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Ex Contradictione Non Sequitur Quodlibet

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We summarize here the main arguments, basic research lines, and results on the foundations of the logics of formal inconsistency. These involve, in particular, some classes of well-known paraconsistent systems. We also present their semantical interpretations by way of possible-translations semantics and their applications to human reasoning and machine reasoning.[1]

1. Do we need to worry about inconsistency?

Classical logic, as we all know, cannot survive contradictions. Among the principles that were gradually incorporated into the "properties of correct reasoning" since Aristotle, the *Principle of Pseudo-Scotus* (*PPS*), also known since medieval times as *ex contradictione sequitur quodlibet* (and also called the *Principle of Explosion* by some contemporary logicians), states that in any theory exposed to the enzymatic character of a contradiction A and ¬A one can derive any other arbitrary sentence B, so that the theory would turn out to be trivial. Another principle called the *Principle of Non-Contradiction* (*PNC*) states that there should be theories from which no such contradictions are derivable. To those principles, one could add the *Principle of Non-Triviality* (*PNT*), stating that there should be at least one theory and one sentence B such that B is not derivable from this theory.

In order to fully understand what those principles mean, what the relationship is between them and what their importance is for modeling the concept of inconsistency let us introduce some formalism. This formalism will be apt for the syntactical approach to the logics of formal inconsistency we discuss in the first three sections of this paper. However, the reader should be aware that it is possible to start from a purely syntactical account, as we do in Sections 4 and 5.

Let **For** be a collection of formulas of a certain language, and call a *theory* any subset of **For**. Let a consequence relation $\vdash$ over **For** be a relation between theories and formulas of **For**, that is, $\vdash \subseteq (\wp(\text{For}) \times \text{For})$, where $\wp(\text{For})$ denotes the power set of **For**. If Γ is a subset of **For**, we write $\Gamma \vdash A$ when $\langle\Gamma, A\rangle \in \vdash$. We write $\Gamma \nvDash A$ when it is not the case that $\Gamma \vdash A$. We define a *logic* **L** to be a structure constituted of **For** and the relation $\vdash$. The consequence relation of a given logic is often defined by its axioms and rules, or else from some semantical interpretation associated with the logic.

[1] This material was discussed during the II World Congress of Paraconsistency (WCP'2000) in Juquehy, SP, Brazil, and will appear more fully in W. A. Carnielli, and J. Marcos, "A taxonomy of C-systems."

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Some basic assumptions on what the relation $\vdash$ would have to obey in order to be considered a consequence relation, known as the Tarskian conditions, are the following:

(1) *Reflexivity* If $A \in \Gamma$, then $\Gamma \vdash A$.
(2) *Monotonicity* If $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash A$.
(3) *Transitivity* If $\Gamma \vdash A$ and $\{\Delta, A\} \vdash B$, then $\Gamma \cup \Delta \vdash B$.

These conditions allow for the characterization of an immense number of distinct logics, but they still can be made more permissive (as, for example, weakening the requirement of monotonicity in order to characterize non-monotonic logics). In our present study, we will stick to these three basic assumptions and consider the effect of some additional properties of the relation $\vdash$.

Fix some logic **L** for the following discussion. A theory Γ of **L** is said to be:

(1) *Contradictory with respect to* $\neg$ (or simply *contradictory*)
If there exists a formula A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.
(2) *Trivial*
If for every A, we have $\Gamma \vdash A$.
(3) *Explosive*
If for every A, we have $\Gamma \cup \{A, \neg A\} \vdash B$.

These definitions are important for distinguishing theories from their underlying logic: A logic **L** is *contradictory, trivial,* or *explosive* if, respectively, all of its theories are contradictory, trivial, or explosive.

We can now restate *PNC, PNT,* and *PPS* in more formal terms.

• **The Principle of Non-Contradiction** (**PNC**) for a logic **L**
L should have non-contradictory theories, that is, there should be some theory Γ such that for no A, $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.

• **The Principle of Non-Triviality** (**PNT**) for a logic **L**
L should have a non-trivial theory, that is, there must exist a theory Γ and some formula A such that $\Gamma \nvdash A$.

• **The Principle of Pseudo-Scotus** (**PPS**) for a logic **L**
L should have only explosive theories, that is, for every theory Γ, $\Gamma \cup \{A, \neg A\}$ is trivial.

For classical logic, of course, all these principles hold. It seems intuitively acceptable that **PNT** should be taken as the most important of these three principles —after all, if **PNT** does not hold for a certain logic, then every Γ would entail every A and the relation $\vdash$ would be total, that is, $\vdash \; = (\wp(\text{For}) \times \text{For})$. In that case, $\vdash$ would not be an interesting *deductive* relation, for it would stop "making the difference," failing to give, we argue, any special meaning to the notion of *derivability*. We shall, accordingly, accept **PNT** throughout this study, avoiding the consideration of trivial logics.

It is also interesting to remark that, using the property of monotonicity mentioned above, it is sufficient to consider **PPS** valid for just the empty theory: Call $\mathbf{PPS}_{\varnothing}$ the statement "{A, ¬A} is trivial." Then, in view of monotonicity, we conclude that $\mathbf{PPS}_{\varnothing}$ implies **PPS**. The converse (**PPS** implies $\mathbf{PPS}_{\varnothing}$) is obvious. Similar reasoning applies to **PNC** and **PNT**. Call $\mathbf{PNT}_{\varnothing}$ the statement "the empty theory is not deductively trivial" and call $\mathbf{PNC}_{\varnothing}$ the statement "the empty theory is non-contradictory." By monotonicity, **PNT** implies $\mathbf{PNT}_{\varnothing}$, and the converse is obvious, and the same for **PNC** and $\mathbf{PNC}_{\varnothing}$. Thus, for monotonic logics we do not need to care about the distinctions between **PPS**, **PNT**, **PNC** and their respective counterparts, $\mathbf{PPS}_{\varnothing}$, $\mathbf{PNT}_{\varnothing}$, $\mathbf{PNC}_{\varnothing}$. Dealing with non-monotonic logics, the distinctions would have to be taken into consideration, but our primary treatment of the question would still apply.

We intend to discuss here the *logics of formal inconsistency,* which constitute a large class of paraconsistent logics where the notion of inconsistency can be linguistically expressed. For those logics, we may adopt or not a *cautious* position to the effect that not only **PNT** but also **PNC** should be required—although we want our logics to be able to support contradictory theories, we may not want that our logics derive contradictions.

Paraconsistent logics are often misunderstood as logics that inevitably derive contradictions. This is clearly a mistake. Although there exist some logics (the so-called *dialectical logics,* or *logics of impossible objects*) that violate both **PPS** and **PNC** and have theses which are not classical theses, this particular case of paraconsistent logics will *not* be studied here.[2] The logics surveyed in this study just support contradictions and permit reasoning with them, but neither engender contradictions nor validate any bizarre form of reasoning. To the contrary: As we shall see, the logics of inconsistency are, in a sense, "more conservative" than classical logic.

While **PNT** and **PNC** can be regarded as *ontological* principles in that they presuppose the existence of certain theories inside our logics, **PPS** can be seen as a kind of flexible, *operative* principle: It describes how the logic works when its theories are exposed to contradictory formulas, and thus it seems likely to be changed or even discarded by some logics and situations in which the trivializing operation that **PPS** describes is not justified. So, the starting point to our approach to paraconsistency will be the cautious one, and we will ask ourselves: (1) While maintaining **PNT** and **PNC**, is there any good reason why we should challenge **PPS**? (2) If so, should we modify or simply abandon this principle?

The first is a factual question: We should try to challenge **PPS** if there is sufficient demand, and we will argue that there is. The second is a foundational question of logico-mathematical character, and we will also argue that it is plainly possible to construct a great variety of interesting logics alternative to (propositional and quantified) classical logic maintaining **PNT** and **PNC**, while modifying only **PPS**.

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[2] Compare C. Mortensen, "Aristotle's thesis in consistent and inconsistent logics" and N. C. A. da Costa, and R. G. Wolf, "Studies in paraconsistent logic I: The dialectical principle of the unity of opposites".

An interesting example of an intellectual activity where holding contradictory or inconsistent hypotheses is more the rule than the exception is abductive reasoning, conceived of as reasoning that looks for explanatory hypotheses and the evaluation of such hypotheses. Scientific activity, and in particular medical diagnosis, generally uses abduction when looking for explanations. According to P. Thagard, and C. Shelley, this is inevitable:

> We are not urging inconsistency as a general epistemological strategy, only noting that it is sometimes necessary to form hypotheses inconsistent with what is currently accepted in order to provoke a general belief revision that can restore consistency.[3]

A second example of how contradictions can easily be incorporated into reasoning (independent of any ontological commitments to the actual existence of concrete inconsistent objects in the world) concerns the domain of machine intelligence and the efforts to find rules for automated reasoning. The following example given by A. Rose in "Remarque sur les notions d'indépendance et de non-contradiction" was intended to show that the concepts of independence and triviality in a formal system are themselves independent of each other, but it can also be used as an example of how certain rules and procedures, widely used in the formalization of machine reasoning, must face the problem of contradictions. One such procedure is the closed-world assumption, largely used in databases and logic programming, which proposes that, if from a certain (knowledge-based) system **S** one cannot infer information A and know nothing about its negation $\neg$A, then one would be entitled to add $\neg$A to **S** by default. Consider now the following fragment **S** of classical propositional logic closed under modus ponens and the substitution rule:

1. $(((A\to\neg A)\to A)\to A)$
2. $(A\to(B\to A))$
3. $((\neg B\to\neg A)\to(A\to B))$
4. $((A\to B)\to((B\to C)\to(A\to C))$

Consider also the formula $D = (\neg A\to A)\to A$. It is not hard to show that **S** does not entail either D or $\neg$D, but if $\neg$D is adjoined to **S** the resulting system becomes trivial. Since the closed-world assumption has exactly the effect of adjoining $\neg$D to **S**, it is clear that that assumption may produce contradictions, which will result in triviality due to **PPS**. It is, in fact, not hard to see that **PPS** holds in **S**:

a. From A and $\neg$A, by axiom 2 derive $((B\to\neg B)\to A)$ and $(\neg B\to\neg A)$.
b. From $(\neg B\to\neg A)$ by axiom 3 derive $(A\to B)$.
c. From $((B\to\neg B)\to A)$ and $(A\to B)$ by axiom 4 derive $((B\to\neg B)\to B)$.
d. From $((B\to\neg B)\to B)$ by axiom 1 derive B .

Hence for every Γ, A, and B, we have $\Gamma\cup\{A,\neg A\}\vdash_{\mathbf{S}} B$.

[3] "Abductive reasoning: logic, visual thinking, and coherence."

As we shall see, such a trivialization does not occur in the logics of formal inconsistency that we will consider, for axiom 3 above will in all cases be valid only under the proviso that A is known to be not inconsistent.

2. The contradictory and the inconsistent

The concepts of contradiction and inconsistency need not be taken to be equivalent. That classical logicians take them as equivalent is explained by two facts: first, for classical logicians, "contradiction," "inconsistency" and "triviality" are usually seen as synonymous, and, second, **PNC** and **PNT** are indeed equivalent for several logics. Yet this equivalence is far from necessary.

It is clear that trivial theories, closed or not, are necessarily contradictory (if there is a symbol for negation in the language) since they derive all sentences, negated and non-negated. From the definitions of **PNC** and **PNT** above we obtain the following result, which will hold in all of our logics of formal inconsistency:

Metatheorem I Every trivial theory is contradictory, that is, **PNC** implies **PNT**.

Proof: Suppose **PNC** holds. Then, there exists a theory Γ such that, for every A, either $\Gamma \nvdash A$ or $\Gamma \nvdash \neg A$. Hence there exist Γ and A such that $\Gamma \nvdash A$, and this is what **PNT** states.

The converse, however, does not hold, because an appeal to **PPS** is necessary in order for a contradictory theory to be trivial. We may add, as our second founding result:

Metatheorem II Assuming **PPS**, contradictory theories and trivial theories coincide; that is, **PPS** implies: **PNT** if and only if **PNC**.

Proof: Suppose that **PPS** holds. It follows then from the transitivity of $\vdash$ that if a theory is contradictory it is trivial, that is, that **PNT** implies **PNC**.

This last metatheorem can be understood as *ex contradictione non sequitur quodlibet,* in the sense that what follows from a contradiction depends on other underlying principles (in this case, it depends on **PPS**). So, controlling the explosive power of contradictions is necessary to gain control over the destructive effects of trivialization. But is that enough?

It is clear what contradictory and trivial theories are, as those concepts can be defined in terms of purely set-theoretic conditions. It is also clear what the role of **PPS** is, since, if **PPS** holds, every contradictory theory turns out to be trivial.

If we can only modify **PPS**, the sets of consequences of contradictory theories will not necessarily be trivial. But what is a consistent theory? While contradiction and triviality are definable, respectively, in terms of negation (hence as linguistic) and in set-theoretical terms (hence as metamathematical), consistency is better seen as a semantic concept.

The novelty of our approach is that, although acknowledging the semantic character of the notion of consistency, we endeavor to internalize it in the language and treat it then from a purely abstract point of view, independently from contradiction and triviality. Considering that it is **PPS** that forces negation to behave in such absolute terms that contradictory theories collapse into triviality, it is possible, as we shall see, to define forms of "careful" negation in several ways so as to avoid such collapse.

In traditional logic, inconsistency and contradiction are taken to be just two ways of referring to the same concept. For example, neither Aristotle nor Łukasiewicz, in his celebrated analysis of Aristotelian logic, ever mentioned consistency or inconsistency, but always contradictions, since in classical logic the distinction is immaterial—for, in this case, the metamathematical concept of inconsistency is internalized into the object language by means of conjunctions of contradictory statements.[4] This is so, however, due to **PPS** (Metatheorems I and II). If **PPS** is not to be accepted as dogma, for the reasons already presented, such an internalization will not be guaranteed.

It is possible, nonetheless, to conceive of the concept of inconsistency in such a way that, while contradictory theories are certainly inconsistent, the reverse might not be true. In a similar way to the concept of point, which is taken in geometry as a primitive notion only describable through its relationship to other concepts, inconsistency can be taken in logic to be a primitive notion as well. From this point of view, inconsistent and contradictory theories do not coincide, as our logics of formal inconsistency **bC** and **Ci** will make clear.

Contradictory theories depend strictly on negation and its properties, while inconsistent theories do not. So, for example, if we say "It is raining in Ghent" and "It is not raining in Ghent," those are contradictory statements, but should not necessarily lead to trivialization. They may lead to trivialization if we add some extra information, for example, that we are talking about the same instant of time, and that the concept of instant of time is sharp enough so as to exclude the possibility of rain and absence of rain at a given instant of time. Or we may say "It is raining at 11h 34m22s" and say "It is not raining at 11h34m22s," which again are contradictory statements, but do not necessarily lead to trivialization. In this case, they may become so if we add extra information, for example that we talking about the same point in space, and that the concept of point is strict enough so as to exclude rain and the absence of rain.

Von Wright suggests that the Kantian conceptions of space and time in the *Critique of Pure Reason*[5] (though with different aims) are not alien to this kind of intuition:

> If this representation [he refers to time] were not an a priori (inner) intuition, no concept, no matter what it might be, could render comprehensible the possibility of an alteration, that is, of a combination of contradictorily opposed predicates in one and the same object, for instance, the being and the not-being of one and the same thing in one and the same place. Only in time

[4] J. Łukasiewicz, "On the principle of contradiction in Aristotle."

[5] Compare I. Kant, *Critique of Pure Reason,* Transcendental Aesthetic II, §4.

can two contradictorily opposed predicates meet in one and the same object, namely, one after the other.[6]

The point here is not whether or not one accepts the transcendental ideality of time, but the role of time (and also space) as examples of entities that bind contradictory statements together and make them inconsistent. In other words, contradictory statements A and ¬A by themselves will not be sufficient to entail any other statement, unless we require an extra condition—in our approach, that A is consistent. The consistency here could be understood as introduced by a Kantian rendering of space and time as concepts which exclude the possibility of the concomitant existence of opposed predicates at the same point of space or time.

Based on this intuition, as an addendum to the results in the **Metatheorem I** and **Metatheorem II**, we propose the following as our basic metaprinciple:

Metaprinciple I
No contradictory theory is consistent, but a contradictory non-consistent theory need not be trivial.

To this metaprinciple we can consider the addition of another (a kind of converse of **Metaprinciple I**):

Metaprinciple II
Every inconsistent theory is contradictory, but not necessarily trivial.

As we will show, distinct classes of paraconsistent calculi arise, depending on whether we take **Metaprinciple II** in conjunction with **Metaprinciple I**. It is noteworthy that virtually all known paraconsistent systems in the literature do assume **Metaprinciple II**.

It is possible to give models for inconsistent theories, even if those might be regarded as epistemologically puzzling. Obtaining models and understanding their role is an extraordinarily important mathematical enterprise: It required enormous efforts of the most brilliant minds, and more than twenty centuries, until mathematicians would allow themselves to consider models where, given a straight line S and a point P outside of it, one could draw not just one line, but infinitely many or no parallel lines to S passing through P, as in the well-known case of non-Euclidean geometries.

3. A logic for the illogical?

The challenge is to find mathematically interesting systems that can provide a foundational sense for what contradictions and inconsistency are and suggest an acceptable semantic interpretation with which people would feel comfortable while reasoning with contradictions. The case of (imaginary) complex numbers seems to make a good comparison: Even if we do not know what they are, and may even suspect there is little sense in insisting on which way they can exist in the "real" world, the most important aspect is that it is possible to calculate with

6 G. H. von Wright, "Time, change and contradiction".

them. Girolamo Cardano, who first had the idea of computing with such numbers, seems to have seen this point clearly—he failed, however, to acknowledge the importance of it. In 1545 he wrote in his *Ars Magna*:

> Dismissing mental tortures, and multiplying $5+\sqrt{-15}$ by $5-\sqrt{-15}$, we obtain $25-(-15)$. Therefore the product is 40. . . . And thus far does arithmetical subtlety go, of which this, the extreme, is, as we have said, so subtle that it is useless.[7]

His discovery that one could operate with a mathematical concept independently of what our intuition says, and that utility (or something else) could be a guiding criterion for accepting or rejecting experimentation with mathematical objects, certainly contributed to the proof of the Fundamental Theorem of Algebra by C. F. Gauss in 1799, before which complex numbers were not fully accepted.

The idea of proposing logics that enable one to operate with what does not appear to be "rational" goes in the same direction: Good underlying mathematical theory plus usefulness would have to constitute the only criteria to evaluate a mathematical formalism that deals with inconsistency or contradictions.

It is time now to give a more precise definition for the logics of formal inconsistency (**LFI**-*systems*): An **LFI** is any logic where a syntactic notion of formal consistency can be defined in a syntactical way in such a way that this new notion of formal consistency and the already known notion of contradiction can be related in the light of Metaprinciples I and II. In particular, as we discuss below, in many cases this can be done by endowing the language with a new connective $\circ$ and considering new appropriate axioms.

Among **LFI**-systems it is possible to identify a subclass of the so-called **C**-s*ystems* that preserve the positive fragment of some other logic and in which consistency or inconsistency are expressible by means of new connectives. As a subclass of the **C**-systems, we define the **dC**-*systems* to be those in which the notion of formal consistency can be introduced as a defined connective. The **dC**-systems include several classes of paraconsistent systems, including the ones in the hierarchy C_n of N. C. A. da Costa.[8]

The main axioms we will consider here for the study of an interesting class of **C**-systems based on classical logic are the following: Call C_{min} an appropriate axiomatization of the positive fragment of classical propositional logic in the language $\neg, \rightarrow, \wedge, \vee, \circ$, plus the axioms $(\neg\neg A \rightarrow A)$ and $(A \vee \neg A)$ and closed under the rules of *modus ponens* and substitution.[9] Define the basic logic of formal inconsistency, **bC**, as C_{min} plus the following deduction scheme, where $\circ$ is a new unary operator meant to model "A is formally inconsistent":

The Gentle Principle of Explosion $\circ A, A, \neg A \vdash B$

Note that this axiom is in line with **Metaprinciple I**: A contradictory theory (one containing A and $\neg$A) is not consistent and is not necessarily trivial. It would

[7] See J. O'Connor, and E. Robertson, "The MacTutor History of Mathematics Archive".
[8] See N. C. A. da Costa, *Inconsistent Formal Systems*.
[9] This system was studied in our "Limits for paraconsistent calculi".

become trivial if, besides being contradictory, it were formally consistent. In such a case its very consistency becomes contradictory, and this situation leads to triviality.

Examples of dervived consequences of **bC** are:

(1) $A, \neg A \vdash \neg\circ A$
If A is contradictory, then A is not formally consistent.

(2) $\circ A \vdash \neg(A\wedge \neg A)$
If A is formally consistent, then A is non-contradictory. (1st form)

(3) $\circ A \vdash \neg(\neg A\wedge A)$
If A is formally consistent, then A is non-contradictory. (2nd form) [10]

It is clear that the theorems above are variations on Metaprinciple I. A very important observation is that in **bC** the notions of "not consistent" and "inconsistent" do *not* coincide. Indeed, even if we introduce the concept of "consistent" as internal to the language through a new symbol $\bullet$, understanding $\bullet A$ to model "A is formally inconsistent," $\circ A$ and $\neg\bullet A$, and $\neg\circ A$ and $\bullet A$ would not be interdefinable, contrary to what one might assume. This will be further clarified below.

In **bC** a new negation, called *strong negation,* can be defined as:

$$\sim A \equiv_{Def} \neg A \wedge \circ A$$

This recovers several features of classical negation, though not all. We have, for instance, $A, \sim A \vdash_{\mathbf{bC}} B$, and thus **PPS** holds relative to this strong negation (that is, this negation is explosive). But $\nvdash_{\mathbf{bC}} (A\vee \neg A)$ and $\nvdash_{\mathbf{bC}} (A\rightarrow \sim\sim A)$. Consequently, the strong negation ~A is not classical, even though it is explosive (intuitively, ~A is somehow analogous to intuitionistic negation). But another strong negation, this one having all the properties of classical negation, is definable in **bC** by letting:

$$\dot{\sim} A \equiv_{Def} A\rightarrow (A\wedge \sim A)$$

Yet another logic of formal inconsistency, **Ci**, is defined by first defining $\bullet A \equiv_{Def} \neg\circ A$ and adjoining to **bC** the deduction scheme:

$$\bullet A \vdash (A\wedge \neg A)$$

In the system **Ci**, inconsistency and contradictions are equivalent to each other, due to the fact that $(A\wedge \neg A) \vdash_{\mathbf{bC}} \neg\circ A$ (as noted before) and the axiom just introduced, plus the definitions. Omitting the definitions, however, we obtain intermediate logics between **bC** and **Ci** that realize **Metaprinciple II.**

Moreover, in **Ci** both strong negations mentioned above are equivalent and acquire all the properties of classical negation—but it is still possible to define non-classical strong negations in **Ci**, via for example $\neg\neg\sim A$ and $\neg\neg\dot{\sim} A$. Some properties of this logic are:

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[10] In many systems of paraconsistent logic, although $(A\wedge \neg A)$ and $(\neg A\wedge A)$ are equivalent, $\neg(A\wedge \neg A)$ and $\neg(\neg A\wedge A)$ are not. See, e.g., . João Marcos, *Possible-Translations Semantics*.

(4) $\bullet A, \neg\bullet A \vdash_{\mathbf{Ci}} B$ and $\circ A, \neg\circ A \vdash_{\mathbf{Ci}} B$
The *Principle of Explosion* holds for formally consistent (or formally inconsistent) formulas.

(5) $\vdash_{\mathbf{Ci}} \circ\circ A$ and $\vdash_{\mathbf{Ci}} \circ\bullet A$
Both consistent and inconsistent formulas are consistent.

(6) $\circ A \vdash_{\mathbf{Ci}} \circ\neg A$
If a formula is consistent, its negation is also consistent.

(7) $\bullet\neg A \vdash_{\mathbf{Ci}} \bullet A$
A formula is inconsistent if its negation is inconsistent.

What doesn't hold in this logic? **PPS** still does not hold, that is, $A, \neg A \nvdash_{\mathbf{Ci}} B$ for some A and B. De Morgan's Laws and the rule of contraposition only hold in restricted forms, e.g., $(A\to B) \nvdash_{\mathbf{Ci}} (\neg B\to \neg A)$, though $(A\to \circ B) \vdash_{\mathbf{Ci}} (\neg \circ B\to \neg A)$. Also, **Ci** does not prove any formulas to be consistent, unless they already refer to consistency or inconsistency. That is, $\circ A$ is provable in **Ci** if and only if A is itself of the form $\circ\neg B$, $\circ B$, $\bullet B$ or $\bullet\neg B$.

The converses of (2) and (3) are still not valid, and thus we may consider the addition of some other deduction schema to **Ci**, as for example:

Levo-based scheme for contradictoriness (1):[11]
$\neg(A\wedge \neg A) \vdash \circ A$.
Dextro-based scheme for contradictoriness (d):
$\neg(\neg A\wedge A) \vdash \circ A$.
Bi-directional scheme for contradictoriness (b):
$\neg(A\wedge \neg A) \vee \neg(\neg A\wedge A) \vdash \circ A$.
Global scheme for contradictoriness (g):
$\vdash (B\leftrightarrow(A\wedge \neg A)) \to (\neg B\leftrightarrow\neg(A\wedge \neg A)$.

In formal terms, a **dC**-system is any **C**-system where $\circ$ and $\bullet$ can be defined in terms of the usual connectives of the language $\neg, \wedge, \vee, \to$. In the case of da Costa's C_1, the strongest calculus of his hierarchy of paraconsistent logics in his *Inconsistent Formal Systems,* $\circ A$ is defined as $\neg(A\vee \neg A)$ and $\bullet A$ as $\neg\circ A$, so that this logic is an extension of **Cil**, the logic obtained by the addition of the axiom (l) to **Ci**.

Several other distinct **dC**-systems can be defined by choosing adequate axioms of "propagation" for consistency and inconsistency. Many different choices are possible. Depending on the particular ones we choose, we may obtain finite many-valued paraconsistent logics or infinite-valued paraconsistent logics.

Some choices which have been tried for the axioms of propagation are the following:

First Choice $(\circ A\wedge \circ B) \to \circ(A\#B)$ for every binary connective # .

Second Choice $(\circ A\vee \circ B) \to \circ(A\#B)$ for every binary connective # .

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[11] This axiom holds for the system C_1 of da Costa, for example, as discussed in W. A. Carnielli, "Possible-translations semantics for paraconsistent logics" and J. Marcos, *Possible-Translations Semantics*.

Third Choice $(\circ A \# \circ B)$ for every binary connective # .

Fourth Choice $\bullet(A \wedge B) \leftrightarrow ((\bullet A \wedge B) \vee (\bullet B \wedge A))$
$\bullet(A \wedge B) \leftrightarrow ((\bullet A \wedge \neg B) \vee (\bullet B \wedge \neg A))$
$\bullet(A \to B) \leftrightarrow (\bullet B \wedge A)$

Fifth Choice $\bullet(A \# B) \leftrightarrow (\bullet A \wedge \bullet B)$ for every binary connective # .

Sixth Choice $\circ(\neg A)$ for every formula A .

Seventh Choice $\circ A$ for every formula A .

The first choice plus (l) above (p. 97) defines the calculus C_1 of the hierarchy of C_n; the second choice plus (l) defines the calculus C_1^+ of N. C. A. da Costa, J.-Y. Béziau, and O. Bueno, "Aspects of paraconsistent logic." Neither of these are many-valued. The third and sixth choices plus (b) define the three-valued maximal logic $\mathbf{P}^1$, introduced in A. M. Sette, "On the propositional calculus $\mathbf{P}^1$." The third choice plus (b) and $(A \to \neg\neg A)$ defines another maximal three-valued logic.[12] The fourth and fifth choices define, respectively, the three-valued maximal logics **LFI1** and **LFI2** studied in W. A. Carnielli, J. Marcos, and S. de Amo, "Formal inconsistency and evolutionary databases." And finally the seventh choice defines classical propositional logic. Many other combinations are possible.[13]

The fourth choice gives yet another axiomatization for the three-valued paraconsistent calculus $\mathbf{J_3}$, bearing some resemblance to the axiomatization given in R. Epstein's *Propositional Logics*. There, the axiom $(A \wedge (\neg A \wedge ©A)) \to B$ is considered among ten other axioms, where ©A is defined as $\neg(\Diamond A \wedge \Diamond \neg A)$ for a primitive modal operator $\Diamond$. The matrix interpretation of ©A in that book coincides with our matrix interpretation for $\circ A$, as discussed in the next section. This logic also coincides with the logic **CLuNs**, to be found, for instance, in D. Batens' "A survey of inconsistency-adaptive logics," and it has appeared quite often in the literature.

So much for the syntactical part of the logics of formal inconsistency. Semantical interpretations are a complicated issue for paraconsistency in general. The first **C**-systems were introduced only in proof-theoretical terms, and only some years later were semi-truth-functional bivalued semantics proposed for their interpretation. Those semantics, however, offer a very weak and debatable "meaning" to paraconsistent logics, and we describe in the next section an attractive alternative semantics called possible-translations semantics.

4. The Rosetta stone analogy

Found by Napoleon's troops in 1799 near the town of Rashid (Rosetta) in Lower Egypt, the Rosetta Stone is a piece of basalt which after several unclear episodes happens to be found today at the British Museum in London, containing

12 See C. Mortensen,"Paraconsistency and C_n" and J. Marcos, *Possible-Translations Semantics*.

13 See W. A. Carnielli and J. Marcos, "A taxonomy of C-systems" and J. Marcos, "8K solutions and semi-solutions to a problem of da Costa".

inscriptions that were the key to deciphering Egyptian hieroglyphic writing.[14] The deciphering was possible only due to the inscriptions appearing in three forms: hieroglyphic, Demotic, and Greek. By comparing the hieroglyphic and Demotic scripts with the Greek version, and assuming that they contained the same text, the British physicist Thomas Young and the French Egyptologist Jean François Champollion were able to decipher the hieroglyphic and Demotic versions in 1822. Moreover, by further comparing the hieroglyphic text to equivalents of the better known Coptic language they could attach a phonetics to the hieroglyphic writings, which were supposed to be only symbolic. Our semantic approach to paraconsistent logics, known as *possible-translations semantics*, is in many aspects similar to the deciphering of the Rosetta stone.[15]

In very general terms, a *translation* from a logic system **L** into a logic system **L´** is just a language homomorphism that preserves derivability, that is, if A is provable in **L** from premises Γ, and * is a translation from **L** into **L´**, then A* should be provable in **L'** from premises $\Gamma^* = \{ B^*: B \in \Gamma\}$, that is, if $\Gamma \vdash A$ then $\Gamma^* \vdash A^*$. Several specializations and variations of this notion have been studied in the literature,[16] but this general definition is adequate to our present purposes.

In intuitive terms, the idea is to project a given "hieroglyphic" logic by means of translations of it into simpler (usually many-valued) systems, and combine their respective forcing relations in order to obtain a sound and complete semantical interpretation to the initial complicated system. The simpler systems would thus play the role of Greek and Coptic in the Rosetta stone analogy. We may think of this process as working in two distinct directions[17]: When analyzing a complicated logic in terms of simpler components, we call the process *splitting logics*; but it is also possible to think of this process in the direction of synthesis, by defining a complex logic starting from simpler ones, and in this case we call the process *splicing logics*. Possible-translations semantics can be seen as a kind of *distributed semantics* where the meaning of a sentence in the "hieroglyphic" logic is made clear by way of a suitable combination of all the translations of that sentence into the component logics.

We give here an example of how this kind of semantics can help to give meaning to contradictions. For the basic **C**-systems **bC** and **Ci** the logics playing the role of auxiliary languages in the Rosetta stone analogy will be copies of the three-valued logic pictured below, with truth-values T, t and F, of which T and F are absolute "true" and "false," while t can be understood as "provisionally true."

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[14] See E. A. W. Budge, *The Rosetta Stone*.

[15] As in W. A. Carnielli, "Possible-translations semantics for paraconsistent logics" and J. Marcos, *Possible-Translations Semantics*. The analogy is due to J. Marcos, who presented it first in his Master Thesis defense at the University of Campinas, Brazil.

[16] See W. A. Carnielli and I. M. L. D'Ottaviano, "Translations between logical systems: a manifesto".

[17] As introduced in W. A. Carnielli and M. E. Coniglio, "A categorial approach to the combination of logics".

$\wedge$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\rightarrow$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\neg_w$	$\neg_s$
T	F	F
t	t	F
F	T	T

	$\circ_w$	$\circ_s$
T	T	T
t	T	F
F	T	T

For **Ci**, as we show below, the meanings of $\wedge$, $\vee$, $\rightarrow$ are fixed, but the meanings of $\neg$ and $\circ$ vary: Each of them will be assigned two distinct interpretations, namely a weak and a strong one. For negation $\neg$, the weak interpretation $\neg_w$ regards the value t as careful truth, and assigns to the negation of t also the value t. On the other hand, the strong interpretation $\neg_s$ makes no distinction between t and T, and assigns F to the negation of t . For $\circ$, the weak interpretation $\circ_w$ forgets the distinction between t and T, while the strong interpretation $\circ_s$ recognizes the value t as "provisionally true," and thus potentially inconsistent.

For the system **Ci** the set of all recursive possible translations to be considered is definable by the following clauses, to be obeyed by any translation * in this set:

Tr1 For atomic p, $p^* = p$.
For atomic p, $(\neg p)^* = \neg_w p$.
Tr2 For non-atomic A, either $(\neg A)^* = \neg_s A$ or $(\neg A)^* = \neg_w A$.
Tr3 For # any of $\wedge$, $\vee$, $\neg$, $(A \# B)^* = A^* \# B^*$.
Tr4 If $(\neg A)^* = \neg_w A^*$, then $(\circ A)^* = \circ_s A$.
If $(\neg A)^* = \neg_s A^*$, then $(\circ A)^* = \circ_w A$.

As an example, a formula of the form $\neg\circ\neg A$ will have eight possible distinct translations, according to the above clauses: If $(\neg\neg A)^* = \neg_w(\neg A)^*$, then $(\neg\circ\neg A)^*$ will be either $\neg_w \circ_s \neg_w A^*$, or $\neg_s \circ_s \neg_w A^*$, or $\neg_w \circ_s \neg_s A^*$ or $\neg_s \circ_s \neg_s A^*$. If $(\neg\neg A)^* = \neg_s(\neg A)^*$, then $(\neg\circ\neg A)^*$ will be either $\neg_w \circ_w \neg_w A^*$, or $\neg_s \circ_w \neg_w A^*$, or $\neg_w \circ_w \neg_s A^*$ or $\neg_s \circ_w \neg_s A^*$.

In other words, the syntax will be interpreted in different semantic scenarios, and here of course we have infinitely many distinct translations interpreting the formulas of **Ci** into distinct fragments of the above three-valued logics, according to the choices for the interpretation of the connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, and $\circ$.

Possible-translations semantics are a powerful tool for combining logics, complementary to other methods such as fibring. In what concerns the two main directions for combining logics cited before (splitting logics and splicing logics[18]) possible-translations semantics initially seem to be more apt for splitting while fibring methods are more apt for splicing.

Possible-translations semantics have already been given for the calculi in the

[18] See W. A. Carnielli and M. E. Coniglio, "A categorial approach to the combination of logics".

hierarchy C_n and for a slightly stronger version of C_n, offering thus a solution to the difficult problem of finding good semantics for paraconsistent logics.[19] It is also possible to give such semantics for some many-valued logics.[20] Based on these semantics, connections to other views on logics like dialogical logic could also be established, as suggested in Shahid Rahman and Carnielli, "The dialogical approach to paraconsistency." A categorial treatment of possible-translations semantics is presented in W. A. Carnielli and M. E. Coniglio, "A categorial approach to the combination of logics," where completeness with respect to possible-translations semantics is characterized by means of limits of categorial diagrams.

5. Modeling human and computer reasoning, or why we can't reason without contradictions

A good example of why the logic of ordinary language reasoning would have to abhor the *Principle of Explosion* is the following. Suppose that, in the course of an investigation, you receive information on some given subject (for example, as a response by two or more people to "Does Dick live in Arizona?") in the form of two sentences A and B of type "yes" or "no." Now there are exactly three possibilities (in a two-valued logic) concerning the truth-values of those sentences: Either they are both true, or both false, or they are contradictory, which occurs only in case they are distinct. Now it happens that the contradictory answer is very opportune, because it is the only case in which you're sure you received wrong information! Human reasoners profit from this possibility, and we would not exaggerate in saying that this ability is even essential for survival: In many cases, signs of danger are recognized in this way. So, reasoning with contradictories, instead of deriving anything else from them, seems to be an essential trait of human thinking. By itself, this would be a strong case in favor of developing a logic where **PPS** is not accepted as universal.

Interesting applications of paraconsistent logics can be found in the domain of automated reasoning and knowledge-based systems. We mention two examples of applications. The first refers to automated proof methods and logic programming. The resolution method and the tableau method are known to be equivalent for classical logic, and the programming language PROLOG which is based on such methods permits computer programs to be written almost axiomatically, directly in a sort of logic programming language, though for non-classical logics those methods do not necessarily coincide or have comparable computational complexity. In W. A. Carnielli and M. Lima-Marques, "Reasoning under inconsistent knowledge" we proposed a signed tableau method for the paraconsistent calculus C_1 of da Costa (and the method is immediately generalizable to other calculi of the C_n hierarchy) which differs from the classical

[19] See J. Marcos, *Possible-Translations Semantics* and W. A. Carnielli, "Possible-translations semantics for paraconsistent logics".

[20] A particular case of possible-translations semantics called "society semantics" has been studied in W. A. Carnielli and M. Lima-Marques, "Society semantics and multiple-valued logics".

signed tableaux with respect to the rules for negation (all other notions, like closed signed tableaux, etc, being the same as the classical ones).

Recalling that in C_1 the formula A^o (which plays exactly the same role of the formula $\circ A$ in the axiomatization presented in Section 3 above) abbreviates $\neg(A \wedge \neg A)$, the new rules at the propositional level are the following, where # stands for any of the binary connectives $\wedge, \vee, \rightarrow$. As usual in tableau proofs, rules containing the symbol | represent an "or" tree, with branching nodes.

$$\frac{T(\neg A)}{F(A) \mid F(A^o)}$$

If $\neg A$ is true, then either A is false or it is false that A is consistent, i.e., A is inconsistent/contradictory.

$$\frac{F(\neg A)}{T(A)}$$

If $\neg A$ is false, then A is true.

$$\frac{F((A\# B)^o)}{F(A^o) \mid F(B^o)}$$

If A # B is inconsistent, then either A or B is inconsistent.

Notice that the rule for $T(\neg A)$ differs from the classical one just by adding an alternative branch $F(A^o)$. The rule for $F((A\#B)^o)$ does not exist in the classical case, however both could be added as redundant rules to the classical tableau rules.

Completeness of these rules for the propositional calculus C_1 (and for its first-order version, C_1^* with additional rules) was proven in W. A. Carnielli and M. Lima-Marques, "Reasoning under inconsistent knowledge," where other examples for automated reasoning were also given.

A simple yet illustrative example is the well-known "Nixon Diamond," often repeated in the literature. Suppose we have the following statements:

Nixon is a Quaker: $Q(n)$.
If Nixon is a Quaker, then Nixon is a pacifist: $Q(n) \rightarrow P(n)$.
Nixon is not a pacifist: $\neg P(n)$.

These statements are contradictory, assuming that there is some person having all such properties. At this point, a human reasoner would suspect that one of them should be disqualified, but this maneuver would be blocked if all of the claims had equal confidence status. If all of them are to be taken as true, there is no other rational possibility besides some predicate being inexact, or vague, or subject to contradictions. Let's suppose that it can always be clarified who is a Quaker and who is not. Then the only candidate for inexactness or possible contradictoriness is "pacifist." This is exactly the conclusion given by our system[21]: We can run a tableau for the set of propositions $S = \{T(Q(n)),$

[21] In W. A. Carnielli and M. Lima-Marques, "Reasoning under inconsistent knowledge."

$T(Q(n) \rightarrow P(n)), T(\neg P(n))\}$. The system concludes, instead of becoming blocked, that $F(P(n)^{o})$, that is, P is contradictory.

The second example is for designing an implementation of databases that are robust enough to work in the presence of contradictions. There are several ways in which a database can be contradictory. Different users having equal access to some given database may introduce new claims, and even new rules or constraints, which, despite being consistent from the point of view of each user, can still be globally contradictory. Traditional databases may detect contradictory information and then start a complicated, and computationally extremely inefficient, procedure for "restoring consistency," but by no means can they afford modification of constraints, for reasons explained below. It is thus natural to embed databases in logical environments that permit reasoning with contradictory information, while maintaining all other desirable features of traditional logic, such as reasoning with the law of excluded middle, reasoning by cases, and reasoning by means of quantifiers.

In general, information stored in databases must be checked to verify some previous conditions (called integrity constraints) in order to be safely integrated in the database. Integrity constraints are expressed by (fixed) first-order sentences. For example, a database storing information about books may contain the requirement that no book in the collection can have more than one title, a condition that could be expressed by the following first-order formula, where Title(x, y) means that y is a string that is the title of book x:

$$\forall x\, \forall y\, \forall z\, ((\text{Title}(x, y) \wedge \text{Title}\,(x, z)) \rightarrow y = z)$$

Updates in traditional databases are only performed if the new database would satisfy the integrity constraints; if not, the database maintains its previous state. So, in a traditional database system, an imperative control never allows contradictory information to be considered.

The situation would be worse for traditional databases if integrity constraints themselves could be changed in time, instead of remaining fixed forever. Such evolving databases, which we call *evolutionary databases,* seem very interesting for the domain of artificial reasoning. The fact that, traditionally, integrity constraints are defined by the database designer and remain fixed for the users during the database lifetime is a severe limitation on databases, due solely to the logical foundations of classical database theory.

In W. A. Carnielli, J. Marcos, and S. de Amo, "Formal inconsistency and evolutionary databases" we have introduced new logics that axiomatize a formal representation of inconsistency in classical logic, starting from a purely semantical standpoint. Though in that paper we take inconsistency to be equivalent to contradictoriness, that is not necessary according to the observations made in Section 3. That simplification does not affect the treatment of information. Starting from an intuitive semantic account of what contradictory (or inconsistent) data should be, and taking into consideration some basic requirements, we provided two distinct sound and complete axiomatics for such semantics, **LFI1** and **LFI2**, as well as for their first-order extensions, **LFI1*** and **LFI2***, depending on which additional requirements are considered.

These two formal systems are examples of Logics of Formal Inconsistency (**LFI**) and form part of a much larger family of maximal paraconsistent three-valued logics.[22] It is important to note that **LFI1*** and **LFI2*** are proper subsystems of classical logic extended with a consistency operator, and they entail thus, in intuitive terms, fewer theorems than this extended form of classical logic. We have shown, however, that they can codify any classical or paraconsistent reasoning, because there exist grammatically faithful conservative translations from classical and paraconsistent first-order logics into **LFI1*** and **LFI2***.[23]

We repeat here an example of an evolutionary database considered in W. A. Carnielli, J. Marcos, and S. de Amo, "Formal inconsistency and evolutionary databases." Suppose that a claim P is proposed by a certain source. It may enter the database either with the token √ or the token × appended to it. In case not-P is proposed, the claim P enters with the token × or does not enter at all. In case that we know nothing about P, nothing is added to the database. As a consequence, in case P and not-P are simultaneously proposed (for instance, by different sources), then P enters the database with the token ×.

As a concrete example, suppose we have a database schema **DS** containing three relations: Author(Name, Country), Title(Book, String), and Translated(Book, Language). Suppose also that two different sources, I and II, provide information to our database, telling us:

Source I

1. Joaquim Maria Machado de Assis was born in Brazil.
2. Gabriel García Marquez was born in Colombia.
3. Machado de Assis is author of *Dom Casmurro*.
4. *Dom Casmurro* has not been translated into Polish.

Source II

5. Gabriel García Marquez was not born in South America.
6. Gabriel García Marquez is author of *One Hundred Years of Solitude*.
7. *One Hundred Years of Solitude* has been translated into Polish.

Now claim (4) is negative,and may be stored in **DS** with a × or not stored at all. On the other hand, claims (2) and (5) are contradictory, given certain already stored knowledge about geography, and in this case those claims are added with the token × appended to it. The remaining positive information may be added liberally either having √ or × as a suffix.

So we have already at least one piece of inconsistent information stored in **DS**. Besides that one, another one may appear if, for instance, Source I adds a new constraint asserting that "No South American author has ever been translated into Polish." After **DS** has been updated taking this new constraint into consider-

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[22] See J. Marcos, "8K solutions and semi-solutions to a problem of da Costa".

[23] See W. A. Carnielli, J. Marcos, and S. de Amo, "Formal inconsistency and evolutionary databases".

ation, the relation “Translated” will also contain contradictory information. While traditional databases cannot support this situation, our model permits us to reason with these contradictions, even taking advantage of this controversy to get better knowledge about the sources, much in the same way a human reasoner would do.

6. Closure

In this paper we have tried to defend the idea that reasoning with contradictions is not only useful but perfectly well-founded from the logico-mathematical standpoint. We have also discussed some underlying questions of paraconsistency, indicating some applications of it to human and automated reasoning and to database models. We have shown how several axiomatic systems of paraconsistent logics can be formulated, and how possible-translations semantics can be assigned to some of them. We also have tried to show how those interpretations are intuitively clear and philosophically appealing, despite some common objections that could be raised.

Often philosophers make predictions about what is mathematically possible. Some writers, in their efforts to find definitive arguments in favor of their own beliefs, try to blame logic, or more widely, mathematics as a whole, risking their reputation on what could or could not be done in the domain of mathematical possibilities. Some “throw the baby out with the bathwater,” prematurely dismissing the possibility of constructing logic systems which are robust enough to carry a good portion of classical reasoning. Karl Popper, while objecting against dialectics, acknowledges the intrinsic interest of contradictions:

> Dialecticians say that contradictions are fruitful, or fertile, or productive of progress, and we have admitted that this is, in a sense, true.[24]

However, he cautiously warns the reader that from a pair of contradictory premises any conclusion may be deduced in classical logic, and he asserts:

> The question may be raised whether this situation holds good in any system of logic, or whether we can construct a system of logic in which contradictory statements do not entail every statement. I have gone into this question, and the answer is that such a system can be constructed. The system turns out, however, to be an extremely weak system. Very few of the ordinary rules of inference are left, not even the *modus ponens* which says that from a statement of the form ‘If *p* then *q*’ together with *p* we can infer *q*. In my opinion, such a system is of no use for drawing inferences, although it may perhaps have some interest for those who are specially interested in the construction of formal systems as such.

Popper’s remarks can be seen as a good defense of our position: He concludes not only that contradictions can be seen positively, but also that other systems of logic could be constructed so that **PPS**, at least as taken for granted in classical logic, would not be valid. Unfortunately, he stops too soon, blocked by

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[24] *Conjectures and Refutations: The Growth of Scientific Knowledge,* p. 316.

his own misfortune.[25] But the fact that a great philosopher did not succeed in the mathematical enterprise of constructing a system of this sort, powerful enough to incorporate most of ordinary rules of inference including *modus ponens,* certainly does not mean that this task is impossible. Several mathematicians and philosophers in the course of history have experienced similar difficulties imagining non-Euclidean geometries, complex numbers and the like, but fortunately science does not survive on the imagination (or lack of it) of one individual.

By taking seriously the task of constructing logics for formal inconsistency we may also be helping to understand other philosophical questions: according to S. Knuuttila and A. I. Lehtinen in "Change and contradiction: a fourteenth-century controversy," the possibility of accepting that contradictory sentences could be true at the same instant of time was already considered (for theological arguments) almost seven centuries ago. Instead of blaming logic or mathematics and risking sibylline prophecies on what cannot be done, it is often better to investigate what can be achieved—even if what is achieved in the end are negative results, as the mathematical impossibilities of solving the classical Greek problems of elementary geometry, the theorems of Gödel, the indecidability of first-order logic and many other results familiar to contemporary logic.

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[25] His system was introduced in K. R. Popper, "On the theory of deduction. I. Derivation and its generalizations" and "On the theory of deduction. II. The definitions of classical and intuitionist negation".

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