An ordinary differential equation defined by a computable function whose maximal interval of existence is non-computable

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Abstract. Let $(\alpha, \beta) \subset \mathbb{R}$ denote the maximal interval of existence of solution for the initial-value problem

$$\begin{cases} \frac{dx}{dt} = f(t, x), & f : E \to \mathbb{R}^m, E \text{ is an open subset of } \mathbb{R}^{m+1} \\ x(t_0) = x_0, & \text{with } (t_0, x_0) \in E. \end{cases}$$

We show that (α, β) is r.e. (recursively enumerable) open and the solution $x(t)$ defined on (α, β) is computable, provided that (a) f is computable and effectively locally Lipschitz, and (b) (t_0, x_0) is a computable point. We also prove that this result is the best in the sense that, for some initial-value problems satisfying (a) and (b), their maximal intervals of existence are non-recursive.

1 Introduction

Differential equations are fundamental in modeling physical processes, including nonlinear systems of ordinary differential equations (ODEs for short) $\dot{x} = f(t, x)$, where $f : E \to \mathbb{R}^n$, E is an open subset of $\mathbb{R}^{n+1}$, $x = x(t)$ is a function of t, and $\dot{x}$ denotes the derivative of x with respect to t. The well-posedness of such systems has long been established in the fundamental existence-uniqueness theorem [CL55], which states that, under appropriate regularity conditions for f, the initial value problem

$$\dot{x} = f(t, x(t)), \; x(t_0) = x_0 \tag{1}$$

(for short, we sometimes write $f(t, x)$ instead of $f(t, x(t))$) has a unique solution $x(t)$ defined on a maximal interval of existence $(\alpha, \beta) \subset \mathbb{R}$. In general, however, it is not possible to solve the above nonlinear system analytically with an explicit solution formula. Many of those nonlinear systems can only be solved numerically on computers and indeed computers are playing an ever larger role in studying differential equations. Numerical methods are usually tailor-made for individual problems and often depend on certain assumptions, for example, the existence of some time interval where the solution is defined. This requirement is crucial but in general hard to verify. In practice, there are several approaches to deal with this problem. One possible approach is to take some insight from the physical counterpart of the ODE. For example, if an ODE is intended to model the orbit of the Earth around the Sun, then we may assume that the solution is defined for $t \in [t_0, \infty)$. However, from a mathematical point of view, this certainly is not a very satisfactory solution.

The satisfactory solution of course is to have an "automated method" that determines the maximal interval (α, β) and computes the solution on (α, β) from the data defining the IVP (1). Thus, it becomes useful to know whether it is possible to derive such "automated method"? In this note, we present a negative answer to the question. We show that the maximal interval where the solution of the IVP (1) is defined may not be recursive, even when the function f is computable and of class C^∞. In such circumstances, there is no algorithm to decide whether or not $[t_0, t]$ is contained in (α, β) from the information that $t_0 \in (\alpha, \beta)$ for arbitrary $t \geq t_0$. The undecidability indicates that the limit behavior of the IVP (1) may not be determined by "general numerical recipes". In other words, such undecidability suggests possible limitations concerning numerical methods for solving ODEs.

There are other noncomputability results related to the initial value problems of differential equations. For example, Pour-El and Richards [PER79] showed that the IVP (1) defined with computable data may have noncomputable solutions. In [PER81], [WZ02] it is shown that there is a three-dimensional wave equation, defined with computable data, such that the unique solution is nowhere computable. However, in these examples, noncomputability is not "genuine" in the sense that the problems in the study are ill-posed: either the solution is not unique or the solution is not stable. In other words, ill-posedness generated noncomputability in those examples. In contrast, all IVPs studied in this note are classically well-posed. For reference we also mention the existence of other results about computability of ODEs that can be found in [Abe70], [Abe71], [BB85], [Ko91], [Ruo96].

The computational model used in this paper is the Turing machine-based "bit" model [PER89], [Ko91], [Wei00]. This approach is based on the classical theory of computability, where an approximation of the output with arbitrary precision is computed from a suitable approximation of the input.

The paper is organized as follows. Section 2 introduces necessary concepts and results from computable analysis and the theory of ODEs. Section 3 presents a theorem stating that the maximal interval $(\alpha, \beta) \subset \mathbb{R}$ of (1), where the solution is defined, is recursively enumerable and the solution is computable there, if the data defining the initial value problem is computable. Section 4 provides a counterexample showing that (α, β) is r.e. but non-computable. Due to the page limit, proofs are either sketchy or omitted.

2 Preliminaries

This section introduces necessary concepts and results from computable analysis and from the theory of ODEs. For more details the reader is referred to [PER89], [Ko91], [Wei00] for computable analysis and [CL55], [Lef65] for ODEs. The idea underlying these definitions is as follows. Consider the number π. This should be a "computable number" from an intuitive point of view, since we can design an algorithm that gives us any number of digits of its decimal expansion. This was the idea of Turing in his seminal paper [Tur36]. However, as he soon recognized, the decimal expansion is not adequate to define computable real functions (for instance, it can be shown [Wei00] that the function $x \mapsto 3x$ is not computable in this framework). Instead, we need a different approach. The classical procedure is based on the following idea. It is known that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$. Hence, each real can be approximated by a sequence of rational numbers. Therefore, if one can find an algorithm that, given some precision 2^{-n} as input (i.e. we want the output to have n significant bits), gives a rational q satisfying $|q - x| < 2^{-n}$, one says that $x \in \mathbb{R}$ is computable. We now precise this and other notions.

Definition 1. *1. A sequence $\{r_n\}$ of rational numbers is called a ρ-name of a real number x if there are four functions a, b, c, d from $\mathbb{N}$ to $\mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers including 0, such that for all $n \in \mathbb{N}$, $r_n = (-1)^{a(n)} \frac{b(n)}{c(n)+1}$ and*

$$j > d(n) \quad \Rightarrow \quad |r_j - x| \leq \frac{1}{2^n}. \tag{2}$$

2. A real number x is called computable if a, b, c and d are computable (recursive) functions.

3. A sequence $\{x_k\}_{k\in\mathbb{N}}$ of real numbers is computable if there are four computable functions a, b, c, d from $\mathbb{N}^2$ to $\mathbb{N}$ such that, for all $k, n \in \mathbb{N}$,

$$j > d(k,n) \quad \Rightarrow \quad \left|(-1)^{a(j,k)} \frac{b(j,k)}{c(j,k)+1} - x_k\right| \leq \frac{1}{2^n}.$$

Similarly, we can define computable points and sequences over $\mathbb{R}^m$, $m > 1$, by assuming that each component is computable. Next we present a notion of computability for open and closed subsets of $\mathbb{R}^m$, which can be found in [Wei00].

Definition 2. *1. An open set $E \subseteq \mathbb{R}^m$ is called recursively enumerable (r.e. for short) open if there are computable sequences $\{a_n\}$ and $\{r_n\}$, $a_n \in E \cap \mathbb{Q}^m$ and $r_n \in \mathbb{Q}$ such that*

$$E = \cup_{n=0}^{\infty} B(a_n, r_n)$$

and for any $n \in \mathbb{N}$, the closure of $B(a_n, r_n)$, denoted as $\overline{B(a_n, r_n)}$, is contained in E, where $B(a_n, r_n) = \{x \in \mathbb{R}^m : |x - a_n| < r_n\}$.

2. A closed subset $K \subseteq \mathbb{R}^m$ is called r.e. closed if there exist computable sequences $\{b_n\}$ and $\{s_n\}$, $b_n \in \mathbb{Q}^m$ and $s_n \in \mathbb{Q}$, such that $\{B(b_n, s_n)\}_{n\in\mathbb{N}}$ lists all rational balls intersecting K.

3. *An open set $E \subseteq \mathbb{R}^m$ is call computable (or recursive) if E is r.e. open and its complement is r.e. closed.*

It is well known that a bounded open interval $(\alpha, \beta) \subset \mathbb{R}$ is computable if and only if α and β are computable real numbers. Throughout the paper we assume that $E \subseteq \mathbb{R}^{m+1}$ is r.e. open, $\{a_n\}$, $\{r_n\}$ and $B(a_n, r_n)$ are defined as above. Let $C^k(E)$ denote the set of all continuously differentiable, up to order k, functions defined on E.

Definition 3. *A function $f : E \to \mathbb{R}^m$ is computable if there is a Type-2 Turing machine which translates any input ρ-name of $x \in E$ to a ρ-name of $f(x)$. Or equivalently, there is an oracle Turing machine such that for any input $n \in \mathbb{N}$ (accuracy) and any ρ-name of $x \in E$ given as an oracle, the machine will output a rational number r satisfying $|r - x| \leq 2^{-n}$.*

We recall that a Type-2 Turing machine is an ordinary Turing machine that allows infinite sequences of symbols from a finite alphabet as input as well as output (see, for example, [Wei00] for more details). It can be proved that if f is computable on E, then there exists a computable modulus function $e : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which is locally effective in the sense that $|f(x) - f(y)| \leq 2^{-k}$ whenever $x, y \in \cup_{j=0}^{n} \overline{B(a_j, r_j)}$ and $|x - y| \leq 2^{-e(k,n)}$. In particular, this implies that f must be continuous.

Recall that a function $f : E \to \mathbb{R}^m$ is said to be locally Lipschitz if f satisfies a Lipschitz condition on every compact set $V \subset E$. The following definition gives a computable analysis analog of the Lipschitz condition.

Definition 4. *Let $E = \cup_{n=0}^{\infty} B(a_n, r_n)$ be a r.e. open set. A function $f : E \to \mathbb{R}^m$ is called effectively locally Lipschitz on E if there exists a computable sequence $\{K_n\}$ of positive integers such that*

$$|f(x) - f(y)| \leq K_n \, |x - y| \ \ \textit{whenever } x, y \in \overline{B(a_n, r_n)}.$$

Definition 5. *Let $E = \cup_{n=0}^{\infty} B(a_n, r_n)$ be a r.e. open set. A function $f : E \to \mathbb{R}^m$, with $E \subseteq \mathbb{R}^{j+1}$ is effectively locally Lipschitz on the last j variables if there exists a computable sequence $\{K_n\}$ of positive integers such that if t denotes the first variable of f, then*

$$|f(t, x) - f(t, y)| \leq K_n \, |y - x| \ \ \textit{whenever } (t, x), (t, y) \in \overline{B(a_n, r_n)}.$$

Notice that an effectively locally Lipschitz function $f : E \to \mathbb{R}^m$, where $E \subseteq \mathbb{R}^{m+1}$, is also effectively locally Lipschitz on the m last variables. Also it is clear that if $f \in C^1(E)$ then it is locally Lipschitz on E, so that continuous differentiability is a strictly stronger condition than being locally Lipschitz. This fact extends to computable functions in the following way.

Theorem 1. *Assume that E is r.e. open and that $f : \mathbb{R}^m \to \mathbb{R}^k$ is a computable function in $C^1(E)$ (meaning that both f and its derivative f' are computable). Then f is effectively locally Lipschitz on E.*

Proof. Let K_n be an integer greater or equal to $\max_{x\in\overline{B(a_n,r_n)}}|f'(x)|$. Since f', a_n, r_n are computable, the real number $\max_{x\in\overline{B(a_n,r_n)}}|f'(x)|$ is also computable (notice that, because f' is computable, it has a modulus of continuity that can be used to get the maximum over $\overline{B(a_n,r_n)}$ within any preassigned precision). Moreover, since f' has a locally effective modulus of continuity, subsequently we may assume that the sequence $\{K_n\}$ is a computable sequence of positive integers. Now, for any $x, y \in \overline{B(a_n,r_n)}$, let $u = y - x$. Then $x + su \in \overline{B(a_n,r_n)}$ for $0 \le s \le 1$ because $\overline{B(a_n,r_n)}$ is a convex set. Define $F : [0,1] \to \mathbb{R}^n$ by $F(s) = f(x + su)$. Then by the chain rule,

$$F'(s) = f'(x + su) \cdot u = f'(x + su) \cdot (y - x).$$

Therefore,

$$|f(x)-f(y)| = |F(1)-F(0)| = \left|\int_0^1 F'(s)ds\right| \le \int_0^1 |f'(x+su)\cdot(y-x)|ds \le K_n|x-y|.$$

Next we turn our attention to some results concerning initial value problems defined with ODEs. Let us consider the following initial value problem

$$\begin{cases} \dot{x} = f(t,x), \\ x(t_0) = x_0, \end{cases} \tag{3}$$

where $(t_0, x_0) \in E \subset \mathbb{R}^{m+1}$ and $f : E \to \mathbb{R}^m$ is a continuous function and satisfies a local Lipschitz condition in the second variable. The following is an immediate consequence of the fundamental existence-uniqueness theory for the initial value problem (3) [CL55], [Lef65].

Theorem 2 (Maximal interval of existence). *Let E be an open subset of $\mathbb{R}^{m+1}$ and assume that $f : E \to \mathbb{R}^m$ is continuous and locally Lipschitz in the second argument. Then for each $(t_0, x_0) \in E$, the problem (3) has a unique solution $x(t)$ defined on a maximal interval (α, β) with the following property: if $\beta < \infty$, either $(t, x(t))$ approaches the boundary of E, or $x(t)$ is unbounded as $t \to \beta^-$ (similar conditions hold for α).*

3 Computability of the maximal interval

Theorem 3. *Let $E \subseteq \mathbb{R}^{m+1}$ be a r.e open set and $f : E \to \mathbb{R}^m$ be a computable function that is also effectively locally Lipschitz on the last m variables. Let (α, β) be the maximal interval of existence of the solution $x(t)$ of the initial-value problem (3) where (t_0, x_0) is a computable point in E. Then (α, β) is a r.e. open interval and x is a computable function on (α, β).*

Proof. (Sketch) We consider the right maximal interval (t_0, β) and prove (t_0, β) is r.e. open and x is computable on it. The same argument applies to the left maximal interval (α, t_0). For simplicity, we assume that E is an open subset of 2-dimensional Euclidean space $\mathbb{R}^2$.

Since $\{a_n\}$ and $\{r_n\}$ are computable sequences and f is a computable function on E, both sequences $\{M_n\}$, $M_n = \max_{z \in \overline{B(a_n, r_n)}} |f(z)|$, and $\{K_n\}$, as defined in Def. 5 are computable. Then, using the classical proof of the existence of the solution of a given ODE (*cf.* [CL55], [Lef65]), one can use the following algorithm to compute the maximal interval

1. Set $n = 0$
2. Compute an index l_n such that $x_n \in B(a_{l_n}, r_{l_n})$
3. Compute a time interval $[t_n, t_{n+1}]$ where the solution of $\dot{x} = f(t, x)$, $x(t_n) = x_n$ is defined
4. Set $x_{n+1} = x(t_{n+1})$ and increment n
5. Go to step 2

Notice that the time interval $[t_n, t_{n+1}]$ referred to in step 3 can be obtained from the proof of the existence of the solution of a given ODE. For instance, one can take $t_{n+1} = t_n + \min\{2^{-K_{l_n}}/M_{l_n}, 2^{-K_{l_n}}\}$ [CL55], [Lef65]. Then it is possible to show, by a contradiction argument, that the maximal interval is given by $(t_0, \beta) = \cup_{n=0}^{\infty}(t_0, t_n)$, i.e. that $t_n \to \beta$ as $n \to \infty$. The idea is the following. If t_n does not converge to β, it must converge to some $\gamma < \beta$. Then, for some index $j \in \mathbb{N}$, one has $(\gamma, x(\gamma)) \in B(a_j, r_j)$. Since t_n converges increasingly to γ and it is known classically that $x : [t_0, \beta) \to E$ is continuous, one can find a sufficiently large n_0 such that $(t_{n_0}, x(t_{n_0})) \in B(a_j, r_j)$ and $t_{n_0} + \min\{2^{-K_j}/M_j, 2^{-K_j}\} > \gamma$. But, by construction, $t_{n_0+1} = t_{n_0} + \min\{2^{-K_j}/M_j, 2^{-K_j}\}$. Thus $t_{n_0+1} > \gamma$. We have a contradiction.

The solution x is computable because, given $t \in (t_0, \beta)$, we can: (i) get an index $n \in \mathbb{N}$ such that $t \leq t_n$ (ii) compute numerically the solution over $[t_0, t_n]$ to get the value of $x(t)$.

4 Non-recursiveness of the maximal interval

In this section, we present some undecidability results concerning ODEs. In particular, for the initial-value problem (3), we show that the maximal interval can be non-computable. Although our result is for the case where f is continuous (more precisely, f is piecewise linear), nevertheless, the construction can be "smoothed" so that f becomes C^{∞}. Moreover, an explicit expression can be written for such an f. The construction depends on the following lemma, which can also be used to prove other results concerning undecidability.

Lemma 1. *Let $a : \mathbb{N} \to \mathbb{N}$ be a computable function. Then there exists a computable and effectively locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that the unique solution of the problem*

$$\dot{x} = f(x), \;\; x(0) = 0 \tag{4}$$

is defined on a maximal interval $(-\alpha, \alpha)$ with

$$\alpha = \sum_{i=0}^{\infty} \frac{1}{2^{a(i)}}.$$

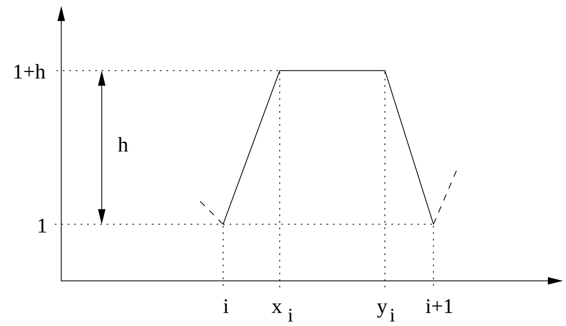

Fig. 1. Sketch of the function f on the interval $[i, i+1]$, for $i \in \mathbb{N}$.

Proof. (Sketch) The idea is as follows: f is constructed piecewisely on intervals of the form $[i, i+1]$, $i \in \mathbb{N}$ (for negative values, we take $f(x) = f(|x|)$) in such a way that the solution of the initial-value problem

$$\dot{x} = f(x), \ x(0) = i \tag{5}$$

satisfies $x(2^{-a(i)}) = i+1$, which implies that the solution of the problem $\dot{x} = f(x)$ and $x(0) = 0$ will satisfy $x(2^{-a(0)}) = 1$, $x(2^{-a(0)} + 2^{-a(1)}) = 2$, ..., or more generally

$$x\left(\sum_{i=0}^{n} 2^{-a(i)}\right) = n + 1, \quad \text{for all } n \in \mathbb{N}.$$

Notice that f does not depend on t and therefore the solution is invariant under time translations. If we take $\alpha = \sum_{i=0}^{\infty} 2^{-a(i)}$, then $x(t) \to \infty$ as $t \to \alpha^-$. For $t < 0$, we require that $x(-2^{-a(i)}) = -(i+1)$, then $x(t) \to -\infty$ as $t \to -\alpha^+$. Therefore the maximal interval must be $(-\alpha, \alpha)$.

The function f is defined on each interval $[i, i+1]$ as suggested by Fig. 1. Since f must be continuous, we need to glue the values of f at the endpoints of these intervals. This is achieved by assuming that $f(i) = 1$ for $i \in \mathbb{N}$. It can be shown that if we define the function f on the interval $[i, i+1]$ as follows:

$$f(x) = \begin{cases} 1 + (x - i)2^{a(i)}/(x_i - i) & \text{if } x \in [i, x_i) \\ 1 + 2^{a(i)} & \text{if } x \in [x_i, y_i), \\ 1 + 2^{a(i)} - (x - y_i)2^{a(i)}/(i + 1 - y_i) & \text{if } x \in [y_i, i+1), \end{cases}$$

where

$$x_i = i + \frac{1 - \Delta_i}{2}, \qquad y_i = i + \frac{1 + \Delta_i}{2},$$

and

$$0 < \Delta_i = \frac{2^{-a(i)} - 2^{-a(i)} \ln(2^{a(i)} + 1)}{(1 + 2^{a(i)})^{-1} - 2^{-a(i)} \ln(2^{a(i)} + 1)} < 1,$$

the solution of (5) satisfies $x(2^{-a(i)}) = i + 1$ as requested. Moreover, this f is easily seen to be computable and effectively locally Lipschitz.

Theorem 4. *There exists an effectively locally Lipschitz computable function* $f : \mathbb{R} \to \mathbb{R}$ *such that the unique solution of the problem*

$$\dot{x} = f(x), \ x(0) = 0$$

is defined on a non-computable maximal interval.

Proof. In [PER89, Sec. 0.2], it is shown that if $a : \mathbb{N} \to \mathbb{N}$ is a one to one recursive function generating a recursively enumerable nonrecursive set A, then $\alpha = \sum_{i=0}^{\infty} 2^{-a(i)}$ is a non computable real number. Consequently, the open interval $(-\alpha, \alpha)$ is non-computable. The theorem now follows immediately from the previous lemma.

The function f in Theorem 4 can be constructed so that f is of class C^{∞} and all its derivatives are computable functions, and hence f is also effectively locally Lipschitz. This condition matches the assumption set down in Theorem 3. Thus, Theorem 3 gives rise to the best possible result concerning computability of a maximal interval.

5 Conclusion

In this paper we studied some computational issues regarding Initial Value Problems defined with ODEs. In particular we showed that IVPs (1), where f and (t_0, x_0) are recursive can have a nonrecursive maximal interval, thus suggesting some fundamental limitations on the design of numerical methods for solving ODEs. Note that this result is valid for the case when f is of class C^{∞}, but we didn't cover the case where f is analytic. It would be interesting to know what happens in this case.

Acknowledgments. DG wishes to thank Manuel Campagnolo for helpful discussions. DG was partially supported by the *Fundação Calouste Gulbenkian* through the *Programa Gulbenkian de Estímulo à Investigação*, and by the *Fundação para a Ciência e a Tecnologia* and EU FEDER POCTI/POCI via CLC, grant SFRH/BD/17436/2004, and the project ConTComp POCTI/MAT/45978 /2002. NZ was partially supported by University Cincinnati's Taft Summer Research Fellowship. JB was partially supported by *Fundação para a Ciência e a Tecnologia* and EU FEDER POCTI/POCI via CAMGSD.

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