

The robust minimal controllability problem for switched linear continuous-time systems

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Abstract

In this paper, we address the robust minimal controllability problem for switched linear continuous-time systems. The problem is to determine the minimal subset of state variables to actuate such that the switching linear system is controllable, under the scenario where a set of actuators may fail along the time. Two variations of this problem are considered, depending on whether we want to design an input matrix for each mode (*i.e.*, a different set of actuators that may fail in each mode), or if we want to design a common input matrix common across all the modes, and a set of actuators may fail. In both cases, we want to ensure that, given an initial condition, we can drive the system towards any desired state. For both problems, we characterize the sparsest input matrices which ensure that the system is controllable, whenever the autonomous dynamics' matrix of each mode is simple, and for which a left-eigenbasis is available. We reduce these problems to set multi-covering problems, showing that using a sufficient condition for controllability, the first is NP-complete. These allow us to deploy known, close-to-optimal, polynomial algorithms approximating the solutions of the problems we study.

I. INTRODUCTION

Switched systems are paramount in an extensive number of applications, such as control of mechanical systems, process control, automotive industry, power systems, aircraft/traffic control, see for instance [1], [2]. The systems belonging to the subclass of switched systems whose subsystems are described by linear differential equations are called *switched linear systems*. These systems alone consist of a line of research with growing attention [1], and several works aim to study the properties of this class such as controllability, observability and reachability [2]–[5].

Recent works studied controllability under the scope of uncertain switched linear systems, where the state matrices' entries of each mode are only known to be zero or non-zero [6]. A switched linear system is said to be structurally controllable whenever there exists a numerical realization of the non-zero entries of the state matrices leading to a controllable switched linear system. In [7], the authors introduced a framework to model check structural properties of switched linear systems, and propose its use to check the structural controllability of each subsystem. In [8], the authors addressed the structural minimal controllability problem for switched linear continuous-time systems, finding the minimum number of inputs that need to be considered to attain structural controllability of the system. However, the state matrices' entries may be linearly dependent, and the system is structurally controllable but not controllable by the same set of actuators. In contrast, in this paper, we propose to address the scenario where we have knowledge of the state matrices entries and that these matrices are simple. We aim to ensure the controllability of the system, the ability to drive the system from an initial state to the desired state, extending the results in [9].

We assume that either we have access to common transitions and knowledge of the existing modes of the switching system, or that the controller is equipped with supervisory capabilities enabling the system to switch between modes, as considered in same engineering applications as in [8], [10]. More specifically, given a switched linear system with continuous time, we address the problem of finding

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the minimum number of inputs/actuators and state variables we need to actuate, ensuring the system's controllability under two scenarios (when a specified number of inputs may fail):

(i) design an input matrix, for each system's mode, that controls the system actuating a small number of state variables;

(ii) design a common input matrix that controls the system actuating a small number of state variables.

Main contributions of this paper consist in addressing (i) and (ii) while providing insights on how the obtained solutions can be exploited to improve the computational complexity of the proposed algorithms. We reduce both problems to the well studied set multi-covering problem [11]. Also, we show that (i) is NP-complete when we use a sufficient condition for controllability. These results allow us to use known polynomial complexity algorithms that approximate the set multi-covering problem, to get approximations for (i) and (ii). However, due to the combinatorial nature of (ii), only (i) may be approximated in polynomial time with those approximation algorithms.

Paper structure. Section II states the problems we aim to solve and Section IV illustrates the main results with examples. Finally, Section V concludes the paper.

A. Notation

We denote vectors by small font letters (e.g. v, w and b), matrices by capital letters (e.g. A, B and C), and their entries by one and two subscripts, respectively. We designate sets by calligraphic letters (e.g. \mathcal{S}, \mathcal{I} , and \mathcal{J}), and the number of elements of a set \mathcal{S} by $|\mathcal{S}|$. We represent a collection of vectors by $\{v^j\}_{j \in \mathcal{J}}$ or $\{\bar{v}_i^j\}_{\substack{i \in \mathcal{I} \\ j \in \mathcal{J}}}$, depending on the number of indices used to enumerate the elements of the collection. We use square brackets in vectors or matrices, to separate an enumeration of those from their entries, e.g., $[B_k]_{i,j}$ stands for the j th column of the i th row of the k th matrix of an enumeration B_k , or $[v_i^j]_k$ stands for the k th entry of the vector v_i^j . Given a matrix A , we denote by $\sigma(A)$ its *spectrum*, i.e., the set of its eigenvalues. If $\mathcal{I} = \{i_1, \dots, i_k\}$ and $B \in \{0, 1\}^{n \times m}$, with $m, n \geq k$, $B(\mathcal{I})$ is the matrix where $[B]_{j,i} = 1$ for $i \in \mathcal{I}$ and the remaining entries of B are equal to zero. The structural pattern of a vector v and a matrix A , or a *structural vector and matrix*, is denoted by \bar{v} and \bar{A} , where the entries are 0, or a real entry that we denote by \star . The map $\cdot : \mathbb{C}^n \times \mathbb{C}^n$ is the usual inner product between vectors, i.e., $v \cdot w = v^\dagger w$ the product of v by the conjugate transpose of w . We extend the inner product map $\cdot : \{0, \star\}^n \times \{0, \star\}^n$ to structural vectors, where $\bar{v} \cdot \bar{w} \neq 0$ if and only if there is $i \in \{1, \dots, n\}$ s.t. $\bar{v}_i = \bar{w}_i = \star$. Similarly, given $\bar{u}, \bar{v} \in \{0, \star\}^n$, we extend the plus operation, $+$: $\{0, \star\}^n \times \{0, \star\}^n$, to be $\bar{w} = \bar{u} + \bar{v}$, with $\bar{w}_i = 0$ if $\bar{u}_i = \bar{v}_i = 0$ and $\bar{w}_i = \star$ otherwise, for $i = 1, \dots, n$. The map $\|\cdot\|_0$ counts the number of non-zero entries of vectors and matrices. By considering that $\star > 0$ and $\star = \star$, we extend inequalities to structural vectors. A *multiset* is a set where each element may occur more than once. Consider multisets \mathcal{X} and \mathcal{Y} . The multiset $\mathcal{X} \sqcup \mathcal{Y}$ is such that, if $a \in \mathcal{X}$ or $a \in \mathcal{Y}$ then $a \in \mathcal{X} \sqcup \mathcal{Y}$. If a occurs n_1 times in \mathcal{X} and n_2 times in \mathcal{Y} , then a occurs $\max\{n_1, n_2\}$ times in $\mathcal{X} \sqcup \mathcal{Y}$. Although not very intuitive, this ‘unusual’ union will be useful to address (ii), in Section III.

II. PROBLEMS STATEMENT

Consider a large-scale dynamical system with dynamics modeled by a switched linear continuous-time system (SLCS). Conceptually, we can see a switched linear continuous-time system (SLCS) as a set of linear continuous-time systems (LCS), where each element of the set is called a *mode*, together with a set of discrete events that cause the system to switch between modes. Subsequently, an SLCS for which some actuators may fail, due to either a malicious entity tempering with the actuators or natural phenomena reasons, may be described as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}^{\mathcal{M} \setminus \mathcal{A}_{\sigma(t)}}u(t), \quad (1)$$

where $\sigma : \mathbb{R}^+ \rightarrow \mathbb{M} = \{1, \dots, m\}$ is a piecewise switching signal, that only switches once in a given dwell-time, $x(t) \in \mathbb{R}^n$ the state of the system, and $u(\sigma(t)) \in \mathbb{R}^p$ is a piecewise continuous input signal.

Moreover, $B_{\sigma(t)}^{\mathcal{M} \setminus \mathcal{A}_{\sigma(t)}}$ consists of the subset of columns with indices in $\mathcal{M} \setminus \mathcal{A}_{\sigma(t)}$, the set $\mathcal{M} = \{1, \dots, p\}$ is the set of inputs' labeling indices and $\mathcal{A}_{\sigma(t)}$ the set of indices of affected (*i.e.*, malfunctioning) inputs, for each mode $\sigma(t)$. Additionally, as discussed in the introduction, we focus on the scenario where we have the knowledge of the switching signal, as well as dwell-time, as in [8], [10] and references therein.

To ease the notation, we refer to the system in (1) by the pair $(A_{\sigma(t)}, B_{\sigma(t)}^{\mathcal{M} \setminus \mathcal{A}_{\sigma(t)}})$. Each mode of the system corresponds to the time interval where the switching signal is constant, $\sigma(t) = i$ and $i \in \{1, \dots, m\}$. In other words, it corresponds to an LCS system, which we denote by the pair $(A_i, B_i^{\mathcal{M} \setminus \mathcal{A}_i})$. It is worth noticing that in each mode the dynamics' matrix could have a different dimension. For instance, we may want to model a power system such that some of its components (*e.g.*, the generators) may be working, depending on the mode the system is in. Hence, for a mode where some generators are not working, the dynamics' matrix may be designed with a small number of state variables. Moreover, we can include this behavior taking n as the maximum of the dimensions of each mode's dynamics' matrix A_i , $i \in \mathbb{M}$, and assuming fixed the system variables' order. Hence, when a variable does not play a role in a mode, its dynamics' matrix has zeros in the respective row and column.

Furthermore, from a systems' engineering perspective, we often want to ensure that the systems possess properties such as controllability. The SLCS (1) is controllable if, for any initial state $x(0) = x_0$ and any desired state x_d there exists a time instance $t_f > 0$, a switching signal $\sigma : [0, t_f[\rightarrow \mathbb{M}$ and an input $u : [0, t_f[\rightarrow \mathbb{R}^p$, s.t. $x(t_f) = x_d$. In other words, we can always design a control law that drives the system from an initial state to any desired state in a finite amount of time. Thus, for each mode, an extra set of actuators must be in place to ensure that the system is still controllable if some inputs fail. Besides, due to economic restrictions, *i.e.*, since more actuation capabilities incur in higher cost, it is of utmost importance to deploy the minimum number of actuators that can still control (1), whenever some specified maximum number of actuators may fail for each mode as in (1). We refer to this problem as robust minimal controllability problems for SLCS (rMCPS). Subsequently, given the system (1), the rMCPS can be posed as follows:

Problem 1: Determine matrices $B_1 \in \mathbb{R}^{n \times (s_1+1)n}, \dots, B_m \in \mathbb{R}^{n \times (s_m+1)n}$ that are a solution to the minimization problem

$$\begin{aligned} \min_{B_1, \dots, B_m} \quad & \sum_{i=1}^m \|B_i\|_0 \\ \text{s.t.} \quad & (A_i, B_i^{\mathcal{M} \setminus \mathcal{A}_i}) \text{ is controllable for all } i \in \mathbb{M}, \text{ and} \\ & |\mathcal{A}_i| \leq s_i \text{ and } \mathcal{A}_i \subset \mathcal{M} \text{ for all } i \in \mathbb{M}, \end{aligned} \quad (2)$$

where the dimension of B_i is $n \times (s_i + 1)n$ to guarantee that there exists a solution to the problem. \diamond

Similarly, we can model the case where we want to design a common input matrix, *i.e.*, $B_i = B_j = B$ for all $i, j \in \mathbb{M}$, that controls the SLCS and for which a certain number of inputs may fail, as finding an input matrix $B^{\mathcal{M} \setminus \mathcal{A}}$ such that

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B^{\mathcal{M} \setminus \mathcal{A}}u(t) \quad (3)$$

is controllable. Notice that \mathcal{A} can be seen as $\mathcal{A}_{\sigma(t)}$, *i.e.*, the inputs that may fail are the same across dynamic's switching. Therefore, another problem we are interested in is as follows: given a common actuator placement, *i.e.*, (3), the *common* rMCPS (crMCPS) consists in the following problem.

Problem 2: Determine matrix B from the minimization problem

$$\begin{aligned} \min_{B \in \mathbb{R}^{n \times (s+1)n}} \quad & r \quad \|B\|_0 \\ \text{s.t.} \quad & (A_i, B^{\mathcal{M} \setminus \mathcal{A}}) \text{ is controllable for all } i \in \mathbb{M}, \text{ and} \\ & |\mathcal{A}| \leq s \text{ and } \mathcal{A} \subset \mathcal{M} \text{ for all } i \in \mathbb{M}, \end{aligned} \quad (4)$$

where the dimension of B is $n \times (s + 1)n$ to assure that there exists a solution to the problem. \diamond

Note that, concatenating s times the identity matrix results in an input matrix that is a feasible solution to both problems, where the only relevant columns of B are the non-zero ones. Although problems (1) and (3) seem to be very similar, the proposed solutions are quite diverse, and in fact, they exhibit different computational complexity issues.

Additionally, to solve Problem 1 and Problem 2, we require two technical assumptions that we now detail.

Assumption 1: The dynamics matrix of each mode, A_i with $i \in \mathbb{M}$, is simple, i.e., A_i has distinct eigenvalues.

Note that, many applications have dynamics' matrices satisfying Assumption 1, e.g., dynamical systems modeled as random networks of the Erdős-Rényi type [12], or benchmark dynamical systems in control system engineering [13], [14].

Assumption 2: A left-eigenbasis of A_i is known for each mode $i \in \mathbb{M}$ (the set of the left-eigenvectors of A_i).

Assumption 2 is a technical restriction. In general, the left-eigenbasis is acquired by numerical methods and, hence, we obtain approximated eigenvectors up to a floating-point error.

III. THE ROBUST MINIMAL CONTROLLABILITY PROBLEM

In this section, we present the main results of the paper. We address problem rMCPS (2) and crMCPS (4), by 'decoupling' the problems into their discrete and continuous optimization properties. We start by identifying the structure of the solutions and, after, a numerical realization of them that ensures controllability under the possible input failures. We introduce the *minimum set multi-covering problem*, that we use in Algorithm 1 to build a solution to the rMCPS and the crMCPS. Then, we find a dedicated solution to the rMCPS, in Theorem 1, that is used together with Algorithm 3 to find a general solution to the problem, in Theorem 2. A solution to the crMCPS is constructed, in Corollary 2, using Algorithm 2. Finally, we show that the rMCPS is NP-complete, in Theorem 3.

We start by noticing that the Popov-Belevitch-Hautus (PBH) eigenvalues controllability test gives us a sufficient controllability condition for SLCS (1) or (3).

Proposition 1: Given an SLCS, if for each mode $i \in \mathbb{M}$ the (A_i, B_i) is controllable, then the SLCS is controllable.

Hence, Proposition 1 bestow a polynomial method (in m and n) to check a sufficient condition for the controllability of an SLCS. In other words, for each mode $i \in \mathbb{M}$, and for each eigenvalue $\lambda \in \sigma(A_i)$, we only need to compute the rank of $[A_i - \lambda I_n \ B_i]$.

However, this criterion does not inform about which entries of each B_i should be non-zero, as well as with which particular values, to ensure the rank condition. That is, we can verify in polynomial time that each B_i is a solution. Notwithstanding, we notice that the rMCPS is computationally challenging to solve since a particular instance of the rMCPS, i.e., when (1) has one mode, we get the minimal controllability problem (MCP) that is known to be NP-hard, we can polynomially reduce an NP-hard problem to this problem, see [9]. Thus, the rMCPS is as difficult as the latter, which leads to the following result.

Corollary 1: The rMCPS (2) and the crMCPS (4) are NP-hard. ◇

Instead of a naïve usage of the PBH eigenvalue test that leads to a strictly combinatorial procedure for solving the SLCS, we may consider the PBH test for controllability using eigenvectors, which allow us to design a sufficient condition for the controllability of an SLCS.

Proposition 2: Given (1), if for each mode $i \in \mathbb{M}$ and for each left-eigenvector v of A_i we have that $v^\dagger B_i \neq 0$, then the system is controllable.

Proposition 2 plays a central role in this paper's main results.

As previously mentioned, we first address the discrete part, which requires us to introduce the following constructs.

Definition 1: (Minimum Set Multi-covering Problem [11]) Given a universe with m elements, $\mathcal{U} = \{1, 2, \dots, m\}$, a collection of n sets $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_n\}$, where $\mathcal{S}_j \subset \mathcal{U}$, for $j \in \{1, \dots, n\}$, s.t. $\bigcup_{j=1}^n \mathcal{S}_j =$

\mathcal{U} , and a demand function $d : \mathcal{U} \rightarrow \mathbb{N}$ that imposes the number of times the element i needs to be covered. The minimum set multi-covering problem consists in finding a smallest set of indices $\mathcal{I}^* \subseteq \{1, \dots, n\}$ s.t. $\cup_{j \in \mathcal{I}^*} \mathcal{S}_j = \mathcal{U}$ and every element $i \in \mathcal{U}$ is covered, at least, $d(i)$ times, i.e.,

$$\begin{aligned} \mathcal{J}^* = \arg \min_{\mathcal{J} \subseteq \{1, \dots, n\}} & |\mathcal{J}| \\ \text{s.t.} & |\{j \in \mathcal{J} : i \in \mathcal{S}_j\}| \geq d(i) . \end{aligned} \quad \diamond$$

A particular case of the problem in Definition 1 is when each element needs to be covered once, $d(i) = 1$ for all $i \in \mathcal{U}$, called the *minimum set covering problem*, see [11]. These two problems are ubiquitous in the fields of combinatorics, computer science, and complexity theory. They are NP-complete problems for which efficient approximation algorithms are known and well studied [15].

These problems are particularly useful to leverage the controllability criterion using the PHB criterion to ensure the feasibility of the sparsest input. In particular, for the LTI case, we have the following approach.

Lemma 1 ([9]): Given a non-empty collection of non-zero vectors $\{v^j\}_{j \in \mathcal{J}}$, with $v^j \in \mathbb{R}^n$, the procedure of finding $b^* \in \mathbb{C}^n$ that is a solution to

$$\begin{aligned} b^* = \arg \min_{b \in \mathbb{R}^n} & \|b\|_0 \\ \text{s.t.} & v^k \cdot b \neq 0, \text{ for all } j \in \mathcal{J} \end{aligned} \quad (5)$$

is polynomially reducible (in $|\mathcal{J}|$ and n) to a minimum set covering problem. \diamond

Next, we can build upon this problem and the notion of controllability for the SLCS, and under the assumptions posed in Section II, to find the sparsest set of vectors that ensure controllability. Specifically, since we have different modes, the goal is to consider the sparsest sets of inputs across the different eigenvectors of the left-eigenbasis.

Subsequently, we start by presenting an algorithm (Algorithm 1) that receives a collection of structural vectors and outputs a setup for a set-covering problem. The sets we build, \mathcal{S}_k^i , have two indices, i and k , the first matches the mode the structural vectors belong to, and the second ranges from 1 up to the number of such vectors in that mode. A pair (i, j) belongs to \mathcal{S}_k^i whenever the j th structural vector of mode i is non-zero at index k .

Algorithm 1 Polynomial reduction of the structural optimization problem (5), to a set-covering problem

Input: Consider the eigenvectors of the left eigenbasis of the different modes, $\{v_i^j\}_{\substack{i \in \mathbb{M} \\ j \in \{1, \dots, |\mathcal{J}_i|\}}}$, and $\mathcal{J} = \cup_{i \in \mathbb{M}} \mathcal{J}_i$.

Output: The setup for a set multi-covering problem, $\mathcal{S} = \{\mathcal{S}_j^i\}_{\substack{i \in \mathbb{M} \\ j \in \{1, \dots, |\mathcal{J}_i|\}}}$ and \mathcal{U} , a set with n sets, and the universe of these sets, respectively.

- 1: **set** $\mathcal{S}_i^j = \emptyset$, for $i \in \mathbb{M}$ and $j \in \{1, \dots, |\mathcal{J}_i|\}$
 - 2: **for** $i \in \mathbb{M}$
 - for** $j = 1, \dots, |\mathcal{J}_i|$
 - for** $k = 1, \dots, n$
 - if** $[\bar{v}_j^i]_k \neq 0$ **then**

$$\mathcal{S}_k^i = \mathcal{S}_k^i \cup \{(i, j)\}$$
 - 3: **set** $\mathcal{S} = \{\mathcal{S}_j^i\}_{\substack{i \in \mathbb{M} \\ j \in \{1, \dots, |\mathcal{J}_i|\}}}$ and $\mathcal{U} = \cup_{\mathcal{V} \in \mathcal{S}} \mathcal{V}$.
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To make our approach easier to follow, we first characterize the dedicated solutions to the rMCPS, i.e., the solution where each input actuates a unique state variable.

Theorem 1: Given a left-eigenbasis, v_1^i, \dots, v_n^i , for each A_i and for each mode $i \in \mathbb{M}$, and given the number of possible input failures for each mode, s_1, \dots, s_m , consider the multi-set covering problem $(\mathcal{S}, \mathcal{U}; d)$, where

- $\mathcal{S} = \cup_{i \in \mathbb{M}} \{\mathcal{S}_1^i, \dots, \mathcal{S}_{(s_i+1)n}^i\}$;
- $\mathcal{U} = \cup_{i \in \mathbb{M}} \mathcal{U}_i$, with $\mathcal{U}_i = \{(i, 1), \dots, (i, n)\}$;

- $d(i, j) = s_i + 1$, for $(i, j) \in \mathcal{U}$;

where $\mathcal{S}_k^i = \{(i, j) : [v_j^i]_l \neq 0 \text{ and } l - 1 = k \pmod n\}$, for $i \in \mathbb{M}$ and $k \in \mathcal{K} \equiv \{1, \dots, (s_i + 1)n\}$. Then, the following conditions are equivalent:

- (i) \mathcal{I}^* is a solution to the set multi-covering problem $(\mathcal{S}, \mathcal{U}; d)$ and $\mathcal{I}_i^* = \{(i, a) : (i, a) \in \mathcal{I}^*\}$;
- (ii) the set of matrices $\{B_1(\mathcal{I}_1^*), \dots, B_m(\mathcal{I}_m^*)\}$, where $B_i \in \{0, 1\}^{n \times (s_i + 1)n}$, is a dedicated solution to the rMCPS (2), with $[B_i(\mathcal{I}_i^*)]_{j,k} = 1$ for $k = j \pmod n$ and $(i, j) \in \mathcal{I}^* \subset \mathcal{K}$, and $[B_i(\mathcal{I}_i^*)]_{j,k} = 0$ otherwise.

Proof: By Theorem 5 in [9], $B_i(\mathcal{I}_i^*)$ is an optimal feasible dedicated solution to controls mode i , equivalent to a solution \mathcal{I}_i^* that covers \mathcal{U}_i , for demand $d(i, j) = s_i + 1$. By Proposition 2, the set of matrices $\{B_1(\mathcal{I}_1^*), \dots, B_m(\mathcal{I}_m^*)\}$ is a feasible dedicated solution to rMCPS (2), equivalent to a solution of the multi-set covering problem $(\mathcal{S}, \mathcal{U}, d)$. ■

Note that the first entry of the pairs constituting the universe of the set multi-covering problem is what identifies the mode of the input matrix when recovering a solution to the set multi-covering problem to a solution of the rMCPS (2).

The crMCPS (4) requires a different approach because we want to design an input matrix that is equal to every mode. Intuitively, this is computationally more demanding, since we want to minimize the size of the union of the solutions for each mode, *i.e.*, we want to actuate a small number of state variables that, across the modes, ensure the system to be controllable. The first step to solve the problem consists in applying Algorithm 1, to build the sets of \mathcal{S} . The second step is to find a solution for the crMCPS (4), using \mathcal{S} , \mathcal{U} and the demand function $d(i) = s + 1$. We need to, carefully, choose one solution of indices of the state variables that we need to actuate, which ensures each mode to be, not only, controllable, but also robust to s input failures, maximizing the state variables in common across the modes. To achieve this, we build an algorithm that needs to find all the solutions to several set multi-covering problems, translating to all possible solution to the crMCPS (4) and, afterward, we need to select one that actuates the smallest number of state variables across all modes. We summarize this procedure in Algorithm 2 that selects a minimal number of state variables across all modes that we need to actuate such that Proposition 2 holds, yielding a solution to the crMCPS (4). Besides, note that Algorithm 2 has worst-case complexity exponential (in m and n), since each set multi-covering problem may have an exponential number of solutions.

Algorithm 2 Find a minimal set of state variables of a problem (4) that need to be actuated

Input: An instance of the problem (4), a collection of sets \mathcal{S} and a universe \mathcal{U} , the output of Algorithm 1 applied to the set of eigenvector for each dynamics matrix A_i , and demand function d , with $\left\{ \bar{v}_i^j \right\}_{\substack{i \in \mathbb{M} \\ j \in \{1, \dots, n\}}}$ and $d(i) = s + 1$, respectively.

Output: A set of state variables' indices to actuate s.t. the problem instance is controllable, $\mathcal{I} \subseteq \{1, \dots, n\}$.

- 1: **for** $i \in \mathbb{M}$
 - set** $\mathcal{S}_i = \{S_k^t : t = i \text{ and } k \in \{1, \dots, n\}\}$
 - set** $\mathcal{U}_i = \{k : (i, k) \in \mathcal{U}\}$
 - for** $j = 1, \dots, n$
 - set** \mathcal{B}_i as the set of all possible covers for \mathcal{U}_i
with the collection of sets \mathcal{S}_i and demand d
 - 2: **set** $\{\mathcal{X}_1^*, \dots, \mathcal{X}_m^*\} = \arg \min_{\mathcal{X}_1 \in \mathcal{B}_1, \dots, \mathcal{X}_m \in \mathcal{B}_m} |\bigsqcup_{i \in \mathbb{M}} \mathcal{X}_i|$
 - 3: **set** $\mathcal{I} = \bigsqcup_{i \in \mathbb{M}} \mathcal{X}_i^*$ and $\mathcal{B} = \{\mathcal{B}_i\}_{i \in \mathbb{M}}$
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Now, we go further and characterize the general solutions to the rMCPS (2) and the crMCPS (4), not only dedicated ones. We derive the global solutions based on the dedicated solutions by combining them. Towards this goal, we propose a *merging* procedure [9] that we describe in Algorithm 3. The procedure tries to combine in the smallest possible number of inputs, the entries of the dedicated inputs while ensuring that the PHB eigenvectors criterion holds. Specifically, Algorithm 3 picks two *compatible* inputs, *i.e.*, with different structure and structural inner-product 0. In this procedure, when we combine

two compatible inputs, we set the first one to actuate the variables that both actuate, and we discard the second one (the respective column is set to zero).

Algorithm 3 Merging procedure

Input: An input matrix $\bar{B} \in \{0, \star\}^{n \times m}$.

Output: The matrix $\bar{B} = [\bar{b}_1 \dots \bar{b}_m]$ with inputs merged.

1: **while** $\exists_{i,j} : \bar{b}_i \neq \bar{b}_j \neq 0$ and $\bar{b}_i \cdot \bar{b}_j = 0$
 set $\bar{b}_i = \bar{b}_i + \bar{b}_j$ and $\bar{b}_j = 0$

Up to this point, Algorithm 3 is about the structure of the input matrix \bar{B} , and we now build a numerical realization of it. Subsequently, we need to solve the following problem to perform the second step required to obtain a solution to the rMCPS (2) and the crMCPS (4), *i.e.*, a parametrization B^* of the structural matrix \bar{B} (a feasible solution to be orthogonal to a given set of $m = |\mathcal{J}|$ vectors, $\{v^j\}_{j \in \mathcal{J}}$):

$$\begin{aligned} B^* = \arg \min_{B \in \mathbb{R}^{n \times m}} & 0 \\ \text{s.t.} & B \cdot v^j \neq 0, \text{ for all } j \in \mathcal{J} \text{ and} \\ & B \text{ has the structure of } \bar{B}. \end{aligned} \quad (6)$$

As a consequence, we can obtain a solution for the rMCPS (2) as described in the following result.

Theorem 2: Let $\{B_i(\mathcal{I}_i^*)\}_{i=1}^m$, where $B_i \in \{0, 1\}^{n \times (s_i+1)n}$, be a dedicated solution to the rMCPS (2), obtained with Theorem 1. Further, let $\{\bar{B}_i\}_{i=1}^m$, where $\bar{B}_i \in \{0, \star\}^{n \times (s_i+1)n}$ be the sparsities of the matrices resulting from the merging procedure, Algorithm 3, between any of the effective inputs, for each $B_i(\mathcal{I}_i^*)$. Then, the set of matrices $\{B_i^*\}_{i=1}^m$, where each matrix $B_i^* \in \mathbb{R}^{n \times n}$ is obtained using the optimization task (6) for inputs \bar{B}_i and left-eigenbasis of A_i is a solution to the rMCPS (2). \diamond

Proof: By Theorem 1, then we have a feasible solution to the rMCPS (2), and, by construction, Algorithm 3, we preserve the feasibility by merging only compatible inputs. Also, by construction, the optimization problem (6) ensures that the obtained numerical realization of the solution's sparsity verifies the PHB eigenvector criterion. Hence, we obtain a solution to the rMCPS (2). \blacksquare

In fact, the above also applies to the crMCPS (4), considering a matrix \bar{B} , instead of the set of matrices $\{\bar{B}_i\}_{i=1}^m$. Hence, the next result readily follows.

Corollary 2: Let $B(\mathcal{I}_i^*)$, with $B \in \{0, 1\}^{n \times (s+1)n}$, be a dedicated solution to the crMCPS (4), obtained with Theorem 1. Further, let $\bar{B} \in \{0, \star\}^{n \times (s+1)n}$ be the sparsity of the matrix resulting from the merging procedure, Algorithm 3, between any of the effective inputs. Then, the matrix B^* obtained using the optimization task (6) for inputs \bar{B} and left-eigenbasis of A_1, \dots, A_m is a solution to the rMCPS. \diamond

It is worth noticing that the decision version of the rMCPS (2), when using the sufficient condition of Proposition 2, is equivalent to the set multi-covering problem and the rMCPS (2) is NP-complete for that controllability sufficient condition. For the crMCPS (4), since we lose the information about the modes when we build the input matrix B , the decision version of the problem is not equivalent to the decision version of the set multi-covering problem when using the sufficient condition of Proposition 2.

Theorem 3: By using the sufficient condition for controllability in Proposition 2, the rMCPS (2) is NP-complete. \diamond

Proof: Using Proposition 2 as the controllability condition, the rMCPS (2) is equivalent to the set covering problem, by Theorem 1 and Theorem 2, the result follows. \blacksquare

IV. ILLUSTRATIVE EXAMPLES

In this section, we illustrate the main results from this paper. In order to do so, we fix an SLCS with two modes and dynamics matrices A_1 and A_2 given by $A_1 = \begin{bmatrix} 2 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 2 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ -2 & -1 & -3 & -7 & 1 \end{bmatrix}$, and $A_2 =$

$$\begin{bmatrix} 2 & -1 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 2 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ -2 & -1 & -3 & -7 & 1 \end{bmatrix}$$
. The dynamics matrices' left-eigenvalues are, respectively, $V_1 = \begin{bmatrix} | & | \\ v_1^1 & \dots & v_1^5 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$,

and $V_2 = \begin{bmatrix} | & | \\ v_2^1 & \dots & v_2^5 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$. The eigenvalues of each mode are $\sigma(A_1) = \sigma(A_2) = \{1, 2, 3, 4, 5\}$.

By design, the spectrum is equal, but do not need to be, as the only assumption is that the state matrices need to be simple.

A. Example I

Next, we explore this example when we consider that no input can fail ($s = 0$) as an instance of both the rMCPS (2) and the crMCPS (4), and when, in both cases, some inputs may fail.

1) *The rMCPS (2) scenario:* First, we consider we want to design input matrices for each mode that actuate the minimal number of inputs that, by Proposition 2, ensures the system to be controllable. Applying Algorithm 1, we obtain the following set: $\mathcal{S}_1^1 = \{(1, 3), (1, 4), (1, 5)\}$, $\mathcal{S}_2^1 = \{(1, 1), (1, 4)\}$, $\mathcal{S}_3^1 = \{(1, 2)\}$, $\mathcal{S}_4^1 = \{(1, 1), (1, 2), (1, 3)\}$ and $\mathcal{S}_5^1 = \{(1, 4)\}$, that correspond to the first mode of the system. For the second mode of the system, we get sets $\mathcal{S}_1^2 = \{(2, 3), (2, 4), (2, 5)\}$, $\mathcal{S}_2^2 = \{(2, 1), (2, 4)\}$, $\mathcal{S}_3^2 = \{(2, 2), (2, 5)\}$, $\mathcal{S}_4^2 = \{(2, 1), (2, 3)\}$ and $\mathcal{S}_5^2 = \{(2, 4)\}$. This yields the universe $\mathcal{U} = \cup_{i=1}^2 \{(i, 1), (i, 2), (i, 3), (i, 4), (i, 5)\}$. Now, solving the associated set (multi-)covering problem $(\mathcal{S}, \mathcal{U}; d = 1)$, we obtain as a solution that $\mathcal{U} = \mathcal{S}_1^1 \cup \mathcal{S}_4^1 \cup \mathcal{S}_1^2 \cup \mathcal{S}_2^2 \cup \mathcal{S}_3^2$.

This leads to a structure of a solution to the rMCPS (2) of $\bar{B}_1 = [\ast 0 0 \ast 0]^\top$ and $\bar{B}_2 = [\ast \ast 0 0 0]^\top$. Subsequently, we can check that by setting every non-zero entry of \bar{B}_1 and \bar{B}_2 as 1, by Proposition 2, we obtain a solution to the problem.

2) *The crMCPS (4) scenario:* Now, our aim is to design a common input matrix that controls the system. First, we need to compute all possible solutions, for each mode of the system. For both modes, the universe of the associated set covering problem is $\mathcal{U} = \{1, 2, 3, 4, 5\}$. Now, we apply Algorithm 2 and we get, for the first mode, $\mathcal{S}_1 = \{\{1, 4, 5\}, \{1, 4\}, \{2\}, \{1, 2, 3\}, \{4\}\}$, and, thus, each of the following sets of indices constitute a solution that covers the universe $\mathcal{B}_1 = \{\{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 4, 5\}, \{1, 2, 3, 4\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$, that is, for each set $\mathcal{X} \in \mathcal{B}_1$, we have that $\mathcal{U} = \bigsqcup_{i \in \mathcal{X}} [\mathcal{S}_1]_i$. Note that 5 only belongs to the first set of \mathcal{S}_1 , and 3 is only in the fourth set of \mathcal{S}_1 . Hence, these two sets need to belong to the solution. Analogously, for the second mode, we have that $\mathcal{S}_2 = \{\{3, 4, 5\}, \{1, 2, 4\}, \{2, 5\}, \{1, 3\}, \{4\}\}$, and all solutions of the associated set covering problem are $\mathcal{B}_2 = \{\{1, 2, 3\}, \{1, 3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$. By combining the two sets of possible solutions, we obtain the result that consists in selecting, for instance, $\{1, 4\} \in \mathcal{B}_1$ together with $\{1, 3, 4\} \in \mathcal{B}_2$. Thus, $\{1, 3, 4\}$ are the set of state variables that we need to actuate, in both modes, to attain controllability. Then, an input matrix with dedicated inputs may have the following sparsity $\bar{B} = \begin{bmatrix} \ast & 0 & 0 & 0 & 0 \\ 0 & 0 & \ast & 0 & 0 \\ 0 & 0 & 0 & \ast & 0 \end{bmatrix}^\top$. Now, resorting to Algorithm 3, we get the input matrix pattern with merged columns $\bar{B} = [\ast 0 \ast \ast 0]^\top$, and we can check that $B = [1 0 1 1 0]^\top$ is a solution to the problem. Note that the solutions to the problem (2) and (4) instances are distinct. In fact, if we set B_1 and B_2 in Section IV-A.1 as B , we get a solution to the problem (2) instance, that is not minimal in each mode.

B. Example II

Now, we explore the scenario where a set of inputs may malfunction in the switching system. We want to account for them when designing the inputs, and the respective variables that they control, to still be able to control the system.

1) *The rMCPS (2) scenario:* Suppose now that in the first mode, there are not inputs that may fail, but, in the second mode, one input may fail. In other words, $s_1 = 0$ and $s_2 = 1$, which translates to have, in the corresponding set multi-covering problem, $d(1, j) = 1$ and $d(2, j) = 2$ for $j \in \mathcal{U}$, with \mathcal{U} and collection of sets \mathcal{S} as in IV-A.1.

Now, a solution to the problem is $\mathcal{U} = \mathcal{S}_1^1 \cup \mathcal{S}_4^1 \cup \mathcal{S}_1^2 \cup \mathcal{S}_2^2 \cup \mathcal{S}_3^2 \cup \mathcal{S}_4^2$, translating to the pattern for the input matrices, when considering dedicated inputs, $\bar{B}_1 = [* 0 0 * 0]^\top$, and $\bar{B}_2 = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \end{bmatrix}^\top$. We apply Algorithm 3 and obtain that a single input controls the second mode, and each non-zero value of the inputs matrices being 1 verifies Proposition 2, i.e., the input matrices $B_1 = [1 0 0 1 0]^\top$ and $B_2 = [1 1 1 1 0]^\top$ are a solution to the problem instance.

2) *The crMCPS (4) scenario*: Finally, suppose the objective is to design a single input matrix that not only ensures that the system is controllable, but also that the system remains controllable whenever, at most, one input fails. In other words, $s = 1$, which we reflect in the demand function for the set multi-covering problems of Algorithm 2, $d = 2$.

By recalling the sets \mathcal{B}_1 and \mathcal{B}_2 from Section IV-A.2, we know that a solution for the first mode must have twice the indices 1 and 4 so that we cover elements 3 and 5 twice. Hence, a possible and minimal solution is the set of indices $\mathcal{I} = \{1, 1, 2, 3, 4, 4\}$. In fact, we can check that this also produces a solution for the second mode. Hence, the pattern of the solution, when considering dedicated inputs, and a solution, after applying Algorithm 3, are, respectively, $\bar{B} = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$, and $B = [1 1 0 1 1 0]^\top$.

We can find others solutions by changing the merging order. If we set B_1 and B_2 in Section IV-B.2 as B , we get a solution to the problem (2) instance, the same we obtained. However, we want to ensure the system to be robust to one input failure for each mode, while in Section IV-B.2, we want to ensure that the system is robust to one input failure only in the second mode.

V. CONCLUSIONS

In this work, we addressed two robust minimal controllability problems for switched linear continuous-time systems. The first is to design an input matrix for each mode that ensures the switched system to be controllable. The second is to design a common input matrix guaranteeing the switched system to be controllable. We showed that the first problem is NP-complete. We reduce both problems to set multi-covering problems in a two-step, solving first the discrete nature of the problem and afterward the continuous one. These reductions allow deploying approximated and efficient algorithms to solve set multi-covering problem instances, to get solutions to rMCPS (2) or the crMCPS (4) instances that are feasible and have sub-optimality guarantees.

Future work involves exploring the structure of the problems to attain optimal solutions, leveraging the eigenbasis' structure. Further, we want to consider, besides the number of inputs for controllability, obtaining a certain controllability index, minimizing the number of inputs and the number of times that we need to actuate the system. Last, we want to relate macroscopic interconnections between dynamical systems, leading to a modular approach to the actuation placement that ensures controllability.

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