Locally tabular $\neq$ locally finite

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Abstract. We show that for an arbitrary logic being locally tabular is a strictly weaker property than being locally finite. We describe our hunt for a logic that allows us to separate the two properties, revealing weaker and weaker conditions under which they must coincide, and showing how they are intertwined. We single out several classes of logics where the two notions coincide, including logics that are determined by a finite set of finite matrices, self-extensional logics, algebraizable and equivalential logics. Furthermore, we identify a closure property on models of a logic that, in the presence of local tabularity, is equivalent to local finiteness.

Mathematics Subject Classification (2010). Primary 03B22; Secondary 03G27.

Keywords. Locally tabular, locally finite, selfextensional logics, matrix semantics.

1. Introduction

This paper is part of an ongoing research which aims at characterizing classes of logics via abstract properties of their consequence relations. The first result this enterprise, obtained by the Polish school of logicians (Tarski, Łukasiewicz, Lindenbaum and others), is arguably the correspondence between the standard Tarskian notion of a logic as a structural consequence relation on formulas and the semantical consequence relation determined by a class of logical matrices. Another fundamental advance was the characterization of logics defined by a single matrix as precisely those structural consequence relations that satisfy the so-called *cancellation* property [12].

Our paper [5] is also a contribution to this line of research. There we identify properties that characterize various families of logics which can be

Work done under the scope of R&D Unit 50008, financed by the applicable financial framework (FCT/MEC through national funds and when applicable co-funded by FEDERPT2020) and with the support of EU FP7 Marie Curie PIRSES-GA-2012-318986 project *GeTFun: Generalizing Truth-Functionality*. The first author also acknowledges the FCT postdoc grant SFRH/BPD/76513/2011.

regarded as finite-valued in a broad sense, that is logics which can be endowed with a semantics involving some finite structure or a finite number of finite structures. We showed in particular that logics that are characterizable by a family of n-generated matrices are precisely those that are $\mathbf{Fm}_n$-*determined* (the logics determined by all instances with formulas in n variables). Adding the property of *local tabularity*, we obtain the class of logics that are characterizable by a finite set of finite matrices, and adding cancellation, we have the logics characterizable by a single finite matrix.

Local tabularity is a well-studied property in the context of modal and super-intuitionistic logics (see e.g. [6, p. 19]). Interestingly, although all results of [5] rely on local tabularity, the property that actually helped us construct semantics based on finite matrices is a slightly different one that we called *local finiteness*. We were able to use this notion because the two are equivalent in the presence of $\mathbf{Fm}_{\mathsf{fin}}$-determinedness (which was always assumed in our results). At the time it was not clear whether this equivalence held true in a wider context, maybe even for arbitrary logics. The present paper focuses precisely on the relation between local tabularity and local finiteness, and in particular on determining under which conditions the two coincide.

For a given logic, both local tabularity and local finiteness state that there are only a finite number of equivalence classes over any set of formulas built with a finite number of variables, but the two concern a different equivalence relation on formulas. Local tabularity refers to the *Frege relation* (logical equivalence or inter-derivability) of a logic, while local finiteness refers to its *Tarski congruence*, which is the largest congruence of the formula algebra that is contained in the Frege relation (it follows that local finiteness always implies local tabularity).

In many well-known logics, the Tarski congruence actually coincides with the Frege relation: this happens for instance with classical and intuitionistic logic, local consequences associated to normal modal logics, Gödel-Dummett infinite-valued logic, the Belnap-Dunn four valued logic. These are called *selfextensional logics*. It is equally easy to find natural examples of non-selfextensional logics: for instance the global consequences associated to normal modal logics, Łukasiewicz's many-valued logics, Kleene's strong three-valued logic, bilattice logics, as well as most extensions of the Belnap-Dunn four valued logic [1].

Thus the question arises whether, for arbitrary logics, local finiteness is in fact strictly stronger than local tabularity, that is, is there a logic that is locally tabular but not locally finite?

If such an example exists, it will not be easily found among the known logics in the literature. This is because we have to rule out not only all selfextensional as well as (taking into account [5]) all finite-valued logics, but also, though this is not immediate to see, all the logics that are algebraizable in the sense of [2]. For instance, none of the fuzzy logics considered since [10], which would otherwise offer a vast array of non-locally finite logics, will provide a counterexample. Even more, one can show that any logic that is equivalential (a weaker notion than algebraizable: see [7]) will not do either, which crosses out many modal logics, even if we consider the local consequence.

Put in other terms, a locally tabular but non-locally finite logic must be quite peculiar even in the landscape of non-classical (non-equivalential, non-selfextensional) logics. The main contribution of the present paper is precisely the construction of such a logic, witnessing that local tabularity does not imply local finiteness; in this endeavour, the greatest technical effort lies in the description of the equivalence classes of the Frege relation, which is crucial in ensuring the logic is locally tabular.

We moreover single out and discuss in detail a number of sufficient conditions that ensure that the two notions coincide, thus somehow negatively suggesting the features that any other potential counterexample logic must possess. The weakest of these conditions, which is already quite close to the algebraic definition of local finiteness, concerns the closure of reduced g-models of a logic under (infinite) finitely-generated submodels–see Theorem 3.2 (3). In the presence of local tabularity, it is in fact equivalent to local finiteness of the logic.

The paper is organized as follows. After the preliminary Section 2 which fixes the notation and basic definitions, in Section 3 we present and discuss a number of properties which ensure that local tabularity and local finiteness coincide. Theorem 3.2 sums up the results in this section. Finally, Section 4 constructs the logic that witnesses independence of the two properties (Theorem 4.3).

2. Preliminaries

In this section we fix the notation and introduce the definitions that are used throughout the paper.

2.1. Logics

(See [13] for further details). We denote by Var the (countable) set of *propositional variables*. Given an algebraic signature (that we usually leave implicit), we denote by $\mathbf{Fm}$ the absolutely free algebra built over Var. We use the same notation for the underlying carrier of $\mathbf{Fm}$, the set of formulas in the signature. A *logic* defined over $\mathbf{Fm}$, which we denote by $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, is a structural consequence relation. For a finite set of variables $\{p_1, \ldots, p_n\} \subseteq \mathsf{Var}$, we denote by $\mathbf{Fm}_n$ the absolutely free algebra built over $\{p_1, \ldots, p_n\}$, which we view as a subalgebra of $\mathbf{Fm}$. For $\varphi \in \mathbf{Fm}$ let $\mathsf{sbf}(\varphi)$ denote the set of subformulas φ, and and $\mathsf{head}(\varphi)$ the outermost connective of φ, or φ if it is a variable.

2.2. Algebras and matrices

(See [4, 8] for further details). As usual, an algebra $\mathbf{A}$ is a set A equipped with a finite number of finitary operations. Given an algebra $\mathbf{A}$ and a set $X \subseteq A$, we say that $\mathbf{A}$ is *generated by* X when, for every $a \in A$, there is a term $t(p_1,\dots,p_k)$ in the algebraic language of $\mathbf{A}$ and elements $a_1,\dots,a_k \in X$ such that $a = t_{\mathbf{A}}(a_1,\dots,a_k)$. In this case we write $A = \mathbb{S}(X)$. If $|X| \leq n \in \mathbb{N}$.

We will be dealing with matrix semantics for logics. A *matrix* is a pair $M = \langle \mathbf{A}, D\rangle$ where $\mathbf{A}$ is an algebra and $D \subseteq A$ is a subset of *designated elements*. We extend the above definitions on algebras to matrices, saying that $M = \langle \mathbf{A}, D\rangle$ is generated by some $X \subseteq A$ (n-generated, finitely-generated) when its algebraic reduct $\mathbf{A}$ is.

Each matrix $M = \langle \mathbf{A}, D\rangle$ determines a logic $\mathcal{L}_M = \langle \mathbf{Fm}, \vdash_M\rangle$ in the above sense by defining $\Gamma \vdash_M \varphi$ if and only if, for all homomorphism $v\colon \mathbf{Fm} \to \mathbf{A}$, we have that $v(\Gamma) \subseteq D$ implies $v(\varphi) \in D$. This extends to a class of matrices $\mathcal{M} = \{M_i : i \in I\}$, which defines a logic $\mathcal{L}_{\mathcal{M}} = \langle \mathbf{Fm}, \vdash_{\mathcal{M}}\rangle$ by setting $\Gamma \vdash_{\mathcal{M}} \varphi$ if and only if $\Gamma \vdash_{M_i} \varphi$ for all $i \in I$. A logic is said *strongly finite* if it is characterized by a finite set of finite matrices.

The *Leibniz congruence* of a matrix $M = \langle \mathbf{A}, D\rangle$ is the largest congruence of $\mathbf{A}$ (usually denoted $\boldsymbol{\Omega}_{\mathbf{A}}(F)$) that is compatible with D, meaning that, for all elements $a, b \in D$, if $a \in D$ and $\langle a, b\rangle \in \boldsymbol{\Omega}_{\mathbf{A}}(D)$, then $b \in D$. The *reduction* of $M = \langle \mathbf{A}, D\rangle$ is the matrix $M^* = \langle \mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}}(D), D/\boldsymbol{\Omega}_{\mathbf{A}}(D)\rangle$, where $\mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}}(D)$ is the usual quotient algebra and $D/\boldsymbol{\Omega}_{\mathbf{A}}(D) = \{[a]_{\boldsymbol{\Omega}_{\mathbf{A}}(D)} : a \in D\}$. A matrix is $M = \langle \mathbf{A}, D\rangle$ *reduced* when $\boldsymbol{\Omega}_{\mathbf{A}}(D)$ is the identity, that is, when M is isomorphic to its reduction M^*. Any matrix M defines the same logic as its reduction M^*, which makes reduced matrices particularly important in the semantical study of logics.

Any logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash\rangle$ is (trivially) complete with respect to the class of matrices $\mathcal{M}_{\mathcal{L}} = \{\langle \mathbf{Fm}, \mathcal{T}\rangle : \mathcal{T} \text{ is a theory of } \mathcal{L}\}$. This class can itself be reduced in the following way. $M_{\mathcal{L}}$ is an example of a *generalized matrix* (g-matrix), that is, a pair $\langle \mathbf{A}, \mathcal{C}\rangle$ where $\mathbf{A}$ is an algebra and $\mathcal{C}$ is a closure system on $\mathbf{A}$ (i.e. a family $\mathcal{C} \subseteq P(A)$ closed under arbitrary intersections, with $A \in \mathcal{C}$). The *Tarski congruence* of a g-matrix $\langle \mathbf{A}, \mathcal{C}\rangle$ is the greatest congruence θ of $\mathbf{A}$ that is compatible with each $D \in \mathcal{C}$ (i.e. D is a union of equivalence classes of θ), and can be obtained as $\theta = \bigcap\{\boldsymbol{\Omega}_{\mathbf{A}}(D) : D \in \mathcal{C}\}$. The *reduction* of $\langle \mathbf{A}, \mathcal{C}\rangle$ is the g-matrix $\langle \mathbf{A}/\theta, \mathcal{C}/\theta\rangle$, where $\mathcal{C}/\theta = \{D/\theta : D \in \mathcal{C}\}$. Notice that a g-matrix $\langle \mathbf{A}, \mathcal{C}\rangle$ is just a particular class of matrices that share the same underlying algebra $\mathbf{A}$, hence all definitions about classes of matrices are extended to g-matrices. We will use this fact whenever it is conveniente throughout the paper.

Given a logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash\rangle$, we denote by $\equiv$ the Tarski congruence of the g-matrix $M_{\mathcal{L}} = \{\langle \mathbf{Fm}, \mathcal{T}\rangle : \mathcal{T} \text{ is a theory of } \mathcal{L}\}$. Applying the reduction process to this g-matrix, we obtain the *Lindenbaum-Tarski g-matrix*

$$M^*_{\mathcal{L}} = \langle \mathbf{Fm}^*, \{\mathcal{T}^* : \mathcal{T} \text{ is a theory of } \mathcal{L}\}\rangle,$$

where $\mathbf{Fm}^* = \mathbf{Fm}/{\equiv}$ and $\mathcal{T}^* = \mathcal{T}/{\equiv}$. This can be restricted to formulas in n-variables, defining $\equiv^n$ as the restriction $\equiv \cap\, (\mathbf{Fm}_n \times \mathbf{Fm}_n)$ and denoting by $\mathbf{Fm}^*_n$ the corresponding quotient $\mathbf{Fm}_n/{\equiv^n}$.

For a logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash\rangle$, the class of $\mathcal{L}$-*algebras*

$$\mathbf{Alg}(\mathcal{L}) := \{\mathbf{A} : \langle \mathbf{A}, \mathcal{C}\rangle \text{ is a reduced g-matrix for } \mathcal{L}\}$$

consists of all algebras $\mathbf{A}$ (in the same algebraic language as $\mathcal{L}$) that are the reducts of some reduced g-matrix for $\mathcal{L}$. So, in particular, we have that $\mathbf{Fm}^* \in \mathbf{Alg}(\mathcal{L})$.

2.3. Local tabular and local finite logics

Given a logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, the inter-derivability relation $\dashv\vdash$ is known as the *Frege relation* of $\mathcal{L}$. This is always an equivalence relation, but not necessarily a congruence of the formula algebra $\mathbf{Fm}$. When $\dashv\vdash$ is a congruence of $\mathbf{Fm}$, then it coincides with the Tarski congruence $\equiv$ and the logic is called *selfextensional*. In general, the Tarski congruence is always contained in the Frege relation: $\equiv$ is in fact the largest congruence of $\mathbf{Fm}$ contained in $\dashv\vdash$. The following characterization of the Tarski congruence [8, p. 29] will be especially useful for us: for any logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ and for all $\varphi, \psi \in \mathbf{Fm}$,

$$\varphi \equiv \psi \quad \text{iff} \quad \gamma(\varphi, \vec{q}) \dashv\vdash \gamma(\psi, \vec{q}) \quad \forall \gamma(p, \vec{q}) \in \mathbf{Fm}$$

where $\gamma(p, \vec{q})$ is any formula with parameters p and $\vec{q} = q_1, \dots, q_k$. We say that a set of formulas with parameters $\Delta(p, \vec{q}) \subseteq \mathbf{Fm}$ is *($\mathcal{L}$-)separating* when, for all $\varphi, \psi \in \mathbf{Fm}$,

$$\varphi \equiv \psi \quad \text{iff} \quad \gamma(\varphi, \vec{q}) \dashv\vdash \gamma(\psi, \vec{q}) \quad \forall \gamma(p, \vec{q}) \in \Delta.$$

Using this terminology, the above characterization of the Tarski congruence can be rephrased by saying that the set of all formulas $\mathbf{Fm}$ is $\mathcal{L}$-separating for any logic $\mathcal{L}$.

A logic is called *locally tabular*[1] when, for every $n \in \mathbb{N}$, $\mathbf{Fm}_n$ is partitioned in finitely many classes by the Frege relation (i.e. any set of formulas built over finitely many propositional variables contains only finitely many nonequivalent formulas). Locally finiteness is defined in a similar way, swapping the Frege relation by the Tarski congruence. We say that a logic is *locally finite* when, for every $n \in \mathbb{N}$, $\mathbf{Fm}_n$ is partitioned in finitely many classes by the Tarski congruence. From the definitions it immediately follows that local finiteness and local tabularity coincide on self-sextensional logics, and that local finiteness implies local tabularity.

2.4. Local finiteness in algebra

Local finiteness is a well-known notion from universal algebra. An algebra $\mathbf{A}$ is said to be locally finite when all finitely-generated subalgebras of $\mathbf{A}$ are finite [4, p. 76, Definition 10.14]. Likewise, a class of algebras is locally finite when all algebras in the class are locally finite.

Our application of the term to logics is justified by the following fact, which will be often used in our proofs.

Remark 2.1. For any logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, the following are equivalent:

(a) $\mathcal{L}$ is locally finite,
(b) $\mathbf{Fm}^*$ is locally finite,
(c) $\mathbf{Alg}(\mathcal{L})$ is locally finite.

[1] Our terminology is inspired by [6, p. 19], which however does not distinguish between the two notions.

It is clear that (b) implies (a). To show that (a) implies (b), it is sufficient to notice that every finitely-generated subalgebra of $\mathbf{Fm}^*$ is a subalgebra of $\mathbf{Fm}^*_n$ for some $n \in \mathbb{N}$. It is obvious that (c) implies (b) because $\mathbf{Fm}^* \in \mathbf{Alg}(\mathcal{L})$. To see that (b) implies (c), recall that by [8, p. 30] the algebra $\mathbf{Fm}^*$ is free in $V(\mathbf{Fm}^*)$, where $V(\mathbf{Fm}^*)$ denotes the variety generated by $\mathbf{Fm}^*$. This means, by Theorem 10.15 in [4, p. 76], that $V(\mathbf{Fm}^*)$ is locally finite if and only if $\mathbf{Fm}^*$ is locally finite. Hence, for any class of algebras $\mathbb{K}$ satisfying $\mathbf{Fm}^* \in \mathbb{K} \subseteq V(\mathbf{Fm}^*)$, we have that $\mathbb{K}$ is locally finite if and only if $\mathbf{Fm}^*$ is locally finite. The result then follows from $\mathbf{Alg}(\mathcal{L}) \subseteq V(\mathbf{Alg}(\mathcal{L})) = V(\mathbf{Fm}^*)$, which is proved in [8, Proposition 2.26].

3. When locally tabular = locally finite

As mentioned earlier, for a large class of logics, being locally tabular is equivalent to being locally finite. In this section we identify some very general conditions that force the two properties to coincide. Being that the other implication always holds, we are looking for conditions such under which, by assuming local tabularity, we obtain local finiteness. This will point us to where we need to look for a possible counterexample (to be constructed in the next section), and will help us understand why such a logic must exhibit a somewhat peculiar behaviour.

To begin with, notice that all finite-valued logics are both locally tabular and locally finite. That is, a logic is both locally tabular and locally finite whenever the following condition is met:

(1) *$\mathcal{L}$ is strongly finite.*

This follows from [5, Theorem 3.14] according to which a logic is strongly finite if and only if it is locally tabular and $\mathbf{Fm}_{\mathsf{fin}}$-determined, and [5, Lemma 3.8], which states that under $\mathbf{Fm}_{\mathsf{fin}}$-determinedness local tabularity is equivalent to local finiteness.

As mentioned in the preceding section, the Tarski congruence of any logic can be defined in terms of its Frege relation by means of what we called a *separating set*. Many logics known in the literature admit a finite separating set, and all logics that we are aware of (except ad hoc examples) admit a separating set that is built over a finite number of variables. For example, a separating set that works for bilattice logics [3], as well as for all the (non-selfextensional) extensions of Belnap-Dunn logic [11, 1], is given by $\Delta = \{p \vee q, \neg p \vee q\}$, so that $\varphi \equiv \psi$ if and only if $\varphi \vee q \dashv\vdash \psi \vee q$ and $\neg\varphi \vee q \dashv\vdash \neg\psi \vee q$. Some logics, for example certain modal systems, require an infinite separating set such as $\Delta = \{\Box p, \Box\Box p, \ldots\}$, and it is easy to build ad hoc examples with this feature. For us, what matters is that, in all of these cases, Δ only uses a finite number of variables, for we can prove the following result. Let $\mathcal{L}$ be a locally tabular logic. Then $\mathcal{L}$ is locally finite when the following condition holds:

(2) *the Tarski congruence of $\mathcal{L}$ is defined from the Frege relation using a separating set built over a finite number of variables.*

This fact, which we proved in [5, Lemma 3.7], is perhaps in practice the most useful condition that we currently know of, and has important consequences. In particular, it tells us that our search for a counterexample can be narrowed down to the class of logics that are not equivalential. The class of equivalential logics is one of the best known and studied in algebraic logic: using our terminology, we can say that a logic is *equivalential* when it admits a separating set built over exactly two variables [7, p. 3].

Equivalential logics include as a proper subclass all algebraizable logics, for example intuitionistic logic together with its extensions, the global consequence of many normal modal logics, most so-called truth-preserving fuzzy logics, and so on. Using the algebraic logic terminology, we can then say that in the best case our counterexample will belong to the class of "weakly algebraizable" logics [7, p. 6]. As a matter of fact, at the moment we do not know whether a weakly algebraizable logic can be built that would serve as a counterexample for our purpose (see the discussion at the end of this section).

The last condition that we consider is somewhat more technical, and we were led to single it out by the very process of construction of our counterexample. Let us then retrace some of the steps that we followed in the construction.

We were looking for a locally tabular but non-locally finite logic; the easier way to obtain such a logic seemed to be by considering the logic defined by a single logical matrix $\langle \mathbf{A}, D \rangle$. Taking into account (1) above, we know that $\mathbf{A}$ must be infinite. In fact, the matrix $\langle \mathbf{A}, D \rangle$ must be "essentially infinite", that is, it must yield a logic that cannot be characterized by any other finite matrix, or even any finite set of finite matrices. For example, taking $\mathbf{A}$ to be an infinite Boolean algebra would obviously not do the job. This "essential infinity" is captured by the fact that our matrix $\langle \mathbf{A}, D \rangle$ must be *reduced* in the sense introduced in Section 2, that is, it must not be possible to reduce its size obtaining another finite matrix that defines the same logic.

However, even if we have a reduced matrix $\langle \mathbf{A}, D \rangle$, with $\mathbf{A}$ infinite, the logic of $\langle \mathbf{A}, D \rangle$ is not guaranteed to be non-locally finite. For example, taking $\mathbf{A}$ to be a denumerable linearly ordered Heyting algebra (with, say, $D = \{1\}$, where 1 denotes the top element of $\mathbf{A}$) would not work either. In fact, although $\langle \mathbf{A}, \{1\} \rangle$ is reduced, the logic it defines (known as Gödel-Dummett logic) is known to be locally finite[2].

The safest way to proceed seemed thus to be to consider a reduced matrix $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is not locally finite, which is justified by the formulation of local finiteness given in Remark 2.1 (c). One could, for instance, let $\mathbf{A}$ itself to be infinite and finitely generated. This, however, would not work either. In fact, we proved in [5] that, when $\mathbf{A}$ is a finitely-generated algebra, the logic of any matrix $\langle \mathbf{A}, D \rangle$ is $\mathbf{Fm}_{\mathsf{fin}}$-determined, and therefore will be locally finite if and only if it is locally tabular. A better option (which is what we

[2]Though this is not essential for the reasoning that we wanted to illustrate here, we note that Gödel-Dummett logic is also algebraizable and selfextensional, each of these reasons being sufficient to make it an unsuitable candidate for us.

actually do in the next section) is to build a matrix $\langle \mathbf{A}, D\rangle$ such that $\mathbf{A}$ itself is not finitely-generated, but has a proper subalgebra that is infinite and finitely-generated.

Let us now introduce the last and weakest of the properties. This one is telling us that, in order to separate local tabularity from local finiteness, we should consider the logic of a reduced matrix which has an infinite finitely-generated submatrix that is not reduced. As we shall see, the following property is equivalent to local finiteness when assuming local tabularity.

Proposition 3.1. *A locally tabular logic $\mathcal{L}$ is locally finite if the following holds:*

(3) every infinite finitely-generated sub-g-matrix of any reduced g-matrix for $\mathcal{L}$ is reduced.

Proof. Let $\mathcal{L}$ be a locally tabular logic that satisfies (3). Using the fact that $\mathcal{L}$ is locally finite if and only if its Lindenbaum-Tarski algebra $\mathbf{Fm}^*$ is locally finite (Remark 2.1), it will be enough to show that any finitely-generated subalgebra $\mathbf{A}$ of $\mathbf{Fm}^*$ is finite. Given such $\mathbf{A}$, let $M = \langle \mathbf{A}, \mathcal{C}\rangle$ be the sub-g-matrix of $M^*_{\mathcal{L}}$ whose algebraic reduct is $\mathbf{A}$, and let $\mathcal{L}_M$ be the corresponding logic. By assumption M is reduced, so $\mathbf{A} \in \mathbf{Alg}(\mathcal{L}_M)$. If $\mathbf{A}$ were infinite, then $\mathbf{Alg}(\mathcal{L}_M)$ would not be locally finite, which means (by Remark 2.1) that the logic $\mathcal{L}_M$ would not be locally finite. At the same time, $\mathcal{L}_M$ is stronger than $\mathcal{L}$ and is therefore locally tabular. As $\mathbf{A}$ is finitely-generated (thus $\mathbf{Fm}_{\mathsf{fin}}$-determined) and locally tabular, we have, by [5, Lemma 3.8], that $\mathcal{L}_M$ must be locally finite, thus obtaining a contradiction. We conclude that $\mathbf{A}$ must be finite as required. $\square$

Notice that Proposition 3.1 actually establishes an equivalence: if a logic $\mathcal{L}$ is locally tabular, then $\mathcal{L}$ satisfies (3) if and only if it is locally finite. This is clear since the formulation of local finiteness given in Remark 2.1 (b) implies that any locally finite logic $\mathcal{L}$ will vacuously satisfy (3). This fact can cause some discomfort regarding the property (3), it would be neater to be able to replace it by: every finitely generated sub-g-matrices of reduced g-matrix is reduced. Although these last property surely implies (3), we do not know if these are equivalent, even assuming locally tabularity. However, showing that a locally tabular logic $\mathcal{L}$ is also locally finite is equivalent to checking that all finitely generated sub-g-matrix of reduced g-matrices of $\mathcal{L}$ are either finite or reduced.

We already knew from (2) that a locally tabular and equivalential logic $\mathcal{L}$ must also be locally finite. We can now provide an alternative proof: observe that, by [9, Proposition 2.8], if $\mathcal{L}$ is equivalential, then the class of its reduced g-matrices enjoys the so-called *subreduction property* [9, Definition 2.6]. This implies in particular that every (not necessarily infinite or finitely-generated) sub-g-matrix of a reduced g-matrix for $\mathcal{L}$ is reduced, and so by (3) we have that $\mathcal{L}$ must be locally finite.

The next theorem sums up the results explained so far.

Theorem 3.2. *For a locally tabular logic $\mathcal{L}$, each of the following conditions is sufficient to conclude that $\mathcal{L}$ is also locally finite.*

(1) *$\mathcal{L}$ is strongly finite.*
(2) *The Tarski congruence of $\mathcal{L}$ is defined from its Frege relation using a separating set built over a finite number of variables.*
(3) *Any infinite finitely-generated sub-g-matrix of a reduced g-matrix for $\mathcal{L}$ is reduced.*

Of the above conditions, (1) is the strongest and (3) the weakest. In fact, as we have seen, for a locally tabular logic (3) is actually equivalent to being locally finite. Note aso that (1) alone, even without assuming local tabularity, entails that $\mathcal{L}$ is locally finite and therefore locally tabular. Taking into account what we observed above, this means that (1) implies (3). Also, we proved in [5] that (1) implies (2). A variety of examples show that neither (2) nor (3) implies (1). The only implication which is still open is thus whether (3) implies (2). We state two more open problems below.

Problem 1. As announced, in the next section we are going to define a logic that is locally tabular and non-locally finite. Our example is a non-protoalgebraic logic, that is, it falls outside the so-called Leibniz hierarchy (shown e.g. in [7, p. 7]). The above-discussed results imply that one cannot find, as an alternative example, a logic that is equivalential or higher up in the hierarchy. However, we do not know whether it might be possible to find, for example, a weakly algebraizable, or even a regularly weakly algebraizable logic, that serves the purpose.

Problem 2. Parallel to the above question, and in the opposite direction, besides the equivalential: is it possible to find a class of logics in the Leibniz hierarchy for which condition (3) holds? Or even, perhaps, a class of logics that corresponds exactly to those that satisfy (3)? This would provide an alternative characterization of logics for which local tabularity and local finiteness must coincide. Examples of such classes would be truth-equational logics, protoalgebraic and (regularly) weakly algebraizable logics. Attacking this problem should benefit from solving the above-mentioned question of finding neater alternative formulations of (3). Is it true that, at least under local tabularity, (3) is equivalent to: any infinite finitely-generated sub-g-matrix of a reduced g-matrix for $\mathcal{L}$ is reduced?

4. The separating example

In this section we separate the two properties by building a logic that is locally tabular but fails to be locally finite. We introduce the logic semantically via the matrix $M = \langle \mathbf{A}, D\rangle$ defined below. Notice that our logic will have to lack all the properties mentioned in the preceding section, in particular, it cannot be self-extensional nor equivalential (hence, also not algebraizable).

The algebra $\mathbf{A} = \langle A, \oplus, \Box\rangle$ has the set $A = \mathbb{N} - \{0\}$ of positive natural numbers as universe and two operations, a binary $\oplus$ and a unary $\Box$. In order to define the operations of $\mathbf{A}$, it is convenient to introduce some auxiliary notation. A natural number n is a *perfect square* if $n = k^2$ for some natural

k. Given a natural number n, the operation *upper square* (us) returns the smallest perfect square greater or equal to n, that is

$$\mathsf{us}(n) := \min\{k^2 \in \mathbb{N} : n \leq k^2\},$$

and *lower square* (ls) the greatest perfect square smaller than n, that is

$$\mathsf{ls}(n) := \max\{k^2 \in \mathbb{N} : k^2 < n\}.$$

The binary operation $\oplus$ picks two natural numbers and, if second is the successor of the first and both lie between the same two consecutive perfect squares, it returns the successor of the second; otherwise it returns 1. That is, for $m, n \in A$,

$$m \oplus n := \begin{cases} n & \text{if } \mathsf{us}(m) = \mathsf{us}(n) \text{ and } n = m + 1 \\ 1 & \text{otherwise.} \end{cases}$$

We also write $\oplus(m_1) = \oplus(m_1, m_1)$, $\oplus(m_1, m_2, m_3) = \oplus(\oplus(m_1, m_2), m_3)$ and for $k, m_1, \ldots, m_k \in \mathbb{N}$,

$$\oplus(m_1, \ldots, m_{k+1}) = \oplus(\oplus(m_1, \ldots, m_k), m_{k+1}).$$

Notice that $\oplus(m, m + 1, \ldots, \mathsf{us}(m)) = \mathsf{us}(m)$.

The unary operation[3] $\Box$ returns the successor of the upper square of its input if the input is already the successor of a perfect square, otherwise it returns 1. For $m \in A$,

$$\Box m := \begin{cases} \mathsf{us}(m) + 1 & \text{if } 1 < m = \mathsf{ls}(m) + 1 \\ 1 & \text{otherwise.} \end{cases}$$

Equivalently,

$$\Box m := \begin{cases} (n + 1)^2 + 1 & \text{if } m = n^2 + 1 \text{ for some } n \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

We obtain our matrix $M = \langle \mathbf{A}, D\rangle$ by choosing as distinguished elements the set of perfect squares, that is, $D := \{n^2 : n > 1\}$.

We mentioned in the previous section that we want to depart from an "essentially infinite" matrix. Indeed the next lemma shows that that M is reduced.

Lemma 4.1. *M is reduced.*

Proof. For each pair of elements $m, n \in A$, there is an algebraic term that takes one to D while leaving the other outside. Let $\alpha_m(n, m+1, \ldots, \mathsf{us}(m)) = \oplus(n, m + 1, \ldots, \mathsf{us}(m))$. Then it is easy to check that $\alpha_m(m) = \mathsf{us}(m) \in D$ and $\alpha_m(n) = 1 \notin D$. $\Box$

The next proposition shows that $\mathcal{L}_M$ is not locally finite, as desired.

Proposition 4.2. *$\mathcal{L}_M$ is not locally finite.*

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[3]For $n \in \mathbb{N}$ we write $\Box^n(a)$ for n applications of $\Box$ to a, that is, $\Box^0 a = a$ and $\Box^{n+1}a = \Box\Box^n a$.

Proof. It is easy to see that $\mathbb{S}(2,4) = \{1,4\} \cup \{n^2+1 : n \geq 1\}$. The algebra having $\mathbb{S}(2,4)$ as universe is infinite yet finitely generated. By Remark 2.1 and 4.1, we obtain that $\mathcal{L}_M$ is not locally finite. $\square$

Notice that $x \oplus y = 1$ for all $x, y \in \mathbb{S}(2,4)$. Consider the matrix $\langle \mathbf{B}, D \cap \mathbb{S}(2,4)\rangle = \langle \mathbf{B}, \{4\}\rangle$, where $\mathbf{B}$ is the algebra having $\mathbb{S}(2,4)$ as universe. This matrix is (by definition) a submatrix of $\langle \mathbf{A}, D\rangle$. It is easy to see that $\langle \mathbf{B}, \{4\}\rangle$ is not reduced, and its Leibniz congruence has only two classes, namely $[4]$ and $[1] = [n^2+1]$ with $n \geq 1$. Its reduction is the two-element matrix $\langle\{[1],[4]\},\{[4]\}\rangle$. This witnesses the failure of condition (3) from Section 3, which suggests that $\mathcal{L}_M$ may be a good candidate for separating local tabularity and local finiteness.

The rest of this section is devoted to the proof of the remaining part of our main theorem below, i.e that $\mathcal{L}_M$ is indeed locally tabular.

Theorem 4.3. *$\mathcal{L}_M$ is locally tabular and not locally finite.*

In order to prove that that $\mathcal{L}_M$ is locally tabular (Proposition 4.7), we need to show that for every natural n, there are only a finite number of Frege classes over $\mathbf{Fm}_n$. For that we will have to describe explicitly the shape of all Frege classes of $\mathcal{L}$. Lemmas 4.5 and 4.6 are crucial in this effort. We start with the following remark stating a few basic facts which will be used in subsequent proofs.

Remark 4.4. Let $v\colon \mathbf{Fm} \to \mathbf{A}$ be a valuation and $\varphi \in \mathbf{Fm}$. Then,

(i) If $v(\varphi) > 1$, then $v(\psi) > 1$ for all $\psi \in \mathsf{sbf}(\varphi)$.
(ii) If $v(\Box\psi) > 1$ then $\mathsf{head}(\psi) \neq \oplus$.
(iii) $\Box^k(v(p)^2+1) = (v(p)+k)^2+1$.

Proof. (i) It is sufficient to observe that, for all $a \in \mathbf{A}$, we have $1 \oplus a = a \oplus 1 = \Box 1 = 1$. This means that, for any algebraic term $t(p_1, \dots, p_k)$, letting $p_i = 1$ for an arbitrary $1 \leq i \leq k$, we have $t(p_1, \dots, p_k) = 1$ as well.
(ii) For all $a, b \in \mathbf{A}$, we have $\Box(a \oplus b) = 1$.
(iii) Follows from the definition of the $\Box$ by an easy inductive argument. $\square$

Remark 4.4 (i) has interesting consequences on the logical level. Namely, two satisfiable formulas φ, ψ can be equivalent in $\mathcal{L}_M$ only if $\mathsf{var}(\varphi) = \mathsf{var}(\psi)$. Otherwise there would be, for example, some $p \in \mathsf{var}(\varphi) - \mathsf{var}(\psi)$. Then, taking a valuation v such that $v(\psi) \in D$ and $v(p) = 1$, we would have $v(\varphi) = 1 \notin D$, contradicting our assumption. This means that over n variables there are at least 2^n equivalence classes modulo the Frege relation. However, as we are going to show, we can also find a finite upper bound to the number of nonequivalent formulas, therefore concluding that $\mathcal{L}_M$ is locally tabular.

In order to characterize the Tarski congruence relation we introduce some auxiliar functions.

For every $\psi \in \mathbf{Fm}$, we define a map $d_\psi : \mathbf{Fm} \to \mathcal{P}_{\mathsf{fin}}(\mathbb{N})$, where $d_\psi(\varphi)$ is meant to capture the positions of the occurrences of φ outside $\Box$ in ψ. The definition is as follows:

$$d_\psi(\varphi) = \begin{cases} \{0\} & \text{if } \psi = \varphi \\ \emptyset & \text{if } \psi \notin \mathsf{sbf}(\varphi) \text{ or } (\psi \neq \varphi \text{ and } \mathsf{head}(\varphi) = \Box) \\ d_\psi(\varphi_1)^{+1} \cup d_\psi(\varphi_2) & \text{if } \psi \neq \varphi = \varphi_1 \oplus \varphi_2. \end{cases}$$

where $B^{+1} = \{n+1 : n \in B\}$ for each $B \subseteq \mathbb{N}$.

Example. Consider, for instance, the formula $\varphi = p \oplus (\Box q \oplus r)$. Then we have

$$d_p(\varphi) = d_p(p)^{+1} \cup d_p(\Box q \oplus r) = \{1\} \cup d_p(q)^{+1} \cup d_p(r) = \{1\}$$
$$d_q(\varphi) = d_q(p)^{+1} \cup d_q(\Box q \oplus r) = \emptyset \cup d_q(\Box q)^{+1} \cup d_q(r) = \emptyset$$
$$d_r(\varphi) = d_r(p)^{+1} \cup d_r(\Box q \oplus r) = \emptyset \cup d_r(\Box q)^{+1} \cup d_r(r) = \{0\}.$$

If we take $\varphi = (p \oplus q) \oplus p$, we have instead

$$\begin{aligned} d_p(\varphi) &= d_p(p \oplus q)^{+1} \cup d_p(p) \\ &= d_p(q)^{+2} \cup d_p(q)^{+1} \cup d_p(p) \\ &= \{2\} \cup \emptyset \cup \{0\} = \{0, 2\}. \end{aligned}$$

△

We shall also consider $\mathsf{sbf}^+(\varphi)$, the subformulas of φ that occur at least once inside the scope of a $\Box$

$$\mathsf{sbf}^+(p) = \emptyset$$
$$\mathsf{sbf}^+(\Box\varphi) = \mathsf{sbf}(\varphi)$$
$$\mathsf{sbf}^+(\varphi_1 \oplus \varphi_2) = \mathsf{sbf}^+(\varphi_1) \cup \mathsf{sbf}^+(\varphi_2)$$

and $\mathsf{sbf}^-(\varphi)$, the subformulas of φ that occur at least once outside the scope of a $\Box$

$$\mathsf{sbf}^-(p) = \emptyset$$
$$\mathsf{sbf}^-(\Box\varphi) = \{\Box\varphi\}$$
$$\mathsf{sbf}^-(\varphi_1 \oplus \varphi_2) = \{\varphi_1 \oplus \varphi_2\} \cup \mathsf{sbf}^+(\varphi_1) \cup \mathsf{sbf}^+(\varphi_2).$$

Example. For $\varphi = \Box p \oplus p$ we have $\mathsf{sbf}^+(\varphi) = \{p\}$ and $\mathsf{sbf}^-(\varphi) = \{\varphi, p\}$. △

Thus it can happen that $\mathsf{sbf}^+(\varphi) \cap \mathsf{sbf}^-(\varphi) \neq \emptyset$. However, it immediately follows from (a) in the next lemma that, when φ is satisfiable, then $\mathsf{sbf}^+(\varphi) \cap \mathsf{sbf}^-(\varphi) = \emptyset$. Furthermore, the next two lemmas contain a series of such observations about the different shape patterns of formulas having particular possible values, that will be instrumental in counting the Frege equivalence classes of $\mathcal{L}_M$.

Lemma 4.5. *Let $\varphi \in \mathbf{Fm}$ be a formula, and let $v : \mathbf{Fm} \to \mathbf{A}$ be a valuation such that $v(\varphi) > 1$, i.e. $v(\varphi) = \mathsf{us}(v(\varphi)) - n$ for some $n \geq 0$ and*

$$\mathsf{ls}(v(\varphi)) = (m-1)^2 < v(\varphi) = m^2 - n \leq m^2 = \mathsf{us}(v(\varphi))$$

for some $m \geq 2$. Then, for all $\psi \in \mathbf{Fm}$,

(a) $\mathsf{sbf}^+(\psi) \cap \mathsf{sbf}^-(\psi) = \emptyset$,
(b) *if* $\psi \in \mathsf{sbf}^-(\varphi)$, *then* $(m-1)^2 < v(\psi) = v(\varphi) - i$ *where* $d_\psi(\varphi) = \{i\}$,
(c) *if* $\psi \in \mathsf{sbf}^+(\varphi)$, *then* $\psi \in \mathsf{sbf}(\Box^k p) - \{\Box^k p\}$ *for some* $1 \leq k \leq m-2$ *and*
$p \in \mathsf{Var}$ *with* $v(p) = (m-1-k)^2 + 1$,
$\Box^k p \in \mathsf{sbf}^-(\varphi)$ *and*
$d_{\Box^k p}(\varphi) = \{2m - n - 2\}$.
(d) *If* $\Box^{k_1} p, \Box^{k_2} p \in \mathsf{sbf}^-(\varphi)$ *then* $k_1 = k_2$.

Proof. We prove items (a), (b) and (c) by induction on the structure of φ. For the base case, $\varphi = p \in \mathsf{Var}$, the result follows immediately from $\mathsf{sbf}(p) = \mathsf{sbf}^-(p) = \{p\}$ and $d_p(p) = \{0\}$. For the step we have to consider two cases, either $\varphi = \Box\varphi_1$ or $\varphi = \varphi_1 \oplus \varphi_0$.

Case 1: $\varphi = \Box\varphi_1$. To establish (a) and (b) it is sufficient to note that $\mathsf{sbf}^-(\varphi) = \{\varphi\}$ and $\mathsf{sbf}^+(\varphi) = \mathsf{sbf}(\varphi) - \{\varphi\} = \mathsf{sbf}(\varphi_1)$. We are going to show that, for any $\psi \in \mathsf{sbf}^+(\varphi) = \mathsf{sbf}(\varphi_1)$, item (c) holds with $\varphi = \Box^k p$. The assumption $v(\varphi) > 1$ and Remark 4.4 imply that $\varphi = \Box^k p$ for some $k \geq 1$ and $p \in \mathsf{Var}$. We also must have $v(\varphi) = (m-1)^2 + 1 = m^2 - n$ and $v(p) = j^2 + 1$ for some $j \geq 1$. Hence, $(m-1)^2 + 1 = v(\varphi) = v(\Box^k p) = \Box^k v(p) = \Box^k (j^2+1) = (k+j)^2 + 1$, where the last equality also holds by Remark 4.4. This means that $k = m-1-j$, i.e. $1 \leq k \leq m-2$ and $v(p) = (m-1-k)^2 + 1$ as required. From $v(\varphi) = (m-1)^2 + 1 = m^2 - n$ we have $n = 2m-2$, that is $2m-n-2 = 0$, which establishes $d_{\Box^k p}(\varphi) = d_{\Box^k p}(\Box^k p) = \{2m-n-2\} = \{0\}$.

Case 2: $\varphi = \varphi_1 \oplus \varphi_0$. The assumption that $v(\varphi) > 1$ implies $v(\varphi) = v(\varphi_1) \oplus v(\varphi_0) = v(\varphi_0) = m^2 - n$ and $v(\varphi_1) = v(\varphi_0) - 1 = m^2 - n - 1 > (m-1)^2$, that is $(m-1)^2 < v(\varphi_j) = v(\varphi) - j \leq m^2 - n$ for $0 \leq j \leq 1$. By induction hypothesis (a) we have $\mathsf{sbf}^+(\varphi_j) \cap \mathsf{sbf}^-(\varphi_j) = \emptyset$ for $0 \leq j \leq 1$. We then have

$$\mathsf{sbf}^+(\varphi) \cap \mathsf{sbf}^-(\varphi) = (\mathsf{sbf}^+(\varphi_1) \cup \mathsf{sbf}^+(\varphi_0)) \cap (\{\varphi\} \cup \mathsf{sbf}^-(\varphi_1) \cup \mathsf{sbf}^+(\varphi_0))$$

Clearly, $\psi \in \mathsf{sbf}^+(\varphi) \cap \mathsf{sbf}^-(\varphi)$ can only happen if either $\psi \in \mathsf{sbf}^+(\varphi_1) \cap \mathsf{sbf}^-(\varphi_0)$ or $\psi \in \mathsf{sbf}^+(\varphi_0) \cap \mathsf{sbf}^-(\varphi_1)$. Assume the former. $\psi \in \mathsf{sbf}^+(\varphi_0)$ implies, by induction hypothesis (c), that $\psi \in \mathsf{sbf}(\Box^k p)$ for some $1 \leq k \leq m-2$ and $v(p) = (m-1-k)^2 + 1$, which means that $v(\psi) < v(\Box^k p) = (m-1)^2 + 1$. On the other hand, $\psi \in \mathsf{sbf}^-(\varphi_1)$ implies, by induction hypothesis (b), that $v(\psi) \geq (m-1)^2 + 1$, thus yielding a contradiction. A similar reasoning shows that $\psi \in \mathsf{sbf}^+(\varphi_0) \cap \mathsf{sbf}^-(\varphi_1)$ also leads to a contradiction. Hence (a) holds.

To prove (b), let $\psi \in \mathsf{sbf}^-(\varphi) = \{\varphi\} \cup \mathsf{sbf}^-(\varphi_1) \cup \mathsf{sbf}^+(\varphi_0)$. The case of $\psi = \varphi$ is easily dealt with as before. Let then $\psi \in \mathsf{sbf}^-(\varphi_1)$. Then by induction hypothesis (b) we have $(m-1)^2 < v(\psi) = v(\varphi_1) - i = v(\varphi_0) - 1 - i = v(\varphi) - (i+1)$ with $d_\psi(\varphi_1) = \{i\}$. If also $\psi \in \mathsf{sbf}^-(\varphi_0)$, then again by induction hypothesis (b) we have $(m-1)^2 < v(\psi) = v(\varphi_0) - k = v(\varphi) - k$ where $d_\psi(\varphi_0) = \{k\}$. Hence, $v(\varphi) - 1 - i = v(\varphi) - k$, that is $k = i+1$, and $d_\psi(\varphi) = d_\psi(\varphi_1 \oplus \varphi_0) = d_\psi(\varphi_1)^{+1} \cup d_\psi(\varphi_0) = \{i+1\} \cup \{k\} = \{k\} \cup \{k\} = \{k\}$ as required. If instead $\psi \notin \mathsf{sbf}^-(\varphi_0)$, then $d_\psi(\varphi_0) = \emptyset$, and so

$$d_\psi(\varphi) = d_\psi(\varphi_1 \oplus \varphi_0) = d_\psi(\varphi_1)^{+1} \cup d_\psi(\varphi_0) = \{i+1\} \cup \emptyset = \{i+1\}$$

as required.
Finally, assuming $\psi \in \mathsf{sbf}^-(\varphi_0) - \mathsf{sbf}^-(\varphi_1)$ we have $d_\psi(\varphi_1) = \emptyset$ and, by induction hypothesis (b), $(m-1)^2 < v(\psi) = v(\varphi_0) - k = v(\varphi) - k$ with $d_\psi(\varphi_0) = \{k\}$. Since, as we have seen, $\psi \notin \mathsf{sbf}^+(\varphi_1)$, we have $d_\psi(\varphi) = d_\psi(\varphi_1 \oplus \varphi_0) = \emptyset^{+1} \cup d_\psi(\varphi_0) = \emptyset \cup \{k\} = \{k\}$ as required.

To prove (c), let $\psi \in \mathsf{sbf}^+(\varphi) = \mathsf{sbf}^+(\varphi_1) \cup \mathsf{sbf}^+(\varphi_0)$. If $\psi \in \mathsf{sbf}^+(\varphi_0)$, then, by induction hypothesis (c), we have $\psi \in \mathsf{sbf}(\Box^k p)$ for some $1 \leq k \leq m-2$ and $p \in \mathsf{Var}$ with $v(p) = (m-1-k)^2 + 1$, $\Box^k p \in \mathsf{sbf}^-(\varphi_0) \subseteq \mathsf{sbf}^-(\varphi)$ and $d_{\Box^k p}(\varphi_0) = \{2m-n-2\}$. If $\psi \notin \mathsf{sbf}^+(\varphi_1)$, then a fortiori $\Box^k p \notin \mathsf{sbf}^+(\varphi_1)$. Then the formula $\Box^k p$ allows us to establish the statement, because $d_{\Box^k p}(\varphi) = d_{\Box^k p}(\varphi_1 \oplus \varphi_0) = \emptyset^{+1} \cup d_{\Box^k p}(\varphi_0) = \emptyset \cup \{2m-n-2\} = \{2m-n-2\}$. Otherwise, suppose $\psi \in \mathsf{sbf}^+(\varphi_1)$. Then $\psi \in \mathsf{sbf}(\Box^{k_1} p_1)$ for some $1 \leq k_1 \leq m-2$ and $p_1 \in \mathsf{Var}$ with $v(p_1) = (m-1-k_1)^2 + 1$, $\Box^{k_1} p_1 \in \mathsf{sbf}^-(\varphi_1) \subseteq \mathsf{sbf}^-(\varphi)$ and $d_{\Box^{k_1} p_1}(\varphi_1) = \{2m-(n+1)-2\}$ because $v(\varphi_1) = m^2 - (n+1)$. From $\psi \in \mathsf{sbf}(\Box^k p) \cap \mathsf{sbf}(\Box^{k_1} p_1)$ we obtain $p = p_1$ and $v(p_1) = (m-1-k_1)^2 + 1 = (m-1-k)^2 + 1 = v(p)$, i.e. $k = k_1$. Hence the statement will hold with witness $\Box^k p$ if we can show that $d_{\Box^k p}(\varphi) = \{2m-n-2\}$. We have $d_{\Box^k p}(\varphi) = d_{\Box^k p}(\varphi_1 \oplus \varphi_0) = d_{\Box^k p}(\varphi_1)^{+1} \cup d_{\Box^k p}(\varphi_0) = \{2m-(n+1)-2\}^{+1} \cup \{2m-n-2\} = \{2m-n-1-2+1\} \cup \{2m-n-2\} = \{2m-n-2\}$, as required. Finally, suppose $\psi \in \mathsf{sbf}^+(\varphi_1) - \mathsf{sbf}^+(\varphi_0)$. From the first assumption we have, as before, $\psi \in \mathsf{sbf}(\Box^{k_1} p_1)$ for some $1 \leq k_1 \leq m-2$ and $p_1 \in \mathsf{Var}$ with $v(p_1) = (m-1-k_1)^2 + 1$, $\Box^{k_1} p_1 \in \mathsf{sbf}^-(\varphi_1) \subseteq \mathsf{sbf}^-(\varphi)$ and $d_{\Box^{k_1} p_1}(\varphi_1) = \{2m-(n+1)-2\}$. Analogously as before, $\psi \notin \mathsf{sbf}^+(\varphi_0)$ implies $\Box^{k_1} p_1 \notin \mathsf{sbf}^+(\varphi_0)$. Then $d_{\Box^{k_1} p_1}(\varphi) = d_{\Box^{k_1} p_1}(\varphi_1 \oplus \varphi_0) = d_{\Box^{k_1} p_1}(\varphi_1)^{+1} \cup d_{\Box^{k_1} p_1}(\varphi_0) = \{2m-(n+1)-2\}^{+1} \cup \emptyset = \{2m-n-1-2+1\} = \{2m-n-2\}$. That is, the formula $\Box^{k_1} p_1$ allows us to establish the statement.

Item (d) follows directly from (b) and (c). In fact, from $\Box^{k_1} p, \Box^{k_2} p \in \mathsf{sbf}^-(\varphi)$ we obtain that $p \in \mathsf{sbf}^+(\varphi)$. Hence by (c), there is k such that

$$v(p) = (m-1-k)^2 + 1.$$

From $\Box^{k_1} p, \Box^{k_2} p \in \mathsf{sbf}^-(\varphi)$ we have, by (b),

$$(m-1)^2 + 1 \leq v(\Box^{k_1} p), v(\Box^{k_2} p) \leq m^2 + 1.$$

Since, for $j \in \{1, 2\}$, we have $v(\Box^{k_j} p) = (m-1-k+k_j)^2 + 1$, we conclude that $k_1 = k_2 = k$. $\Box$

Observe that, by Lemma 4.5, if a formula $\varphi \in \mathbf{Fm}$ is satisfiable, then $d_\psi(\varphi)$ is a singleton for all $\psi \in \mathsf{sbf}^-(\varphi)$. Moreover, the following result reveals specific patterns in the Frege classes among satisfiable formulas in terms of the values $d_\psi(\varphi)$ for $\psi \in \mathsf{sbf}^-(\varphi)$. Let us denote $\mathsf{var}^-(\varphi) := \mathsf{sbf}^-(\varphi) \cap \mathsf{Var}$ and $\mathsf{var}^+(\varphi) := \mathsf{sbf}^+(\varphi) \cap \mathsf{Var}$.

Lemma 4.6. *Let $\varphi \in \mathbf{Fm}$ be satisfiable. Then*

(a) *there is* $\ell \geq 0$ *such that*

$$\bigcup_{p \in \mathsf{var}^-(\varphi)} d_p(\varphi) = \{j : 0 \leq j \leq \ell\},$$

(b) *for all* $\psi \in \mathsf{sbf}(\varphi)$*, if* $\Box\psi \in \mathsf{sbf}^-(\varphi)$*, then* $d_{\Box\psi}(\varphi) \subseteq \{0, \ldots, \ell + 1\}$.

Proof. By induction on the structure of φ. Notice that, since φ is satisfiable, by Remark 4.4 we have that either $\varphi = p \in \mathsf{Var}$ or $\varphi = \varphi_1 \oplus \varphi_0$ with $\varphi_1, \varphi_0 \in \mathbf{Fm}$.
(a) For the base case, $\varphi = p \in \mathsf{Var}$, we have $\mathsf{sbf}^-(p) = \{p\}$ and

$$\bigcup_{p \in \mathsf{var}^-(p)} d_p(p) = d_p(p) = \{0\}.$$

For the step, $\varphi = \varphi_1 \oplus \varphi_0$, we have that, for $i \in \{0, 1\}$, there is $\ell_i \geq 0$ such that

$$\bigcup_{p \in \mathsf{var}^-(\varphi_i)} d_p(\varphi_i) = \{j : 0 \leq j \leq \ell_i\}.$$

Recall that $d_\psi(\varphi) = (d_\psi(\varphi_1))^{+1} \cup d_\psi(\varphi_0)$ for all $\psi \in \mathsf{sbf}^-(\varphi)$ such that $\psi \neq \varphi$ and $d_\varphi(\varphi) = \{0\}$. Hence, applying the induction hypothesis,

$$\begin{aligned}
\bigcup_{p \in \mathsf{sbf}^-(\varphi)} d_p(\varphi) &= d_\varphi(\varphi) \cup (\bigcup_{p \in \mathsf{sbf}^-(\varphi_1)} d_p(\varphi_1))^{+1} \cup \bigcup_{p \in \mathsf{sbf}^-(\varphi_0)} d_p(\varphi_0) \\
&= (\bigcup_{p \in \mathsf{var}^-(\varphi_1)} d_p(\varphi_1))^{+1} \cup \bigcup_{p \in \mathsf{var}^-(\varphi_0)} d_p(\varphi_0) \\
&= \{j : 0 \leq j \leq \ell_1\}^{+1} \cup \{j : 0 \leq j \leq \ell_0\} \\
&= \{j : 0 \leq j \leq \max\{\ell_1 + 1, \ell_0\}\}.
\end{aligned}$$

(b) The base case is trivial. For the step, $\varphi = \varphi_1 \oplus \varphi_0$, consider $\Box\psi \in \mathsf{sbf}^-(\varphi)$. Then $\Box\psi \in \mathsf{sbf}^-(\varphi_1) \cup \mathsf{sbf}^-(\varphi_0)$. By induction hypothesis, if $\Box\psi \in \mathsf{sbf}^-(\varphi_i)$ for $i \in \{1, 0\}$, then $d_{\Box\psi}(\varphi_i) \subseteq \{0, \ldots, \ell_i + 1\}$. Then

$$\begin{aligned}
d_{\Box\psi}(\varphi) &\subseteq \{j : 0 \leq j \leq \ell_1 + 1\}^{+1} \cup \{j : 0 \leq j \leq \ell_0 + 1\} \\
&= \{j : 0 \leq j \leq \max\{\ell_1 + 1, \ell_0\} + 1\} \\
&= \{j : 0 \leq j \leq l + 1\}.
\end{aligned}$$

□

As observed after Remark 4.4, two (Frege-)equivalent formulas φ_1, φ_2 must satisfy $\mathsf{var}(\varphi_1) = \mathsf{var}(\varphi_2)$. This means that over n variables there are at least 2^n equivalence classes modulo the Frege relation. However, we can also find an upper bound to the number of nonequivalent formulas, therefore concluding that the logic is locally tabular, as show by the following proposition.

Proposition 4.7. $\mathcal{L}_M$ *is locally tabular.*

Proof. First note that all unsatisfiable formulas are equivalent, so we only need to prove that, for each finite set of variables $V = \{p_1, \ldots, p_n\}$, there are only finitely many satisfiable nonequivalent formulas whose variables are exactly V. For this purpose, for every satisfiable $\varphi \in \mathbf{Fm}$ such that $\mathsf{var}(\varphi) = V$, we are going to define a map $f_\varphi \colon V \to \mathbb{N}$ with the property that, for any $\varphi_1, \varphi_2 \in \mathbf{Fm}$ such that $\mathsf{var}(\varphi_1) = \mathsf{var}(\varphi_2) = V$ and $f_{\varphi_1} = f_{\varphi_2}$, we have that φ_1 and φ_2 are equivalent modulo the Frege relation. The result will then follow by showing that there are only finitely many different maps $f_\varphi \colon V \to \mathbb{N}$.

Let then φ be a satisfiable formula, i.e. there is $v \colon \mathbf{Fm} \to \mathbf{A}$ such that $v(\varphi) = m^2$ for some $m \geq 2$. By Lemma 4.5, for every $p \in \mathsf{var}(\varphi)$, either $p \in \mathsf{var}^-(\varphi)$ or $p \in \mathsf{var}^+(\varphi)$ but not both. Define a map $f_\varphi \colon \mathsf{var}(\varphi) \to \mathbb{N}$ as follows:

$$f_\varphi(p) = \begin{cases} i_p & \text{if } p \in \mathsf{var}^-(\varphi) \\ k_p & \text{if } p \in \mathsf{var}^+(\varphi) \end{cases}$$

where i_p and k_p are given as in Lemma 4.5 (b) and (c), that is $d_p(\varphi) = \{i_p\}$ for $p \in \mathsf{var}^-(\varphi)$ and $\Box^{k_p} p \in \mathsf{sbf}^-(\varphi)$ for $p \in \mathsf{var}^+(\varphi)$. Lemma 4.5 tells us that $v(\varphi) = m^2 - 0 \in D$ if and only if for each $p \in \mathsf{var}(\varphi)$,

$$v(p) = \begin{cases} m^2 - f_\varphi(p) & \text{if } p \in \mathsf{var}^-(\varphi) \\ (m - 1 - f_\varphi(p))^2 + 1 & \text{if } p \in \mathsf{var}^+(\varphi). \end{cases}$$

This implies that, if $f_{\varphi_1} = f_{\varphi_2}$, then φ_1 and φ_2 are equivalent modulo the Frege relation. This holds because, by definition, for two formulas φ_1, φ_2, being equivalent modulo the Frege relation means that, for every valuation $v \colon \mathsf{Var} \to \mathbf{A}$, $v(\varphi_1) \in D$ if and only if $v(\varphi_1) \in D$.

We claim that $f_\varphi(\mathsf{var}(\varphi)) \subseteq \{0, \ldots, n-1\}$. In fact, by Lemma 4.6 we know that

$$\{f_\varphi(p) : p \in \mathsf{Var}^-(\varphi)\} = \{0, \ldots, \ell\}$$

for some $\ell \leq n - 1$. Hence $f_\varphi(\mathsf{var}^-(\varphi)) \subseteq \{0, \ldots, n-1\}$. Let $p \in \mathsf{var}^+(\varphi)$, and notice that this implies $n > 1$. By Lemma 4.5 (c) we have

$$\begin{aligned} v(\Box^{f_\varphi(p)} p) &= (m-1)^2 + 1 \\ f_\varphi(p) &\leq m - 2 \\ d_{\Box^{f_\varphi(p)} p}(\varphi) &= 2m - 2 \end{aligned}$$

and by Lemma 4.6 (b) we have $d_{\Box^{f_\varphi(p)} p}(\varphi) = 2m - 2 \leq \ell + 1$. From $2m - 2 \leq \ell + 1$ we obtain $m - 1 \leq (\ell + 1)/2$. Then, $f_\varphi(p) < m - 1 < (\ell + 1)/2 \leq n$ because $\ell \leq n - 1 \geq 1$. Thus we also have $f_\varphi(\mathsf{var}^+(\varphi)) \subseteq \{0, \ldots, n-1\}$, which means that $f_\varphi(\mathsf{var}(\varphi)) \subseteq \{0, \ldots, n-1\}$. The statement then follows because there are only a finite number of maps $f : V \to \{0, \ldots, n-1\}$. $\Box$

Acknowledgment

We would like to thank colleagues and friends with whom we had helpful discussions, in particular Carlos Caleiro, João Marcos, Ramon Jansana and Josep Maria Font.

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