ is a **fibration in** $\mathcal{C}at_P$ if it has p-cartesian liftings of morphisms in B and such liftings belong to E.

7.9. Proposition.

- *If* $p : \mathbb{E} \to \mathbb{B}$ *is a fibration of paracategories, then* $\mathsf{EC}p : \mathsf{EC}(\mathbb{E}) \to \mathsf{EC}(\mathbb{B})$ *is a fibration in* $\mathcal{C}at_P$*.*
- *If* $p : (\mathbb{C}, C) \to (\mathbb{D}, D)$ *is a fibration in* $\mathcal{C}at_P$*, then* $U(p) : U(\mathbb{C}, C) \to U(\mathbb{D}, D)$ *is a fibration of paracategories.*

As a particular instance of the above indexed-family-of-paracategories/fibration correspondence, notice that for presheaves

$$\mathcal{P}ar\mathcal{C}at(\mathbb{C}, \mathcal{S}et) \simeq \mathcal{C}at_P(\mathsf{EC}(\mathbb{C}), \mathcal{S}et) \simeq \mathcal{C}at(\underline{\mathsf{U}}\mathsf{EC}(\mathbb{C}), \mathcal{S}et)$$

thus a paracategory $\mathbb{C}$ and its envelope do have the same 'modules' indeed.

8 Examples from Probabilistic Automata Theory

Several results on automata theory can be elegantly explained using categorical language [AM74, AT91], since an automaton is essentially an action over a monoid [EW67]. Bearing this in mind, it is easy to explain combination and realization of automata as universal constructions [Gog72, WN95].

Most of these results can be extended to richer structures, such as linear, internal and non-deterministic automata [Adá76] . However, probabilistic automata tend to be an exception to this elysian setting, specially when we need to restrict the probabilistic transitions of an automata [GSST95]. The reason for this inconvenience lies on the fact that there is no good notion of morphism for probability spaces such that *conditional probability* would correspond to *subojects* and *measure preserving transformations* would correspond to some kind of *epimorphisms*. As we shall see, the natural notion of morphism for probabilistic automata fails to be closed under composition, yielding a proper paracategory.

Since probability theory can be formulated with Borel measure spaces, we start by organising these within a convenient paracategory $\mathcal{B}or$. This paracategory serves as a basis for more elaborated structures: random variables, stochastic processes and probabilistic automata, which provide examples of fibrations and pullbacks in $\mathcal{P}ar\mathcal{C}at$. We show the central result of the behaviour/realisation adjunction of probabilistic automata as an example of a fibred adjunction of paracategories (Theorem 8.6).

Finally, we show how the aggregation of automata from [Mat00] can be construed in terms of representability for a suitable partial multicategory of cones (Corollary 8.10).

8.1 Borel probability measures

Given a topological space T, its *Borel algebra* $\mathcal{B}T$ is the smallest σ-algebra[2] containing all open sets of T. Measures, and probabilities in particular, can be

[2] A σ-algebra $\mathcal{F}$ over a set Ω is a collection of subsets of Ω closed under complements and countable (including empty) unions and disjunctions.

defined over Borel sets, with the usual requirement that T be Hausdorff, locally compact and with a countable base, in order to adapt results smoothly from the classical Borel algebra over the real numbers [Hal69]. Here we consider all topological spaces.

A well established result asserts that given a probability measure $\mathcal{P}$ over a Borel algebra on T and a (special kind of) continuous map $f : T' \to T$, we can, canonically, obtain a probability measure on T'. We will set out this result as a fibration on paracategories, but first we provide some basic background.

- A **Borel probability measure** $\mathcal{P}$ is a topological space T endowed with a probability measure over the Borel σ-algebra of T. It is useful to consider the **degenerated** $\mathcal{P}^\infty$ Borel probability measure over some topological space T, with value $\mathcal{P}^\infty(A) = \infty$ (this is not even a measure since $\mathcal{P}^\infty(\emptyset) = \infty$);
- A **measure preserving transformation** between $\mathcal{P}$ and $\mathcal{P}'$ is a continuous map $t : T \to T'$ between the underlying topological spaces such that $P(t^{-1}(B')) = P'(B')$, for all $B' \in \mathcal{B}T'$.

The above notion of measure preserving transformation is often too strict to be useful in practice. In general, when we deal with probability spaces we need to consider subobjects so that the inclusion $\iota : \mathcal{P} \hookrightarrow \mathcal{P}'$ should be a morphism. This requirement is important, for instance, when we combine/restrict probabilistic automata or stochastic processes [Mat00]. We are thus led to consider the following notion of morphism between probability measures:

- A **subspace** $\mathcal{P}$ of a Borel probability measure $\mathcal{P}'$ is the Borel probability measure over the relative topology of T in T' where $P(B) = P'(B|T) = P'(B)/P'(T)$. Note that the if $P(T) = 0$ we obtain, by convention, the degenerated probability measure over T. So the empty set (with its topology) endowed with the degenerated measure is always a subspace of any Borel probability measure.
- A **morphism** between Borel probability measures $f : \mathcal{P} \to \mathcal{P}'$ is a measure preserving transformation between $\mathcal{P}$ and a subspace of $\mathcal{P}'$. More precisely:
  - $f : T \to T'$ is continuous;
  - $f(T)$ is a Borel set of T'[3];
  - $P(f^{-1}(B')) = P'(B'|f(T))$ for all Borel sets B' of T'.
- Borel probability measures endowed with their morphisms constitute the (proper) paracategory $\mathcal{B}or$, with partial composites induced by function composition in $\mathcal{S}et$ (or $\mathcal{T}op$).

The following proposition allows us to change the underlying topological space of a probability measure. It is a simple corollary of the unique extension theorem of a measure from a semi-algebra [Hal69].

[3]Whenever T is locally compact, has a countable base and T' is Hausdorff, $f(T)$ is a Borel set of T'

8.1. Proposition. *Given a continuous map* $f : T \to T'$ *and Borel probability measure* $\mathcal{P}'$ *over* T' *such that:*

1. $\{f^{-1}(O')\}_{O' \in T}$ *is a subbase of* T*;*
2. $f(T)$ *is a Borel set of* T'*;*

there exists a Borel measure $\mathcal{P}$ *over* T *such that* $f : \mathcal{P} \to \mathcal{P}'$ *is a morphism in* $\mathcal{B}or$*.*

Recall that the functor $U : \mathcal{T}op \to \mathcal{S}et$, which forgets the topology of a space, is a fibration[4] Hence the construction pertaining to Proposition 8.1.(1) yields cartesian morphisms:

8.2. Corollary. *Let* $\mathcal{T}op_C$ *be the broad*[5] *subcategory of* $\mathcal{T}op$ *whose morphisms satisfy conditions (1) and (2) of Proposition 8.1 and let* $\mathcal{B}or_C$ *be the analogous broad subparacategory of* $\mathcal{B}or$*. The forgetful functor* $U : \mathcal{B}or_C \to \mathcal{T}op_C$ *is a fibration of paracategories.*

As an immediate consequence topological subspaces (which are measurable) lift to probability subspaces (conditional probabilities), which was one of our desiderata.

8.2 Random variables and probabilistic automata

Random variables and stochastic processes are defined in terms of Borel probability measures. They provide examples of fibrations of paracategories. We build them in indexed terms, using the following basic concepts:

- A **measurable space** is a set Ω endowed with a σ-algebra $\mathcal{F}$ over it. A **measurable map** between measurable spaces $m : \langle \Omega, \mathcal{F} \rangle \to \langle \Omega', \mathcal{F}' \rangle$ is a function $m : \Omega \to \Omega'$ such that $m^{-1}(B') \in \mathcal{F}$ for all $B' \in \mathcal{F}'$. Measurable set and measurable maps constitute the category $\mathcal{M}es$.
- Each Borel probability measure $\mathcal{P}$ becomes a measurable set $\mathcal{B}_{\mathcal{P}}$ via its Borel σ-algebra. Furthermore a morphism $f : \mathcal{P} \to \mathcal{P}'$ induces a measurable map $f : \mathcal{B}_{\mathcal{P}} \to \mathcal{B}_{\mathcal{P}'}$ and we get thus a functor $M : \mathcal{B}or \to \mathcal{M}es$.
- A **random variable** X over a Borel probability measure $\mathcal{P}$ is a measurable map $X : M(\mathcal{P}) \to \mathbb{R}$ (that is, an element of $\mathcal{M}es(M(\mathcal{P}), \mathbb{R})$).
- A **random quantity** Q is an object of $M/\mathcal{M}es$, that is, an element of $\mathcal{M}es(M(\mathcal{P}), S)$ for some $\mathcal{P}$ and S. The measure space S is called the *state space of* Q.

[4]Given a topoloical space $(\Omega, \mathcal{U})$ and a function $f : \Omega' \to \Omega$, its U-cartesian lifting is $\overline{f} : (\Omega', \mathcal{U}') \to (\otimes, \mathcal{U})$ where $\mathcal{U}'$ is the topology generated $f^{-1}(\mathcal{U})$, which is the smallest one making f continuous.

[5]having the same objects as the ambient category

- A **stochastic process** is a family of random variables $\{X_i\}_{i \in I}$ over some probability space $\mathcal{P}$. I is called the *index parameter*. When I is a total order, it is called the *time structure*, otherwise it is called the *spatial structure*. The range of the random variable is called the *state space* of the process.
- A **general stochastic process** is a family of random quantities $\{Q_i\}_{i \in I}$ over some probability space $\mathcal{P}$. The process is called *space stable* iff all random quantities range over the same state space S. The terminology on the index parameter I is similar to the previous case. We shall be interested in state stable processes with spatial index structure.

We define the paracategories of random variables $\mathcal{R}ndV$ and random quantities $\mathcal{R}ndQ$ as the following comma-paracategories:

- $\mathcal{R}ndV \equiv M/\mathbb{R}$;
- $\mathcal{R}ndQ \equiv M/\mathcal{M}es$.

The functor that associates each random variable to the underlying Borel space $P : \mathcal{R}ndV \to \mathcal{B}or$ is a fibration, and therefore there is a indexed paracategory $D : \mathcal{B}or^{op} \to \mathcal{P}ar\mathcal{C}at$ where $D(\mathcal{P}) = \mathcal{R}ndV(\mathcal{P})$. We are now able to define the stochastic processes paracategory $\mathcal{S}tcP$:

- $\mathcal{S}tcP$ is the Grothendieck paracategory $\int F$, where $F : (\mathcal{B}or \times \mathcal{S}et)^{op} \to \mathcal{P}ar\mathcal{C}at$ is given as $F(\mathcal{P}, I) = \{I, \mathcal{R}ndV(\mathcal{P})\}$ (cotensor by the discrete category I).

Similarly, the functor that associates to a random quantity its underlying Borel space $P : \mathcal{R}ndQ \to \mathcal{B}or$ is a fibration. Thus, we obtain the Grothendieck paracategory $\mathcal{G}\mathcal{S}tcP = \int G$, where $G : (\mathcal{B}or \times \mathcal{S}et)^{op} \to \mathcal{P}ar\mathcal{C}at$ is given as $G(\mathcal{P}, I) = \{I, \mathcal{R}ndQ(\mathcal{P})\}$.

The paracategory of space stable processes is introduced by constraining $\mathcal{G}\mathcal{S}tcP$ to families of random quantities with the same space state. Hence, $\mathcal{S}sp$ is given by the following pullback in $\mathcal{P}ar\mathcal{C}at$

$$\begin{tikzcd}
\mathcal{S}sp \arrow[r, dotted] \arrow[d, dotted] & \mathcal{G}\mathcal{S}tcP \arrow[d, "St"] \\
\mathcal{S}et \times \mathcal{M}es \arrow[r, "Cn"'] & \int In
\end{tikzcd}$$

where $In : \mathcal{S}et^{op} \to \mathcal{P}ar\mathcal{C}at$ is $In(I) = \mathcal{M}es^I$, St maps each process to its family of state spaces and $Cn(I, S)$ is the constant $I-$indexed family with mesurable space S.

8.3. Proposition. *The functor $\pi_G : \mathcal{S}sp \hookrightarrow \mathcal{G}\mathcal{S}tcP \to \mathcal{B}or \times \mathcal{S}et$ is a fibration.*

Proof. Since $\pi_G : \mathcal{G}\mathcal{S}tcP \to \mathcal{B}or \times \mathcal{S}et$ is a fibration, for any object $T = (\mathcal{P}, I, \{Q_i : \Omega \to D\}_{i \in I})$ in $\mathcal{S}sp$ and morphism $(p, f) : (\mathcal{P}', I') \to \pi_G T$ there is a strong π_G-cartesian morphism $\overline{(p,f)}$ in $\mathcal{GS}tcP$ such that $\pi_G\overline{(p,f)} = (p,f)$. Since $dom\overline{(p,f)} = (\mathcal{P}', I', \{Q'_{i'} : \Omega \to D\}_{i' \in I'})$ where $Q'_{i'} = Q_{f(i')}$, we conclude that $dom\overline{(p,f)}$ is stable and so $\overline{(p,f)} = (p, f, \{id_D\}_{i \in I}) \in \mathcal{S}sp$.

It remains to show the universal property of $\overline{(p,f)}$ in $\mathcal{S}sp$. Consider morphisms $(q,g,c) : (\mathcal{P}'', I'', \{Q''_{i''} : \Omega'' \to D''\}_{i'' \in I''}) \to T$ and $(r,h) : (\mathcal{P}'', I'') \to (\mathcal{P}', I')$ such that

$$\begin{tikzcd}
(\mathcal{P}', I') \arrow[rr, "(p,f)"] & & \pi_G T \\
& (\mathcal{P}'', I'') \arrow[ul, "(r,h)"] \arrow[ur, "(q,g)"'] &
\end{tikzcd}$$

There exists a unique morphism $\widetilde{(r,h)} = (r,h,c)$ which makes the following diagram commute in $\mathcal{S}sp$

$$\begin{tikzcd}
(\mathcal{P}', I', \{Q'_{i'} \to D\}_{i' \in I'}) \arrow[rr, "\overline{(p,f)}"] & & T \\
& (\mathcal{P}'', I'', \{Q''_{i''} : \Omega'' \to D''\}_{i'' \in I''}) \arrow[ul, "\widetilde{(r,h)}"] \arrow[ur, "(q,g,c)"'] &
\end{tikzcd}$$

□

A probabilistic automaton with input set I and state space $S \in \mathcal{M}es$ is a state stable process indexed by $S \times I$ and with state space S (this process is usually called the *probability transition function*) plus an element from S (the *initial state*). Therefore, we obtain the paracategory $\mathcal{P}aut$ of probabilistic automata as the following pullback in $\mathcal{P}ar\mathcal{C}at$:

$$\begin{tikzcd}
\mathcal{P}aut \arrow[r, dotted] \arrow[d, dotted] & \mathcal{S}et \times \mathcal{M}es_* \arrow[d, "X"] \\
\mathcal{S}sp \arrow[r, "\langle Ind, St \rangle"'] & \mathcal{S}et \times \mathcal{M}es
\end{tikzcd}$$

where X is such that $X(I,S) = (I, S \times I)$ (imposing the index parameter to be a product of the state space with a set), Ind is the functor that associates each state stable process to its index parameter and Mes_* is the category of pointed measurable spaces such that for the point $s \in S$ its singleton $\{s\}$ is measurable.

Finally, in order to augment probabilistic automata with outputs (in Moore style [LMSS56]), we consider an *output labeling function* which assigns to each state an output, that is, an object Λ of $S/\mathcal{M}es$. The *set of outputs* of the automata is the codomain of Λ. We obtain then the paracategory of *Moore probabilistic automata* $\mathcal{MP}aut$ as the the following pullback:

$$\begin{tikzcd}
\mathcal{MP}aut \arrow[r, dashed] \arrow[d, dashed] & \mathcal{M}es^{\to} \arrow[d, "Dom"] \\
\mathcal{P}aut \arrow[r, "St"'] & \mathcal{M}es
\end{tikzcd}$$

More explicitly, $\mathcal{MP}aut$ is consists of:

objects tuples of the form $(\mathcal{P}, S, I, O, \delta, s_0, \Lambda)$ where:

- $\mathcal{P} \in \mathcal{B}or$; $S, O \in \mathcal{M}es$; $I \in \mathcal{S}et$;
- $\delta = \{\delta_{si}\}_{si \in S \times I}$, δ_{si} a random quantity over $\mathcal{P}$ with state space S;
- $s_0 \in S$;
- $\Lambda : S \to O$ is a morphism in $\mathcal{M}es$.

morphisms tuples of the form (p, f, g, h) where:

- $p \in \mathcal{B}or(\mathcal{P}, \mathcal{P}'); f \in \mathcal{M}es(S, S'); g \in \mathcal{S}et(I, I'); h \in \mathcal{M}es(O, O')$;

such that the following diagram commutes in $\mathcal{M}es$ for all $si \in S \times I$

$$\begin{tikzcd}
1 \arrow[r, "s_0"] \arrow[dr, "s'_0"'] & S \arrow[d, "f"] \\
& S'
\end{tikzcd}
\qquad
\begin{tikzcd}
M(\mathcal{P}) \arrow[r, "\delta_{si}"] \arrow[d, "M(p)"'] & S \arrow[r, "\Lambda"] \arrow[d, "f"] & O \arrow[d, "h"] \\
M(\mathcal{P}') \arrow[r, "\delta_{f(s)g(i)}"'] & S' \arrow[r, "\Lambda'"'] & O'
\end{tikzcd}$$

composition inherited from $\mathcal{B}or \times \mathcal{M}es \times \mathcal{S}et \times \mathcal{M}es$.

We conclude this section by pointing out that $\langle P, Inp\rangle : \mathcal{MP}aut \to \mathcal{B}or \times \mathcal{S}et$ is a fibration:

8.4. Proposition. *The functor $\langle P, Inp\rangle : \mathcal{MP}aut \to \mathcal{B}or \times \mathcal{S}et$ is a fibration.*

Proof. Given an automaton $A = (\mathcal{P}, S, I, O, \{\delta_{si}\}_{si \in SI}, s_0, \Lambda)$ and a morphism $(p, f) : (\mathcal{P}', I') \to \langle P, Inp\rangle A$ its $\langle P, Inp\rangle$-cartesian lifting is $\overline{(p, f)} = (p, id_S, f, id_O) : (\mathcal{P}', S, I', O, \{\delta'_{si'}\}_{si' \in SI'}, s_0, \Lambda) \to A$, with $\delta'_{si'} = \delta_{sf(i)}$.
To verify its universal property, given another morphism $(q, t, g, c) : (\mathcal{P}'', S'', I'', O'', \{\delta''_{s''i''}\}_{s''i'' \in S''I''}, s''_0, \Lambda'') \to A$ and a factorisation

$$\begin{tikzcd}
(\mathcal{P}', I') \arrow[rr, "(p,f)"] & & \pi_G T \\
& (\mathcal{P}'', I'') \arrow[ul, "(r,h)"] \arrow[ur, "(q,g)"'] &
\end{tikzcd}$$

it lifts to a factorisation

$$\begin{tikzcd}
(\mathcal{P}', S, I', O, \{\delta'_{si'}\}_{si' \in SI'}, s_0, \Lambda) \arrow[rr, "\overline{(p,f)}"] & & A \\
& & (\mathcal{P}'', S'', I'', O'', \{\delta''_{s''i''}\}_{s''i'' \in S''I''}, s''_0, \Lambda'') \arrow[ull, "\widetilde{(r,h)}"] \arrow[u, "(q,t,g,c)"']
\end{tikzcd}$$

with $\widetilde{(r, h)} = (r, t, h, c)$. □

The above fibration allows us to change the probabilities of the transition process of an automaton canonically, along a morphism in $\mathcal{B}or$. In particular, the cartesian lifting of a monomorphism in $\mathcal{B}or$provides us with our required restriction of an automata to a subset of outcomes.

8.3 Behaviour and realisation

The behaviour of a Moore probabilistic automaton with input set I and output set O is a state stable process indexed by I^* and with state space O. The process gives the probability of obtaining some output o after performing a sequence of actions $\sigma \in I^*$. We define the paracategory of behaviours $\mathcal{B}eh$ via the following pullback:

$$\begin{tikzcd}
\mathcal{B}eh \arrow[r, dashed] \arrow[d, dashed] & \mathcal{S}sp_* \arrow[d, "Ind"] \\
\mathcal{S}et \arrow[r, "M"'] & \mathcal{S}et
\end{tikzcd}$$

where $M(I) = I^+$, the set of non-empty sequences of I, and $\mathcal{S}sp_*$ is the subparacategory of $\mathcal{S}sp$ where the state space is pointed (and the morphisms preserve the points). More explicitly, $\mathcal{B}eh$ consists of:

objects tuples $(\mathcal{P}, I, O, o, \{Q_\sigma\}_{\sigma \in I^+})$;

morphisms tuples (p, g, h) such that the following diagram commutes in $\mathcal{M}es$ for all $\sigma \in I^*$:

$$\begin{tikzcd}
M(\mathcal{P}) \arrow[r, "Q_\sigma"] \arrow[d, "M(p)"'] & O \arrow[d, "h"] & 1 \arrow[l, "o"'] \arrow[dl, "o'"] \\
M(\mathcal{P}') \arrow[r, "Q_{g^*(\sigma)}"'] & O' &
\end{tikzcd}$$

composition inherited from $\mathcal{B}or \times Set \times \mathcal{M}es$.

Just as for automata, we have the following result:

8.5. Proposition. *The functor $\langle P, Inp\rangle : \mathcal{B}eh \to \mathcal{B}or \times \mathcal{S}et$ is a fibration.*

Proof. We note that $\mathcal{B}eh$ can be identified with the pullback

$$\begin{tikzcd}
\mathcal{B}eh \arrow[r, dashed] \arrow[d, dashed, "\langle P, Inp\rangle"'] & \mathcal{S}sp_* \arrow[d, "\langle P, Ind\rangle"] \\
\mathcal{B}or \times \mathcal{S}et \arrow[r, "id_{\mathcal{B}or} \times M"'] & \mathcal{B}or \times \mathcal{S}et
\end{tikzcd}$$

Adapting Proposition 8.3, we see that $\langle P, Ind\rangle : \mathcal{S}sp_* \to \mathcal{B}or \times \mathcal{S}et$ is a fibration, and hence, $\langle P, Inp\rangle : \mathcal{B}eh \to \mathcal{B}or \times \mathcal{S}et$ is a fibration as well, since these are stable under pullback (*change of base*). $\square$

Given a Moore probabilistic automaton $(\mathcal{P}, I, S, \delta, s_0, \Lambda)$ we can extract its **behaviour** $(\mathcal{P}, I, O, o, \{Q\}_{\sigma \in I^+})$, where:

- $o = \Lambda(s_0)$;

- $Q_\sigma(\omega) = \Lambda(\delta^*_{s_0\sigma}(\omega))$;
- δ^* is a state stable process over $\mathcal{P}$, with state space S, indexed by $S \times I^*$, with $\delta^*_{s\epsilon}(\omega) = s$ and $\delta^*_{s\sigma i}(\omega) = \delta_{\delta^*_{s\sigma}(\omega)i}(\omega)$.

This construction can be extended in the obvious way to a (Kleene) functor $B : \mathcal{MP}aut \to \mathcal{B}eh$.

8.6. Theorem. *The functor B has a full and faithful left adjoint (the free realisation), fibred over $\mathcal{B}or \times \mathcal{S}et$:*

$$\begin{tikzcd}
 & \mathcal{B}or \times \mathcal{S}et & \\
\mathcal{MP}aut \arrow[ur, "\langle P{,}Ind\rangle"] \arrow[rr, bend right, "B"'] \arrow[rr, phantom, "\perp"] & & \mathcal{B}eh \arrow[ul, "\langle P{,}Ind\rangle"'] \arrow[ll, bend right, "R"']
\end{tikzcd}$$

Proof. Set $R(\mathcal{P}, I, O, o, \{Q\}_{\sigma \in I^+}) = (\mathcal{P}, S, I^+, \delta, (\epsilon, o), \Lambda)$, where

- S is the smallest measurable space containing the space $M(\mathcal{P}) \times I^+ \times O$ and $\{(\epsilon, o)\}$, where I^+ is the free measurable space over the non-empty sequences of I;
- $\delta_{(\omega,\sigma,o)i}(\omega') = (\omega', \sigma i, Q_{si}(\omega'))$;
- $\Lambda(\omega, \sigma, o) = o$.

The counit is $\epsilon_{(\mathcal{P},S,I,O,\{\delta_{si}\}_{si\in SI},s_0,\Lambda)} = (id_{\mathcal{P}}, f, id_I, id_O)$ with

- $f(\epsilon, o) = s_0$;
- $f(\omega, \sigma, u) = \delta^*_{s_0\sigma}(\omega)$.

while the unit is $\eta_{(\mathcal{P},I,O,o,\{Q_\sigma\}_{\sigma\in I^+})} = (id_{\mathcal{P}}, id_I, id_O)$ (hence R is full and faithful). $\square$

8.7. Remark. When B is restricted to the fibre over $\langle \mathcal{P}, Inp, Out\rangle$ (Out being the functor that associates to an automaton its output set), it has a right adjoint called the *Nerode* or *minimal realisation* functor.

8.4 Tensor product of probabilistic automata

In this section we show that the free aggregation of probabilistic automata in [Mat00] arises as a tensor product associated to a representable partial multicategory. First, we set up the corresponding structure at the level of probability spaces $\mathcal{B}or^{\tilde{\partial}}_{\blacktriangleright}$ and define the relevant multicategory of probabilitstic automata $\mathcal{MP}aut^{\tilde{\partial}}_{\blacktriangleright}$ upon it.

Tensor product of probability spaces

We consider the full subparacategory $\mathcal{B}or^{\eth}$ of $\mathcal{B}or$ whose underlying topological spaces are locally compact, with a countable base and Hausdorff.

Define a multicategory $\mathcal{B}or^{\eth}_{\blacktriangleright}$ as follows:

objects those of $\mathcal{B}or^{\eth}$;

morphisms $\langle f_1, ...f_n\rangle : \langle \mathcal{P}_1, \ldots, \mathcal{P}_n\rangle \to \mathcal{P}$ where $f_i : \mathcal{P} \to \mathcal{P}_i$ is a continuous map between the underlying topological spaces;

composition pointwise composition of cones:

$$\langle f_1, \ldots, f_n\rangle\langle\langle f_{11}, \ldots, f_{1m_1}\rangle \ldots \langle f_{n1}, \ldots, f_{nm_n}\rangle\rangle = \langle f_{11}f_1, \ldots, f_{nm_n}f_n\rangle$$

Consider the following class of morphisms in $\mathcal{B}or^{\eth}_{\blacktriangleright}$:

$$D = \{\langle f_1, ...f_n\rangle : \langle \mathcal{P}_1, \ldots, \mathcal{P}_n\rangle \to \mathcal{P} \mid \forall_{i\in 1,\ldots,n}\forall_{B_i\in\mathcal{F}_i} P(\bigcap_{i=1}^{n} f_i^{-1}(B_i)) = \prod_{i=1}^{n} P_i(B_i|f_i(\Omega_i))\}$$

8.8. Theorem. *The partial multicategory $(\mathcal{B}or^{\eth}_{\blacktriangleright}, D)$ is representable.*

Proof. Set $\otimes_{i=1}^{n}\mathcal{P}_i = (\prod_{i=1}^{n}\Omega_i, \mathcal{F}, P)$ where

- $\mathcal{F}$ is the σ-algebra generated by $\mathcal{S} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$
- P is the unique measure such that $P(B_1\times, \ldots, \times B_n) = \prod_{i=1}^{n} P_i(B_i)$, which exists because $\mathcal{S}$ is a semi-algebra for $\mathcal{F}$.

The cone of projections out of $(\prod_{i=1}^{n}\Omega_i, \mathcal{F}, P)$ is the required universal morphism. Further details can be found in [Mat00]. □

We have thus endowed the paracategory $\mathcal{B}or^{\eth} = \overline{(\mathcal{B}or^{\eth}_{\blacktriangleright}, D)}$ with a monoidal structure.

8.9. Remark. The multicategory $\mathcal{B}or^{\eth}_{\blacktriangleright}$ is essentially the same construction as the multicategory of discrete (co)cones $\mathbb{C}_{\blacktriangleright}$ associated to any category $\mathbb{C}$ in [Her00, Ex. 2.2(2)], whose representability corresponds to the existence of (co)products in $\mathbb{C}$.

Aggregation of probabilistic automata

We consider the full subparacategory $\mathcal{MP}aut^{\eth}$ of $\mathcal{MP}aut$ whose underlying probability spaces are in $\mathcal{B}or^{\eth}$.

Define a multicategory $\mathcal{MP}aut^{\eth}_{\blacktriangleright}$ as follows:

objects those of $\mathcal{MP}aut^{\eth}$;

morphisms $\langle f_1, ...f_n\rangle : \langle M_1, \ldots, M_n\rangle \to M$
where $f_i = (f_i^1, f_i^2, f_i^3, f_i^4) : M = (\mathcal{P}, S, I, O, \delta, s_0, \Lambda) \to M_i$ is such that

- f_i^1 is a continuous map between the underlying topological spaces of $\mathcal{P}$ and $\mathcal{P}_i$;
- f_i^2, f_i^3, f_i^4 are like in $\mathcal{MP}aut$.

composition pointwise composition of cones as in $\mathcal{B}or_{\blacktriangleright}^{\eth}$

Consider the following class of morphisms in $\mathcal{MP}aut_{\blacktriangleright}^{\eth}$:

$$\underline{D} = \{\langle f_1, ...f_n\rangle : \langle M_1, \ldots, M_n\rangle \to M \mid \langle f_1^1, ...f_n^1\rangle \in D\}$$

8.10. Corollary. *The partial multicategory* $(\mathcal{MP}aut_{\blacktriangleright}^{\eth}, \underline{D})$ *is representable.*

We have thus produced yet another monoidal paracategory $\mathcal{MP}aut^{\eth} = (\mathcal{MP}aut_{\blacktriangleright}^{\eth}, \underline{D})$.

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