The connection between computability of a nonlinear problem and its linearization: the Hartman-Grobman theorem revisited

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Abstract

As one of the seven open problems in the addendum to their 1989 book *Computability in Analysis and Physics* [21], Pour-El and Richards asked, "What is the connection between the computability of the original nonlinear operator and the linear operator which results from it?" Yet at present, systematic studies of the issues raised by this question seem to be missing from the literature. In this paper, we study one problem in this direction: the Hartman-Grobman linearization theorem for ordinary differential equations (ODEs). We prove, roughly speaking, that near a hyperbolic equilibrium point x_0 of a nonlinear ODE $\dot{x} = f(x)$, there is a computable homeomorphism H such that $H \circ \phi = L \circ H$, where ϕ is the solution to the ODE and L is the solution to its linearization $\dot{x} = Df(x_0)\,x$.

1 Introduction

The technique of linearization is important in the study of nonlinear systems and their structural stability. In his 1879 doctoral thesis [20], Poincaré showed that under certain conditions on the eigenvalues of $Df(x_0)$, the nonlinear vector field $f(x)$ is conjugate to its linearization $Df(x_0)\,x$ in a neighborhood of the equilibrium point x_0. Later, a number of researchers contributed to progress on related problems, but the most significant results stem from the work of D.M. Grobman [6] and P. Hartman [9], [10] (in 1959 and 1960, respectively), who showed the conjugacy of solutions as described in the paragraph above, but without constructive proofs. Their results—whether for flows or maps—go loosely under the name of the Hartman-Grobman theorem. This theorem (or collection of theorems) remains important, since it shows the structural stability of hyperbolic equilibria in sufficiently smooth dynamical systems. Our result shows that the conjugacy guaranteeing this stability is computable.

2 Definitions and notation

Before giving a precise statement of our main result (Theorem 1), some definitions and notational conventions are in order. First the notion of computability. To carry out computations on infinite objects such as real numbers, we encode those objects as infinite sequences of rational numbers (or equivalently, sequences of any finite or countable set Σ of symbols), using representations (see [24] for a complete development). A represented space is a pair $(X; \delta)$ where X is a set, $\mathrm{dom}(\delta) \subseteq X$, and $\delta : \Sigma^{\mathbb{N}} \to X$ is an onto map. Every $q \in \mathrm{dom}(\delta)$ such that $\delta(q) = x$ is called a δ-name of x (or a name of x when δ is clear from context). Naturally, an element $x \in X$ is computable if it has a computable name. In this paper, we use the following particular representations for points in $\mathbb{R}^n$; for open subsets of $\mathbb{R}^n$; and for functions in $C^k(\mathbb{R}^n; \mathbb{R}^m)$, the set of all continuously k times differentiable functions defined on open subsets of $\mathbb{R}^n$ with ranges in $\mathbb{R}^m$:

(1) For every point $x \in \mathbb{R}^n$, a name of x is a sequence $\{r_k\}$ of points with rational coordinates satisfying $|x - r_k| < 2^{-k}$. Thus x is computable if there is a Turing machine (or a computer program or an algorithm) that outputs a rational n-tuple r_k on input k such that $|r_k - x| < 2^{-k}$. A matrix is computable if every entry of the matrix is a computable real or complex number;

(2) For every function $f \in C^k(\mathbb{R}^n; \mathbb{R}^m)$, a name of f is a sequence $\{P_l\}$ of polynomials with rational coefficients satisfying $d(P_l, f) < 2^{-l}$, where P_l are defined on $\mathbb{R}^n$ with ranges in $\mathbb{R}^m$,

$$d(P_l, f) = \sum_{j=0}^{\infty} 2^{-j} \frac{\|P_l - f\|_{k,j}}{\|P_l - f\|_{k,j} + 1},$$

$\|P_l - f\|_{k,j} = \sup_{0 \leq i \leq k, x \in \mathrm{dom}(f), |x| \leq j} |P_l^{(i)}(x) - f^{(i)}(x)|$, and $f^{(i)}$ is the i^{th} order derivative of f. Thus, f is computable if there is an (oracle) Turing machine that outputs P_l (more precisely coefficients of P_l) on input l satisfying $d(P_l, f) < 2^{-l}$;

(3) A name of an open set $O \subset \mathbb{R}^n$ is a sequence of polynomials that is a name of the distance function $d_{\mathbb{R}^n \setminus O} : \mathbb{R}^n \to \mathbb{R}$, where $d_{\mathbb{R}^n \setminus O}(x) = \inf_{y \in \mathbb{R}^n \setminus O} |x - y|$ for all $x \in \mathbb{R}^n$. Consequently, the open set O is computable if the distance function $d_{\mathbb{R}^n \setminus O}$ is computable.

The notion of computable maps between represented spaces now arises naturally. A map $\Phi : (X; \delta_X) \to (Y; \delta_Y)$ between two represented spaces is computable if there is a computable map $\phi : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ such that $\Phi \circ \delta_X = \delta_Y \circ \phi$. Informally speaking, this means that there is a Turing machine that outputs a name of

$\Phi(x)$ when given a name of x as input.

Next some notations. Let $\mathcal{A}_H$ denote the set of all hyperbolic $n\times n$ matrices, where a matrix A is hyperbolic if all of its eigenvalues have nonzero real part. The operator norm is used for $A \in \mathcal{A}_H$; i.e., $\|A\| = \sup_{|x|\neq 0} |Ax|/|x|$. For Banach spaces X and Y, let $C^k(X;Y)$ denote the set of all continuously k times differentiable functions defined on open subsets of X with ranges in Y, and $\mathcal{L}(X;Y)$ the set of all bounded linear maps from X to Y. Let $\mathcal{O}$ denote the set of all open subsets of $\mathbb{R}^n$ containing the origin of $\mathbb{R}^n$, $\mathcal{I}$ the set of all open intervals of $\mathbb{R}$ containing zero, and $\mathcal{F}$ the set of all functions $f \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ such that the domain of f is in $\mathcal{O}$, $f(0) = 0$, and $Df(0) \in \mathcal{A}_H$. In other words, for any $f \in \mathcal{F}$, 0 is a hyperbolic equilibrium point of f. (We recall that if $f(x_0) = 0$ and $Df(x_0) \in \mathcal{A}_H$, then $x_0 \in \mathbb{R}^n$ is called a hyperbolic equilibrium point, or hyperbolic fixed point, of f.)

3 The main result

We now state our main result.

Theorem 1 *There is a computable map* $\Theta : \mathcal{F} \to \mathcal{O}\times\mathcal{O}\times C(\mathbb{R}^n;\mathbb{R})\times C(\mathbb{R}^n;\mathbb{R}^n)$ *such that for any* $f \in \mathcal{F}$, $f \mapsto (U, V, \mu, H)$, *where*

(a) $H : U \to V$ *is a homeomorphism;*

(b) the unique solution $x(t, \tilde{x}) = x(\tilde{x})(t)$ *to the initial value problem* $\dot{x} = f(x)$ *and* $x(0) = \tilde{x}$ *is defined on* $(-\mu(\tilde{x}), \mu(\tilde{x})) \times U$*; moreover,* $x(t, \tilde{x}) \in U$ *for all* $\tilde{x} \in U$ *and* $-\mu(\tilde{x}) < t < \mu(\tilde{x})$ *;*

(c) $H(x(t, \tilde{x})) = e^{Df(0)t}H(\tilde{x})$ *for all* $\tilde{x} \in U$ *and* $-\mu(\tilde{x}) < t < \mu(\tilde{x})$ *.*

Recall that for any $\tilde{x} \in \mathbb{R}^n$, $e^{Df(0)t}\tilde{x}$ is the solution to the linear problem $\dot{x} = Df(0)x$, $x(0) = \tilde{x}$. So the theorem shows that the homeomorphism H, computable from f, maps trajectories near the origin, a hyperbolic equilibrium point, of the nonlinear problem $\dot{x} = f(x)$ onto trajectories near the origin of the linear problem $\dot{x} = Df(0)x$. In other words, H is a conjugacy between the linear and nonlinear trajectories near the origin. We note that classical proofs of the Hartman-Grobman linearization theorem are not constructive, and so the effective version of the theorem cannot be obtained from a classical proof.

The proof of Theorem 1 is presented at the end of this section. In that proof, we make use of a number of lemmas and auxiliary results, beginning with the following:

Lemma 2 *There is a computable map* $\mathcal{F} \to \mathbb{R}^+ \times \mathbb{R}^+ \times C(\mathbb{R}^n; C^1(\mathbb{R};\mathbb{R}^n))$, $f \mapsto (\alpha_f, \epsilon_f, u_f)$, *such that for any* $\tilde{x} \in B(0, \epsilon_f)$, $u_f(\tilde{x})(\cdot)$ *is the solution to the initial value problem* $\dot{x} = f(x)$ *and* $x(0) = \tilde{x}$ *on the interval* $(-\alpha_f, \alpha_f)$ *satisfying* $u_f(\cdot)(\cdot) \in C^1((-\alpha_f, \alpha_f) \times B(0, \epsilon_f))$, *where* $\mathbb{R}^+$ *is the set of all positive real numbers and* $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$.

Proof. In [4], it was proved that the solution map $(f, \tilde{x}) \mapsto (\alpha_{(f,\tilde{x})}, u(\cdot, \tilde{x}))$ is computable, where $u(t, \tilde{x})$ is the solution to the initial value problem $\dot{x} = f(x)$

and $x(0) = \tilde{x}$ on the interval $(-\alpha_{(f,\tilde{x})}, \alpha_{(f,\tilde{x})})$. The lemma can be proved by a similar argument using successive approximations. ■

The contraction mapping principle is used in computing the homeomorphism H. To apply the contraction mapping principle, it is essential to decompose the solution $e^{At}x_0$ to the linear problem $\dot{x} = Ax$, $x(0) = x_0$ into the sum of a contraction and an expansion (an expansion is the inverse of a contraction). In classical proofs, the decomposition is obtained by transforming A into a block-diagonal matrix $RAR^{-1} = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ such that e^{Bt} is a contraction and e^{Ct} is an expansion, where R is the matrix consisting of generalized eigenvectors corresponding to the eigenvalues of A. Although the eigenvalues of A are computable from A (see the proof of Lemma 3 below), the process of finding the eigenvectors is not in general continuous, thus it is a non-computable process. This is one of the non-constructive obstacles appearing in classical proofs. To overcome this, we make use of an analytic approach that does not require finding eigenvectors. This approach is based on a function-theoretical treatment of the resolvents (see, e.g., [18], [12], [13], and [23]). Lemma 3 below gives an algorithm that decomposes e^{At} as desired.

Lemma 3 *There is a computable map* $\Upsilon : \mathcal{A}_H \to (0,1)\times\mathbb{N}\times C(\mathbb{R};\mathcal{L}(\mathbb{R}^n;\mathbb{R}^n))\times C(\mathbb{R};\mathcal{L}(\mathbb{R}^n;\mathbb{R}^n))$, $A \mapsto (\sigma_A, K_A, A_S, A_U)$, *such that* $e^{At} = A_S(t) + A_U(t)$ *for all* $t \in \mathbb{R}$. *Moreover,* $\|A_S(t)\| \leq K_A e^{-\sigma_A t}$ *for* $t \geq 0$ *and* $\|A_U(t)\| \leq K_A e^{\sigma_A t}$ *for* $t \leq 0$.

Proof. It is clear that one can compute the eigenvalues of A from the input $A \in \mathcal{A}_H$ since the eigenvalues are zeros of the characteristic polynomial $\det(A - \lambda I_n)$ of A, where $\det(A - \lambda I_n)$ denotes the determinant of $A - \lambda I_n$ and I_n is the $n \times n$ unit matrix. Assume that $\lambda_1, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_n$ (counting multiplicity) are eigenvalues of A with $Re(\lambda_j) < 0$ for $1 \leq j \leq k$ and $Re(\mu_j) > 0$ for $k+1 \leq j \leq n$, where $Re(z)$ denotes the real part of a complex number z. Then a rational number $\sigma_A > 0$ can be computed from the eigenvalues of A such that $Re(\lambda_j) < -\sigma_A$ for $1 \leq j \leq k$ and $Re(\mu_j) > \sigma_A$ for $k+1 \leq j \leq n$.

Let M be a natural number such that $M > \max\{\sigma_A, 1, |\lambda_j| + 1, |\mu_l| + 1 : 1 \leq j \leq k, k+1 \leq l \leq n\}$. We now construct two simple piecewise-smooth closed curves Γ_1 and Γ_2 in $\mathbb{R}^2$: Γ_1 is the boundary of the rectangle with vertices $(-\sigma_A, M)$, $(-M, M)$, $(-M, -M)$, and $(-\sigma_A, -M)$, while Γ_2 is the boundary of the rectangle with vertices (σ_A, M), (M, M), $(M, -M)$, and $(\sigma_A, -M)$. Then Γ_1 with positive (counterclockwise) orientation encloses in its interior all the λ_j for $1 \leq j \leq k$, and Γ_2 with positive orientation encloses all the μ_j for $k+1 \leq j \leq n$ in its interior. We observe that for any $\xi \in \Gamma_1 \bigcup \Gamma_2$, the matrix $A - \xi I_n$ is invertible. Since the function $g : \Gamma_1 \bigcup \Gamma_2 \to \mathbb{R}$, $g(\xi) = ||(A - \xi I_n)^{-1}||$ is computable (see, for example, [26]) from A, where $(A - \xi I_n)^{-1}$ is the inverse of the matrix $A - \xi I_n$, it follows that the maximum of g on $\Gamma_1 \bigcup \Gamma_2$ is computable from A. Let $K_1 \in \mathbb{N}$ be an upper bound of this maximum. Now for any $t \in \mathbb{R}$, from (5.47) of [14],

$$\begin{aligned} e^{At} &= -\frac{1}{2\pi i}\int_{\Gamma_1} e^{\xi t}(A - \xi I_n)^{-1} d\xi - \frac{1}{2\pi i}\int_{\Gamma_2} e^{\xi t}(A - \xi I_n)^{-1} d\xi \qquad (1) \\ &= A_S(t) + A_U(t)\,. \end{aligned}$$

It is clear that $A_S,\ A_U \in C(\mathbb{R};\mathcal{L}(\mathbb{R}^n;\mathbb{R}^n))$. Since integration is a computable operator, it follows that A_S and A_U are computable from A. Moreover, a simple calculation shows that $|| - \frac{1}{2\pi i}\int_{\Gamma_1} e^{t\xi}(A-\xi I_n)^{-1}d\xi|| \le 4K_1 M e^{-\sigma_A t}/\pi$ for $t \ge 0$ and $|| - \frac{1}{2\pi i}\int_{\Gamma_2} e^{t\xi}(A-\xi I_n)^{-1}d\xi|| \le 4K_1 M e^{\sigma_A t}/\pi$ for $t \le 0$. Let

$$K_A = \max\{4MK_1, 1\}\,. \tag{2}$$

Then

$$\|A_S(t)\| \le K_A e^{-\sigma_A t} \text{ for } t \ge 0 \text{ and } \|A_U(t)\| \le K_A e^{\sigma_A t} \text{ for } t \le 0\,. \tag{3}$$

The proof is complete. ∎

In the following we fix a function $f \in \mathcal{F}$ and show that the corresponding (U, V, μ, H) described in Theorem 1 is computable from any name of $f \in \mathcal{F}$. For the nonlinear problem

$$\dot{x} = f(x), \quad x(0) = \tilde{x} \tag{4}$$

let us denote by $x(t,\tilde{x})$ or $\phi_t(\tilde{x})$ the unique solution of (4), that is, $x(t,\tilde{x}) = \phi_t(\tilde{x}) = u_f(\tilde{x})(t)$ in terms of Lemma 2. Also for the fixed function f let us use α and ϵ to denote the corresponding positive real numbers α_f and ϵ_f, computable from f, in Lemma 2; that is,

$$\alpha = \alpha_f, \quad \epsilon = \epsilon_f\,. \tag{5}$$

Equation $\dot{x} = f(x)$ can be written in the form

$$\dot{x} = Ax + F(x)\,, \tag{6}$$

where $A = Df(0)$ and $F(x) = f(x) - Ax$. Since $f \in \mathcal{F}$, $A = Df(0) \in \mathcal{A}_H$. It follows from Lemma 3 that there are $0 < \sigma_A < 1$, $K_A \in \mathbb{N}$, A_S, $A_U \in C(\mathbb{R};\mathcal{L}(\mathbb{R}^n;\mathbb{R}^n))$ computable from A satisfying (1) and (3). Thus a number t_0 can be computed from A such that

$$0 < t_0 < \alpha, ||A_S(t_0)|| < 1/(K_A+1) < 1, ||A_U(-t_0)|| < 1/(K_A+1) < 1\,. \tag{7}$$

Recall that, by Lemma 2, both $x = x(t,\tilde{x})$ and its derivative are computable from f on $(-\alpha,\alpha)\times B(0,\epsilon)$; subsequently, $x(t_0,\cdot)$ and its derivative are computable from f on $B(0,\epsilon)$ since $0 < t_0 < \alpha$ and t_0 is computable from $A = Df(0)$.

Let us denote $A_S(t_0)$, $A_U(t_0)$, $A_S(0)$, and $A_U(0)$ by B, C, P_S and P_U, respectively. Then, by (3), $\|P_S\| \le K_A$ and $\|P_U\| \le K_A$. Moreover, $\mathbb{R}^n = \mathbb{E}^s \bigoplus \mathbb{E}^u$, where $\mathbb{E}^s = P_S\mathbb{R}^n$ and $\mathbb{E}^u = P_U\mathbb{R}^n$. The linear manifolds $\mathbb{E}^s$ and $\mathbb{E}^u$ are the stable subspace and the unstable subspace of A, respectively, which are invariant under A (see, e.g., [23]).

From the construction it is readily seen that the linear maps B, C, P_S, and P_U are computable from A, hence also from f, since $A = Df(0)$. Moreover, since they are linear maps, their derivatives are also computable from f. The following lemma summarizes some properties of these linear maps which will be used repeatedly in the remainder of the proof. For simplicity we abbreviate $P \circ Q$ by PQ for maps $P, Q : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 4 *The following hold for the linear maps B, C, P_S, and P_U:*

(i) $P_S P_U = P_U P_S = 0$*,* $P_S P_S = P_S$*, and* $P_U P_U = P_U$*.*

(ii) $P_S + P_U = id$ *and* $\mathbb{R}^n = \mathbb{E}^s \bigoplus \mathbb{E}^u$*, where id is the identity map.*

(iii) $BP_U = P_U B = 0$ *and* $CP_S = P_S C = 0$*.*

(iv) $P_S B = BP_S = B$ *and* $CP_U = P_U C = C$*.*

(v) The linear manifold $\mathbb{E}^s$ is invariant under B and the linear manifold $\mathbb{E}^u$ is invariant under C.

Proof. The proof of (i) can be found in Section 1.5.3 of [14], the proof of (ii) in Section 4.6 of [23], and proofs of (iii) and (v) in [5]. The equalities in (iv) follow from (ii), (iii), and the fact that B and C are linear maps. ■

Remark 5 *As a consequence of the above lemma, for any $x_0 \in \mathbb{R}^n$, if we use y_0 and z_0 to denote $P_S x_0 \in \mathbb{E}^s$ and $P_U x_0 \in \mathbb{E}^u$ respectively, then $x_0 = (P_S + P_U)x_0 = y_0 + z_0$. Since $\mathbb{R}^n = \mathbb{E}^s \bigoplus \mathbb{E}^u$, we may identify (y_0, z_0) with $y_0 + z_0$. Similarly, for any function $x : \mathbb{R}^{n+1} \to \mathbb{R}^n$, if we denote $P_S x(t, x_0)$ by $y(t, y_0, z_0)$ and $P_U x(t, x_0)$ by $z(t, y_0, z_0)$, then*

$$x(t, x_0) = y(t, y_0, z_0) + z(t, y_0, z_0) = (y(t, y_0, z_0), z(t, y_0, z_0)) .$$

These notations will be used throughout the rest of the paper.

Lemma 6 $BB^{-1}P_S = B^{-1}BP_S = P_S$ *and* $CC^{-1}P_U = C^{-1}CP_U = P_U$*, where* $B^{-1} = A_S(-t_0)$ *and* $C^{-1} = A_U(-t_0)$*. In other words, B^{-1} is the inverse of B on the stable manifold $\mathbb{E}^s$ of the linear map A while C^{-1} is the inverse of C on the unstable manifold $\mathbb{E}^u$ of A. Moreover, both B^{-1} and C^{-1} are computable from f.*

Proof. It is sufficient to prove that $CC^{-1}P_U = C^{-1}CP_U = P_U$. The same argument can be used to show that $BB^{-1}P_S = B^{-1}BP_S = P_S$. Define two holomorphic functions $\phi_1(\xi) = e^{t_0\xi}$ and $\phi_2(\xi) = e^{-t_0\xi}$, $\xi \in \mathbb{R}^2$. Then the function $\phi(\xi) = \phi_1(\xi)\phi_2(\xi) \equiv 1$ is holomorphic. If one uses the Dunford-Taylor integrals to define the operator $\phi(A)$ as

$$\phi(A) = -\frac{1}{2\pi i}\int_{\Gamma_2} \phi(\xi)R(\xi)d\xi = -\frac{1}{2\pi i}\int_{\Gamma_2} \phi(\xi)(A - \xi I)^{-1}d\xi \,,$$

then the following property holds [14]:

$$\phi(\xi) = \phi_1(\xi)\phi_2(\xi) \quad \Rightarrow \quad \phi(A) = \phi_1(A)\phi_2(A) \,.$$

In other words,

$$\begin{aligned} & \phi_1(A)\phi_2(A) \\ = & \left(-\frac{1}{2\pi i}\int_{\Gamma_2} e^{t_0\xi}R(\xi)d\xi\right)\left(-\frac{1}{2\pi i}\int_{\Gamma_2} e^{-t_0\xi}R(\xi)d\xi\right) \\ = & \phi(A) \\ = & -\frac{1}{2\pi i}\int_{\Gamma_2} R(\xi)d\xi \\ = & P_U \,. \end{aligned}$$

Consequently, $CC^{-1}P_U x = P_U(P_U x) = P_U x$, $x \in \mathbb{R}^n$. Similarly, $C^{-1}CP_U x = P_U(P_U x) = P_U x$, $x \in \mathbb{R}^n$.

It is clear from Lemma 3 that $B^{-1} = A_S(-t_0)$ and $C^{-1} = A_U(-t_0)$ are both computable from A, and thus from f. ■

We note that $(B+C)\tilde{x} = (A_S(t_0)+A_U(t_0))\tilde{x} = e^{At_0}\tilde{x}$ is the solution at time t_0 to the linearization $\dot{x} = Df(0)x = Ax$, $x(0) = \tilde{x}$ of the nonlinear problem $\dot{x} = f(x)$ and $x(0) = \tilde{x}$ (see (1)). It is clear from (7) that B and C^{-1} are contractions on $\mathbb{R}^n$. This fact plays a key role in the construction of the desired homeomorphism H. Another key fact used in the construction is that, near the hyperbolic equilibrium point 0, the (nonlinear) feedback, $x(t_0, \cdot) - (B + C)$, is weak, where $x(t_0, \cdot)$ is the solution to the nonlinear problem $\dot{x} = f(x)$ at time t_0. Thus, if B is a contraction on the stable manifold $\mathbb{E}^s$, then so is $P_S x(t_0, \cdot)$ near 0. This allows one to construct H on $\mathbb{E}^s$, say H_1, by using the contraction mapping principle via a fixed point argument. Using the same argument, one can also construct H on $\mathbb{E}^u$, say H_2. Since $\mathbb{R}^n = \mathbb{E}^s \bigoplus \mathbb{E}^u$, we obtain the desired map H as the sum of H_1 and H_2. Definition 7 together with Lemmas 8 and 9 below give some quantitative bounds for this "weak feedback" at time t_0 in $\mathbb{E}^s$ and $\mathbb{E}^u$.

Definition 7 *Let $\alpha > 0$ and $\epsilon > 0$ be the two computable real numbers defined in (5) and t_0, $0 < t_0 < \alpha$, the computable real number satisfying (7), and let $\phi_{t_0}(\tilde{x}) = x(t_0, \tilde{x})$ be the solution to the initial-value problem $\dot{x} = f(x)$ and $x(0) = \tilde{x}$ at $t = t_0$. Let $\tilde{Y}, \tilde{Z} : B(0, \epsilon) \to \mathbb{R}^n$ be two functions defined as follows: for $x \in B(0, \epsilon)$,*

$$\tilde{Y}(x) = P_S\phi_{t_0}(x) - Bx,\ \tilde{Z}(x) = P_U\phi_{t_0}(x) - Cx.$$

Using the notations introduced in Remark 5, the two functions may also be written in the form $\tilde{Y}(x) = \tilde{Y}(y, z) = y(t_0, y, z) - By$ and $\tilde{Z}(x) = \tilde{Z}(y, z) = z(t_0, y, z) - Cz$. Moreover, it follows from Lemma 4 that $P_S \circ \tilde{Y} = \tilde{Y}$ and $P_U \circ \tilde{Z} = \tilde{Z}$.

The functions $\tilde{Y}$ and $\tilde{Z}$ represent the feedbacks of the nonlinear problem $\dot{x} = f(x)$ at time t_0 on $B(0, \epsilon)$ in $\mathbb{E}^s$ and $\mathbb{E}^u$, respectively. Combining Lemma 2 and the fact that B, C, P_S, and P_U as well as their derivatives are computable from f, it follows that the functions $\tilde{Y}$, $\tilde{Z}$, $D\tilde{Y}$, and $D\tilde{Z}$ are all computable from f. Moreover, $\tilde{Y}(0) = \tilde{Z}(0) = D\tilde{Y}(0) = D\tilde{Z}(0) = 0$. (We note that $\tilde{Y}(0) = y(t_0, 0, 0) - 0 = y(t_0, 0, 0)$ and $\tilde{Z}(0) = z(t_0, 0, 0) - 0 = z(t_0, 0, 0)$. But since $x_0 = 0$ is a hyperbolic fixed point, $x(t, 0) \equiv 0$ for all t, and so $y(t_0, 0, 0) = P_S x(t_0, 0) = 0$ and $z(t_0, 0, 0) = P_U x(t_0, 0) = 0$. On the other hand, since $0 \in \mathbb{R}^n$ is an equilibrium point of f, i.e., $f(0) = 0$, $Dx(t_0, 0) = e^{Df(0)t_0} = e^{At_0} = B + C$ (see, e.g., Section 2.3 of [19]). Thus $D(\tilde{Y} + \tilde{Z})(0) = Dx(t_0, 0) - (B + C) = (B + C) - (B + C) = 0$, which implies that $D\tilde{Y}(0) = 0$ and $D\tilde{Z}(0) = 0$.) Thus the following lemma holds.

Lemma 8 *There exists a function $\eta : (0, 1) \to (0, \infty)$ computable from f such that for any $a \in (0, 1)$,*

$$|D\tilde{Y}(x)| = |D\tilde{Y}(y, z)| \le a/K_A \quad \text{and} \quad |D\tilde{Z}(x)| = |D\tilde{Z}(y, z)| \le a/K_A$$

on $|x| = |y + z| \le \eta(a)$. *Without loss of generality we assume that* $0 < \eta(a) < \min\{a, \epsilon\}$.

The above lemma confirms that the feedbacks are not only weak near the hyperbolic equilibrium point 0 at the (fixed) time t_0, but also change slowly. Now let us fix an $a \in (0,1)$ such that

$$2aK_A(\|B^{-1}\| + \|C\|) < 1\,. \tag{8}$$

For technical reasons, in the next lemma, we extend the functions $\tilde{Y}$ and $\tilde{Z}$ from $B(0, \eta(a)/2)$ to the whole space $\mathbb{R}^n$ in such a way that the extensions are smooth with bounded derivatives.

Lemma 9 *(1) There are two differentiable functions* $\breve{Y}, \breve{Z} : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying the following conditions*

$$\breve{Y}(x) = \begin{cases} \tilde{Y}(x) & |x| < \eta(a)/2 \\ 0 & |x| \ge \eta(a) \end{cases}$$

$$\breve{Z}(x) = \begin{cases} \tilde{Z}(x) & |x| < \eta(a)/2 \\ 0 & |x| \ge \eta(a)\,. \end{cases}$$

Moreover, $|D\breve{Y}(x)| \le a/K_A$ *and* $|D\breve{Z}(x)| \le a/K_A$ *for every* $x \in \mathbb{R}^n$, *and the functions* $\breve{Y}$, $\breve{Z}$, $D\breve{Y}$, *and* $D\breve{Z}$ *are all computable from* f.

(2) Let $Y = P_S \circ \breve{Y}$ *and* $Z = P_U \circ \breve{Z}$. *Then* Y, Z, *and their derivatives are computable from* f. *Moreover,* $Y(x) = \tilde{Y}(x)$ *and* $Z(x) = \tilde{Z}(x)$ *for* $|x| < \eta(a)/2$, $|DY(x)| \le a$ *and* $|DZ(x)| \le a$ *for all* $x \in \mathbb{R}^n$.

Proof.

(1) See Lemma 3.1 in Section 9.3 of [11] or Lemma 7.6 in Section 5.7.1 of [23] for the construction of $\breve{Y}$ and $\breve{Z}$. From the construction and the fact that $\tilde{Y}$, $\tilde{Z}$, $D\tilde{Y}$, $D\tilde{Z}$, and $\eta(a)$ are all computable from f, it follows that $\breve{Y}$, $\breve{Z}$ and their derivatives are computable from f.

(2) It is clear that Y, Z, and their derivatives are computable from f. For $|x| < \eta(a)/2$, by Definition 7, we have $Y(x) = P_S \circ \breve{Y}(x) = P_S \circ \tilde{Y}(x) = \tilde{Y}(x)$. Similarly $Z(x) = \tilde{Z}(x)$ for $|x| < \eta(a)/2$. Now for all $x \in \mathbb{R}^n$ satisfying $|x| \le \eta(a)$ (recall that $\eta(a) < \epsilon$ from Lemma 8), since $|DY(x)| = |D(P_S \circ \breve{Y})(x)| \le \|P_S\| \cdot |\breve{Y}(x)| \cdot |D\breve{Y}(x)| \le K_A|\breve{Y}(x) - \breve{Y}(0)| \cdot a/K_A \le a(a/K_A)|x - 0| \le a\eta(a) < a$ (recall $\eta(a) < a < 1$ and $K_A > 1$). On the other hand, for all $x \in \mathbb{R}^n$ satisfying $|x| \ge \eta(a)$, $|DY(x)| = 0$. Thus $|DY(x)| \le a$ for all $x \in \mathbb{R}^n$. Similarly, $|DZ(x)| \le a$ for all $x \in \mathbb{R}^n$.

■

Corollary 10 *The functions* $Y, Z : \mathbb{R}^n \to \mathbb{R}^n$ *are bounded by* $a\eta(a)$, *i.e.,* $|Y(x)| \le a\eta(a)$ *and* $|Z(x)| \le a\eta(a)$ *for all* $x \in \mathbb{R}^n$.

Proof. It suffices to show that Y is bounded by $a\eta(a)$. For any $x \in \mathbb{R}^n$, if $|x| \geq \eta(a)$, then $Y(x) = 0$ by Lemma 9. On the other hand, if $|x| < \eta(a)$, then it follows from Lemma 9 and the mean value theorem that $|Y(x)| = |Y(x) - Y(0)| \leq a|x| < a\eta(a)$. Thus $|Y(x)| \leq a\eta(a)$ holds for all $x \in \mathbb{R}^n$. ∎

The next result shows that if an invertible linear map is perturbed slightly, then the resulting nonlinear map is still invertible. In other words, if the feedback is weak, then the nonlinear map is invertible provided its linearization is.

Lemma 11 *Let $C(\mathbb{R}^n; \mathbb{R}^n)$ be the set of all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, $\mathcal{V}$ the set of all pairs (V, g) such that $V, g : \mathbb{R}^n \to \mathbb{R}^n$, V is linear and invertible, and g is Lipschitz continuous with Lipschitz constant θ_g satisfying $0 < \|V^{-1}\|\theta_g < 1$, and let $\mathcal{S} = \{S \in C(\mathbb{R}^n; \mathbb{R}^n) : S(x) = Vx + g(x), (V, g) \in \mathcal{V}\}$. Then the function $\chi : \mathcal{S} \to C(\mathbb{R}^n; \mathbb{R}^n)$, $S \mapsto S^{-1}$, is computable.*

Proof. Apart from computability, the proof is an imitation of the proof of Lemma 8.2 of Chapter IX in [11], which shows the existence of the function χ. For completeness we include it here.

First notice that for any $(V, g) \in \mathcal{V}$, since V is nonsingular, the maps $V \mapsto V^{-1}$ and $V \mapsto \|V^{-1}\|$ are computable. Thus if one can show that $V^{-1}S \mapsto (V^{-1}S)^{-1}$, $S(x) = Vx + g(x), (V, g) \in \mathcal{V}$, is computable from input (V, g), then $S^{-1} = S^{-1}VV^{-1} = (V^{-1}S)^{-1}V^{-1}$ is computable from input (V, g). (As before we abbreviate $P \circ Q$ by PQ for $P, Q : \mathbb{R}^n \to \mathbb{R}^n$.) Let us denote $y = V^{-1}S(x) = V^{-1}(Vx + g(x)) = x + V^{-1}g(x)$, $x \in \mathbb{R}^n$. Then to compute $(V^{-1}S)^{-1}$ is to solve the equation $y = x + V^{-1}g(x)$ for x, or equivalently, to solve $x = y - V^{-1}g(x)$ for x. We solve the equation by way of successive approximations. Define

$$\begin{aligned} x_0 &= 0 \\ x_n &= y - V^{-1}g(x_{n-1}) \,. \end{aligned} \tag{9}$$

We observe that for $n \geq 2$,

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|V^{-1}(g(x_{n-1}) - g(x_{n-2}))\| \\ &\leq \|V^{-1}\|\theta_f \|x_{n-1} - x_{n-2}\| \,. \end{aligned}$$

By induction

$$\|x_n - x_{n-1}\| \leq \left(\|V^{-1}\|\theta_g\right)^{n-1} \cdot \|x_1 - x_0\| \,.$$

Since $0 < \|V^{-1}\|\theta_g < 1$, the sequence $x_0, x_1, x_2, \ldots$ is effectively convergent to $\tilde{x}$. Consequently, taking the limit of Eq. (9) shows that $\tilde{x}$ is the solution of the equation $x = y - V^{-1}g(x)$. This shows that the equation $x = y - V^{-1}g(x)$ has a unique solution for every given y and, moreover, the solution is computable from V^{-1}, g, and y. This proves that $V^{-1}S$ is invertible and its inverse $(V^{-1}S)^{-1}$ is computable from (V, g). ∎

Now we define two functions $L, T : \mathbb{R}^n \to \mathbb{R}^n$, where for $x = (P_S x, P_U x) = (y, z) \in \mathbb{R}^n$,

$$L(x) = L(y, z) = (B + C)x = By + Cz = (By, Cz) \tag{10}$$

and

$$T(x) = T(y,z) = (By + Y(y,z), Cz + Z(y,z))\,. \tag{11}$$

Since the maps B, C, Y, and Z are all computable from f, so are the functions L and T.

Remark 12 *(1) It is clear that $L(x) = e^{At_0}x$. Moreover L is invertible on $\mathbb{R}^n$ with inverse $L^{-1} = e^{-At_0} = B^{-1} + C^{-1}$ (see the proof of Lemma 6).*

(2) We also note that $T(x) = \phi_{t_0}(x)$ on $B(0, \eta(a)/2)$ by Definition 7 and Lemma 9. Since $T(x) = (B+C)x + (Y+Z)(x)$, $B + C = e^{At_0}$ is an invertible linear map with inverse e^{-At_0}, $\|e^{-At_0}\| = \|A_S(-t_0)+A_U(-t_0)\| = \|B^{-1}+C^{-1}\| \le \|B^{-1}\|+\|C^{-1}\| \le \|B^{-1}\|+\|C\|$ (recall that $\|C^{-1}\| < \|C\|$), $|D(Y+Z)(x)| \le |DY(x)| + |DZ(x)| \le 2a$ for all $x \in \mathbb{R}^n$ (Lemma 9 (2)), and $2aK_A(\|B^{-1}\| + \|C^{-1}\|) < 2aK_A(\|B^{-1}\| + \|C\|) < 1$ by (8), it follows that $(B+C, Y+Z) \in \mathcal{V}$, and thus Lemma 11 implies that T is invertible on $\mathbb{R}^n$ and its inverse T^{-1} is computable from f, as is T. Moreover, if we set $y_1 = By_0 + Y(y_0, z_0)$ and $z_1 = Cz_0 + Z(y_0, z_0)$, then $(y_0, z_0) = T^{-1}(y_1, z_1) = (B^{-1}y_1 + Y_1(y_1, z_1), C^{-1}z_1 + Z_1(y_1, z_1))$, where $Y_1(y_1, z_1) = -B^{-1}Y(T^{-1}(y_1, z_1))$ and $Z_1(y_1, z_1) = -C^{-1}Z(T^{-1}(y_1, z_1))$.

Lemma 13 *Let $L, T : \mathbb{R}^n \to \mathbb{R}^n$ be the two maps defined by (10) and (11). For any map $H : \mathbb{R}^n \to \mathbb{R}^n$, let $H_1 = P_S \circ H$ and $H_2 = P_U \circ H$. Then $H \circ T = L \circ H$ if and only if H_1 and H_2 satisfy the following two equations:*

$$H_1(By + Y(y,z), Cz + Z(y,z)) = B(H_1(y,z)) \tag{12}$$

and

$$H_2(By + Y(y,z), Cz + Z(y,z)) = C(H_2(y,z))\,. \tag{13}$$

We recall that $P_S + P_U$ is the identity map on $\mathbb{R}^n$, thus $H = (P_S + P_U) \circ H = H_1 + H_2$.

Proof. It follows from the definitions of L, T, H_1, and H_2 that

$$H \circ T(y,z) = (H_1(By + Y(y,z), Cz + C(y,z)), H_2(By + Y(y,z), Cz + C(y,z)))$$

and

$$L \circ H(y,z) = (B(H_1(y,z), H_2(y,z)), C(H_1(y,z), H_2(y,z)))\,.$$

Since B is a linear map, $(H_1(y,z), H_2(y,z)) = H_1(y,z) + H_2(y,z) = P_S H(y,z) + P_U H(y,z)$, and $BP_U = 0$, one obtains that $B(H_1(y,z), H_2(y,z)) = BH_1(y,z)$. Similarly, $C(H_1(y,z), H_2(y,z)) = CH_2(y,z)$. Therefore, it follows from (12) and (13) that $H \circ T = (H_1(By + Y(y,z), Cz + C(y,z)), H_2(By + Y(y,z), Cz + C(y,z))) = (B(H_1(y,z)), C(H_2(y,z))) = L \circ H$.

On the other hand, if $H \circ T = L \circ H$, then it follows from Lemma 4 that

$$\begin{aligned} H_1(By + Y(y,z), Cz + Z(y,z)) &= P_S(H(T(y,z))) \\ &= P_S(L(H(y,z))) \\ &= [P_S(B+C)](H_1(y,z) + H_2(y,z)) \\ &= B(H_1(y,z)). \end{aligned}$$

Thus Eq. (12) is satisfied. The same argument applies to establish Eq. (13). ∎

Lemma 14 *There is an algorithm that computes from f a unique homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ and its inverse H^{-1} such that $H = id + \tilde{H}$, id is the identity map on $\mathbb{R}^n$, $\tilde{H} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded with respect to the sup norm (i.e. $\|\tilde{H}\|_\infty = \sup_{(y,z)\in\mathbb{R}^n} |\tilde{H}(y,z)| < \infty$), and*

$$H \circ T = L \circ H\,.$$

Proof. The proof follows the idea of the proof of Lemma 8.3 in Chapter IX of [11]. From Lemma 13, it suffices to construct an algorithm that computes H_1 and H_2 from T and L which satisfy Eqs. (12) and (13). Let us first compute $H_2 = z + \tilde{H}_2$. Clearly it suffices to compute $\tilde{H}_2$. From Eq. (13) it follows that

$$Cz + Z(y,z) + \tilde{H}_2(By + Y(y,z), Cz + Z(y,z)) = C(z + \tilde{H}_2(y,z)),$$

which in turn implies that

$$\tilde{H}_2(y,z) = C^{-1}(Z(y,z) + \tilde{H}_2(By + Y(y,z), Cz + Z(y,z))). \tag{14}$$

We can then compute $\tilde{H}_2$ using successive approximations on Eq. (14). Define

$$\begin{aligned} \tilde{H}_2^0(y,z) &= 0 \\ \tilde{H}_2^{k+1}(y,z) &= C^{-1}(Z(y,z) + \tilde{H}_2^k(By + Y(y,z), Cz + Z(y,z)),\ k \geq 1. \end{aligned} \tag{15}$$

Thus $\tilde{H}_2^k$ are defined and continuous for all $(y,z) \in \mathbb{R}^n$. They are also bounded because $\tilde{H}_2^0$ is bounded and $|Z(y,z)| \leq a\eta(a)$ for all $(y,z) \in \mathbb{R}^n$ by Corollary 10. Now since

$$\begin{aligned} & \|\tilde{H}_2^k - \tilde{H}_2^{k-1}\|_\infty \\ = & \sup_{(y,z)\in\mathbb{R}^n} |C^{-1}(Z(y,z) + \tilde{H}^{k-1}(By + Y(y,z), Cz + Z(y,z))) - \tilde{H}_2^{k-1}(y,z)| \\ = & \sup_{(y,z)\in\mathbb{R}^n} |C^{-1}(\tilde{H}^{k-1}(By + Y(y,z), Cz + Z(y,z)) - \tilde{H}_2^{k-2}(By + Y(y,z), Cz + Z(y,z)))| \\ \leq & \|C^{-1}\| \|\tilde{H}_2^{k-1} - \tilde{H}_2^{k-2}\|_\infty \\ \leq & \|C^{-1}\|^{k-1} \|\tilde{H}_2^1 - \tilde{H}_2^0\|_\infty = \|C^{-1}\|^{k-1} \|\tilde{H}_2^1\|_\infty \end{aligned}$$

and

$$\begin{aligned} & \|\tilde{H}_2^1\|_\infty \\ = & \sup_{(y,z)\in\mathbb{R}^n} |C^{-1}(Z(y,z) + \tilde{H}_2^0(By + Y(y,z), Cz + Z(y,z)))| \\ = & \sup_{(y,z)\in\mathbb{R}^n} |C^{-1}(Z(y,z))| \leq \|C^{-1}\| a\eta(a)\,, \end{aligned}$$

it follows that $\|\tilde{H}_2^k - \tilde{H}_2^{k-1}\|_\infty \leq \|C^{-1}\|^k a\eta(a)$. Since $C^{-1} = A_U(-t_0)$, $\|C^{-1}\| < 1/(K_A + 1) < 1$ by (7); consequently, the $\tilde{H}_2^k$'s are uniformly and effectively convergent to a continuous, bounded, and computable function $\tilde{H}_2$ on $\mathbb{R}^n$ as $k \to \infty$. Taking limits in Eq. (15) shows that $\tilde{H}_2$ satisfies Eq. (14). The uniqueness of $\tilde{H}_2$ follows from the fact that $\tilde{H}_2$ is bounded on $\mathbb{R}^n$. If there is another continuous and bounded function $\tilde{G}_2 : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{G}_2(y,z) = C^{-1}(Z(y,z) + \tilde{G}_2(By + Y(y,z), Cz + Z(y,z)))$ but $\tilde{G}_2 \neq \tilde{H}_2$, then we can choose a positive number $\delta > 0$ such that $\|\tilde{H}_2 - \tilde{G}_2\|_\infty - \delta > 0$ and $0 < \delta < (1 - \|C^{-1}\|)\|\tilde{H}_2 - \tilde{G}_2\|_\infty$. Now let us pick $(y_0, z_0) \in \mathbb{R}^n$ such that

$$|\tilde{H}_2(y_0, z_0) - \tilde{G}_2(y_0, z_0)| > \|\tilde{H}_2 - \tilde{G}_2\|_\infty - \delta\,.$$

But from Eq. (14) it follows that

$$\begin{aligned}
& |\tilde{H}_2(y_0, z_0) - \tilde{G}_2(y_0, z_0)| \\
= \;& |C^{-1}(\tilde{H}_2(By_0 + Y(y_0, z_0), Cz_0 + Z(y_0, z_0)) - \tilde{G}_2(By_0 + Y(y_0, z_0), Cz_0 + Z(y_0, z_0)))| \\
\leq \;& \|C^{-1}\| \cdot \|\tilde{H}_2 - \tilde{G}_2\|_\infty < \|\tilde{H}_2 - \tilde{G}_2\|_\infty - \delta\,.
\end{aligned}$$

This is a contradiction. The uniqueness of $\tilde{H}_2$, thus H_2 as well, is established.

To compute H_1 we use the same argument on the equation $H_1 T^{-1} = B^{-1} H_1$, which is a variation of Eq. (12). Note that T^{-1} exists and is computable by Remark 12.

It remains to show that H^{-1} is also computable from T and L. For this purpose we interchange the roles of T and L and make use of the same construction to compute functions G_1 and G_2 such that $T \circ G = G \circ L$, where $G = G_1 + G_2$. Since $H \circ T = L \circ H$ and $T \circ G = G \circ L$, it follows that $(H \circ G) \circ H \circ T = H \circ G \circ L \circ H = H \circ T \circ G \circ H = L \circ H \circ (G \circ H)$, we have $H \circ G = G \circ H = id$ by virtue of uniqueness. Thus H is a homeomorphism on $\mathbb{R}^n$ and $H^{-1} = G$ is computable from T and L. ■

Finally we come to the **Proof of Theorem 1.** We need to show that two open subsets $U, V \in \mathcal{O}$, a function $\mu : U \to \mathbb{R}^+$, and a homeomorphism $H : U \to V$ can be computed from f such that for any $\tilde{x} \in U$ and $-\mu(\tilde{x}) < t < \mu(\tilde{x})$, $H(\phi_t(\tilde{x})) = e^{At} H(\tilde{x})$.

First we recall that $\alpha = \alpha_f$ and $\epsilon = \epsilon_f$ (see (5)), it then follows from Lemma 2 that the unique solution $x(t, \tilde{x}) = \phi_t(\tilde{x})$ to the initial value problem $\dot{x} = f(x)$ and $x(0) = \tilde{x}$ is defined, computable, and continuously differentiable on $(-\alpha, \alpha) \times B(0, \epsilon)$. We also recall that $A = Df(0)$, $L = e^{At_0}$, and $T(\tilde{x}) = \phi_{t_0}(\tilde{x})$ for all $\tilde{x} \in B(0, \eta(a)/2)$, where $0 < \eta(a) < \epsilon$ and $0 < t_0 < \alpha$ (see (7)). Now for any $0 < t < \alpha - t_0$, since $T \circ \phi_t = \phi_{t_0} \circ \phi_t = \phi_{t_0 + t} = \phi_t \circ \phi_{t_0} = \phi_t \circ T$ and, similarly, $L \circ e^{-At} = e^{-At} \circ L$ (ϕ_{t_0+t} is well defined since $0 < t_0 + t < \alpha$), it follows that

$$\begin{aligned}
(e^{-At} \circ H \circ \phi_t) \circ T &= e^{-At} \circ H \circ T \circ \phi_t \\
&= e^{-At} \circ L \circ H \circ \phi_t \\
&= L \circ (e^{-At} \circ H \circ \phi_t)\,,
\end{aligned}$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$ is the homeomorphism supplied by Lemma 14. By virtue of uniqueness and linearity of L, $e^{-At} \circ H \circ \phi_t = H$; in other words, $H(\phi_t(\tilde{x})) = e^{At} H(\tilde{x})$ for all $\tilde{x} \in B(0, \eta(a)/2)$ and $0 < t < \alpha - t_0$.

Next we show that there is a function $\mu : B(0, \eta(a)/2) \to (0, \alpha)$ computable from f such that $\phi_t(\tilde{x}) \in B(0, \eta(a)/2)$ for $\tilde{x} \in B(0, \eta(a)/2)$ and $-\mu(\tilde{x}) < t < \mu(\tilde{x})$, where $0 < \eta(a) < 1$ is defined in Lemma 8. From the previous paragraph, we must ensure $\mu(\tilde{x}) < \alpha - t_0$ on $B(0, \eta(a)/2)$. We also note that, since $B(0, \eta(a)) \subset B(0, \epsilon)$ and $B(0, \epsilon)$ is contained in the domain of f by Lemma 2, it follows that $N = \max_{\tilde{x} \in \overline{B(0, \eta(a)/2)}} \{|f(\tilde{x})|, |Df(\tilde{x})|\}$ is well defined and computable from (any C^1 name of) f, where $\overline{B(0, \eta(a)/2)}$ denotes the closure of

$B(0, \eta(a)/2)$. Next, we observe that the function

$$\delta : B(0, \eta(a)/2) \to (0, \eta(a)/2), \quad \tilde{x} \mapsto \frac{(\eta(a)/2) - |\tilde{x}|}{2} \tag{16}$$

is computable from f and satisfies $|\tilde{x}| + \delta(\tilde{x}) < \eta(a)/2$ on $B(0, \eta(a)/2)$. Now we come to the definition of the desired function $\mu : B(0, \eta(a)/2) \to (0, \alpha)$: for any $\tilde{x} \in B(0, \eta(a)/2)$,

$$\mu(\tilde{x}) = \min \left\{ \frac{\alpha - t_0}{2}, \frac{(\eta(a)/2) - |\tilde{x}| - \delta(\tilde{x})}{2N} \right\}. \tag{17}$$

It is clear that μ is computable from f and $\mu(\tilde{x}) < \alpha - t_0$ on $B(0, \eta(a)/2)$. It remains to show that, for any $\tilde{x} \in B(0, \eta(a)/2)$ and $-\mu(\tilde{x}) < t < \mu(\tilde{x})$, $\phi_t(\tilde{x}) \in B(0, \eta(a)/2)$. For this purpose we make use of the Picard method of successive approximations; the method is based on the fact that $x(t) = x(t, \tilde{x}) = \phi_t(\tilde{x}) = \phi_t$ is the solution to the initial value problem $\dot{x} = f(x)$ and $x(0) = \tilde{x}$ if and only if $x(t)$ is a continuous function satisfying the integral equation

$$x(t) = \tilde{x} + \int_0^t f(x(s))ds.$$

Define a sequence of functions

$$u_0(t) = \tilde{x}, \qquad u_{k+1}(t) = \tilde{x} + \int_0^t f(u_k(s))ds, \quad k \geq 0. \tag{18}$$

It is clear that for any $\tilde{x} \in B(0, \eta(a)/2)$, $|u_0(t)| = |\tilde{x}| < \frac{\eta(a)}{2} - \delta(\tilde{x})$. Assume now that $\max_{[-\mu(\tilde{x}), \mu(\tilde{x})]} |u_k(t)| < \frac{\eta(a)}{2} - \delta(\tilde{x})$. Then for $-\mu(\tilde{x}) < t < \mu(\tilde{x})$,

$$|u_{k+1}(t)| \leq |\tilde{x}| + \left| \int_0^t f(u_k(s))ds \right| \leq |\tilde{x}| + N\mu(\tilde{x}),$$

which implies that

$$\max_{|t| < \mu(\tilde{x})} |u_{k+1}(t)| < \frac{\eta(a)}{2} - \delta(\tilde{x}). \tag{19}$$

It is well known from the classical proofs of the fundamental existence and uniqueness theorem for the nonlinear system $\dot{x} = f(x)$ that if $|t| \leq 1/N$, then the sequence (18) is contracting and converges to the unique solution $\phi_t(\tilde{x})$ of the problem $\dot{x} = f(x)$ and $x(0) = \tilde{x}$ (see, e.g., Section 2.2 of [19]). In particular, it follows from (19) that $\max_{|t| < \mu(\tilde{x})} |\phi_t(\tilde{x})| \leq (\eta(a)/2) - \delta(\tilde{x}) < \eta(a)/2$. This shows that μ has the desired property.

Finally, let $U = B(0, \eta(a)/2)$ and $V = H(U)$. Since $\eta(a)$ is computable from f (see Lemma 8 and Eq. (8)), so is the open subset U. By Theorem 6.2.4 of [24] the map $(g, O) \mapsto g^{-1}(O)$ (for $g \in C(\mathbb{R}^n; \mathbb{R}^n)$ and open subset O of $\mathbb{R}^n$) is computable from g and O; it then follows that $V = H(U) = (H^{-1})^{-1}(U)$ is computable from H^{-1} and U. Since both H^{-1} and U are computable from f, V is also computable from f. The proof of Theorem 1 is complete. ■

Theorem 1 can be generalized in a uniform way to include the case where the hyperbolic equilibrium points are not necessarily 0. Let $\mathcal{U}$ be the set of all open subsets of $\mathbb{R}^n$.

Theorem 15 *There is a computable map* $\Theta : C(\mathbb{R}^n;\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{U} \times \mathcal{U} \times C(\mathbb{R}^n;\mathbb{R}) \times C(\mathbb{R}^n;\mathbb{R}^n)$, $(f,p) \mapsto (U,V,\mu,H)$, *where* p *is a hyperbolic equilibrium point of* f *(i.e.* $f(p)=0$ *and* $Df(p) \in \mathcal{A}_H$*), such that the following holds true:*

(a) $p \in U$, $p \in V$, *and* $H : U \to V$ *is a homeomorphism;*

(b) the unique solution $x(t,\tilde{x})$ *to the initial value problem* $\dot{x} = f(x)$ *and* $x(0) = \tilde{x}$ *is defined on* $(-\mu(\tilde{x}),\mu(\tilde{x})) \times U$*; moreover,* $x(t,\tilde{x}) \in U$ *for all* $\tilde{x} \in U$ *and* $-\mu(\tilde{x}) < t < \mu(\tilde{x})$*;*

(c) $H(x(t,\tilde{x})) = e^{Df(p)t}H(\tilde{x})$ *for all* $\tilde{x} \in U$ *and* $-\mu(\tilde{x}) < t < \mu(\tilde{x})$.

The proof of Theorem 1 can be extended straightforwardly to prove Theorem 15.

4 An open problem

As a problem for further investigation, it would be interesting to determine the computational complexity of the computable homeomorphism H. For a nonlinear problem $\dot{x} = f(x)$ and $x(0) = 0$, the solution L to its linearization is obviously polynomial-time computable; on the other hand it was shown recently that there exists a polynomial-time (global) Lipschitz computable function f such that the solution to the nonlinear problem $\dot{x} = f(x)$ and $x(0) = 0$ is polynomial-space hard (see [15] and [16]). Yet we note that the conjugacy H applies near a hyperbolic equilibrium point, and it is known that hyperbolicity may help to reduce computational complexity. For instance, although many Julia sets are non-computable, hyperbolic Julia sets are not only computable but polynomial-time computable (see, for example, [2], [3], [22], [25]).

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