Characterizing finite-valuedness*

Carlos Caleiro, Sérgio Marcelino
{smarcel,ccal}@math.tecnico.ulisboa.pt
SQIG - Instituto de Telecomunicações
Dep. Matemática - Instituto Superior Técnico
Universidade de Lisboa, Portugal

Umberto Rivieccio
urivieccio@dimap.ufrn.br
Departamento de Informática e Matemática Aplicada
Universidade Federal do Rio Grande do Norte
Natal – Brasil

2017/03/15

Abstract

We introduce properties of consequence relations that provide abstract counterparts of different notions of finite-valuedness in logic. In particular, we obtain characterizations of logics that are determined (i) by a single finite matrix, (ii) by a finite set of finite matrices, and (iii) by a class of n-generated matrices for some natural number n. A crucial role is played in our proofs by two closely related notions, *local tabularity* and *local finiteness*.

Keywords: Matrix semantics, Many-valued logic, Finite-valued logic, Strongly finite logic, Locally tabular, local finiteness, Cancellation, Finite-determinedness.

1 Introduction

The aim of this paper is to give abstract characterizations of different notions of finite-valuedness in logic, that is, logics whose semantics involve a finite

*This work was done under the scope of R&D Unit 50008, financed by the applicable financial framework (FCT/MEC through national funds and when applicable co-funded by FEDERPT2020). Sérgio Marcelino also acknowledges the FCT postdoc grant SFRH/BPD/76513/2011. The authors further acknowledge the support of EU FP7 Marie Curie PIRSES-GA-2012-318986 project GeTFun: Generalizing Truth-Functionality. Last but not least, the authors are indebted to João Marcos and Ramon Jansana for fruitful discussions on the topics of this paper.

number of truth-values. It is well known since the work of Lindenbaum that structural Tarskian logics correspond exactly to the semantical consequence relations determined by a family of logical matrices [23]. More recently, building on the seminal results obtained by Łoś and Suszko [15], and by Wójcicki [21, 23], on the abstract notions of uniformity and couniformity, Shoesmith and Smiley [20] singled out the cancellation property, which captures exactly the class of structural logics that can be characterized by a single (possibly infinite) logical matrix. This last property is often considered in the literature as the defining feature of logics that are called *many-valued*, a family that includes fuzzy logics (finite- and infinite-valued Łukasiewicz logics, Gödel-Dummett logics, the logics of continuous t-norms), the logics of Post, Belnap-Dunn four-valued logic, etc.

With the present paper, that we consider a natural continuation of this line of research, we hope to shed further new light on the very notion of finite-valuedness. An obvious starting point for a discussion of finite-valuedness is the notion of *tabularity* which, although introduced in the context of modal and super-intuitionistic logics [8, p. 49], can be extended to arbitrary logics. One may say that a logic is *tabular* when it is defined by a truth table, or, more generally, when it is determined by some finite structure (frame, logical matrix, etc.). This definition still leaves out certain logics that one would intuitively regard as finite-valued, such as Kleene's logic of order, which is defined by two three-element matrices and yet, crucially, cannot be characterized by any single matrix. In the context of matrix semantics, this example suggests that another reasonable notion would be to say that a logic is finite-valued when it is given by a finite set of finite matrices, dubbed *strongly finite* logics in **??**. Other generalizations of finite-valuedness may also be fruitful to consider, for instance that of being characterizable by a class of matrices each of which is generated by at most n elements (for a certain $n \in \mathbb{N}$). One may then wonder whether or not these properties hold for a given consequence relation. When is a logic characterizable by a single finite matrix? When is it characterizable by a finite set of finite matrices? When is it characterizable by a class of n-generated matrices for some given n?

In the present paper we provide an answer to the above questions, that is, we give necessary and sufficient conditions for a logic to be characterizable by a finite matrix, by a finite set of finite matrices, and by a class of n-generated matrices. It is worthwhile emphasizing that these conditions are expressed at the same level of abstraction as the above-mentioned notion of cancellation, or even as the properties that define the Tarskian notion of logic itself. To the best of our knowledge, only the second of the above-stated questions has already found an answer in the literature [22, Theorem 3.9]; we believe, however, to have improved the result of [22] by providing a simpler and more workable characterization.

The paper is organized as follows. Section 2 introduces the basic definitions and overviews the relevant known results. Section 3 presents our main characterization results (Theorems 3.12, 3.14 and 3.16), and includes two subsections. In Subsection 3.1 we show that the three essential properties involved in our characterizations are independent, and in Subsection 3.2 we illustrate the usefulness of the characterization results using some simple but informative examples. Finally, Section 4 discusses the obtained results alongside with possible directions of future research.

2 Preliminaries

In this section we fix the notation and introduce the definitions that are used throughout the paper.

Algebras (see [7] for further details). As usual, an algebra $\mathbb{A}$ is a set A equipped with a finite number of finitary operations. Given an algebra $\mathbb{A}$ and a set $X \subseteq A$, we say that $\mathbb{A}$ is *generated by* X when, for every $a \in A$, there is a term $t(p_1, \ldots, p_k)$ in the algebraic language of $\mathbb{A}$ and elements $a_1, \ldots, a_k \in X$ such that $a = t_{\mathbb{A}}(a_1, \ldots, a_k)$. If $|X| = n \in \mathbb{N}$, we say that $\mathbb{A}$ is *n-generated*, and in general we say that an algebra is *finitely generated* when it is n-generated for some $n \in \mathbb{N}$. When $\mathbb{A}$ is n-generated, we denote the generators by $\{a_1, \ldots, a_n\}$ and, for each $a \in A$, we fix a term (in at most n variables), denoted $\alpha_a = t(p_1, \ldots, p_n) \in \mathbf{Fm}_n$, such that $a = t_{\mathbb{A}}(a_1, \ldots, a_n)$. When $\mathbb{A}$ is finite with $A = \{a_1, \ldots, a_n\}$, we take $\alpha_{a_i} = p_i$.

Logics (see [23] for further details). We denote by Var the (countable) set of *propositional variables*. Given an algebraic signature (that we often leave implicit), we denote by $\mathbf{Fm}$ the absolutely free algebra built over Var. A *logic* defined over $\mathbf{Fm}$, is denoted by $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, where $\vdash$ is a structural consequence operator. We say that a set $\Gamma \subseteq \mathbf{Fm}$ is an $\mathcal{L}$-*theory* when $\Gamma^{\vdash} := \{\psi \in \mathbf{Fm} : \Gamma \vdash \psi\} = \Gamma$. An $\mathcal{L}$-theory Γ is *consistent* when $\Gamma \neq \mathbf{Fm}$. We say $\mathcal{L}$ is *finitary* if $\Gamma \vdash \varphi$ implies there is finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$. Let the finite set of variables $\{p_1, \ldots, p_n\} \subseteq \mathsf{Var}$ be denoted Var_n, we denote by $\mathbf{Fm}_n$ the absolutely free algebra built over Var_n, which we view as a subalgebra of $\mathbf{Fm}$.

Matrices (see [12] for further details). We will be dealing with matrix semantics for logics. A *matrix* is a pair $M = \langle \mathbb{A}, D \rangle$ where $\mathbb{A}$ is an algebra and $D \subseteq A$ is a subset of *designated elements*. We extend the above definitions on algebras to matrices, saying that $M = \langle \mathbb{A}, D \rangle$ is generated by some $X \subseteq A$ (n-generated, finitely generated) when its algebraic reduct $\mathbb{A}$ is.

Each matrix $M = \langle \mathbb{A}, D \rangle$ gives rise to a logic $\mathcal{L}_M = \langle \mathbf{Fm}, \vDash_M \rangle$ in the above sense by defining $\Gamma \vDash_M \varphi$ if and only if, for all homomorphism $v \colon \mathbf{Fm} \to \mathbb{A}$, we have that $v(\Gamma) \subseteq D$ implies $v(\varphi) \in D$. In this case we say M is the characteristic matrix of $\mathcal{L}_M$. This extends to a class of matrices $\mathcal{M} = \{M_i : i \in I\}$, which defines a logic $\mathcal{L}_{\mathcal{M}} = \langle \mathbf{Fm}, \vdash_{\mathcal{M}} \rangle$ by setting $\Gamma \vDash_{\mathcal{M}} \varphi$ if and only if $\Gamma \vDash_{M_i} \varphi$ for all $i \in I$. A logic is said *strongly finite* if it is characterized by a finite set of finite matrices.

Local finiteness and local tabularity. Given a logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, the interderivability relation $\dashv\vdash$ is known as the *Frege relation* of $\mathcal{L}$. This is always an equivalence relation, but not necessarily a congruence of the formula algebra

Fm. The *Tarski congruence* of $\mathcal{L}$ is the largest congruence relation $\equiv_{\mathcal{L}}$ that is contained in $\dashv\vdash$ (we denote it by $\equiv$ instead of $\equiv_{\mathcal{L}}$ when $\mathcal{L}$ is clear from the context). Restricting to formulas in n variables, we define $\equiv^n$ as $\equiv \cap (\mathbf{Fm}_n \times \mathbf{Fm}_n)$ and denote the corresponding quotient $\mathbf{Fm}_n/\equiv^n$ by $\mathbf{Fm}_n^*$. When the Frege relation $\dashv\vdash$ is a congruence of **Fm**, then it coincides with $\equiv$ and the logic is called *self-extensional*. The following characterization of the Tarski congruence [12, p. 29] will be particularly useful for us: for any logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ and for all $\varphi, \psi \in \mathbf{Fm}$,

$$\varphi \equiv \psi \quad \text{iff} \quad \gamma(\varphi, \vec{q}) \dashv\vdash \gamma(\psi, \vec{q}) \quad \forall \gamma(p, \vec{q}) \in \mathbf{Fm}$$

where $\gamma(p, \vec{q})$ is any formula with parameters p and $\vec{q} = q_1, \ldots, q_k$. We say that a set of formulas with parameters $\Gamma(p, \vec{q}) \subseteq \mathbf{Fm}$ is *$\mathcal{L}$-separating* when, for all $\varphi, \psi \in \mathbf{Fm}$,

$$\varphi \equiv \psi \quad \text{iff} \quad \gamma(\varphi, \vec{q}) \dashv\vdash \gamma(\psi, \vec{q}) \quad \forall \gamma(p, \vec{q}) \in \Gamma.$$

We say that a logic is *locally finite* when, for every $n \in \mathbb{N}$, the algebra $\mathbf{Fm}_n^*$ is finite, that is, $\mathbf{Fm}_n$ is partitioned in finitely many classes by the Tarski congruence. A logic is called *locally tabular*[1] when, for every $n \in \mathbb{N}$, $\mathbf{Fm}_n$ is partitioned in finitely many classes by the Frege relation, that is, there is no infinite set of formulas $\{\psi_i : i \in \mathbb{N}\} \subseteq \mathbf{Fm}_n^*$ such that $\psi_i \not\dashv\vdash \psi_j$ for $i \neq j$ (i.e., any set of formulas built over finitely many propositional variables contains only finitely many $\dashv\vdash$-nonequivalent formulas). The definitions immediately imply that local finiteness and local tabularity coincide on self-extensional logics, and that local finiteness implies local tabularity[2]. It is known that classical logic is locally tabular, and therefore also locally finite, but intuitionistic logic fails to have both properties.

Cancellation ([20]). A logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ has *cancellation* if

$$\Gamma, \bigcup\{\Gamma_i : i \in I\} \vdash \varphi \quad \text{implies} \quad \Gamma \vdash \varphi$$

for all $\Gamma \cup \{\varphi\}, \bigcup\{\Gamma_i : i \in I\} \subseteq \mathbf{Fm}$ such that

(i) $\mathsf{var}(\Gamma \cup \{\varphi\}) \cap \mathsf{var}(\bigcup\{\Gamma_i : i \in I\}) = \emptyset$

(ii) $\mathsf{var}(\Gamma_i) \cap \mathsf{var}(\Gamma_j) = \emptyset$ for $i \neq j \in I$

(iii) $(\Gamma_i)^{\vdash} \neq \mathbf{Fm}$ for all $i \in I$.

It is easy to see that every logic given by a single matrix (for instance classical logic) has the cancellation property; the following result shows that this is actually an equivalence.

Theorem 2.1 ([15, 20, 23]). *A logic $\mathcal{L}$ has cancellation if and only if $\mathcal{L} = \mathcal{L}_M$ for some matrix M.*

[1] Our terminology is inspired by [8, p. 19], which however does not distinguish between the two notions.

[2] The converse is not true in general, as shown in [17].

A natural example of a logic lacking cancellation is Kleene's logic of order $\mathsf{K}_{\leq}$ (see, e.g., [11]), which is defined by two three-element matrices. To see that cancellation fails, it is enough to observe that $p \wedge \neg p \vdash q \vee \neg q$ holds in $\mathsf{K}_{\leq}$ but $\emptyset \nvdash q \vee \neg q$.

Producing logics that do not have the cancellation property is easy, and will be useful in Section 3.1. Given a logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, we denote by $\mathcal{L}^- = \langle \mathbf{Fm}, \vdash^- \rangle$ the *theoremless companion of* $\mathcal{L}$ defined as follows. We add to any matrix semantics for $\mathcal{L}$ a matrix $M_\emptyset$ with an empty set of designated elements (the underlying algebra of $M_\emptyset$ is irrelevant). It is easy to see that $\mathcal{L}^-$ has no theorems, since no formula can be satisfied in $M_\emptyset$. However, the presence of $M_\emptyset$ does not affect consequences of non-empty sets of premisses. For a formula $\varphi \in \mathbf{Fm}$ that was valid in $\mathcal{L}$ ($\emptyset \vdash \varphi$), we have that $\nvdash^- \varphi$ but $\psi \vdash^- \varphi$ for any formula ψ (independently of the variables appearing in ψ). Hence, for any $\mathcal{L}$, the theoremless companion $\mathcal{L}^-$ never has cancellation.

3 Finite-valuedness

This section contains a characterization of the different semantical notions of finite-valuedness under consideration. In order to obtain them we introduce a new abstract property, which is given in the following definition.

Definition 3.1. Let $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ be a logic and $\Delta \subseteq \mathbf{Fm}$. We say that:

- $\mathcal{L}$ is Δ-*determined* if, for all $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$, whenever $\Gamma \nvdash \varphi$, there is a substitution $\sigma\colon \mathsf{Var} \to \Delta$ such that $\Gamma^\sigma \nvdash \varphi^\sigma$.
- $\mathcal{L}$ is *finitely determined* when $\mathcal{L}$ is Δ-determined for some finite $\Delta \subseteq \mathbf{Fm}$, and if $|\Delta| \leq n \in \mathbb{N}$, we say that $\mathcal{L}$ is n-*determined*.
- $\mathcal{L}$ is $\mathbf{Fm}_{\mathsf{fin}}$-*determined* when $\mathcal{L}$ is $\mathbf{Fm}_n$-determined for some $n \in \mathbb{N}$.

Notice that, by structurality of $\mathcal{L}$, the first item in the preceding definition could be equivalently stated as: for all $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$,

$$\Gamma \vdash \varphi \text{ if and only if } \Gamma^\sigma \vdash \varphi^\sigma \text{ for every substitution } \sigma\colon \mathsf{Var} \to \Delta.$$

The next lemmas state some basic properties of Δ-determinedness. Recall that $\mathsf{Var}_n = \{p_1, \ldots, p_n\}$.

Lemma 3.2. *If $\mathcal{L}$ is n-determined, then it is Var_n-determined.*

Proof. If $\sigma : \mathsf{Var} \to \Delta = \{\psi_1, \ldots, \psi_n\}$ and $\Gamma^\sigma \nvdash \varphi^\sigma$, then if we define the substitution $\tau : \mathsf{Var} \to \mathsf{Var}_n$ such that $\tau(p) = p_i$ whenever $\sigma(p) = \psi_i$, we obtain that $\Gamma^\tau \nvdash \varphi^\tau$ by structurality of $\mathcal{L}$. Hence, Γ If $\mathcal{L}$ is Δ-determined and the cardinality of Δ is $n \in \mathbb{N}$, then $\mathcal{L}$ is Var_n-determined. □

Lemma 3.3. *Let ∇ be the set formed by choosing a formula in each $\Delta/{\equiv_\mathcal{L}}$ class. If $\mathcal{L}$ is Δ-determined, then it is ∇-determined.*

Proof. Follows directly from the fact that $\equiv_{\mathcal{L}}$ is a congruence of $\mathbf{Fm}$ contained in $\dashv\vdash$. □

In the next definition we introduce a class of matrices that provides a complete semantics for $\mathbf{Fm}_{\mathsf{fin}}$-determined logics.

Definition 3.4. Given a logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, the class of n-generated matrices[3]

$$\mathcal{M}_n^{\mathcal{L}} = \{M_{\mathcal{T}} : \mathcal{T} \text{ is a consistent } \mathcal{L}\text{-theory}\}$$

is defined by $M_{\mathcal{T}} = \langle \mathbf{Fm}_n^*, \mathcal{T}^* \rangle$ and $\mathcal{T}^* = [\mathcal{T} \cap \mathbf{Fm}_n]_{\equiv^n}$.

Lemma 3.5. *The class $\mathcal{M}_n^{\mathcal{L}}$ characterizes the $\mathbf{Fm}_n$-fragment of any logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$. That is, for all $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}_n$, we have $\Gamma \vdash \varphi$ if only if $\Gamma \models_{\mathcal{M}_n^{\mathcal{L}}} \varphi$.*

Proof. The rightward implication follows easily, since $\mathcal{M}_n^{\mathcal{L}}$ is $\mathcal{L}$-sound by construction. In fact, for every $\mathcal{L}$-theory $\mathcal{T}$, the matrix $\langle \mathbf{Fm}_n, \mathcal{T} \rangle$ is $\mathcal{L}$-sound and defines the same logic as $M_{\mathcal{T}} = \langle \mathbf{Fm}_n^*, \mathcal{T}^* \rangle$ because $\equiv$ is a congruence contained in $\dashv\vdash$. For the other direction we need to prove that if $\Gamma \nvdash \varphi$ then there exists a consistent $\mathcal{T}$ and a valuation $v\colon \mathsf{Var} \to \mathbf{Fm}_n^*$ such that $v(\Gamma) \subseteq \mathcal{T}^*$ and $v(\varphi) \notin \mathcal{T}^*$. Then, we can choose the $\mathcal{L}$-theory $\mathcal{T} = \Gamma^{\vdash}$, which is consistent since $\varphi \notin \mathcal{T}$, and v such that $v(p) = [\sigma(p)]_{\equiv^n}$ for all $p \in \mathsf{var}(\Gamma \cup \varphi)$. Then $v(\gamma) = [\gamma^\sigma]_{\equiv^n}$ for all $\gamma \in \mathbf{Fm}$. Clearly $v(\Gamma) = [\Gamma]_{\equiv^n} \subseteq \mathcal{T}^*$ and $v(\varphi) = [\varphi]_{\equiv^n} \notin \mathcal{T}^*$. □

In the above proof we might have considered just the class of matrices having as algebraic reducts the free algebras over n generators, without taking the quotient. This class indeed also characterizes the $\mathbf{Fm}_n$-fragment of any logic. However, it is on the class $\mathcal{M}_n^{\mathcal{L}}$ that we will mainly focus in the remainder of the paper.

In the following lemmas we establish a number of sufficient conditions for a logic to be locally finite. This last property is particularly useful to us, as we will use it to construct finite matrices that characterize a given logic.

Lemma 3.6. *If $\mathcal{L}$ is a locally tabular logic and has a finite $\mathcal{L}$-separating set Γ, then $\mathcal{L}$ is locally finite.*

Proof. We need to show that $\mathbf{Fm}_n^*$ is finite for every $n \in \mathbb{N}$. Let Γ an $\mathcal{L}$-separating set. By assumption, for all $\varphi, \psi \in \mathbf{Fm}$, we have $\varphi \equiv \psi$ if and only if $\gamma(\varphi, \vec{q}) \dashv\vdash \gamma(\psi, \vec{q})$ for every $\gamma \in \Gamma$. Let k be the number of parameters used in Γ. Since $\mathcal{L}$ is locally tabular, $\mathbf{Fm}_{n+k}$ is partitioned by $\dashv\vdash$ into finitely many classes. Thus there cannot be an infinite set $\{\varphi_i : i \in \mathbb{N}\} \subseteq \mathbf{Fm}_n$ such that $\gamma(\varphi_i, \vec{q}) \not\dashv\vdash \gamma(\varphi_j, \vec{q})$ for $i \neq j$. □

Lemma 3.7. *If $\mathcal{L}$ is locally tabular and has an $\mathcal{L}$-separating set Γ built over finitely many variables, then $\mathcal{L}$ is locally finite.*

[3] This family can also be seen as a *generalized matrix* (also called *abstract logic*), i.e., an algebra with a family of distinguished sets (e.g., see [12]).

Proof. Let Γ be an $\mathcal{L}$-separating set built over a finite set of variables. Since $\mathcal{L}$ is locally tabular, there are only finitely many $\dashv\vdash$-nonequivalent formulas in Γ, so we can pick one representative for each equivalence class modulo $\dashv\vdash$, say $\{\gamma_1, \ldots, \gamma_k\} = \Gamma_0 \subseteq \Gamma$. Let us check that Γ_0 is a $\mathcal{L}$-separating set. For each $\gamma_i \in \Gamma \setminus \Gamma_0$, we consider $\gamma_j \in \Gamma_0$ the chosen representative of the $\dashv\vdash$-equivalence class of γ_i and obtain

$$\gamma_i(\varphi, \vec{q}) \dashv\vdash \gamma_i(\psi, \vec{q}) \text{ if and only if } \gamma_j(\varphi, \vec{q}) \dashv\vdash \gamma_j(\psi, \vec{q}),$$

for by assumption $\gamma_i(\varphi, \vec{q}) \dashv\vdash \gamma_j(\varphi, \vec{q})$, $\gamma_i(\psi, \vec{q}) \dashv\vdash \gamma_j(\psi, \vec{q})$ and $\dashv\vdash$ is transitive. The desired result then follows from Lemma 3.6. $\square$

Lemma 3.8. *If $\mathcal{L}$ is $\mathbf{Fm}_{\mathsf{fin}}$-determined and locally tabular, then $\mathcal{L}$ is locally finite.*

Proof. The proofs follow from the fact that if $\gamma(p, \vec{q})$ separates two formulas $\varphi, \psi \in Fm$, then there is $\sigma \colon \mathsf{Var} \to \mathbf{Fm}_n$ such that $\gamma(p, \sigma(\vec{q}))$ also separates them and $\gamma(p, \sigma(\vec{q}))$ has at most n variables as parameters,. That is, if $\gamma(\varphi, \vec{q}) \not\dashv\vdash \gamma(\psi, \vec{q})$, then, since $\mathcal{L}$ is $\mathbf{Fm}_n$-determined, there is $\sigma : \mathsf{Var} \to \mathbf{Fm}_n$ such that

$$\sigma(\gamma(\varphi, \vec{q})) = \gamma(\sigma(\varphi), \sigma(\vec{q})) \not\dashv\vdash \gamma(\sigma(\psi), \sigma(\vec{q})) = \gamma(\sigma(\varphi), \sigma(\vec{q})).$$

Using the structurality of $\mathcal{L}$, it is not hard to show that this fact implies that $\gamma(\varphi, \sigma(\vec{q})) \not\dashv\vdash \gamma(\psi, \sigma(\vec{q}))$. $\square$

In the next lemma we show that in the presence of local tabularity, it is equivalent to a logic to be $\mathbf{Fm}_{\mathsf{fin}}$-determined (note that in general $\mathbf{Fm}_{\mathsf{fin}}$ is infinite) is equivalent to being finitely determined. So, in particular, even if it would seem less general, the above lemma could be equivalently stated by replacing the assumption of $\mathbf{Fm}_{\mathsf{fin}}$-determinedness by finite-determinedness. In most subsequent statements we shall use finite-valuedness when in presence of local tabularity.

Lemma 3.9. *If $\mathcal{L}$ is $\mathbf{Fm}_{\mathsf{fin}}$-determined and locally tabular, then $\mathcal{L}$ is finitely determined, and Var_n-determined for some n.*

Proof. As $\mathcal{L}$ is $\mathbf{Fm}_{\mathsf{fin}}$-determined, there is $n \in \mathbb{N}$ such that $\mathcal{L}$ is $\mathbf{Fm}_n$-determined. By Lemma 3.8 we know that $\mathcal{L}$ is locally finite, and therefore $\mathbf{Fm}_n^*$ is finite. Hence the result follows from Lemmas 3.2 and 3.3. $\square$

The next definition and lemmas aim at characterizing the logics whose semantics can be given by a (not necessarily finite) class of n-generated matrices, for some given $n \in \mathbb{N}$.

Definition 3.10. Let $M = \langle \mathbb{A}, D \rangle$ be a matrix generated by $\{a_1, \ldots, a_n\} \subseteq A$. Define $v_{\mathsf{gen}} : \mathsf{Var}_n \to \{a_1, \ldots, a_n\}$ by

$$v_{\mathsf{gen}}(p_i) = a_i,$$

and, for each $v\colon \mathsf{Var} \to \mathbb{A}$, define the substitution $\sigma_v\colon \mathsf{Var} \to \mathbf{Fm}_n$ as

$$\sigma_v(p) = \alpha_{v(p)}$$

where $\alpha_a = t(p_1, \ldots, p_n) \in \mathbf{Fm}_n$ is the term defined earlier satisfying $v(p) = t_{\mathbb{A}}(a_1, \ldots, a_n)$.

Lemma 3.11. *For all valuations $v\colon \mathsf{Var} \to \mathbb{A}$, we have $v = v_{\mathsf{gen}} \circ \sigma_v$.*

Proof. It is sufficient to observe that $v_{\mathsf{gen}} \circ \sigma_v(p) = v_{\mathsf{gen}}(\alpha_{v(p)}) = v(p)$ for all $p \in \mathsf{Var}$. $\square$

We are now ready to prove the first of our characterization results.

Theorem 3.12. *A logic $\mathcal{L}$ is defined by a class of n-generated matrices for some $n \in \mathbb{N}$ if and only if $\mathcal{L}$ is $\mathbf{Fm}_{\mathsf{fin}}$-determined. In particular, $\mathcal{L}$ is characterizable by the class $\mathcal{M}_n^{\mathcal{L}}$ if and only if $\mathcal{L}$ is $\mathbf{Fm}_n$-determined.*

Proof. We first prove that if $\mathcal{M}$ is a class of n-generated matrices, then $\mathcal{L}_{\mathcal{M}}$ is $\mathbf{Fm}_n$-determined. We need to show that, for every $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$ such that $\Gamma \not\models_{\mathcal{M}} \varphi$, there is a substitution $\sigma\colon \mathsf{Var} \to \mathbf{Fm}_n$ such that $\sigma(\Gamma) \not\models_{\mathcal{M}} \sigma(\varphi)$. From $\Gamma \not\models_{\mathcal{M}} \varphi$ we have that there is $M = \langle \mathbb{A}, D\rangle \in \mathcal{M}$ and a valuation $v\colon \mathsf{Var} \to \mathbb{A}$ such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$. The result then follows by Lemma 3.11 if we let $\sigma := \sigma_v$.
As to the other direction, we know by Lemma 3.5 that $\mathcal{M}_n^{\mathcal{L}}$ chacterizes the $\mathbf{Fm}_n$-fragment of $\mathcal{L}$. We show that if $\mathcal{L}$ is $\mathbf{Fm}_n$-determined, then the family of matrices $\mathcal{M}_n^{\mathcal{L}}$ chacterizes the whole $\mathcal{L}$. All matrices in $\mathcal{M}_n^{\mathcal{L}}$ are at most n-generated and $\mathcal{L}$-sound by construction. In order to prove completeness, we make use of the fact that $\mathcal{L}$ is $\mathbf{Fm}_n$-determined. If $\Gamma \not\vdash \varphi$, we know that there is $\sigma\colon \mathsf{Var} \to \mathbf{Fm}_n$ such that $\Gamma^\sigma \not\vdash \varphi^\sigma$. But then $\Gamma^\sigma \not\models_{\mathcal{M}_n^{\mathcal{L}}} \varphi^\sigma$ and so $\Gamma \not\models_{\mathcal{M}_n^{\mathcal{L}}} \varphi$. $\square$

Our next result (Theorem 3.14) provides a characterization of one of the possible notions of finite-valuedness, namely that of a logic being definable by a finite set of finite matrices. We need a preliminary lemma.

Lemma 3.13. *If $\mathcal{L}_{\mathcal{M}}$ is strongly finite, then it is finitely determined. Moreover, if $\mathcal{M}$ is a class of finite Σ-matrices with cardinality bounded by n, then $\mathcal{L}_{\mathcal{M}}$ is Var_n-determined.*

Proof. We need to show that, assuming $\Gamma \not\models_{\mathcal{M}} \varphi$, there is a substitution $\sigma\colon \mathsf{Var} \to \mathsf{Var}_n$ such that $\sigma(\Gamma) \not\models_{\mathcal{M}} \sigma(\varphi)$. Consider then $M = \langle \mathbb{A}, D\rangle \in \mathcal{M}$ and a valuation $v\colon \mathsf{Var} \to \mathbb{A}$ such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$. The result follows by Lemma 3.11, letting $\sigma := \sigma_v$ where $\alpha_{v(p)} = p_i$ for $v(p) = a_i$. $\square$

Theorem 3.14. *A logic $\mathcal{L}$ is strongly finite if and only if $\mathcal{L}$ is finitely determined and locally tabular.*

Proof. The rightward implication follows from Lemma 3.13. For the converse, notice that if $\mathcal{L}$ is finitely determined and locally tabular, then by Lemma 3.8 $\mathcal{L}$ is also locally finite. Hence $\mathbf{Fm}_n^*$ is finite, which implies that the family of matrices $\mathcal{M}_n^{\mathcal{L}}$ given in Definition 3.4 is also finite. The fact that the family $\mathcal{M}_n^{\mathcal{L}}$ is sound and complete for $\mathcal{L}$ (Theorem 3.12) concludes our proof. □

Theorem 3.14 is a stronger version of [22, Theorem 3.9], which states that a logic is strongly finite (determined by a finite set of finite matrices) if and only if it is $\mathbf{Fm}_{\mathsf{fin}}$-determined and locally finite. Our result refines Wójcicki's in two ways. It uses local finiteness instead of local tabularity, which we now know (by Lemma 3.8) to be equivalent in the presence of $\mathbf{Fm}_{\mathsf{fin}}$-determinedness. A first advantage is then that we refer to the Frege relation (i.e., logical equivalence) instead of the more technically involved notion of Tarski congruence. Secondly, we were able to replace $\mathbf{Fm}_{\mathsf{fin}}$-determinedness by the simpler property of being finitely determined. We also believe that our results better clarify the interdependencies between the various relevant properties.

The fact that strongly finite logics are finitary is well known [23, Theorem 4.1.7]. Next we present a proof of this fact using the above characterization result.

Proposition 3.15. *If $\mathcal{L}$ is $\mathbf{Fm}_{\mathsf{fin}}$-determined and locally tabular then it is finitary.*

Proof. By assumption $\mathcal{L}$ is $\mathbf{Fm}_n$-determined for some $n \in \mathbb{N}$. We first prove that for every $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$ satisfying $\Gamma \vdash \varphi$, there is $m \in \mathbb{N}$ such that $\Gamma_m = \Gamma \cap \mathbf{Fm}_m \vdash \varphi$.

Let us assume this is not the case. Let $m_\varphi \in \mathbb{N}$ be such that $\varphi \subseteq \mathbf{Fm}_{m_\varphi}$. Then, since $\mathcal{L}$ is $\mathbf{Fm}_n$-determined, for each $m > m_\varphi$, there is a $\sigma_m : \mathsf{Var}_m \to \mathbf{Fm}_n$ such that $\Gamma_m^\sigma \nvdash \varphi^\sigma$. As $\Gamma_i \subseteq \Gamma_j$ of $i < j$, we have that $\Gamma_i^{\sigma_j} \nvdash \varphi^{\sigma_j}$ for every $i \leq j$. It is not hard to see that Zorn's lemma guarantees the existence of $\sigma : \mathsf{Var} \to \mathbf{Fm}_n$ satisfying $\Gamma_i^\sigma \nvdash \varphi^\sigma$ for every $i \in \mathbb{N}$. By assumption, we have that $\Gamma \vdash \varphi$, so $\Gamma^\sigma \vdash \varphi^\sigma$. However, $\Gamma^\sigma \subseteq \mathbf{Fm}_n$ and therefore by local tabularity of $\mathcal{L}$ we know there are only finite number of $\mathcal{L}$-equivalence classes in it, hence there is finite $\Delta \subseteq \Gamma$, such that $\Delta^\sigma \dashv\vdash \Gamma^\sigma$. But since Δ is finite, there is m such that $\Delta \subset \Gamma_m$, and therefore $\Gamma_m^\sigma \vdash \varphi^\sigma$, contradicting $\Gamma_i^\sigma \nvdash \varphi^\sigma$ for every $i \in \mathbb{N}$.

We finish the proof by noting that as there is $m \in \mathbb{N}$ such that $\Gamma_m \vdash \varphi$, and $\mathcal{L}$ is locally tabular, there are only finite $\mathcal{L}$-equivalence classes in $\mathbf{Fm}_m$, and therefore there is finite $\Delta \subseteq \Gamma_m \subseteq \Gamma$ such that $\Delta \vdash \Gamma$, and hence $\Delta \vdash \varphi$. □

Lastly, the following theorem characterizes logics that are finite-valued in a stricter sense, that is, logics that can be defined by a single finite matrix.

Theorem 3.16. *A logic $\mathcal{L}$ is defined by a single finite matrix if and only if $\mathcal{L}$ has cancellation, is finitely determined and locally tabular.*

Proof. The rightward implication follows from Theorem 3.14 together with the fact that every logic characterized by a single matrix has cancellation [20]. As

to the other direction, assume $\mathcal{L}$ is Var_n-determined and consider the family of matrices $\mathcal{M}_n^{\mathcal{L}}$ (Definition 3.4). All these matrices share the same underlying algebra $\mathbf{Fm}_n^*$, which is finite because $\mathcal{L}$ is locally tabular (hence, by Lemma 3.8, also locally finite). Thus, the family $\mathcal{M}_n^{\mathcal{L}}$ is finite. Let then $\mathcal{M}_n^{\mathcal{L}} = \{M_1, \ldots, M_m\}$. Referring to Definition 3.4, notice that $\mathcal{T}_1^* = \mathcal{T}_2^*$ implies that $\mathcal{T}_1 \cap \mathbf{Fm}_n = \mathcal{T}_2 \cap \mathbf{Fm}_n$, because $\mathcal{L}$-theories are closed under the derivability relation $\vdash$ and the Tarski congruence $\equiv$ is contained in the Frege relation $\dashv\vdash$. Hence, for $M_i = M_{\mathcal{T}}$, we can unambiguously write $T_i = \mathcal{T} \cap \mathbf{Fm}_n$.
Let for each $1 \leq i \leq m$, $\sigma_i : \mathsf{Var}_n \to \{p_{i\cdot 1}, \ldots, p_{i\cdot n}\}$ such that $\sigma_i(p_j) = p_{i\cdot j}$ and the following matrix: $\mathbb{A} = \mathbf{Fm}_{n\cdot m}/{\equiv^{n\cdot m}}$ and

$$D = [(\bigcup_{1\leq i\leq m} D_i)^{\vdash} \cap \mathbf{Fm}_{n\cdot m}]_{\equiv^{n\cdot m}},$$

where $D_i = (T_i)^{\sigma_i}$. The matrix $M = \langle \mathbb{A}, D\rangle$ is $\mathcal{L}$-sound by construction, moreover $\mathbb{A}$ is finite because $\mathcal{L}$ is locally finite. We are going to see that M is also complete, that is, if $\Delta \models_M \psi$, then $\Delta \vdash \psi$.
Assume $\Delta \nvdash \psi$, and let us prove that $\Delta \not\models_M \psi$. Since $\mathcal{L}$ is Var_n-determined, there is $\sigma\colon \mathsf{Var} \to \mathsf{Var}_n$ such that $\Delta^\sigma \nvdash \psi^\sigma$. Let $1 \leq i \leq m$ be such that $(\Delta^\sigma)^{\vdash} \cap \mathbf{Fm}_n = T_i$. Define a map $v : \mathbf{Fm} \to \mathbb{A}$ given by $v(\varphi) = [(\sigma_i \circ \sigma)(\varphi)]_{\equiv^{n\cdot m}}$ for all $\varphi \in \mathbf{Fm}$. It is easy to check that v is a homomorphism, and that $\mathsf{var}(v(\varphi)) \subseteq \{p_{i\cdot 1}, \ldots, p_{i\cdot n}\}$. Hence, $v(\Delta) \subseteq [D_i]_{\equiv^{n\cdot m}} \subseteq D$ and $v(\psi) \notin [D_i]_{\equiv^{n\cdot m}}$ because $\psi^\sigma \notin T_i = (\Delta^\sigma)^{\vdash} \cap \mathbf{Fm}_n$. We are going to use cancellation to prove that $v(\psi) \notin D$.
From $v(\psi) \notin [D_i]_{\equiv^{n\cdot m}}$, we have $D_i \nvdash \psi^{\sigma_i \circ \sigma}$. Moreover, we have:

- $\mathsf{var}(\psi^{\sigma_i\circ\sigma} \cup D_i) \cap \mathsf{var}(\bigcup_{\substack{0\leq j\leq m\\ i\neq j}} D_j) = \emptyset$,
- $\mathsf{var}(D_{i_1}) \cap \mathsf{var}(D_{i_2}) = \emptyset$ for $0 \leq i_1 \neq i_2 \leq m$,
- $(D_j)^{\vdash} \neq \mathbf{Fm}$, as D_j is a subset of a consistent theory.

Hence, by the cancellation property of $\mathcal{L}$, we have

$$D_i, \bigcup_{\substack{0\leq j\leq m\\ i\neq j}} D_j \nvdash \psi^{\sigma_i\circ\sigma}.$$

Using $\equiv^{n\cdot m} \subseteq \dashv\vdash_{\mathcal{L}}$, it is easy to see that, for each $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}_{n\cdot m}$, we have $\Gamma \vdash \varphi$ if and only if

$$\{\gamma : [\gamma]_{\equiv^{n\cdot m}} \in [\Gamma]_{\equiv^{n\cdot m}}\} \vdash \varphi' \quad \text{for all} \quad \varphi' \in [\varphi]_{\equiv^{n\cdot m}}.$$

Thus, from $\bigcup_{\substack{0\leq j\leq m\\ i\neq j}} D_j \nvdash (\sigma_i \circ \sigma)(\psi)$, we obtain

$$v(\psi) = [(\sigma_i \circ \sigma)(\psi)]_{\equiv^{n\cdot m}} \notin [(\bigcup_{0\leq i\leq m} D_i)^{\vdash} \cap \mathbf{Fm}_{n\cdot m}]_{\equiv^{n\cdot m}} = D.$$

□

$\mathcal{L}$	Canc	$\mathbf{Fm}_{\mathsf{fin}}$	LocTab
CPL	✓	✓	✓
RN	✓	✓	×
G_∞	✓	×	✓
Int	✓	×	×
$\mathsf{K}_\leq$	×	✓	✓
RN^-	×	✓	×
G_∞^-	×	×	✓
Int^-	×	×	×

Figure 1: Separating Canc, $\mathbf{Fm}_{\mathsf{fin}}$ and LocTab.

The construction of the matrix in the proof of the preceding theorem is analogous to those of [15, 20, 23], the essential difference being that we can obtain a finite algebra given that we are working beyond cancellation. This is possible because local tabularity and finite-determinedness guarantee that we can start from a finite number of theories instead of the potentially non-denumerable class of theories of the original construction[4].

Theorem 3.16 clarifies the difference between the two notions of finite valuedness that we have been considering: being characterizable by a finite matrix or by a finite set of finite matrices. The difference is the presence of the cancellation property, which is not related to having a semantics involving only a finite number of truth values. This means that if a logic can be defined by a finite set of finite matrices but not by any finite matrix, then it cannot be defined by any infinite matrix either.

3.1 Separation of properties

In this section we give examples of logics showing that all combinations of the properties considered above are independent.

Classical logic (CPL), which has all three properties listed in Figure 1, is the prototypical example of a logic defined by a single finite matrix.

We call RN the logic defined by the matrix $\langle \mathbb{A}, \{1\}\rangle$, where $\mathbb{A}$ is the free one-generated Heyting algebra, known as the Rieger-Nishimura lattice [3]. RN has obviously cancellation (Canc) and $\mathbf{Fm}_{\mathsf{fin}}$. LocTab fails because RN is not locally finite and thus, by Lemma 3.8, it cannot be locally tabular. Note that in the absence of local tabularity $\mathbf{Fm}_{\mathsf{fin}}$ does not imply finite-determinedness. However, the failure of finite-determinedness of RN is witnessed by the formulas appearing in Gödel's proof that Int is not finite-valued [13]. Gödel considers the formulas $\varphi_n = \bigvee_{1\leq i<j\leq n} (p_i \leftrightarrow p_j)$ for $n \in \mathbb{N}$, and it is easy to check that for every substitution $\sigma : \mathsf{Var} \to \mathsf{Var}_n$ we have $\vdash_{\mathsf{RN}} \varphi_{n+1}^\sigma$ but $\nvdash_{\mathsf{RN}} \varphi_{n+1}$.

[4]Notice that $\mathbf{Fm}$ and $\mathbf{Fm}_n$ have the same cardinality and therefore the theories over them may be of the same cardinality too. Hence, even if a logic is given by a class of n-generated matrices and has cancellation, this construction does not always result in an ω-generated matrix.

G_∞ is the infinite-valued Gödel-Dummett logic, which is defined by the matrix $\langle [\mathbf{0},\mathbf{1}], \{1\}\rangle$ where $[\mathbf{0},\mathbf{1}]$ is the standard real-valued Gödel algebra [14]. It is well-known that G_∞ is not finite-valued but it is locally finite (see e.g. [1]) and hence locally tabular. By Theorem 3.16, this implies that G_∞ cannot be $\mathbf{Fm}_{\mathsf{fin}}$-determined.

Intuitionistic logic (Int) enjoys Canc as shown for instance in [20, Theorem 5] or [23, Theorem 3.2.9]. To see that Int is not locally tabular we just need to invoke the fact that the one-generated Heyting algebra is infinite: this means that Int is not locally finite, and since it is self-extensional, we know that local tabularity must fail. The proof above that RN is not finite-determined works also for Int, but we can further show that Int is not $\mathbf{Fm}_{\mathsf{fin}}$-determined. To prove this we need to invoke some definitions and results from [3, 4]. For each $n \in \mathbb{N}$, the n-universal model of Int (which is unique up to isomorphism), denoted $\mathcal{U}(n)$, captures the $\mathbf{Fm}_n$-fragment of Int, that is, $\emptyset \vdash_{\mathsf{Int}} \psi$ if and only if $\mathcal{U}(n) \vDash \psi$ for every $\psi \in \mathbf{Fm}_n$ [4, Theorem 3.8]. Letting φ_k be the Jankov formula (or even the subframe formula, see e.g. [3, Theorem 2.5]) of $\mathcal{U}(k)$ for $k \in \mathbb{N}$, we have that $\mathcal{U}(n) \vDash \varphi_k$ for $k > n$ and $\mathcal{U}(n) \nvDash \varphi_k$ for $k \leq n$. Hence, $\mathcal{U}(n) \vDash \varphi_{n+1}$, and thus $\mathcal{U}(n) \vDash \varphi^\sigma_{n+1}$ for every $\sigma : \mathsf{Var} \to \mathbf{Fm}_n$. We conclude that $\nvdash_{\mathsf{Int}} \varphi_{n+1}$ but $\vdash_{\mathsf{Int}} \varphi^\sigma_{n+1}$ for every $\sigma : \mathsf{Var} \to \mathbf{Fm}_n$.

As mentioned in Section 2, Kleene's logic of order $\mathsf{K}_\leq$ does not have Canc, however it is locally tabular and $\mathbf{Fm}_{\mathsf{fin}}$-determined by Theorem 3.14. The logics denoted by RN^-, G^-_∞ and Int^- are the theoremless companions of (respectively) RN, G_∞ and Int, defined as in Section 2. These logics complete our table because, as it is easy to show, removing theorems ensures that cancellation fails while preserving $\mathbf{Fm}_{\mathsf{fin}}$ and LocTab.

3.2 Two illustrative examples

We illustrate the advantage of having an abstract characterization of finite-valuedness by analyzing a few examples. We deliberately consider very simple examples which are however sufficiently rich to illustrate the difficulties one can find and how our tools may help in their analysis.

Consider the logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash\rangle$ obtained by adding a nullary connective $\perp$ and the schematic rule $\frac{\perp}{p}$ to the implication fragment of classical logic. One can check that, for all $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$, $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash_{\mathsf{imp}} \varphi$ or $\Gamma \vdash_{\mathsf{imp}} \perp$, where $\vdash_{\mathsf{imp}}$ denotes the consequence relation obtained by using any rule that is valid in the implication fragment of classical logic but never the $\frac{\perp}{p}$ rule[5]. Despite our unfamiliarity with this logic, one easily obtains that $\mathcal{L}$ inherits all of the following properties from the implication fragment of classical logic:

Canc: Assume that Γ, $\bigcup\{\Gamma_i : i \in I\} \vdash \varphi$, and that Γ, φ, Γ_i for $i \in I$ satisfy (i)-(iii). Then Γ, $\bigcup\{\Gamma_i : i \in I\} \vdash_{\mathsf{imp}} \varphi$ or Γ, $\bigcup\{\Gamma_i : i \in I\} \vdash_{\mathsf{imp}} \perp$. Since

[5] $\mathcal{L}$ is the (disjoint) fibring of the logic of classical implication $\mathsf{CPL}_\to$ with the logic of bottom $\mathcal{L}_{\mathsf{bot}}$, i.e., $\mathcal{L} = \mathsf{CPL}_\to \bullet \mathcal{L}_{\mathsf{bot}}$ considered in [16]. Note that $\mathcal{L}$ is still a proper weakening of classical logic as, for instance, $\perp \to p$ is not a theorem. Clearly, $\vdash_{\mathsf{imp}}$ is the same as the consequence relation of $\mathsf{CPL}_\to$ but over a language enriched with $\perp$.

classical logic has cancellation and $\mathsf{var}(\bot) = \emptyset$ we obtain that $\Gamma \vdash_{\mathsf{imp}} \varphi$ or $\Gamma \vdash_{\mathsf{imp}} \bot$, and conclude that $\Gamma \vdash \varphi$.

$\mathbf{Fm}_{\mathsf{fin}}$: We have that $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash_{\mathsf{imp}} \varphi$ or $\Gamma \vdash_{\mathsf{imp}} \bot$. From the obvious 2-determinedness of $\mathsf{CPL}_{\to}$, this is equivalent to $\Gamma^\sigma \vdash_{\mathsf{imp}} \varphi^\sigma$ or $\Gamma^\sigma \vdash_{\mathsf{imp}} \bot^\sigma$ for each $\sigma : \mathsf{Var} \to \mathsf{Var}_2$. Since $\bot^\sigma = \bot$, this is immediately equivalent to $\Gamma^\sigma \vdash \varphi^\sigma$ for each $\sigma : \mathsf{Var} \to \mathsf{Var}_2$, and the 2-determinedness of $\mathcal{L}$ follows.

LocTab: We have that $\varphi \dashv\vdash \psi$ if and only if $\varphi \vdash_{\mathsf{imp}} \psi$ or $\varphi \vdash_{\mathsf{imp}} \bot$, and $\psi \vdash_{\mathsf{imp}} \varphi$ or $\psi \vdash_{\mathsf{imp}} \bot$. Hence, $\dashv\vdash_{\mathsf{imp}} \subseteq \dashv\vdash$ and the local tabularity of $\mathcal{L}$ follows immediately from the local tabularity of $\mathsf{CPL}_{\to}$.

By our characterization we then know that $\mathcal{L}$ must be determined by some finite matrix. A simple search will easily yield the following complete four-valued matrix for $\mathcal{L}$ (see [18] for further details): $M = \langle \mathbb{A}, D \rangle$, where $\mathbb{A} = \mathbf{2} \times \mathbf{2}$ is the $\{\to\}$-reduct of the four-element Boolean algebra, $\bot$ is interpreted as $\langle 1, 0 \rangle$ and $D = \{\langle 1, 1 \rangle\}$.

Let us now consider the *simplest protoalgebraic logic* $\mathcal{I} = \langle \mathbf{Fm}, \vdash \rangle$ of [10]. It is the logic on a single binary connective $\to$, enjoying *modus ponens* and all instances of the axiom $p \to p$. Clearly, for any two formulas φ and ψ, we have that $\varphi \vdash \psi$ if and only if $\vdash \psi$ or $\varphi = \psi$. Since there are infinitely many formulas which are not theorems (already in $\mathbf{Fm}_1$), it follows that the logic is not locally tabular, and hence also not strongly finite.

This argument can be generalized as follows:

Proposition 3.17. *Given $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ such that for all $\varphi, \psi \in \mathbf{Fm}$, $\varphi \vdash \psi$ if and only if $\vdash \psi$ or $\varphi = \psi$, and there are non-theorems of arbitrarily large size, then $\mathcal{L}$ is not strongly finite.*

Proof. If there is n such that there are infinitely many non-theorems in $\mathbf{Fm}_n$, then we immediately conclude that $\mathcal{L}$ is not locally tabular.

Otherwise, assume that for each n there is only a finite number of non-theorems in $\mathbf{Fm}_n$, and therefore their size is bounded. Then, if we pick any of the infinitely many non-theorems ψ_n with size larger than this bound, we have that $\vdash \psi_n^\sigma$ for every $\sigma : \mathsf{Var} \to \mathsf{Var}_n$ but $\nvdash \psi_n$. Therefore, $\mathcal{L}$ is not finitely determined.

In any of the cases we conclude that the logic is not strongly finite. □

Any logic presented only by schema axioms easily satisfies cancellation, as well as the first premiss of Proposition 3.17. However, there are many such logics that are finitely valued: e.g., the logic over the signature with a single n-ary connective c and with the single axiom $\frac{}{c(p_1, \ldots, p_n)}$. In order for a logic of axioms to fail strong finiteness what must happen is that there must be formulas of arbitrary large size that are not instances of the axioms.

This result is not very interesting in itself, neither are the logics it applies to usually considered by logicians. Nonetheless, it shows how, using our characterization results, one can identify relevant connections between the shapes of the schema axioms/rules of a logic and the nature of a possible semantics for it. Although we believe that much richer classes can be covered by similar results, Proposition 3.17 already captures a wide variety of (very weak) logics. It becomes trivial to recognize, for instance, that the logic of a binary connective presented only by (some of) the usual axioms of classic implication (but excluding *modus ponens*) is not strongly finite. Still note that the result also applies to logics with rules, namely the simplest protoalgebraic logic $\mathcal{I}$ mentioned above, or even its axiomatic weakening: the logic of a binary connective having only the rule of *modus ponens*.

4 Conclusions and outlook

We have established new necessary and sufficient conditions for a logic to be characterizable by (i) a single finite matrix (Theorem 3.16), (ii) a finite set of finite matrices (Theorem 3.14), and (iii) a class of n-generated matrices (Theorem 3.12). We accomplished this by making use of three properties: $\mathbf{Fm}_{\mathsf{fin}}$-determinedness, local tabularity and cancellation. We proved that these properties are all independent by giving examples of logics that separate all the possible combinations. While local tabularity and cancellation are easily found in the literature, $\mathbf{Fm}_{\mathsf{fin}}$-determinedness appears only briefly, and unnamed, in [23, p. 256], however the interplay between these properties has never been fully studied.

The property of $\mathbf{Fm}_{\mathsf{fin}}$-determinedness, which reduces, so to speak, the whole logic to what happens in a finite-variable fragment of the language, is the key ingredient to our main results, and is required for characterizing the above-mentioned classes of logics (i–iii). In fact, if a logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ is $\mathbf{Fm}_n$-determined, then $\mathcal{L}$ is the strongest among all the logics $\mathcal{L}'$ in the same signature that coincide with $\mathcal{L}$ over the $\mathbf{Fm}_n$-fragment. That is, for every such logic $\mathcal{L}'$, we have $\mathcal{L}_{\mathbf{Fm}_n} \subseteq \mathcal{L}' \subseteq \mathcal{L}$, where $\mathcal{L}_{\mathbf{Fm}_n}$ is the logic axiomatized by the set of rules $\{\frac{\Delta}{\psi} : \Delta \vdash \psi,\ \Delta \cup \{\psi\} \subseteq \mathbf{Fm}_n\}$. In other words, no such $\mathcal{L}'$ can be $\mathbf{Fm}_{\mathsf{fin}}$-determined, and thus finite-valued, except for $\mathcal{L}$ itself. Hence, the existence of $\mathcal{L}' \neq \mathcal{L}$ reduces to whether $\mathcal{L}$ can be axiomatized using n variables. To see this, one just needs to notice that $\mathcal{L}$ is axiomatizable using at most n variables if and only if $\mathcal{L}_{\mathbf{Fm}_n} = \mathcal{L}$. For instance, since CPL is not axiomatizable using two variables [9], the above argument implies that the logic $\mathsf{CPL}_{\mathbf{Fm}_2}$ generated by the 2-variable fragment of classical logic is not finite-valued.

An interesting problem related to Theorem 3.16 is to come up with an upper bound on the size of a characteristic matrix for a logic $\mathcal{L}$ which has cancellation and is also finitely determined and locally tabular. Indeed, if $\mathcal{L}$ is $\mathbf{Fm}_n$-determined, and $s : \mathbb{N} \to \mathbb{N}$ is the function that assigns to each natural number n the number $s(n)$ of $\equiv^n$-classes of $\mathbf{Fm}_n$, then our result shows that the logic can be given a characteristic matrix whose size is at most $s(n \cdot 2^{s(n)})$. This bound is tight, e.g., for CPL (assuming the language includes at least one nullary connective), as $s(0 \cdot 2^{s(0)}) = s(0) = 2$. It may be interesting to study this tightness (or possible improvements of the upper bound) for other well-known finite-valued logics (e.g., logics determined by a finite Heyting or Łukasiewicz algebra).

As noted earlier, in the absence of local tabularity, the construction in Theorem 3.16 cannot be used to establish a connection between $\mathbf{Fm}_{\mathsf{fin}}$-determinedness and the cardinality of a characteristic matrix for a logic that has cancellation. We wonder whether such a connection exists, and which other abstract property would correspond to a logic having a denumerable characteristic matrix. Int is known to have no denumerable characteristic matrix, but on the other hand it also fails $\mathbf{Fm}_{\mathsf{fin}}$-determinedness.

From the point of view of algebraic logic it would be interesting to study the properties of some algebra-based semantics for a logic that correspond to the various notions of finite-valuedness we considered. For example, it is not hard to show that an algebraizable logic[6] $\mathcal{L}$ is strongly finite if and only if the algebraic counterpart of $\mathcal{L}$ is finitely generated as a generalized quasivariety. Can this result be generalized to wider classes of (or even arbitrary) logics? The answer seems far from obvious, also because a precise formulation of the problem depends on which class of algebras we take as the algebraic counterpart of an arbitrary logic (several well-motivated options are considered in the literature, see [11]).

The characterizations provided in the present paper, together with the result on the independence of the involved properties, give a new perspective on the notion of finite-valuedness. For instance, it is now clear that the divide between a logic admitting a semantics given by a finite set of finite matrices and a single finite matrix is exactly cancellation, a property that is not related to the number of truth values involved. As observed in the introduction, such a strict notion of finite-valuedness leaves out Kleene's logic of order $\mathsf{K}_{\leq}$ [11], which is given by two three-valued matrices that furthermore have the same underlying algebra, and thus differ only on the sets of distinguished elements. This situation corresponds to the notion of generalized matrix: an algebra with a family of sets of distinguished elements instead of just one set, and may be useful for this discussion. In fact, it follows from the proof of Theorem 3.14 that a logic is given by a finite generalized matrix if and only if it is given by a finite set of finite matrices, for all the matrices in the class $\mathcal{M}_n^{\mathcal{L}}$ share the same algebra $\mathbf{Fm}_n^*$ and hence form a generalized matrix. In light of our results, one may thus argue that a more appropriate notion of finite-valuedness for a logic would be to be characterizable by a finite generalized matrix, which is exactly captured by $\mathbf{Fm}_{\mathsf{fin}}$-determinedness plus local tabularity.

We would like to extend these characterization results to non-deterministic matrix semantics (see e.g. [2]). It is not hard to see that any logic given by a finite set of finite non-deterministic matrices is $\mathbf{Fm}_{\mathsf{fin}}$-determined. Furthermore, if n is an upper bound to the cardinality of the non-deterministic matrices, then

[6] Most well-known logics in the literature are algebraizable, for example classical and intuitionistic logic, (the global consequence of) normal modal logics, Łukasiewicz infinite-valued logic (corresponding to, respectively, Boolean algebras, Heyting algebras, modal algebras and MV-algebras). Natural examples of non-algebraizable logics abound, too: for example the above-mentioned Kleene's logic of order is not algebraizable. For a reference, see [6].

the logic is Var_n-determined. We may therefore ask: which additional properties (if any) are needed to capture this broader notion of (non-deterministic) finite-valuedness?

The property of Δ-determinedness seems to deserve further investigation by itself. The fact that classical logic is $\{\top, \bot\}$-determined is an immediate consequence of its two-valued semantics. Something similar can be obtained whenever a logic can express all its truth values, as it happens for example, with the logics in Post's hierarchy [19] (which not only can express the constant functions but also are also functionally complete). From an abstract point of view, one can look at formulas in Δ as syntactical truth values for a Δ-determined logic. It may be interesting to study such a logic by exploring the structure of the set Δ, whenever it is, for example, finite or constituted by formulas with a certain recursive shape pattern.

References

[1] S. Aguzzoli, B. Gerla, and V. Marra. Gödel algebras free over finite distributive lattices. *Annals of Pure and Applied Logic*, 155(3):183–193, 2008.

[2] A. Avron and A. Zamansky. Non-deterministic semantics for logical systems. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, second edition*, volume 16, pages 227–304. Springer, 2011.

[3] G. Bezhanishvili, N. Bezhanishvili, and D. de Jongh. The Kuznetsov-Gerciu and Rieger-Nishimura logics. *Logic and Logical Philosophy*, 17:73–110, 2008.

[4] N. Bezhanishvili and D. de Jongh. Extendible formulas in two variables in intuitionistic logic. *Studia Logica*, 100:61–89, 2012.

[5] N. Bezhanishvili and D. de Jongh. Intuitionistic logic. *Manuscript.* `http://www.cs.le.ac.uk/people/nb118/Publications/ESSLLI%2705.pdf`

[6] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[7] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra.* The Millennium edition, 2000.

[8] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides.* Oxford University Press, 1997.

[9] A. H. Diamond and J. C. C. McKinsey. Algebras and their subalgebras. *Bulletin of the American Mathematical Society*, 53:959–962, 1947.

[10] J. M. Font. The simplest protoalgebraic logic. *Mathematical Logic Quarterly*, 59:435–451, 2013.

[11] J. M. Font. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the I.G.P.L.*, 5(3):413–440, 1997.

[12] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. Springer-Verlag, second edition, 2009.

[13] K. Gödel. Zum intuitionistischen aussagenkalkül. *Anzeiger der Akademie der Wissenschaften in Wien*, 69:65–66, 1932.

[14] P. Hájek. *Metamathematics of fuzzy logic*, volume 4 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 1998.

[15] J. Łoś and R. Suszko. Remarks on sentential logics. *Indagationes Mathematicae*, 20:177–183, 1958.

[16] S. Marcelino and C. Caleiro. Decidability and complexity of fibred logics without shared connectives. *Logic Journal of the IGPL*, 24(5):673–707, 2016.

[17] S. Marcelino and U. Rivieccio. Locally tabular $\neq$ locally finite. Submitted. `http://sqig.math.ist.utl.pt/pub/MarcelinoS/16-MR-localota.pdf`

[18] C. Caleiro, S. Marcelino and U. Rivieccio. Breaking negation. In preparation.

[19] E. L. Post. Introduction to a General Theory of Elementary Propositions. *American Journal of Mathematics*, 43:163–185, 1921.

[20] D. J. Shoesmith and T. J. Smiley. Deducibility and many-valuedness. *The Journal of Symbolic Logic*, 36(4):610–622, 1971.

[21] R. Wójcicki. Logical matrices strongly adequate for structural sentential calculi. *Bulletin de l'Académie Polonaise des Sciences*, Classe III XVII, 333–335, 1969.

[22] R. Wójcicki. Matrix approach in the methodology of sentential calculi. *Studia Logica*, 32:7–37, 1973.

[23] R. Wójcicki. *Theory of logical calculi. Basic theory of consequence operations*, volume 199 of *Synthese Library*. Reidel, Dordrecht, 1988.