Computability and Dynamical Systems: a Perspective

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November 16, 2007

Abstract

In this paper we look at dynamical systems from a computability perspective. We survey some topics and themes of research for dynamical systems and then see how they can be fitted in a computational framework. We will recall some selected results, and enounce problems that might lay possible routes for further research.

1 Introduction

Dynamical systems are extensively used as mathematical models for many phenomena, ranging from the evolution of stock market indexes to the dynamics of a beating heart. However this versatility and wealth of applications comes with a price: dynamical systems can present an incredibly complex behavior, and their understanding is far from complete. Indeed, only recently we started to have a glimpse of the richness that even systems defined with simple rules can present.

In the past few years, with the availability of fast and cheap computers, many dynamical systems have been explored in a more "experimental" manner. Though this approach is usually not mathematically rigorous, it greatly helped to understand concepts like "sensitive dependence on initial conditions", "strange attractors", among others. In the applications domain, it became a standard tool. Today, many experiments are "digitally simulated" in computers as dynamical systems: weather forecast, testing the effects of new drugs in a "digital body", testing the effects of earthquakes in different types of constructions, etc.

However, not much is known about what can computed with a computer and what cannot, at least from the perspective of continuous dynamical systems (the discrete case, where both time and space are integers, has long be settled, by Turing and others).

The purpose of the present paper is to make a brief survey about some selected results and problems concerning computability theory and dynamical systems and to sketch possible routes for further research.

2 Dynamical systems

A dynamical system can be defined as follows.

Definition 1 *A* dynamical system *(DS) defined on the topological space S over $\mathbb{A} = \mathbb{R}_0^+$ ($\mathbb{A} = \mathbb{N}$) is a triple $(S, \mathbb{A}, \phi)$, where $\phi : S \times \mathbb{A} \to S$ is a function satisfying*

1. *Initial condition: $\phi(p, 0) = p$ for any $p \in S$;*
2. *Continuity on both arguments;*
3. *Semigroup property:*

$$\phi(\phi(p, t_1), t_2) = \phi(p, t_1 + t_2),$$

for any point $p \in S$ and any $t_1, t_2 \in \mathbb{A}$.

When $\mathbb{A} = \mathbb{R}_0^+$, we say that we have a *continuous dynamical system.* On the other hand, when $\mathbb{A} = \mathbb{N}$, we say that we have a *discrete dynamical system.* S is the *state space* and points in S are called *states.*

Notice that in the above definition $\mathbb{R}_0^+$ could be replaced by $\mathbb{R}$ and $\mathbb{N}$ by $\mathbb{Z}$. In these cases we say that we have *time-reversible dynamical systems.* For the purpose of this work, we do not distinguish between DSs and time-reversible DSs.

Notice that for discrete DSs, we only need to know $\phi(p, 1)$ for each $p \in S$ to completely define the DSs, since the evolution of system is obtained by iterating $f : S \to S$ given by $f(p) = \phi(p, 1)$. This is why discrete DSs are often identified in the literature with the iteration of a map.

Likewise, for continuous DS's, we can take

$$f(x) = \left.\frac{d}{dt}\phi(x, t)\right|_{t=0}$$

and then see that $\phi(x_0, t)$ is the solution of the initial-value problem

$$\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$$

since

$$\left.\frac{d}{dt}\phi(x_0, t)\right|_{t=t_0} = \left.\frac{d}{dt}\phi(x_0, t_0 + t)\right|_{t=0} = \left.\frac{d}{dt}\phi(\phi(x_0, t_0), t)\right|_{t=0} = f(\phi(x_0, t_0)).$$

Reciprocally, it can be shown that the class of solutions of an initial-value problem defined with ordinary differential equations (ODEs) is a continuous time dynamical system (see the results of Section 8.7 from [HS74]).

In more recent times there has been a renewed interest in the interplay between dynamical systems and computation. There have been basically two types of approach:

1. To consider dynamical systems as computers that perform computation;

2. To study, from a more or less traditional computational perspective, dynamical systems.

An example of an approach corresponding to point 1 is the one presented in [Sie99]. As Siegelmann puts it in [Sie99, p. 147]: "... any physical system observed by an experimenter in a laboratory and any dynamical behavior in nature is perceived as performing a computational process. Beginning from an initial state (input), a physical system evolves in the state space according to an update equation (the computation process) until it reaches some designated state (the output)". Siegelmann even goes further broadly classifying computational models as analog or discrete models depending on the properties of the underlying DS: "The main property that distinguishes analog from digital computational models is the use of a *continuous space state*. In this chapter ... we divide these [analog] models [of computation] into broad categories which partially overlap: discrete time models ..., continuous time models ..., hybrid models ..., and dissipative computational models".

With respect to the second approach, this mainly corresponds to the approach followed in the control theory community, especially in the verification of systems. Basically the idea is to develop algorithms, or to show that they do not exist, such that given the description of a dynamical system into a suitable form, the algorithm states whether the DSs satisfies a given property or not.

A typical verification problem is the reachability problem: given the description of a DS, of a point p in the phase state, and some "simple" subset Γ of the phase space, decide whether the trajectory starting in p will eventually reach, or not, the target set Γ. This is especially helpful to prove that when some device is operating in some range of parameters, its normal evolution will not reach "critical" or "unsafe" states.

There are many results about the reachability problem, most of them establishing the undecidability of the problem. The idea is to encode the Halting problem into the reachability problem. However these results usually rely on the use of exact dynamics, which is not very realistic, since this requires infinite precision. Thus in the hybrid system verification community a folklore conjecture appeared saying that this undecidability is due to non-stability, non-robustness, sensitivity to initial values of the systems, and that it never occurs in "real systems" [AB01]. There were several attempts to formalize and prove (or to disprove) this conjecture: it has been proved that small perturbations of the trajectory still yields undecidability [HR99]. Infinitesimal perturbations of the dynamics for a certain model of hybrid systems has shown to rise to decidability [Frä99]. This has been extended to several models by [AB01]. However the situation is not very clear, and in this point a parallel can be drawn with dynamical systems as we will see.

In this survey we are interested in the second perspective, especially in the long term behavior of continuous-time DS, since it is where most significant problems lie. From a DS point of view there are many interesting topics which are related to this long term behavior and that we now briefly survey.

Invariant sets. These are sets which are maintained under the dynamics of the DS. They are important to characterize the DS, and are related to other concepts like attractors. Formally a set A is *invariant* if $\phi(A,t) = A$ for all t. We can also say that A is (*backward*) *forward*-invariant if $\phi(A,t) \subseteq A$ for all $t \geq 0$ (resp. $t \leq 0$).

Limit sets. Informally a limit set is the set of states a DS approaches assimptotically, by either going forward or backwards in time. Limit sets are important because they can be used to understand the long term behavior of a dynamical system. Examples of limit sets include fixed points, periodic orbits, limit cycles and attractors. Formally, given a point $p \in S$ in the phase state, the (α) ω-limit set $\omega(x)$ ($\alpha(x)$ resp.) is the set of accumulation points of the forward orbit $\{\phi(p,t)\}_{t\geq 0}$ (backward orbit $\{\phi(p,t)\}_{t\leq 0}$ resp.). By other words, ($\overline{A}$ designates the closure of the set A)

$$\omega(x) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{t\geq n} \phi(x,t)}.$$

Attractors, to which a dynamical system evolves after a long enough time. Geometrically an attractor can be a point, a curve, a manifold, or even a complicated set with a fractal structure. Indeed, DSs in the real world tend to be dissipative, which leads the system to some typical behavior. This one part of the phase space of the dynamical system corresponding to the typical behavior is the attracting section. The precise definition of attractor involves some problems which are well described by Buescu [Bue97, p. 1]: "Although the concept of attractor is often taken for granted, the fact is that several distinct, sometimes overlapping, definitions of attractor exist in the literature. Moreover, each of these definitions illuminates specific properties of the dynamical objects under consideration. We interpret this state of affairs as follows. There is no contradiction between the several distinct concepts of attractors — all of them are useful as they reveal slightly different properties. The characterization of the object is given by the set of all these properties, not just by a specific one. We therefore think that there is no such thing as a 'final' or 'correct' definition of attractor". Several different definitions for attractors are given in the Section 1.4 of that book.

In general the attractor is required to satisfy three conditions: (i) be invariant; (ii) be minimal in some sense, i.e. proper subsets of it cannot be attractors; (iii) it should attract most of neighboring trajectories.

Trapping regions. If U is an open set of the phase state such that the closure of U, $\overline{U}$, is compact and $\phi(\overline{U},t) \subseteq U$ for some $t > 0$, then it is called a trapping region. In these conditions

$$\bigcap_{t\geq 0} \phi(U,t)$$

is an attractor. In practice, the existence of attractors is often shown by constructing a trapping region.

Stability. Historically, the study of stable dynamical systems always had a prominent role in the development of the theory of dynamical systems. Since we cannot avoid uncertainty when performing measurements on a physical system, and we cannot fully isolate it, it seems that a good mathematical model for physical phenomena should account for some kind of robustness.

This is the idea underlying *structurally stable systems*, originated from the work of Andronov and Pontryagin [AP37]. A system is structurally stable if small perturbations of it leave the whole orbit structure unchanged, up to a continuous global change of coordinates. Andronov, Pontryagin, and Peixoto showed the following results for the plane $\mathbb{R}^2$: (i) Structurally stable systems form an open and dense subset of the set of dynamical systems with C^1 vector fields; (ii) Attractors for structurally stable systems consist only of fixed points and periodic orbits.

The previous two results say that (i) structurally stable systems in $\mathbb{R}^2$ are generic i.e. they encompass "almost all dynamical systems"; (ii) meaningful attractors in $\mathbb{R}^2$ can be fully characterized into two classes (other unstable attractors, like quasi-periodic orbits obtained by rotations in the circle S^1 are thus not considered).

For some time it was conjectured (cf. the retrospective given in [GH83]) that (i) would hold for higher dimensions. This conjecture would turn out to be false, since structurally stable systems are not dense [Sma66].

For that reason structurally stable systems are no longer seen as the adequate class of "robust system". In our opinion, the situation is probably similar to that depicted by Buescu for attractors: there is no such thing as a "general" or "correct" definition of stability, since this will change from situation to situation. As Guckenheimer and Holmes wrote in their book [GH83, p. 259]: "This principle was embodied in a *stability dogma*, in which structurally unstable systems were regarded as somehow suspect. ... The logic which supports the stability dogma is faulty ... the presumed strange attractors of our examples are not structurally stable, and we are confident that these systems are realistic models for the chaotic behavior of the corresponding (deterministic) physical systems. *But, since the systems are not structurally stable, details of their dynamics which do not persist in perturbations may not correspond to verifiable physical properties of the system...* Thus the stability dogma might be reformulated to state that the only properties of a dynamical system (or a family of dynamical systems) which are *physically relevant* are those which are preserved under perturbations of the system. The definition of physical relevance will clearly depend upon the specific problem".

Classification of attractors. In a given DS, different type of attractors may coexist. In general we will be only concerned with attractors in stable systems. Besides periodic attractors like points or limit circles, several studies starting in the 1960s showed that attractors of a completely different kind exist, usually known as *strange attractors*. While there is no precise definition for this mathematical object (there are a number of different definitions in the literature, see e.g. [Mil85] and [Rue89] and references therein), it has at least two properties: (i) it is robust and nearby trajectories converge to it; (ii) this is not a "simple" attractor (i.e. a point or a periodic orbit). One of the best known of such attractors is Lorenz attractor [Lor63], [Via00] that, while being very robust to changes in the parameters, is not structurally stable.

Bifurcations. Often dynamical systems come in a family, depending on one or more parameters. Usually the phase portrait of the DS changes smoothly as the parameters vary. But sometimes there are "revolutionary" values of the parameter where the phase portrait changes drastically. For instance a sink (attracting fixed point) can become a source (repelling fixed point). These "revolutionary" changes in the phase portrait are known

as bifurcations. Usually a task that one would like to carry is to build a diagram that summarizes the information about the bifurcation values and the behavior of the DS in between bifurcations. These are called *bifurcation diagrams* and, as put in [HW95, p. 269]: "finding the bifurcation diagram for a given family is tantamount to 'understanding the family of [dynamical systems]...'".
For differential equations, bifurcations are rather well understood in two dimensions, where only four different types of exceptional behavior can give rise to bifurcations, yielding for one-parameter families of ODEs the following classification of bifurcations: saddle-node, Andronov-Hopf, saddle connections, and semi-stable limit cycles [HW95].
Other bifurcation scenarios are well-known for discrete-time DSs, like the Ruelle-Takens scenario through quasi-periodicity, the Feigenbaum scenario through period doubling, and the Pomeau-Manneville scenario through intermittency [Rue89].
However, besides these particular cases (including others like crises — see [Ott02, Section 8.3]), we are still very far from any complete classification of possible bifurcation scenarios, even for ODEs in dimension 3.

Basins of attraction. A basin of attraction consists of a set of points in the phase space which evolve to a particular attractor. Formally, if C is an attractor then its basin of attraction is the set $\{p \in S | \omega(p) \subseteq C\}$, where S is the phase state. Basins of attraction can have relatively simple boundaries, or more complicated ones like fractals [GOY87]. Even more complicated behavior can arise when we have *riddled* basins of attraction [AYYK92], [OSA$^+$94], which are thought to exist ubiquitously in high-dimensional dynamics. In this case any point in a riddled basin of attraction of an attractor is arbitrarily close to the basin of attraction of another attractor. Formally, suppose that a system has two attractors which we denote A and C with basins $B(A)$ and $B(C)$. We say that the basin $B(A)$ is riddled if, for every point p in $B(A)$, an ε-radius ball, $B_\varepsilon(p)$ always contains a positive Lebesgue measure of points in $B(C)$ for any $\varepsilon > 0$.
By the way, this concept also illustrates why it is difficult to have a "good" definition of attractor. A common definition of an attractor is to require that there exists some neighborhood around it such that all initial conditions in this neighborhood generate orbits that limit on the attractor. But as we easily see, this definition do not encompasses attractors with riddled basins.

Ergodicity. Ergodic theory is the study of dynamical systems from a statistical perspective. By that reason we need to associate a measure to the state space. The name comes from classical statistical mechanics, where the "ergodic hypothesis" asserts that, asymptotically, the time average of an observable is equal to the space average. This result can be translated to ergodic systems that we will however not define here (see [KB95] or [BS02] for details).
With the help of this theory we are able to define important concepts, especially for dynamical systems where the evolution law is measure-preserving. For instance, one can define notions as *mixing transformations*, or *entropy* [KB95], [BS02].

Wandering sets. This notion is used to formalize a specific idea of movement in DSs. In general, if there is a measure μ associated to the state space, then a set A is wandering if there is a neighborhood U of A and a time T such that for all $t > T$ one has

$$\mu(\phi(U,t) \cap U) = 0.$$

In some sense, the concept of a wandering set is dual to the ideas expressed in the Poincaré recurrence theorem. This theorem says that for (discrete-time) dynamical systems in which the evolution law is measure-preserving (hence the system is conservative, at least if we consider the measure as the physical quantity to be preserved), then almost all points of the phase state, considered as sets, will be non-wandering. Therefore one can define a *dissipative system* as a DS in which there is a wandering set of positive measure.

3 Some results and problems

Provided with the concepts introduced in the previous section, it is not hard to give a list of computational problems one would like to answer. Some examples would be:

1. Given a set A of the phase state, is it invariant?
2. Given a computable point x, is it limit set $\omega(x)$ computable?
3. Given some subset A of the phase state, is there any attractor within this set?
4. Given a family of DSs, can we distinguish those which are stable from those which are not?
5. Can we compute the number of attractors that a DS has? Can we characterize them (e.g. limit circle, strange attractor, etc.) only using fully automatic means?
6. Can we compute the bifurcation diagram for a given family of DSs?
7. Given some point, can we decide to which basin of attraction it belongs?

While some of the previous problems are probably solvable, others are still very far from being settled, especially because of the subtleties and our lack of knowledge about properties of DSs, as mentioned previously. Examples of such questions would be problem 4 (what is "stable"? Most probably depends on the context, as discussed earlier), or problem 5 (we don't have a classification for attractors, and don't know if there will ever be one).

So our point of view regarding dynamical systems is not to "fly too high", i.e. not to aim at some general computational theory for them, that we nonetheless have to restrict to some specific classes of DSs, probably not too representative, but where things are more or less well understood. For instance papers about computational theories involving dynamical systems e.g. [SBHF99], [BHSF02], have always to restrict themselves to particular class of DSs (in the previous case to dissipative systems that converge to a fixed point).

Instead, we believe that the best approach is the one followed in the control theory community, where only problems for specific classes of DSs are studied: linear, sigmoidal, etc., and only where we can formalize exactly what one is searching. For instance, we can try to prove stability for a certain model, by showing that it is robust under the influence of Gaussian noise. Some may argue that this is not realistic modelling, but as we have discussed previously, it's difficult, if not impossible, to give a good definition of "robustness". Only real-world evidence can provide better arguments for claiming that definition A is better than definition B.

Despite the inherent limitations concerning DSs, we can still provide more or less general results from a computational perspective, provided we restrict ourselves to portions of the theory which are well understood.

For instance ODEs model (smooth) continuous-time DSs, and there are some general results which applies to ODEs, namely the general existence-uniqueness theory. We now review some results that were obtained along that line of research. We first need to recall some basic definitions. Here we will mainly deal with the case where $E \subseteq \mathbb{R}^{m+1}$. Hence, when considering a function $f : E \to \mathbb{R}^m$ with argument (t, x), we refer to $t \in \mathbb{R}$ as the first argument and $x \in \mathbb{R}^m$ as the second argument of f.

Definition 2 *Let $E \subseteq \mathbb{R}^{m+1}$ be an open set. A function $f : E \to \mathbb{R}^m$ is called locally Lipschitz in the second argument, on E, if for every compact set $\Lambda \subseteq E$ there is a constant $K_\Lambda \geq 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq K_\Lambda \|x - y\|, \quad \text{for all } (t,x), (t,y) \in \Lambda.$$

The following classical lemma [Hal80] asserts that C^1 functions are locally Lipschitz, and hence locally Lipschitz in the second argument.

Lemma 3 *If $f : E \to \mathbb{R}^m$ is of class C^1 over $E \subseteq \mathbb{R}^l$, then f is locally Lipschitz on E.*

The following result follows as an immediate consequence of the fundamental existence-uniqueness theory for the initial-value problem

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

[CL55], [Lef65], [Hal80] (the expressions $t \to \alpha^+$ and $t \to \beta^-$ mean that t converges to α from above and to β from below, respectively). Basically, it asserts the existence of a maximal interval (in the sense described in the proposition) on which the solution of an initial-value problem with ODEs can be defined.

Proposition 4 *Let E be an open subset of $\mathbb{R}^{m+1}$ and assume that $f : E \to \mathbb{R}^m$ is continuous on E and locally Lipschitz in the second argument. Then for each $(t_0, x_0) \in E$, the problem (1) has a unique solution $x(t)$ defined on a maximal interval (α, β), on which the solution is C^1. The maximal interval is open and has the property that, if $\beta < +\infty$ (resp. $\alpha > -\infty$), either $(t, x(t))$ approaches the boundary of E or $x(t)$ is unbounded as $t \to \beta^-$ (resp. $t \to \alpha^+$).*

Note that, as a particular case, when $E = \mathbb{R}^{m+1}$ and $\beta < \infty$, $x(t)$ is unbounded as $t \to \beta^-$.

We can now ask several questions: given some initial-value problem defined with an ODE, and using only computable data:

1. Will a solution of such initial-value problem (IVP) be computable in the maximal interval?

2. Will the maximal interval itself be computable?

3. Can we at least decide if the maximal interval is bounded or not?

For the first question there are results by Aberth [Abe70], [Abe71], Pour-El and Richards [PER79] that show that if the solution of the IVP is not unique, than it may happen that none of the solutions is computable. There are some differences between the previous two results (Aberth uses constructive methods to study computability. His functions are rational-valued of rational inputs, and so they are not a subclass of real functions. On the other hand, classic reasoning and nonconstructive methods are used by Pour-El and Richards to study computability of functions and operators in analysis, and so the computable functions are a special case of class of real functions), but both show that the problems stem from non-uniqueness of the solution. If the solution is required to be unique, it is computable in any compact set (see any book about the classical theory of ODEs, e.g. [Hal80]). But what about in the maximal interval, which may be non-compact? Results by Moore [DM70], [Moo01], Graça, Zhong, and Buescu [GZB07] show that if the ODE satisfies some (effective) locally Lipschitz condition, then the solution of the IVP (which is unique by the classical theory) is also computable in the *whole* maximal interval.

For the second question the answer is negative, even if the require the ODE to be analytic, as shown in [GZB07]. However it can also be shown that if the ODE satisfies some effective locally Lipschitz condition, then the maximal interval is recursively enumerable.

Finally, for the third question, the answer is also negative even if we require the ODE to be polynomial [GZB07], [GBC07]. We can strengthen this condition to show that the undecidability remains if we consider the class polynomials of degree 56 (or higher).

If we continue to restrict to classical properties of dynamical systems which are more or less well understood, other interesting results were also obtained by Braverman and Yampolsky [BY06]. They have shown that Julia sets, many times used to draw beautiful figures of fractals in computers, can be non-computable even if we stick to the family of quadratic polynomials. This shows that not all fractals can be drawn by computers, even if we discard complexity considerations. However, up to our knowledge, there is no proof that, if we restrict ourselves to *computable* quadratic polynomials, we could also get non-computable Julia sets. This is an interesting question which certainly deserves more attention.

Different approaches have been used to compute classical quantities associated to classical dynamical systems like shifts [Moo90]. For instance in [Spa07] it is shown that the topological entropy can be computed for shift-like dynamical systems.

From our point of view, a direction for further research that might yield some light about the interplay between dynamical systems and the theory of computation is the notion of *shadowing*. Most ODEs cannot be solved directly, and we depend on computer simulations to try to understand the underlying DS. However, the numerical simulation is fraught with truncation errors introduced by the discretization and round-off errors introduced by finite-precision calculation. So a natural question, especially for chaotic DSs is: how reliable are these simulations? One might expect that they only accurately simulate realistic trajectories for short periods of time.

However previous studies [Ano69], [Bow75] have shown the existence of a shadowing property, at least for hyperbolic systems, which states that given a numerical trajectory and some $\varepsilon > 0$, there will always be a "true" trajectory within distance ε from the numerical one, for arbitrarily long times, which is known as the shadowing lemma. While the shadowing lemma is an important result, it is nor widely applicable because many dynamical systems are not hyperbolic. Nonetheless, it can be shown to hold even in non-hyperbolic system, provided some conditions hold (see [Pil99], [GPS$^+$02] and references therein). However, this is not true for all DSs [DGSY94]. A question of interest is the following one from Viana [Via01]: "Is there a useful shadowing lemma for very general non-uniformly hyperbolic systems?"

Another interesting question is: how meaningful are the trajectories computed with the use of the computer? They might get close only to "non-representative" trajectories and thus miss the main features of the DS. If one can prove that computed trajectories are representative of the DS, then a problem that could somehow be related to the shadowing lemma is the following one:

> **Hilbert's sixteenth problem.** Determine an upper bound for the number of limit cycles in polynomial vector fields of order n and investigate their relative positions.

In general, it is known that systems with infinitely many periodic attractors do exist [New79], though nothing is known for polynomial IVPs. As Viana puts it [Via01]: "Nevertheless, three decades after the initial results, this phenomenon remains essentially as little understood as ever. It is not even known whether coexistence of infinitely many periodic attractors may occur *robustly*, that is, for a whole open set of systems ... But, in light of the conjectures mentioned above one does not expect this to be possible. Even more: does coexistence of infinitely many attractors correspond to a zero measure set in the parameter space ...?"

If for polynomial vector fields one may only have a finite number of limit cycles (except, perhaps, for a subset of polynomial vector fields of measure zero) and if the shadowing property provide representative behavior of the underlying DS, one could imagine solving Hilbert's sixteenth problem in the following manner. Construct an algorithm such that given the description of a polynomial vector field, it computes some precision needed to cover the phase space and to have enough representatives of pseudo-trajectories that will approach all attractors. Using these (finite number) of trajectories, one will be able to give an upper bound for the number of limit cycles in polynomial vector field and to know which are their relative positions.

In some sense, in the previous construction, we have used ideas from computational complexity. We implicitly expected that the more "computational information" is need to describe the DS (the length of the description), the more complex can be its behavior. So, if such algorithm exist, we could derive a *computable upper bound* $g(n)$ for the number of limit cycles of a polynomial vector field, where n is not the order of the polynomial vector field as demanded by Hilbert, but rather the *length* of its description.

It would certainly be quite interesting to derive such function g, or to show that it does not exist.

Acknowledgments

The author wishes to thank J. Buescu, M. L. Campagnolo, N. Zhong and O. Bournez for useful discussions. The author was partially supported by *Fundação para a Ciência e a Tecnologia* and EU FEDER POCTI/POCI via CLC, project ConTComp POCTI/MAT/45978/2002, grant SFRH/BD/17436/2004, and within the initiative RealNComp of SQIG - IT. Additional support was also provided by the *Fundação Calouste Gulbenkian* through the *Programa Gulbenkian de Estímulo à Investigação.*

References

[AB01] E. Asarin and A. Bouajjani. Perturbed Turing machines and hybrid systems. In *Logic in Computer Science, 2001. Proc. 16th Annual IEEE Symposium*, pages 269–278, 2001.

[Abe70] O. Aberth. Computable analysis and differential equations. In A. Kino, J. Myhill, and R.E. Vesley, editors, *Intuitionism and Proof Theory*, Studies in Logic and the Foundations of Mathematics, pages 47–52. North-Holland, 1970.

[Abe71] O. Aberth. The failure in computable analysis of a classical existence theorem for differential equations. *Proc. Amer. Math. Soc.*, 30:151–156, 1971.

[Ano69] D. V. Anosov. Geodesic flows on closed riemannian manifolds with negative curvature. *Proc. Steklov Inst. Math. (english translation from the original in Russian)*, 90:1–235, 1969.

[AP37] A. A. Andronov and L. Pontryagin. Systèmes grossiers. *Dokl. Akad. Nauk. SSSR*, 14:247–251, 1937.

[AYYK92] J. Alexander, J. A. Yorke, Z. You, and I. Kan. Riddled basins. *Int. J. Bif. Chaos*, 2:795–813, 1992.

[BHSF02] A. Ben-Hur, H. T. Siegelmann, and S. Fishman. A theory of complexity for continuous time systems. *J. Complexity*, 18(1):51–86, 2002.

[Bow75] R. Bowen. ω-limit sets for axiom A diffeomorphisms. *J. Differential Equations*, 18:333–339, 1975.

[BS02] M. Brin and G. Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, 2002.

[Bue97] J. Buescu. *Exotic Attractors: from Liapunov Stability to Riddled Basins*. Birkhäuser, 1997.

[BY06] M. Braverman and M. Yampolsky. Non-computable Julia sets. *J. Amer. Math. Soc.*, 19(3):551–0578, 2006.

[CL55] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. McGraw-Hill, 1955.

[DGSY94] S. Dawson, C. Grebogi, T. Sauer, and J. A. Yorke. Obstructions to shadowing when a Lyapunov exponent fluctuates about zero. *Phys. Rev. Lett.*, 73(14):1927–1930, 1994.

[DM70] J. W. Daniel and R. E. Moore. *Computation and Theory in Ordinary Differential Equations*. W. H. Freeman & Co Ltd, 1970.

[Frä99] M. Fränzle. Analysis of hybrid systems: An ounce of realism can save an infinity of states. In J. Flum and M. Rodríguez-Artalejo, editors, *Computer Science Logic (CSL'99)*, LNCS 1683, pages 126–140. Springer, 1999.

[GBC07] D. S. Graça, J. Buescu, and M. L. Campagnolo. Boundedness of the domain of definition is undecidable for polynomial ODEs. In R. Dillhage, T. Grubba, A. Sorbi, K. Weihrauch, and N. Zhong, editors, *Proceedings of the 4th International Conference on Computability and Complexity in Analysis (CCA 2007)*, pages 127–135. FernUniversität in Hagen, 2007.

[GH83] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*. Springer, 1983.

[GOY87] C. Grebogi, E. Ott, and J. A. Yorke. Basin boundary metamorphoses: changes in accessible boundary orbits. *Physica D*, 24:243–262, 1987.

[GPS+02] C. Grebogi, L. Poon, T. Sauer, J. Yorke, and D. Auerbach. *Handbook of Dynamical Systems*, volume 2, chapter Shadowability of chaotic dynamical systems, pages 313–344. Elsevier, 2002.

[GZB07] D.S. Graça, N. Zhong, and J. Buescu. Computability, noncomputability and undecidability of maximal intervals of IVPs. *Trans. Amer. Math. Soc.*, 2007. To appear.

[Hal80] J. K. Hale. *Ordinary Differential Equations*. Robert E. Krieger Pub. Co, 2nd edition, 1980.

[HR99] T. Henzinger and J.-F. Raskin. Robust undecidability of timed and hybrid systems. In F. Vaandrager and J. H. van Schuppen, editors, *Hybrid systems: computation and control; Second International Workshop, (HSCC'99)*, LNCS 1569. Springer, 1999.

[HS74] M. W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, 1974.

[HW95] J. H. Hubbard and B. H. West. *Differential Equations: A Dynamical Systems Approach — Higher-Dimensional Systems*. Springer, 1995.

[KB95] A. Katok and B.Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[Lef65] S. Lefshetz. *Differential Equations: Geometric Theory*. Interscience, 2nd edition, 1965.

[Lor63] E. N. Lorenz. Deterministic non-periodic flow. *J. Atmos. Sci.*, 20:130–141, 1963.

[Mil85] J. Milnor. On the concept of attractor. *Comm. Math. Phys.*, 99(2):177–195, 1985.

[Moo90] C. Moore. Unpredictability and undecidability in dynamical systems. *Phys. Rev. Lett.*, 64(20):2354–2357, 1990.

[Moo01] R. E. Moore. Interval analysis: systems of nonlinear equations; interval analysis: differential equations. In *Encyclopedia of Optimization*, pages 8–11, 34–40, 2001.

[New79] S. Newhouse. The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms. *Publ. Math IHES*, 50:101–151, 1979.

[OSA+94] E. Ott, J. C. Sommerer, J. C. Alexander, I. Kan, and J. A. Yorke. The transition to chaotic attractors with riddled basins. *Physica D*, 76:384–410, 1994.

[Ott02] E. Ott. *Chaos in Dynamical Systems*. Cambridge University Press, 2nd edition, 2002.

[PER79] M. B. Pour-El and J. I. Richards. A computable ordinary differential equation which possesses no computable solution. *Ann. Math. Logic*, 17:61–90, 1979.

[Pil99] S. Y. Pilyugin. *Shadowing in Dynamical Systems*. Springer, 1999.

[Rue89] D. Ruelle. *Chaotic Evolution and Strange Attractors*. Cambridge University Press, 1989.

[SBHF99] H. T. Siegelmann, A. Ben-Hur, and S. Fishman. Computational complexity for continuous time dynamics. *Phys. Rev. Lett.*, 83(7):1463–1466, 1999.

[Sie99] H. T. Siegelmann. *Neural Networks and Analog Computation: Beyond the Turing Limit*. Birkhäuser, 1999.

[Sma66] S. Smale. Structurally stable systems are not dense. *Amer. J. Math.*, 88:491–496, 1966.

[Spa07] C. Spandl. Computing the topological entropy of shifts. *Math. Log. Quart.*, 53(4-5):493–510, 2007.

[Via00] M. Viana. What's new on Lorenz strange attractors? *Math. Intelligencer*, 22(3):6–19, 2000.

[Via01] M. Viana. Dynamical systems: Moving into the next century. In B. Engquist and W. Schmid, editors, *Mathematics Unlimited - 2001 and Beyond*, pages 1167–1178. Springer, 2001.