Deciding theoremhood in fibred logics without shared connectives

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Abstract. Fibring is a powerful mechanism for combining logics, and an essential tool for designing and understanding complex logical systems. Abstract results about the semantics and proof-theory of fibred logics have been extensively developed, including general soundness and completeness preservation results. Decidability, however, a key ingredient for the automated support of the fibred logic, has not deserved similar attention.

In this paper, we address the problem of deciding theoremhood in fibred logics without shared connectives. Namely, under this assumption, we provide a full characterization of the mixed patterns of reasoning that lead to theorems in the fibred logic, and use it to prove a general decidability preservation result. The complexity of the decision procedure we obtain is also analyzed.

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1. Introduction

Fibring is a powerful and appealing mechanism for combining logics, a valuable tool for the construction and analysis of complex logics, and thus a key ingredient of the general theory of universal logic [2, 3]. As first proposed by Dov Gabbay in [11, 12], given two logics $\mathcal{L}_1$ and $\mathcal{L}_2$, fibring should combine $\mathcal{L}_1$ and $\mathcal{L}_2$ into the smallest logical system for the combined language which is a conservative extension of both $\mathcal{L}_1$ and $\mathcal{L}_2$. However, it is not hard to see that a conservative extension of the given logics may not always exist. Still, this circumstance does

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not necessarily imply that the construction is meaningless, as one can then aim at being "as conservative as possible". This idea has led to the study of the fibring operation as yielding the smallest logic that extends $\mathcal{L}_1$ and $\mathcal{L}_2$, without worrying about conservativity [8]. It is worth mentioning that the characterization of conservativity in this context is a problem that remains highly unexplored.

Despite the depth of the track of work on fibred logics that led to a substantial understanding of their semantics and proof-theory, including very general soundness and completeness preservation results (see [1, 5, 9, 14, 15, 16, 18], *inter alia*), the question of decidability has not been satisfactorily addressed. The only general result related to, but markedly distinct from, decidability for fibred logics is [10], where the preservation by fibring of the semantic notion of *finite model property* is studied.

In this paper, we focus on the decision problem for theorems of fibred logics, in the case when the logics being combined do not share any connectives. We manage to give a full characterization of the mixed patterns of reasoning that lead to the proofs of theorems in the fibred logic and, as a result, we obtain a general decidability preservation result for theoremhood, the first of this kind. We also analyze the complexity of the decision procedure obtained.

In Section 2 we recall the notions and results needed throughout the paper, namely about fibred logics, and introduce some useful notation. In Section 3 we illustrate the difficulties involved in deciding theoremhood in fibred logics without shared connectives, and provide a thorough analysis of the mixed patterns of reasoning that may occur in the combined logic. Finally, in Section 4 we prove our main result: a decidability preservation result for theoremhood in fibred logics with no shared connectives, and a characterization of the complexity of the decision procedure obtained. We conclude, in Section 5, with an assessment of the results obtained and paths to pursue in future work.

2. Definitions

In this section we recall the essential concepts that we are dealing with in this paper, namely fibring, and introduce some useful notions and notations.

2.1. (Trans)finite sequences

Along the paper, we will need to deal with (not necessarily finite) sequences of objects. Let A be a set (of objects). Given an ordinal η, we use $\overline{a} = \langle a_\kappa \rangle_{\kappa<\eta}$ to denote a η-long sequence of elements of A, or simply a η-sequence, understood as a function from $\{\kappa : \kappa < \eta\}$ to A. The η-sequence $\overline{a}$ is said to be *injective* precisely when it is injective as a function, that is, when $a_i \neq a_j$ for all $i, j < \eta$ with $i \neq j$. As usual, if $\tau \leq \eta$, the sequence $\langle a_\kappa \rangle_{\kappa<\tau}$ is dubbed a *prefix* of $\overline{a}$.

Note that when η is a limit ordinal, a η-sequence does not have a last element. On the contrary, if η is a successor ordinal, and in particular a finite ordinal, then a η-sequence $\overline{a}$ can be understood as $a_0, a_1, \ldots, a_{\eta-1}$, and may also be represented by $\langle a_\kappa \rangle_{\kappa\leq\eta-1}$. The 0-sequence (*empty* sequence) is simply not represented.

2.2. Syntax

A *signature* is a $\mathbb{N}_0$-indexed family $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}_0}$ of sets. The elements of Σ_n are dubbed n-place *connectives*. Being indexed families of sets, the usual set-theoretic notions can be smoothly extended to signatures. We will sometimes abuse notation, and confuse Σ with the set $(\biguplus_{n\in\mathbb{N}_0}\Sigma_n)$ of all its connectives, and write $c \in \Sigma$ when c is some n-place connective and $c \in \Sigma_n$. For this reason, the *empty signature*, with no connectives at all, is simply denoted by $\emptyset$.

Let Σ, Σ' be two signatures. We say that Σ is a *subsignature* of Σ', and write $\Sigma \subseteq \Sigma'$, whenever $\Sigma_n \subseteq \Sigma'_n$ for every $n \in \mathbb{N}_0$. Expectedly, we can also define the *intersection* $\Sigma\cap\Sigma' = \{\Sigma_n\cap\Sigma'_n\}_{n\in\mathbb{N}_0}$, *union* $\Sigma\cup\Sigma' = \{\Sigma_n\cup\Sigma'_n\}_{n\in\mathbb{N}_0}$, and *difference* $\Sigma' \setminus \Sigma = \{\Sigma'_n \setminus \Sigma_n\}_{n\in\mathbb{N}_0}$ of signatures. Clearly, $\Sigma \cap \Sigma'$ is the largest subsignature of both Σ and Σ', and contains the connectives *shared* by Σ_1 and Σ_2. When there are no shared connectives we have that $\Sigma \cap \Sigma' = \emptyset$. Analogously, $\Sigma \cup \Sigma'$ is the smallest signature that has both Σ and Σ' as subsignatures, and features all the connectives from both Σ and Σ' in a *combined signature*. Furthermore, $\Sigma' \setminus \Sigma$ is the largest subsignature of Σ' which does not share any connectives with Σ.

Given a signature Σ and a set P of *variables*, the generated set of *formulas* is the carrier set $L_\Sigma(P)$ of the free Σ-algebra generated by P.

As usual, we define the *size* of a formula as given by the function size such that $\mathsf{size}(p) = 1$ if $p \in P$, and $\mathsf{size}(c(\varphi_1, \ldots, \varphi_n)) = 1 + \sum_{i=1}^{n} \mathsf{size}(\varphi_i)$. For $\Gamma \subseteq L_\Sigma(P)$, we further define $\mathsf{size}(\Gamma) = \sum_{\varphi\in\Gamma} \mathsf{size}(\varphi)$.

If $\varphi \in L_\Sigma(P)$ then we define the *head of* φ to be either $\mathsf{head}(\varphi) = p$ when $\varphi = p \in P$, or $\mathsf{head}(\varphi) = c$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma_n$. Clearly, if $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$ then $L_\Sigma(P) \subseteq L_{\Sigma'}(P')$. Of course, given $\psi \in L_{\Sigma'}(P')$, $\mathsf{head}(\psi)$ may not be in Σ nor P.

We also define the *set of variables occurring in* φ to be either $\mathsf{var}(\varphi) = \{p\}$ when $\varphi = p \in P$, or $\mathsf{var}(\varphi) = \bigcup_{i=1}^{n} \mathsf{var}(\varphi_n)$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma_n$.

In the sequel, we shall assume that signatures are countable and sets of variables are denumerable. We assume fixed a denumerable set P of variables. If Σ is a countable signature then $L_\Sigma(P)$ is clearly denumerable.

Let $\Sigma \subseteq \Sigma'$ be signatures. We shall call a Σ-*monolith of* $\psi \in L_{\Sigma'}(P)$ to any outermost subformula of ψ whose head is in $\Sigma' \setminus \Sigma$. The set $\mathsf{Mon}_\Sigma(\psi)$ of all Σ-monoliths of ψ is defined as follows:

$$\mathsf{Mon}_\Sigma(\psi) = \begin{cases} \emptyset & \text{if } \psi \in P, \\ \bigcup_{i=1}^{n} \mathsf{Mon}_\Sigma(\psi_i) & \text{if } \psi = c(\psi_1, \ldots, \psi_n) \text{ and } c \in \Sigma_n, \\ \{\psi\}, & \text{otherwise.} \end{cases}$$

We extend the notation also to sets of formulas, using $\mathsf{Mon}_\Sigma(\Delta)$ to denote $\bigcup_{\psi\in\Delta} \mathsf{Mon}_\Sigma(\psi)$, given $\Delta \subseteq L_{\Sigma'}(P)$. Clearly, if $\Gamma \subseteq L_\Sigma(P)$ then $\mathsf{Mon}_\Sigma(\Gamma) = \emptyset$.

We shall now consider a reasonable way of defining the perspective, from the point of view of Σ, that one may have of a formula in a given context $\Delta \subseteq L_{\Sigma'}(P)$ (which will most often be left implicit). For the purpose, we shall consider a (bijective) enumeration $e : \{n \in \mathbb{N} : 1 \leq n \leq |\mathsf{Mon}_\Sigma(\Delta)|\} \to \mathsf{Mon}_\Sigma(\Delta)$ of the relevant Σ-monoliths, and use a denumerable set of additional propositional variables $X = \{x_n : n \in \mathbb{N}_0\}$, disjoint from P. We define the function $\mathsf{skel}_\Sigma : \Delta \to L_\Sigma(P \cup X)$ as follows:

$$\mathsf{skel}_\Sigma(\psi) = \begin{cases} \psi & \text{if } \psi \in P, \\ c(\mathsf{skel}_\Sigma(\psi_1), \dots, \mathsf{skel}_\Sigma(\psi_n)) & \text{if } \psi = c(\psi_1, \dots, \psi_n) \text{ and } c \in \Sigma_n, \\ x_{e^{-1}(\psi)}, & \text{otherwise.} \end{cases}$$

We call $\mathsf{skel}_\Sigma(\psi)$ the *Σ-skeleton of* ψ. Clearly, $\mathsf{skel}_\Sigma(\psi)$ is obtained from ψ by substituting each of its Σ-monoliths ψ' by the variable x_n such that $e(n) = \psi'$.

Let $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$. A *Σ′-substitution* is a function $\sigma : P \to L_{\Sigma'}(P')$, which extends freely to a function $\sigma : L_\Sigma(P) \to L_{\Sigma'}(P')$. Given a formula $\varphi \in L_\Sigma(P)$, $\sigma(\varphi)$ is the *instance* of φ by σ, sometimes denoted simply by φ^σ, and is the result of uniformly replacing each variable $p \in P$ occurring in φ by $\sigma(p)$. When $\Gamma \subseteq L_\Sigma(P)$ we use Γ^σ to denote $\sigma[\Gamma] = \{\varphi^\sigma : \varphi \in \Gamma\}$.

Given a context $\Delta \subseteq L_{\Sigma'}(P)$ and an enumeration e of $\mathsf{Mon}_\Sigma(\Delta)$, as well as $\varphi \in L_\Sigma(P \cup X)$ and a η-sequence $\overline{\alpha}$ of $L_{\Sigma'}(P)$ formulas, we will often write $\varphi[\overline{\alpha}]_\Sigma$ to denote the formula $\varphi^{\sigma(\overline{\alpha})}$ where $\sigma(\overline{\alpha}) : P \cup X \to L_{\Sigma'}(P)$ is such that $\sigma(\overline{\alpha})(p) = p$ if $p \in P$, $\sigma(\overline{\alpha})(x_n) = \alpha_{n-1}$ if $0 \leq n-1 < \eta$, and $\sigma(\overline{\alpha})(x_n) = x_n$ otherwise . Hence, we will write $\mathsf{skel}_\Sigma(\psi)[\overline{e}]_\Sigma$ instead of ψ if we want to highlight the Σ-monoliths in the structure of ψ, where $\overline{e}$ is any sufficiently long prefix of the sequence $\langle e(\kappa)\rangle_{\kappa<|\mathsf{Mon}_\Sigma(\Delta)|}$.

Taking advantage of the notation, given two η-sequences $\overline{\alpha}$ and $\overline{\beta}$ of $L_{\Sigma'}(P)$ formulas, with $\overline{\alpha}$ injective, we will also write $\psi[\overline{\alpha}/\overline{\beta}]_\Sigma$ to denote the formula obtained by replacing each occurrence of α_i as a Σ-monolith of ψ by β_i, for all $i < \eta$. It is not difficult to check that $\psi[\overline{\alpha}/\overline{\beta}]_\Sigma = \mathsf{skel}_\Sigma(\psi)[\overline{\gamma}]_\Sigma$ where $\overline{\gamma}$'s length must be bigger than $e^{-1}(\alpha_\kappa)$ for all $\kappa < \eta$, with $\gamma_n = \beta_\kappa$ if $e(n+1) = \alpha_\kappa$, and $\gamma_n = e(n+1)$ if $e(n+1)$ does not occur in $\overline{\alpha}$.

These square bracket notations will be extended to sets of formulas in the obvious manner.

Example 2.1.

Let Σ be the signature with exactly two connectives, a 0-place connective c and a 2-place connective g, that is, $\Sigma_0 = \{c\}$, $\Sigma_2 = \{g\}$ and $\Sigma_n = \emptyset$ for all $n \in \mathbb{N}_0 \setminus \{0, 2\}$. Let Σ′ extend Σ with an additional 1-place connective f, that is, $\Sigma'_0 = \{c\}$, $\Sigma'_1 = \{f\}$, $\Sigma'_2 = \{g\}$ and $\Sigma'_n = \emptyset$ for all $n \in \mathbb{N}_0 \setminus \{0, 1, 2\}$.

Taking the $L'_\Sigma(P)$ formula $\psi = g(f(p), g(c, f(g(f(c), f(p)))))$ we have that

$$\mathsf{Mon}_\Sigma(\psi) = \{f(p), f(g(f(c), f(p)))\}.$$

Note, in particular, that the subformula $f(c)$ is not a Σ-monolith of ψ because if occurs inside the (outermost) monolith $f(g(f(c), f(p)))$. For the same reason, $f(p)$ is only a Σ-monolith of ψ because it also occurs outside $f(g(f(c), f(p)))$.

Hence, we have (in the appropriate context) that $\mathsf{skel}_\Sigma(\psi) = g(x_1, g(c, x_2))$, and thus

$$\psi = g(x_1, g(c, x_2))[f(p), f(g(f(c), f(p)))]_\Sigma.$$

Moreover,

$$\psi[f(p)/\beta]_\Sigma = g(\beta, g(c, f(g(f(c), f(p))))),$$

noting that β only replaces the leftmost occurrence of $f(p)$ in ψ, where it is a Σ-monolith, leaving the second untouched. △

2.3. Logical consequence

A *logic (over signature* Σ*)* is a tuple $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, where $\_^{\vdash} : 2^{L_\Sigma(P)} \to 2^{L_\Sigma(P)}$ is a consequence operator (see [17], for instance), that is, it satisfies the following properties:

$$\begin{array}{ll} \Gamma \subseteq \Gamma^{\vdash} & (\textit{extensiveness}) \\ \Gamma^{\vdash} \subseteq (\Gamma \cup \Delta)^{\vdash} & (\textit{monotonicity}) \\ (\Gamma^{\vdash})^{\vdash} \subseteq \Gamma^{\vdash} & (\textit{idempotence}) \\ (\Gamma^{\vdash})^{\sigma} \subseteq (\Gamma^{\sigma})^{\vdash} & (\textit{structurality}) \end{array}$$

for every $\Gamma, \Delta \subseteq L_\Sigma(P)$ and $\sigma : P \to L_\Sigma(P)$. Note that we do not require, in general, that the logic is *finitary*, i.e., it may happen that $\Gamma^{\vdash}$ properly contains the union of all $\Gamma_0^{\vdash}$ for finite $\Gamma_0 \subseteq \Gamma$.

As usual, we shall confuse the consequence operator with its induced Tarskian consequence relation. Thus, given $\varphi \in L_\Sigma(P)$, we will write $\Gamma \vdash \varphi$ whenever $\varphi \in \Gamma^{\vdash}$. When $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ is finite we write $\varphi_1, \dots, \varphi_n \vdash \varphi$ instead of $\{\varphi_1, \dots, \varphi_n\} \vdash \varphi$. Moreover, as usual, if $\Gamma = \emptyset$ we write $\vdash \varphi$ instead of $\emptyset \vdash \varphi$, and dub φ a *theorem* of $\mathcal{L}$. We also write $\Gamma, \Delta \vdash \varphi$ instead of $\Gamma \cup \Delta \vdash \varphi$.

We shall call any $\Gamma \subseteq L_\Sigma(P)$ such that $\Gamma = \Gamma^{\vdash}$ a *theory* of $\mathcal{L}$, and denote the set of all theories of $\mathcal{L}$ by $\mathbf{Th}(\mathcal{L})$. It is well known that $\mathbf{Th}(\mathcal{L})$ constitutes a complete lattice under the inclusion ordering (see [17], for instance). The top theory of the lattice is $L_\Sigma(P)$, also called the *inconsistent* theory.

A logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ is said to be *consistent* if $\emptyset^{\vdash} \neq L_\Sigma(P)$. Clearly, $\mathcal{L}$ is *inconsistent* (not consistent) precisely when $\vdash p$ for some $p \in P$, or alternatively when its only theory is inconsistent, that is, $\mathbf{Th}(\mathcal{L}) = \{L_\Sigma(P)\}$. $\mathcal{L}$ is said to be *trivial*, if for all non-empty $\Gamma \subseteq L_\Sigma(P)$ we have $\Gamma^{\vdash} = L_\Sigma(P)$. Equivalently, $\mathcal{L}$ is trivial if there exist distinct variables $p, q \in P$ such that $p \vdash q$. Another equivalent characterization is that $\mathcal{L}$ is trivial if $\mathbf{Th}(\mathcal{L}) \subseteq \{\emptyset, L_\Sigma(P)\}$. Of course, all inconsistent logics are trivial.

We say that a logic $\mathcal{L}' = \langle \Sigma', \_^{\vdash'} \rangle$ *extends* $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ if $\Sigma \subseteq \Sigma'$, and $\vdash \subseteq \vdash'$, in the sense that $\Gamma^{\vdash} \subseteq \Gamma^{\vdash'}$ for every $\Gamma \subseteq L_\Sigma(P)$. We say that the extension of $\mathcal{L}$ by $\mathcal{L}'$ is *conservative* if for all $\Gamma \subseteq L_\Sigma(P)$, $\Gamma^{\vdash} = \Gamma^{\vdash'} \cap L_\Sigma(P)$. We also say that the extension of $\mathcal{L}$ by $\mathcal{L}'$ is *weakly conservative* if $\emptyset^{\vdash} = \emptyset^{\vdash'} \cap L_\Sigma(P)$. It is perhaps more common to express these properties in terms of the induced consequence relations. Clearly, $\mathcal{L}'$ extends $\mathcal{L}$ when $\Gamma \vdash \varphi$ implies $\Gamma \vdash' \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq L_\Sigma(P)$.

Furthermore, the extension is conservative precisely when $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash' \varphi$, and weakly conservative when $\vdash \varphi$ if and only if $\vdash' \varphi$.

Given a signature Σ, it is well known that the set of all logics over Σ, $\mathbf{Log}(\Sigma)$, constitutes a complete lattice under the extension ordering defined above (see [17], for instance).

2.4. Hilbert calculi

A *Hilbert calculus* is a pair $\mathscr{H} = \langle \Sigma, R \rangle$ where Σ is a signature, and $R \subseteq 2^{L_\Sigma(P)} \times L_\Sigma(P)$ is a set of *inference rules*. Given $\langle \Delta, \psi \rangle \in R$, we refer to Δ as the set of *premises* and to ψ as the *conclusion* of the rule. When the set of premises is empty, ψ is dubbed an *axiom*. A rule is said to be *finitary* if it has a finite set of premises, and $\mathscr{H}$ is said to be *finitary* of all the rules in R are finitary. An inference rule $\langle \Delta, \psi \rangle \in R$ is often denoted by $\frac{\Delta}{\psi}$, or simply by $\frac{\psi_1 \; \dots \; \psi_n}{\psi}$ if $\Delta = \{\psi_1, \dots, \psi_n\}$ is finite, or even by $\frac{}{\psi}$ if $\Delta = \emptyset$.

Given $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$, a Hilbert calculus $\mathscr{H} = \langle \Sigma, R \rangle$ induces a consequence operator $\_^{\vdash_{\mathscr{H}}}$ on $L_{\Sigma'}(P')$ such that, for each $\Gamma \subseteq L_{\Sigma'}(P')$, $\Gamma^{\vdash_{\mathscr{H}}}$ is the least set that contains Γ and is closed for all applications of instances of the inference rules in R, that is, if $\frac{\Delta}{\psi} \in R$ and $\sigma : P \to L_{\Sigma'}(P')$ is such that $\Delta^\sigma \subseteq \Gamma^{\vdash_{\mathscr{H}}}$ then $\psi^\sigma \in \Gamma^{\vdash_{\mathscr{H}}}$. Of course, this definition induces a logic $\mathcal{L}_{\mathscr{H}} = \langle \Sigma, \_^{\vdash_{\mathscr{H}}} \rangle$.

The definition of $\mathcal{L}_{\mathscr{H}}$ above is arguably too abstract, as it does not highlight the sequence of rule applications that leads one to conclude that $\Gamma \vdash_{\mathscr{H}} \varphi$, when that is the case. Let us be more detailed. Given $\Sigma \subseteq \Sigma'$, $P \subseteq P'$, and $\Gamma \subseteq L_{\Sigma'}(P')$, a *$\mathscr{H}$-derivation from* Γ is a sequence $\overline{\varphi} = \langle \varphi_\kappa \rangle_{\kappa < \eta}$ of formulas in $L_{\Sigma'}(P')$, for some ordinal η, such that, for each $\kappa < \eta$, either:

- $\varphi_\kappa \in \Gamma$, or
- there is $\frac{\Delta}{\psi} \in R$ and $\sigma : P \to L_{\Sigma'}(P')$ with $\psi^\sigma = \varphi_\kappa$ and $\Delta^\sigma \subseteq \{\varphi_\tau : \tau < \kappa\}$.

The fact that $\overline{\varphi}$ is a $\mathscr{H}$-derivation from Γ is denoted by $\Gamma \vdash_{\mathscr{H}} \overline{\varphi}$. We say that such a derivation is a *$\mathscr{H}$-proof from* Γ of each of its formulas, as it is clear that any prefix of a $\mathscr{H}$-derivation from Γ is also a $\mathscr{H}$-derivation from Γ.

Clearly, $\Gamma \vdash_{\mathscr{H}} \varphi$ precisely if φ has a $\mathscr{H}$-proof from Γ, that is, there exists some $\mathscr{H}$-derivation $\langle \varphi_\kappa \rangle_{\kappa < \eta}$ from Γ such that $\varphi = \varphi_\kappa$ for some $\kappa < \eta$, in which case $\langle \varphi_\iota \rangle_{\iota < \kappa + 1}$ is a $\mathscr{H}$-proof of φ from Γ ending in φ.

Example 2.2.

Along the paper, in order to illustrate the problems at hand and the results obtained we will use the following collection of examples:

- $\mathscr{H}_{\mathsf{inc}(\Sigma)} = \langle \Sigma, R_{\mathsf{inc}} \rangle$, for each signature Σ, where R_{inc} has the unique rule

$$\frac{}{p}.$$

- $\mathscr{H}_{\mathsf{tonk}} = \langle \Sigma_{\mathsf{tonk}}, R_{\mathsf{tonk}} \rangle$, where Σ_{tonk} has a unique 2-place connective tonk, and R_{tonk} has the rules

$$\frac{p}{\mathsf{tonk}(p,q)} \qquad \frac{\mathsf{tonk}(p,q)}{q}.$$

- $\mathscr{H}_{\mathsf{cls}} = \langle \Sigma_{\mathsf{cls}}, R_{\mathsf{cls}} \rangle$, where Σ_{cls} has a unique 2-place connective $\Rightarrow$, and R_{cls} has the rules

$$\frac{}{(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))}$$

$$\frac{}{p \Rightarrow (q \Rightarrow p)} \qquad \frac{}{((p \Rightarrow q) \Rightarrow p) \Rightarrow p} \qquad \frac{p \quad p \Rightarrow q}{q}.$$

- $\mathscr{H}_{\mathsf{int}} = \langle \Sigma_{\mathsf{int}}, R_{\mathsf{int}} \rangle$, where Σ_{int} has a unique 2-place connective $\to$, and R_{int} has the rules

$$\frac{}{(p \to (q \to r)) \to ((p \to q) \to (p \to r))}$$

$$\frac{}{p \to (q \to p)} \qquad \frac{p \quad p \to q}{q}.$$

- $\mathscr{H}_{\mathsf{neg}} = \langle \Sigma_{\mathsf{neg}}, R_{\mathsf{neg}} \rangle$, where Σ_{neg} has a unique 1-place connective $\neg$, and R_{neg} has the rules

$$\frac{p}{\neg\neg p} \qquad \frac{\neg\neg p}{p} \qquad \frac{p \quad \neg p}{q}.$$

- $\mathscr{H}_{\mathsf{cnj}} = \langle \Sigma_{\mathsf{cnj}}, R_{\mathsf{cnj}} \rangle$, where $\Sigma_{\wedge}$ has a unique 1-place connective $\wedge$, and R_{cnj} has the rules

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q} \qquad \frac{p \quad q}{p \wedge q}.$$

Clearly, each $\mathscr{H}_{\mathsf{inc}(\Sigma)}$ induces an inconsistent logic, whereas $\mathscr{H}_{\mathsf{tonk}}$ is Prior's infamous tonk system and induces a consistent but trivial logic. The calculi $\mathscr{H}_{\mathsf{cls}}$ and $\mathscr{H}_{\mathsf{int}}$ induce the logics of *classical implication* and *intuitionistic implication*, respectively. Finally, $\mathscr{H}_{\mathsf{neg}}$ induces the logic of (classical or intuitionistic) *negation*, and $\mathscr{H}_{\mathsf{cnj}}$ the logic of (classical or intuitionistic) *conjunction* . Note that with the possible exception of the $\mathscr{H}_{\mathsf{inc}(\Sigma)}$ calculi, all other examples have very simple signatures with one single connective. This is a deliberate choice meant to keep the focus of attention on the relevant problems ahead, and not on the relative complexity of the syntax. △

Given a logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, we can easily associate with it a Hilbert calculus $\mathscr{H}_{\mathcal{L}} = \langle \Sigma, \vdash \rangle$, where the consequence operator $\_^{\vdash}$ in the former is replaced by the induced consequence relation $\vdash$ in the latter. It is easy to check that $\mathcal{L}_{\mathscr{H}_{\mathcal{L}}} = \mathcal{L}$ (see [17], for instance).

For simplicity, we will use $\mathcal{L}_{\mathsf{name}}$ to denote the logic $\mathcal{L}_{\mathscr{H}_{\mathsf{name}}}$ for each of the calculi named in Example 2.2.

2.5. Fibring

Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be two logics. The *fibring* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the smallest logic $\mathcal{L}_1 \bullet \mathcal{L}_2$ over the joint signature $\Sigma_{12} = \Sigma_1 \cup \Sigma_2$ that extends both $\mathcal{L}_1$ and $\mathcal{L}_2$. A direct characterization of this fibred logic can be most easily given by first defining the fibring of Hilbert calculi.

Given Hilbert calculi $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ let their *fibring* be the Hilbert calculus

$$\mathscr{H}_1 \bullet \mathscr{H}_2 = \langle \Sigma_{12}, R_1 \cup R_2 \rangle.$$

Clearly, besides joining their signatures, the fibring of the two calculi consists in simply putting together their rules.

We can now give a simple characterization (see [4]) of the fibring of two logics $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\mathcal{L}_1 \bullet \mathcal{L}_2 = \mathcal{L}_{\mathscr{H}_{\mathcal{L}_1} \bullet \mathscr{H}_{\mathcal{L}_1}}.$$

This means that if $\mathcal{L}_1 \bullet \mathcal{L}_2 = \langle \Sigma_{12}, \_^{\vdash_{12}} \rangle$ then, given $\Gamma \subseteq L_{\Sigma_{12}}(P)$, $\Gamma^{\vdash_{12}}$ is obtained by a (possibly transfinite) sequence of alternate applications of $\vdash_1$ and $\vdash_2$ using substitutions $\sigma : P \to L_{\Sigma_{12}}(P)$.

Both for logics and Hilbert calculi, when there are no shared connectives, i.e. $\Sigma_1 \cap \Sigma_2 = \emptyset$, the fibring is usually said to be *unconstrained*.

3. Theoremhood and mixed reasoning

In this section we will review the notion of theoremhood in fibred logics, illustrated by means of a series of examples, and then, with the focus on unconstrained fibring, we obtain a technical result about the way mixed reasoning can be controlled in the fibred logic.

3.1. Theoremhood in fibred logics

We start by proving a few simple results characterizing the nature of the theoremhood relation in fibred logics. Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be two logics, and $\mathcal{L}_1 \bullet \mathcal{L}_2 = \langle \Sigma_{12}, \_^{\vdash_{12}} \rangle$ be their fibring.

Proposition 3.1. *$\mathcal{L}_1 \bullet \mathcal{L}_2$ has theorems if and only if $\mathcal{L}_1$ has theorems or $\mathcal{L}_2$ has theorems.*

Proof. If $\vdash_{12} \psi$, for some $\psi \in L_{\Sigma_{12}}(P)$, then, by definition of fibring, there exists a $(\mathscr{H}_{\mathcal{L}_1} \bullet \mathscr{H}_{\mathcal{L}_2})$-proof $\overline{\psi} = \langle \psi_\kappa \rangle_{\kappa < \eta+1}$ (from $\emptyset$) such that $\psi_\eta = \psi$. Thus, it is also the case that $\vdash_{12} \psi_0$. Hence, by definition of derivation, there must exist $\frac{\Delta}{\varphi} \in (\vdash_1 \cup \vdash_2)$ and $\sigma : P \to L_\Sigma(P)$ such that $\varphi^\sigma = \psi_0$ and $\Delta^\sigma \subseteq \{\psi_\tau : \tau < 0\} = \emptyset$. Therefore, $\Delta = \emptyset$, and we have that $\vdash_i \varphi$ provided that $\frac{\emptyset}{\varphi} \in \vdash_i$. We conclude that either $\mathcal{L}_1$ has theorems or $\mathcal{L}_2$ has theorems.

Reciprocally, let $i \in \{1, 2\}$ and assume that $\mathcal{L}_i$ has a theorem, that is, $\vdash_i \varphi$ for some $\varphi \in L_{\Sigma_i}(P)$. By definition of fibring, $\vdash_i \varphi$ implies $\vdash_{12} \varphi$, and we conclude that $\mathcal{L}_1 \bullet \mathcal{L}_2$ has theorems. $\square$

This result tells us that the theoremhood relation is empty in logics resulting from the fibring of two logics without theorems. Let us look at an example.

Example 3.2.

Take the logics $\mathcal{L}_{\mathsf{cnj}}$ and $\mathcal{L}_{\mathsf{neg}}$ from Example 2.2. Their fibring $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}}$ does not have theorems, since neither of $\mathcal{L}_{\mathsf{cnj}}$ and $\mathcal{L}_{\mathsf{neg}}$ have theorems. Just note that the calculi defining the two logics do not have any axioms.

The logic $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}}$ is not to be confused with the conjunction and negation fragment of classical logic, which of course has theorems, e.g., $\neg(p \wedge \neg p)$. △

Even if one of the logics has theorems, the theoremhood relation in the fibred logic can still be quite uninteresting.

Proposition 3.3. *If $\mathcal{L}_1$ is inconsistent or $\mathcal{L}_2$ is inconsistent then $\mathcal{L}_1 \bullet \mathcal{L}_2$ is inconsistent.*

Proof. Let $\mathcal{L}_i$ be inconsistent, for some $i \in \{1, 2\}$. Then $\vdash_i p$, which implies $\vdash_{12} p$, and we conclude that $\mathcal{L}_1 \bullet \mathcal{L}_2$ is inconsistent. □

This result shows that every formula is a theorem in logics resulting from fibrings involving a trivial logic. Let us see an example.

Example 3.4.

Take the $\mathcal{L}_{\mathsf{inc}(\Sigma)}$ from Example 2.2, for some signature Σ, and consider any other logic $\mathcal{L}$. Their fibring $\mathcal{L}_{\mathsf{inc}(\Sigma)} \bullet \mathcal{L}$ is inconsistent. Namely, $p \in P$ is a theorem, and therefore, by structurality, every formula is a theorem. △

Still, a fibred logic can be inconsistent even when both the logics being combined are consistent.

Proposition 3.5. *If $\mathcal{L}_1$ is consistent and trivial and $\mathcal{L}_2$ has theorems then $\mathcal{L}_1 \bullet \mathcal{L}_2$ is inconsistent.*

Proof. If $\mathcal{L}_1$ is trivial, we know that $p \vdash_1 q$, where $p, q \in P$ and $p \neq q$. If $\mathcal{L}_2$ has theorems, let $\vdash_2 \varphi$ for some $\varphi \in L_{\Sigma_2}(P)$. Now, $p \vdash_1 q$ implies $p \vdash_{12} q$, and structurality, on its turn, implies that $\varphi \vdash_{12} q$. But $\vdash_2 \varphi$ implies $\vdash_{12} \varphi$, and thus $\vdash_{12} q$. We conclude that $\mathcal{L}_1 \bullet \mathcal{L}_2$ is inconsistent. □

An example follows.

Example 3.6.

Take the logics $\mathcal{L}_{\mathsf{tonk}}$ and $\mathcal{L}_{\mathsf{cls}}$ from Example 2.2, where $\mathcal{L}_{\mathsf{tonk}}$ is consistent but trivial, as $p \vdash_{\mathsf{tonk}} q$, and $\mathcal{L}_{\mathsf{cls}}$ has theorems, namely $\vdash_{\mathsf{cls}} p \Rightarrow p$. Their fibring $\mathcal{L}_{\mathsf{tonk}} \bullet \mathcal{L}_{\mathsf{cls}} = \langle \Sigma_{\mathsf{tonk}} \cup \Sigma_{\mathsf{cls}}, \_^{\vdash} \rangle$ is inconsistent. Take any formula $\varphi \in L_{\Sigma_{\mathsf{tonk}} \cup \Sigma_{\mathsf{cls}}}(P)$. Easily, $p \vdash_{\mathsf{tonk}} q$ implies that $p \vdash q$, and structurality in the fibred logic implies that $p \Rightarrow p \vdash \varphi$. Therefore, as $\vdash_{\mathsf{cls}} p \Rightarrow p$ implies $\vdash p \Rightarrow p$, we have that $\vdash \varphi$. △

Triviality is actually preserved by fibring, leading to uninteresting trivial fibred logics, where the theoremhood relation is therefore total (when the fibred logic becomes inconsistent) or empty (when the fibred logic remains consistent).

Proposition 3.7. *If $\mathcal{L}_1$ is trivial or $\mathcal{L}_2$ is trivial then $\mathcal{L}_1 \bullet \mathcal{L}_2$ is trivial.*

Proof. Let $\mathcal{L}_i$ be trivial, for some $i \in \{1,2\}$. Then $p \vdash_i q$ with $p \neq q$, which implies $p \vdash_{12} q$, and we conclude that $\mathcal{L}_1 \bullet \mathcal{L}_2$ is trivial. □

At this point we know that the theoremhood relationship in $\mathcal{L}_1 \bullet \mathcal{L}_2$ is only interesting if $\mathcal{L}_1$ and $\mathcal{L}_2$ are both non-trivial and at least one of the two has theorems. In that case, how can we decide if a (mixed) formula is a theorem of $\mathcal{L}_1 \bullet \mathcal{L}_2$?

Example 3.8.
Take the logics $\mathcal{L}_{\mathsf{cls}}$ and $\mathcal{L}_{\mathsf{int}}$ from Example 2.2. Both logics are non-trivial, and have theorems. Furthermore, it is known that $\mathcal{L}_{\mathsf{cls}}$ and $\mathcal{L}_{\mathsf{int}}$ are decidable. The fibring $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{int}}$ was studied in [6, 7], and shown to be a conservative extension of both $\mathcal{L}_{\mathsf{cls}}$ and $\mathcal{L}_{\mathsf{int}}$. We will show below that the theorems of $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{int}}$ are indeed decidable. △

3.2. Consequences of non-mixed formulas

Let us now have a more technical look at the patterns of mixed reasoning that occur in fibred logics. We start with a result about syntax.

Lemma 3.9. *Let $\Sigma \subseteq \Sigma'$ and $\Gamma \subseteq L_\Sigma(P)$. Then, for every $\sigma : P \to L_{\Sigma'}(P)$, and every two η-sequences $\overline{\alpha}$ and $\overline{\beta}$ of formulas in $L_{\Sigma'}(P)$, with $\overline{\alpha}$ injective, there exists $\rho : P \to L_{\Sigma'}(P)$ such that*

$$\Gamma^\rho = \Gamma^\sigma[\overline{\alpha}/\overline{\beta}]_\Sigma.$$

Proof. One should observe, to start with, that $\mathsf{Mon}_\Sigma(\Gamma) = \emptyset$. Thus, if $\alpha_\kappa \in \mathsf{Mon}_\Sigma(\varphi^\sigma)$ for some $\varphi \in \Gamma$, then there must exist a variable $p \in P$ occurring in φ such that $\alpha_\kappa \in \mathsf{Mon}_\Sigma(\sigma(p))$. Hence, the substitution defined by $\rho(q) = \sigma(q)[\overline{\alpha}/\overline{\beta}]_\Sigma$ for every $q \in P$ satisfies the conditions of the lemma. □

The previous lemma reflects the fact that the occurrence of Σ-monoliths in instances of $L_\Sigma(P)$ formulas is only possible if they are brought about by the substitution.

Next we prove a (quite) technical lemma, characterizing the irrelevance of certain monoliths in derivations from a set of variables in logics obtained by unconstrained fibring, motivated by the square bracket monolith substitutions introduced earlier. Note that the disjointness of the signatures is instrumental in proving this result.

Lemma 3.10. *Let $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ be Hilbert calculi such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, $V \subseteq P$ and $\overline{\psi} = \langle \psi_\kappa \rangle_{\kappa<\eta}$ a sequence of $L_{\Sigma_{12}}(P)$ formulas.*

If $V \vdash_{\mathscr{H}_{12}} \overline{\psi}$ and $\alpha \in L_{\Sigma_{12}}(P)$ then, either

- *$\alpha = \psi_\kappa$ for some $\kappa < \eta$, or*
- *$V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\eta}$ for every $\beta \in L_{\Sigma_{12}}(P)$ and $i \in \{1,2\}$.*

Proof. Let us assume that $\alpha \neq \psi_\kappa$ for every $\kappa < \eta$. The proof of the second condition follows by complete transfinite induction on the size η of the derivation. For each $\iota < \tau \leq \eta$, we assume, by induction hypothesis, that $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\iota}$, and show that it implies $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$.

If $\tau = 0$ the result is trivial, as the derivation is empty. If τ is a limit ordinal the result is immediate, by definition of derivation. If τ is a successor ordinal, we have to consider two cases.

(1) $\psi_{\tau-1} \in V$.
Then, $\psi_{\tau-1} \in V \subseteq P$, and we have that $\alpha \notin \mathsf{Mon}_{\Sigma_i}(\psi_{\tau-1}) = \emptyset$. Thus, $\psi_{\tau-1}[\alpha/\beta]_{\Sigma_i} = \psi_{\tau-1} \in V$.
By induction hypothesis we have that $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau-1}$ and so, by definition of derivation, we also have $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$.

(2) $\psi_{\tau-1} = \varphi^\sigma$, $\frac{\Delta}{\varphi} \in R_1 \cup R_2$, and $\Delta^\sigma \subseteq \{\psi_\kappa : \kappa < \tau - 1\}$.
Here we have two possibilities, given that $i \in \{1, 2\}$.

(a) $\frac{\Delta}{\varphi} \in R_i$.
Applying Lemma 3.9 to $\Delta \cup \{\varphi\} \subseteq L_{\Sigma_i}(P)$, σ, α and β, we know that there exists ρ such that $\varphi^\rho = \varphi^\sigma[\alpha/\beta]_{\Sigma_i} = \psi_{\tau-1}[\alpha/\beta]_{\Sigma_i}$, and also $\Delta^\rho = \Delta^\sigma[\alpha/\beta]_{\Sigma_i} \subseteq \{\psi_\kappa[\alpha/\beta]_{\Sigma_i} : \kappa < \tau - 1\}$.
By induction hypothesis we have that $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau-1}$ and so, by definition of derivation, we also have $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$.

(b) $\frac{\Delta}{\varphi} \notin R_i$.
If $\delta \in \Delta$ and $\alpha \in \mathsf{Mon}_{\Sigma_i}(\delta^\sigma)$ then (since $\delta \in L_{\Sigma_{3-i}}(P)$) either $\alpha = \delta^\sigma$ or $\mathsf{head}(\delta^\sigma) \in \Sigma_i$. By assumption the former cannot be the case, therefore we must have $\mathsf{head}(\delta^\sigma) \in \Sigma_i$. Hence $\delta \in P$ and $\beta \in \mathsf{Mon}_{\Sigma_i}(\delta^\sigma)$. Consider the substitution defined by $\rho(q) = \sigma(q)[\alpha/\beta]_{\Sigma_i}$ for every $q \in P$. Clearly, as above, $\varphi^\rho = \varphi^\sigma[\alpha/\beta]_{\Sigma_i} = \psi_{\tau-1}[\alpha/\beta]_{\Sigma_i}$, and also $\Delta^\rho = \Delta^\sigma[\alpha/\beta]_{\Sigma_i} \subseteq \{\psi_\kappa[\alpha/\beta]_{\Sigma_i} : \kappa < \tau - 1\}$.
By induction hypothesis we have that $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau-1}$ and so, by definition of derivation, we also have $V \vdash_{\mathscr{H}_{12}} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$. □

In order to finally state and prove our main result about mixed reasoning, we need the following definition, where we are going to collect extra variables to represent some contextual relevant properties.

Definition 3.11. Let $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ be Hilbert calculi, $V \subseteq P$ $\psi \in L_{\Sigma_{12}}(P)$ and $i \in \{1, 2\}$. We define $X^i_V(\psi) \subseteq \{x_0\} \cup \{x_{e^{-1}(\psi')} : \psi' \in \mathsf{Mon}_{\Sigma_t}(\psi)\}$ to be the set such that:

- $x_0 \in X^i_V(\psi)$ whenever $V^{\vdash_{\mathscr{H}_{12}}} \neq \emptyset$, and
- $x_{e^{-1}(\psi')} \in X^i_V(\psi)$ whenever $V \vdash_{\mathscr{H}_{12}} \psi'$.

Example 3.12.
If $\mathscr{H}_1 = \mathscr{H}_{\mathsf{cnj}}$, $\mathscr{H}_2 = \mathscr{H}_{\mathsf{tonk}}$ then, for $i \in \{1,2\}$, we have the following equalities.

$$X^i_{\emptyset}(q) = \emptyset$$
$$X^i_{\{p\}}(q) = \{x_0\}$$
$$X^1_{\emptyset}(p \land \mathsf{tonk}(p,q)) = \{x_{e^{-1}(\mathsf{tonk}(p,q))}\}$$
$$X^1_{\{p\}}(p \land \mathsf{tonk}(p,q)) = \{x_0, x_{e^{-1}(\mathsf{tonk}(p,q))}\}$$
$$X^2_{\emptyset}(p \land \mathsf{tonk}(p,q)) = \{x_{e^{-1}(p\land\mathsf{tonk}(p,q))}\}$$

△

We can finally prove the following result, relating proofs from variables in the fibred logic with proofs from variables in the component logics, in the case of unconstrained fibring.

Proposition 3.13. *Let $\mathscr{H}_1 = \langle \Sigma_1, R_1\rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2\rangle$ be Hilbert calculi such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, $V \subsetneq P$, and $\psi \in L_{\Sigma_{12}}(P)$. Then, for $i, j \in \{1,2\}$ with $i \neq j$, $V \vdash_{\mathscr{H}_{12}} \psi$ if and only if*

$$V, X^i_V(\psi) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi) \ \text{ or } (V \cup (\{x_0\} \cap X^i_V(\psi)))^{\vdash_{\mathscr{H}_j}} = L_{\Sigma_{12}}(P \cup X).$$

Proof. We start by proving the simpler implication from right to left. Let $p \in P\backslash V$. If $V^{\vdash_{\mathscr{H}_{12}}} \neq \emptyset$ fix $\gamma \in V^{\vdash_{\mathscr{H}_{12}}}$, and let $\sigma : P \cup X \to L_{\Sigma_{12}}(P)$ be such that $\sigma(p) = \psi$, $\sigma(q) = q$ if $q \in P$ with $q \neq p$, $\sigma(x_n) = e(n)$ for $n > 0$, and

$$\sigma(x_0) = \begin{cases} x_0 & \text{if } V^{\vdash_{\mathscr{H}_{12}}} = \emptyset, \\ \gamma & \text{if } V^{\vdash_{\mathscr{H}_{12}}} \neq \emptyset. \end{cases}$$

Now, knowing on one hand that $V, X^i_V(\psi) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi)$ and using structurality, we obtain $V^\sigma, (X^i_V(\psi))^\sigma \vdash_{\mathscr{H}_i} (\mathsf{skel}_{\Sigma_i}(\psi))^\sigma$. But clearly, $V^\sigma = V$, $(X^i_V(\psi))^\sigma \subseteq V^{\vdash_{\mathscr{H}_{12}}}$, $(\mathsf{skel}_{\Sigma_i}(\psi))^\sigma = \psi$, and we conclude that $V \vdash_{\mathscr{H}_{12}} \psi$.

If, on the other hand, we know that $(V \cup (\{x_0\} \cap X^i_V(\psi)))^{\vdash_{\mathscr{H}_j}} = L_{\Sigma_{12}}(P \cup X)$, then we have $V \cup (\{x_0\} \cap X^i_V(\psi)) \vdash_{\mathscr{H}_j} p$. But clearly, $V^\sigma = V$, $(\{x_0\} \cap X^i_V(\psi))^\sigma \subseteq V^{\vdash_{\mathscr{H}_{12}}}$, $p^\sigma = \psi$, and we conclude again that $V \vdash_{\mathscr{H}_{12}} \psi$.

Let us now consider the implication from left to right, and assume that we have $V \vdash_{\mathscr{H}_{12}} \psi$. Clearly, $V^{\vdash_{\mathscr{H}_{12}}} \neq \emptyset$ and so $x_0 \in X^t_V(\varphi)$ for every $\varphi \in L_{\Sigma_{12}}(P)$ and every $t \in \{1,2\}$. If $(V \cup (\{x_0\} \cap X^i_V(\psi)))^{\vdash_{\mathscr{H}_j}} = (V \cup \{x_0\})^{\vdash_{\mathscr{H}_j}} = L_{\Sigma_{12}}(P \cup X)$ the statement immediately follows, hence we proceed assuming that we have

$$(V \cup \{x_0\})^{\vdash_{\mathscr{H}_j}} \neq L_{\Sigma_{12}}(P \cup X).$$

The proof follows by complete transfinite induction on the length of $\mathscr{H}_{12}$-derivations. Given that $V \vdash_{\mathscr{H}_{12}} \psi$, there must exist a $\mathscr{H}_{12}$-derivation $\overline{\psi} = \langle \psi_\kappa \rangle_{\kappa<\eta+1}$ from V such that $\psi_\eta = \psi$. We want to show that $V, X^i_V(\psi_\eta) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\eta)$. Thus, we will prove that $V, X^i_V(\psi_\tau) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\tau)$ for any $\tau \leq \eta$ assuming, by induction hypothesis, that the left to right implication holds for any $\mathscr{H}_{12}$-derivation with length lower than τ, and for both i and j.

Note that the case when $\mathsf{head}(\psi_\tau) \in \Sigma_j$ is trivial. Indeed, in that situation, we have that $\mathsf{Mon}_{\Sigma_i}(\psi_\tau) = \{\psi_\tau\}$, and thus $\mathsf{skel}_{\Sigma_i}(\psi_\tau) = x_{e^{-1}(\psi_\tau)} \in X^i_V(\psi_\tau)$. But then, clearly, $V, X^i_V(\psi_\tau) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\tau)$. We assume, henceforth, that either $\psi_\tau \in P$ or $\mathsf{head}(\psi_\tau) \in \Sigma_i$, which implies that $\mathsf{skel}_{\Sigma_j}(\psi_\tau) \in P \cup X$.

We have to consider two cases.

(1) $\psi_\tau \in V$.

We have that $\mathsf{Mon}_{\Sigma_i}(\psi_\tau) = \emptyset$. But then, clearly, $V, X^i_V(\psi_\tau) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\tau)$ as $\mathsf{skel}_{\Sigma_i}(\psi_\tau) = \psi_\tau \in V$.

(2) $\psi_\tau = \varphi^\sigma$, $\frac{\Delta}{\varphi} \in R_1 \cup R_2$, and $\Delta^\sigma \subseteq \{\psi_\kappa : \kappa < \tau\}$.

Here we have two possibilities.

(a) $\frac{\Delta}{\varphi} \in R_i$.

Let $M = \bigcup_{\kappa \le \tau} \mathsf{Mon}_{\Sigma_i}(\psi_\kappa)$. Consider any injective sequence $\overline{\alpha}$ of formulas in M where each formula in M appears exactly once, and define $\overline{\beta}$ to be the same length sequence such that each $\beta_\iota = x_{e^{-1}(\alpha_\iota)}$. Note that $\psi_\kappa[\overline{\alpha}/\overline{\beta}]_{\Sigma_i} = \mathsf{skel}_{\Sigma_i}(\psi_\kappa)$ for every $\kappa \le \tau$.
Hence, applying Lemma 3.9 to $\Delta \cup \{\varphi\} \subseteq L_{\Sigma_i}(P)$, σ, $\overline{\alpha}$ and $\overline{\beta}$, we know that there exists ρ such that $\varphi^\rho = \varphi^\sigma[\overline{\alpha}/\overline{\beta}]_{\Sigma_i} = \psi_\tau[\overline{\alpha}/\overline{\beta}]_{\Sigma_i} = \mathsf{skel}_{\Sigma_i}(\psi_\tau)$, and $\Delta^\rho = \Delta^\sigma[\overline{\alpha}/\overline{\beta}]_{\Sigma_i} \subseteq \{\psi_\kappa[\overline{\alpha}/\overline{\beta}]_{\Sigma_i} : \kappa < \tau\} = \{\mathsf{skel}_{\Sigma_i}(\psi_\kappa) : \kappa < \tau\}$, and we can conclude that $\{\mathsf{skel}_{\Sigma_i}(\psi_\kappa) : \kappa < \tau\} \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\tau)$.
By induction hypothesis we have that $V, X^i_V(\psi_\kappa) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\kappa)$ for each $\kappa < \tau$, and therefore, we have that $V, \bigcup_{\kappa<\tau} X^i_V(\psi_\kappa) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\tau)$.
Consider the substitution $\mu : P \cup X \to L_{\Sigma_i}(P \cup X)$ such that

* $\mu(p) = p$ if $p \in P$
* $\mu(x_n) = x_0$ if $e(n) \notin \mathsf{Mon}_{\Sigma_i}(\psi_\tau)$,
* and $\mu(x_n) = x_n$ otherwise.

Clearly, we have $V^\mu = V$, $(X^i_V(\psi_\kappa))^\mu \subseteq X^i_V(\psi_\tau)$ for each $\kappa < \tau$, and $(\mathsf{skel}_{\Sigma_i}(\psi_\tau))^\mu = \mathsf{skel}_{\Sigma_i}(\psi_\tau)$. Thus, by structurality and monotonicity, $V, X^i_V(\psi_\tau) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\tau)$.

(b) $\frac{\Delta}{\varphi} \in R_j$.

If $\psi_\kappa = \psi_\tau$ for some $\kappa < \tau$, by induction hypothesis we have that $V, X^i_V(\psi_\kappa) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\kappa)$ and so $V, X^i_V(\psi_\tau) \vdash_{\mathscr{H}_i} \mathsf{skel}_{\Sigma_i}(\psi_\tau)$.

We shall finish the proof by showing that no other case is possible. That is, assuming that either $\psi_\tau \in P \setminus V$ (the case when $\psi_\tau \in V$ was covered in **(1)**) or $\mathsf{head}(\psi_\tau) \in \Sigma_i$, and also that $\psi_\kappa \neq \psi_\tau$ for every $\kappa < \tau$, we will derive a contradiction. Let $p \in P \setminus V$. We split in yet another two cases.

(i) $\psi_\tau \in P \setminus V$.

Let $\psi_\tau = p \in P \setminus V$, and $M = \bigcup_{\kappa \le \tau} \mathsf{Mon}_{\Sigma_j}(\psi_\kappa)$. Consider any injective sequence $\overline{\alpha}$ of formulas in M where each formula in M

appears exactly once, and define $\overline{\beta}$ to be the same length sequence such that each $\beta_\iota = x_{e^{-1}(\alpha_\iota)}$. Hence, applying Lemma 3.9 to $\Delta \cup \{\varphi\}$, σ, $\overline{\alpha}$ and $\overline{\beta}$, we know that there exists ρ such that $\varphi^\rho = \varphi^\sigma[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} = \psi_\tau[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} = \mathsf{skel}_{\Sigma_j}(\psi_\tau) = p$, and also $\Delta^\rho = \Delta^\sigma[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} \subseteq \{\psi_\kappa[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} : \kappa < \tau\} = \{\mathsf{skel}_{\Sigma_j}(\psi_\kappa) : \kappa < \tau\}$. We conclude that $\{\mathsf{skel}_{\Sigma_j}(\psi_\kappa) : \kappa < \tau\} \vdash_{\mathscr{H}_j} \mathsf{skel}_{\Sigma_j}(\psi_\tau) = p$.
As $V \vdash_{\mathscr{H}_{12}} \langle\psi_\kappa\rangle_{\kappa<\tau}$, by induction hypothesis, we get to know that $V, X^j_V(\psi_\kappa) \vdash_{\mathscr{H}_j} \mathsf{skel}_{\Sigma_j}(\psi_\kappa)$ for each $\kappa < \tau$. Thus, we also have $V, \bigcup_{\kappa<\tau} X^j_V(\psi_\kappa) \vdash_{\mathscr{H}_j} p$.
Consider the substitution ν such that $\nu(x_n) = x_0$ for all $n \in \mathbb{N}_0$, and $\nu(q) = q$ for all $q \in P$. Clearly $V^\nu = V$, $(\bigcup_{\kappa<\tau} X^j_V(\psi'_\kappa))^\nu \subseteq \{x_0\}$ and $p^\nu = p$. Therefore, by structurality and using ν, we get $V^\nu, \bigcup_{\kappa<\tau} X^j_V(\psi'_\kappa)^\nu \vdash_{\mathscr{H}_j} p^\nu$, which implies $(V \cup \{x_0\})^{\vdash_{\mathscr{H}_j}} = L_{\Sigma_{12}}(P \cup X)$, and we obtain a contradiction.

(ii) $\mathsf{head}(\psi_\tau) \in \Sigma_i$.
Let $p \in P \setminus V$ and define, for all $\kappa \leq \tau$, let $\psi'_\kappa = \psi_\kappa[\psi_\tau/p]_{\Sigma_j}$. Clearly, $\psi'_\tau = \psi_\tau[\psi_\tau/p]_{\Sigma_j} = p$. Let $\mu : P \cup X \to L_{\Sigma_{12}}(P \cup X)$ defined by $\mu(x_{e^{-1}(\psi_\tau)}) = p$, $\mu(x_n) = x_n$ if $n = 0$ or $e(n) \neq \psi_\tau$, and $\mu(q) = q$ for each $q \in P$. Easily, we have that $\mathsf{skel}_{\Sigma_j}(\psi'_\kappa) = (\mathsf{skel}_{\Sigma_j}(\psi_\kappa))^\mu$ for all $\kappa \leq \tau$.
Arguing as in **(i)**, we know that there exists ρ such that $\varphi^\rho = \varphi^\sigma[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} = \psi_\tau[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} = \mathsf{skel}_{\Sigma_j}(\psi_\tau) = x_{e^{-1}(\psi_\tau)}$, and also $\Delta^\rho = \Delta^\sigma[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} \subseteq \{\psi_\kappa[\overline{\alpha}/\overline{\beta}]_{\Sigma_j} : \kappa < \tau\} = \{\mathsf{skel}_{\Sigma_j}(\psi_\kappa) : \kappa < \tau\}$. We conclude that $\{\mathsf{skel}_{\Sigma_j}(\psi_\kappa) : \kappa < \tau\} \vdash_{\mathscr{H}_j} x_{e^{-1}(\psi_\tau)}$. Therefore, by structurality and using μ, it is the case that $(\{\mathsf{skel}_{\Sigma_j}(\psi_\kappa) : \kappa < \tau\})^\mu \vdash_{\mathscr{H}_j} (\mathsf{skel}_{\Sigma_j}(\psi_\tau))^\mu$, that is, $\{\mathsf{skel}_{\Sigma_j}(\psi'_\kappa) : \kappa < \tau\} \vdash_{\mathscr{H}_j} p$.
As $V \vdash_{\mathscr{H}_{12}} \langle\psi_\kappa\rangle_{\kappa<\tau}$ and we assumed that $\psi_\kappa \neq \psi_\tau$ for all $\kappa < \tau$, we can use Lemma 3.10 to conclude that $V \vdash_{\mathscr{H}_{12}} \langle\psi'_\kappa\rangle_{\kappa<\tau}$, and by induction hypothesis, we get that $V, X^j_V(\psi'_\kappa) \vdash_{\mathscr{H}_j} \mathsf{skel}_{\Sigma_j}(\psi'_\kappa)$ for each $\kappa < \tau$. Thus, we also have $V, \bigcup_{\kappa<\tau} X^j_V(\psi'_\kappa) \vdash_{\mathscr{H}_j} p$.
Using the substitution ν as defined in **(i)**, and arguing in the same manner, we arrive at a contradiction. □

4. Decidability

Using the technical results about mixed proofs obtained in the last section, for unconstrained fibring, we can now state and prove our main results concerning theoremhood decidability.

The result of Proposition 3.13 is quite promising, if we want to analyze the decidability of logics obtained by unconstrained fibring. Indeed, under the appropriate circumstances, the result allows us to reduce the problem of checking a theorem (or even a derivation from a finite set of variables) in a fibred logic to the problem of checking a derivation from variables in one of the component logics. Let us have a look at an enlightening example.

Example 4.1.

Consider the unconstrained fibring $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{int}} = \langle \Sigma_{\mathsf{cls}} \cup \Sigma_{\mathsf{int}}, \vdash \rangle$ that we have used in Example 3.8. As direct applications of Proposition 3.13, we can conclude that:

- $\vdash (p \to q) \Rightarrow (p \to q)$, because $(p \to q) \Rightarrow (p \to q) = (x_1 \Rightarrow x_1)[p \to q]_{\Sigma_{\mathsf{cls}}}$, and we have
$$\not\vdash_{\mathsf{int}} p \to q \quad \text{and} \quad x_0 \vdash_{\mathsf{cls}} x_1 \Rightarrow x_1;$$
- $\vdash p \Rightarrow (p \to p)$, because $p \Rightarrow (p \to p) = (p \Rightarrow x_1)[p \to p]_{\Sigma_{\mathsf{cls}}}$, and we have
$$\vdash_{\mathsf{int}} p \to p \quad \text{and} \quad x_0, x_1 \vdash_{\mathsf{cls}} p \Rightarrow x_1;$$
- $\not\vdash (p \to p) \Rightarrow p$, because $(p \to p) \Rightarrow p = (x_1 \Rightarrow p)[p \to p]_{\Sigma_{\mathsf{cls}}}$, and we have
$$\vdash_{\mathsf{int}} p \to p \quad \text{and} \quad x_0, x_1 \not\vdash_{\mathsf{cls}} x_1 \Rightarrow p.$$

Clearly, in this way, to decide theoremood in $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{int}}$, all we need is to have decision procedures for deciding derivation from variables in $\mathcal{L}_{\mathsf{cls}}$ and $\mathcal{L}_{\mathsf{int}}$. △

Recall that a set A is said to be *decidable* if there exists an algorithm D_A that always terminates such that

$$\mathsf{D}_A(w) = \begin{cases} \mathsf{yes} & \text{if } w \in A, \\ \mathsf{no} & \text{if } w \notin A. \end{cases}$$

There are several different flavors of *decidability* that make sense when applied to a logic $\mathcal{L} = \langle \Sigma, {}_{-}^{\vdash} \rangle$, some more standard than others. Assuming that Σ is decidable, we say that $\mathcal{L}$ is:

- *strongly decidable* if there is an algorithm A such that, for each decidable $\Gamma \subseteq L_\Sigma(P)$, $\mathsf{A}(\mathsf{D}_\Gamma)$ terminates and outputs algorithm $\mathsf{D}_{\Gamma^\vdash}$ that decides $\Gamma^\vdash$;
- *finitely decidable* if there is an algorithm A such that, for each finite set $\Gamma \subseteq L_\Sigma(P)$, $\mathsf{A}(\Gamma)$ terminates and outputs an algorithm $\mathsf{D}_{\Gamma^\vdash}$ that decides $\Gamma^\vdash$;
- *var-decidable* if there is an algorithm A such that, for each decidable $V \subseteq P$, $\mathsf{A}(\mathsf{D}_V)$ terminates and outputs an algorithm $\mathsf{D}_{V^\vdash}$ that decides $V^\vdash$;
- *finitely var-decidable* if there is an algorithm A such that, for each finite set $V \subseteq P$, $\mathsf{A}(V)$ terminates and outputs an algorithm $\mathsf{D}_{V^\vdash}$ that decides $V^\vdash$;
- *weakly decidable* if $\emptyset^\vdash$ is decidable.

The decidability of a logic is often identified in the literature with weak decidability, precisely the version that we aimed at in this paper, but one cannot deny the interest in strong or even finite decidability. Note that, due to the shape of Proposition 3.13, we will actually have to deal with finite var-decidability instead of weak decidability.

Of course, these different notions of decidability are related with each other in a simple and clear way, as illustrated in Figure 1. As we know, one can smoothly replace strong decidability with finite decidability when dealing with a finitary logic.

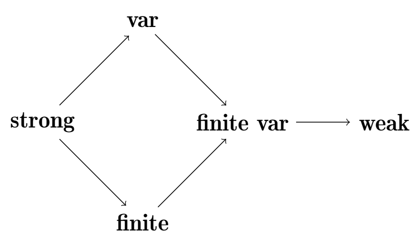

FIGURE 1. The decidability spectrum for a logic $\mathcal{L}$.

Moreover, in the presence of a finitary logic that has an implication-like connective enjoying some form of the *deduction theorem*, weak decidability is equivalent to finite decidability, and also to finite var-decidability.

Proposition 4.2. *If $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ are both finite var-decidable, and $\Sigma_1 \cap \Sigma_2 = \emptyset$, then $\mathcal{L}_1 \bullet \mathcal{L}_2$ is finite var-decidable.*

Proof. As a result of Propositions 3.1, 3.3, 3.5 and 3.7, we can assume that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-trivial (otherwise the fibring is trivial, and either every formula is a theorem, or no formula is a theorem), at least one of the logics has theorems (or the fibring would no have theorems). Let $\mathcal{L}_1$ be finite var-decidable with algorithm A^1, and $\mathcal{L}_2$ be finite var-decidable with algorithm A^2. Consider the following algorithm

A: input $V \subseteq P$
  output D: input $\varphi \in L_{\Sigma_1 \cup \Sigma_2}(P)$
    if $\mathsf{head}(\varphi) \notin \Sigma_2$ then $i \leftarrow 1$ else $i \leftarrow 2$
    let $V_i = V \cup \{x_0\}$, $M = \mathsf{Mon}_{\Sigma_i}(\varphi)$, $\psi = \mathsf{skel}_{\Sigma_i}(\varphi)$
    output $\mathsf{A}^i(V_i \cup \{x_{e^{-1}(\varphi')} : \varphi' \in M, \mathsf{A}(V)(\varphi') = \mathsf{yes}\})(\psi)$.

On input V, A outputs the algorithm $\mathsf{A}(V) = \mathsf{D}$, which uses V to compute $\mathsf{D}(\varphi)$ on input φ. Note that D does some syntactic analysis on the input formula, such as computing its head, monoliths and skeleton. All these constructions are computable, given their definitions, and the assumption that Σ_1 and Σ_2 are decidable. For this reason, one can test membership in a signature, and compute a full enumeration $e : \mathbb{N} \to L_{\Sigma_1 \cup \Sigma_2}(P)$, and its inverse.

We claim that $\mathcal{L}_1 \bullet \mathcal{L}_2$ is finite var-decidable with algorithm A. We show, by induction on the structure of $\varphi \in L_{\Sigma_1 \cup \Sigma_2}(P)$ that, for all finite $V \subseteq P$, $\mathsf{A}(V)(\varphi)$ terminates, and

$$\mathsf{A}(V)(\varphi) = \begin{cases} \mathsf{yes} & \text{if } V \vdash_{12} \varphi, \\ \mathsf{no} & \text{if } V \nvdash_{12} \varphi. \end{cases}$$

For the induction base, let $\varphi = p \in P$. Obviously $\mathsf{head}(p) = p \notin \Sigma_2$ so $i = 1$, and $V_1 = V \cup \{x_0\}$. Moreover, $M = \mathsf{Mon}_{\Sigma_1}(p) = \emptyset$ and $\psi = \mathsf{skel}_{\Sigma_1}(p) = \varphi$. Easily, $V_1 \cup \{x_{e^{-1}(\varphi')} : \varphi' \in M, \mathsf{A}(V)(\varphi') = \mathsf{yes}\} = V_1 = V \cup X_V^1(p)$, as set in Definition 3.11, because the fibred logic has theorems and thus $X_V^1(p) = \{x_0\}$.

Hence, in the last line, $\mathsf{A}^1(V_1)(p)$ tests precisely whether $V, X_V^1(p) \vdash_1 p$. Since $\mathcal{L}_1$ is finite var-decidable with A^1, the algorithm terminates. As $(V \cup \{x_0\})^{\vdash_2} \neq L_{\Sigma_1\cup\Sigma_2}(P\cup X)$, or $\mathcal{L}_2$ would be trivial, Proposition 3.13 guarantees that the answer is yes if $V \vdash_{12} \varphi$, and no if $V \not\vdash_{12} \varphi$.

For the induction step, let $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for some n-place connective $c = \mathsf{head}(\varphi) \in \Sigma_1 \cup \Sigma_2$. Clearly $i = 1$ if $c \in \Sigma_1$, and $i = 2$ if $c \in \Sigma_2$, and $V_i = V \cup \{x_0\}$. Moreover, $M = \mathsf{Mon}_{\Sigma_i}(\varphi) = \bigcup_{j=1}^n \mathsf{Mon}_{\Sigma_i}(\varphi_j)$ and $\psi = \mathsf{skel}_{\Sigma_i}(\varphi)$. By induction hypothesis, for each $\varphi' \in M$, $\mathsf{A}(V)(\varphi')$ terminates and $\mathsf{A}(V)(\varphi') = \mathsf{yes}$ precisely if $V \vdash_{12} \varphi'$ and so, easily, $V_i \cup \{x_{e^{-1}(\varphi')} : \varphi' \in M, \mathsf{A}(V)(\varphi') = \mathsf{yes}\} = V \cup X_V^i(\varphi)$, as set in Definition 3.11, because the fibred logic has theorems and $x_0 \in X_V^i(\varphi)$. Hence, in the last line, $\mathsf{A}^i(V \cup X_V^i(\varphi))(\mathsf{skel}_{\Sigma_i}(\varphi))$ tests whether $V, X_V^i(\varphi) \vdash_i \mathsf{skel}_{\Sigma_i}(\varphi)$. Since $\mathcal{L}_i$ is finite var-decidable with A^i, the algorithm terminates. As $(V \cup \{x_0\})^{\vdash_{3-i}} \neq L_{\Sigma_1\cup\Sigma_2}(P \cup X)$, or $\mathcal{L}_{3-i}$ would be trivial, Proposition 3.13 guarantees that the answer is yes if $V \vdash_{12} \varphi$, and no if $V \not\vdash_{12} \varphi$. □

Let us now have a quick look at the complexity of the decision procedure.

Proposition 4.3. *Let A be the decision algorithm for $\mathcal{L}_1 \bullet \mathcal{L}_2$ defined in the proof of Proposition 4.2, using algorithms A^1 and A^2 for deciding $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The running time of $\mathsf{A}(V)(\varphi)$ for inputs $V \subseteq P$ and $\varphi \in L_{\Sigma_1\cup\Sigma_2}(P)$ is given by a function f such that*

$$f(m, n) \leq n \times \max(f^1(m + n, n), f^2(m + n, n))$$

where $m = \mathsf{size}(V)$, $n = \mathsf{size}(\varphi)$, and f^1, f^2 are the running time functions corresponding to algorithms $\mathsf{A}^1, \mathsf{A}^2$.

Proof. The proof, for a given finite $V \subseteq P$ and $\varphi \in L_{\Sigma_1\cup\Sigma_2}(P)$, is done by induction on the structure of φ.

For the induction base, let $\varphi = p \in P$. Note that the algorithm sets $i = 1$, $V_1 = V \cup \{x_0\}$, $M = \mathsf{Mon}_{\Sigma_1}(p) = \emptyset$, and $\psi = \mathsf{skel}_{\Sigma_1}(p) = p$. Hence, we have only to consider the running time of $\mathsf{A}^1(V_1)(p)$. If $m = \mathsf{size}(V)$, as $\mathsf{size}(p) = 1$, we have that $\mathsf{size}(V_1) \leq m + 1$, and the running time of the algorithm is

$$f(m, 1) \quad = \quad f^1(m + 1, 1) \quad \leq \quad \max(f^1(m + 1, 1), f^2(m + 1, 1)).$$

For the induction step, let $\varphi = c(\varphi_1, \ldots, \varphi_k)$ for some k-place connective $c = \mathsf{head}(\varphi) \in \Sigma_1 \cup \Sigma_2$. Note that the algorithm sets, $i = 1$ if $c \in \Sigma_1$ and $i = 2$ if $c \in \Sigma_2$, and $V_i = V \cup \{x_0\}$. If $m = \mathsf{size}(V)$, and $n = \mathsf{size}(\varphi)$, note that $\mathsf{size}(V_i) \leq m+1$, $\mathsf{size}(M) \leq n-1$, and $\mathsf{size}(\mathsf{skel}_{\Sigma_i}(\varphi)) \leq n$. Note also that $V_i \cup \{x_{e^{-1}(\varphi')} : \varphi' \in M, \mathsf{A}(V)(\varphi') = \mathsf{yes}\} = V \cup X_V^i(\varphi)$, whose size is bounded by $m+1+(n-1) = m+n$. As we have to consider the running time of $\mathsf{A}^i(V \cup X_V^i(\varphi))(\mathsf{skel}_{\Sigma_i}(\varphi))$, and also of all $\mathsf{A}(V)(\varphi')$ for $\varphi' \in M$, using the induction hypothesis and the natural monotonicity of the running time functions, the total running time of the algorithm is given by

$f(m,n) \leq$
$f^i(m+n,n) + \sum_{\varphi' \in M} f(m,n') =$
$f^i(m+n,n) + \sum_{\varphi' \in M} (n' \times \max(f^1(m+n',n'), f^2(m+n',n'))) \leq$
$f^i(m+n,n) + \sum_{\varphi' \in M} (n' \times \max(f^1(m+n,n), f^2(m+n,n))) =$
$f^i(m+n,n) + (\sum_{\varphi' \in M} n') \times \max(f^1(m+n,n), f^2(m+n,n)) =$
$f^i(m+n,n) + \mathsf{size}(M) \times \max(f^1(m+n,n), f^2(m+n,n)) \leq$
$f^i(m+n,n) + (n-1) \times \max(f^1(m+n,n), f^2(m+n,n)) \leq$
$n \times \max(f^1(m+n,n), f^2(m+n,n))$,

where, for improved readability, we used n' as an abbreviation of $\mathsf{size}(\varphi')$. □

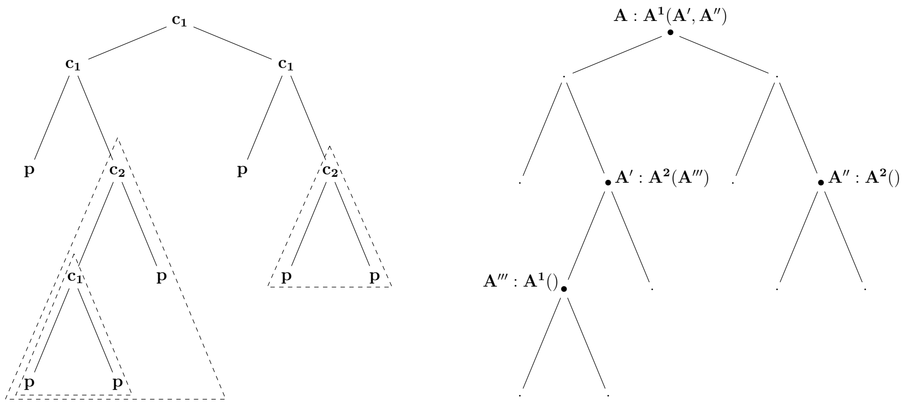

FIGURE 2. Deciding a mixed formula.

The upperbound for the running time of the decision procedure that we established in Proposition 4.3 is better understood if we look at the scheme shown in Figure 2. We are assuming a simple scenario where c_1 is a 2-place connective of $\mathcal{L}_1$, and c_2 a 2-place connective of $\mathcal{L}_2$. On the left side of the figure we have the syntactic tree of the combined formula $c_1(c_1(p, c_2(c_1(p,p),p)), c_1(p, c_2(p,p)))$. The dashed triangles indicate the monolithical structure of the formula. Since the head of the formula is in the signature of logic $\mathcal{L}_1$, triangles appear as soon as one finds the c_2 connective when travelling down the tree. Inside each triangle, the same repeatedly applies. On the right side of the figure we see how the decision algorithm stretches along the tree structure. At the root note, A is executed, recursively deploying other executions of the algorithm at the vertices of the outermost triangles, corresponding to its monoliths, denoted by A' and A''. The outputs of A' and A'' are then used at the root node to execute A^1, which explains the annotation $\mathsf{A} : \mathsf{A}^1(\mathsf{A}', \mathsf{A}'')$. The same structure is propagated downwards. Executing A' deploys A''' at the root of the inner triangle, and then uses its output on A^2. On its turn, A''' simply calls A^1, as there are no further monoliths to be analyzed. The same happens with execution A'', on the right. This example shows well that the running time of the decision procedure is bound by the size of the formula (i.e., the number of nodes of the tree) times the running time of each call to either A^1 or A^2, as at most one such call is necessary at each node.

Clearly, this means that the decision problem for the fibred logic reduces polynomially to the decision problems of the logics given.

Example 4.4.

Consider again the unconstrained fibring $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{int}} = \langle \Sigma_{\mathsf{cls}} \cup \Sigma_{\mathsf{int}}, \vdash \rangle$ that we have used in Example 4.1. It is well known that both logics are weakly decidable, and also finitary, which makes them finitely var-decidable as both enjoy the deduction theorem. The corresponding decision problems are known to be in NP for $\mathcal{L}_{\mathsf{cls}}$, and PSPACE for $\mathcal{L}_{\mathsf{int}}$. As a direct application of Propositions 4.2 and 4.3, we can conclude that $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{int}} = \langle \Sigma_{\mathsf{cls}} \cup \Sigma_{\mathsf{int}}, \vdash \rangle$ is also finitely var-decidable, and that the corresponding decision problem is in PSPACE. $\triangle$

5. Conclusion

We have studied in detail the theoremhood problem for logics obtained by unconstrained fibring, and shown that it is decidable provided that the component logics enjoy the slightly stronger notion of finite var-decidability. The result we obtained is the first of its kind, thus opening the way to formal tool support for fibred logics in a neat and modular way. We also showed that the complexity of the theoremhood decision problem for the fibred logic is essentially the same as the complexity of the finite var-decidability problem of the hardest component logic. Although distinct in nature, it is worth mentioning that our decision algorithm bears some similarities with the Nelson-Oppen approach to deciding joint equational theories [13], a connection that may be worth exploring.

Still, the result we obtained is not fully satisfactory, as it does not allow one to conclude much about the finite decidability of the fibred logic. Indeed, the key result used in the decidability proof, Proposition 3.13, is simply not strong enough. It seems to be possible to extend the proposition to deal with sets of non-mixed hypotheses instead of only sets of variables, but reasoning under arbitrary sets of hypotheses seems to be out of the reach of the techniques we have used. In any case, it should be remarked that extending Proposition 3.13 to deal not just with sets of variables as hypotheses but also with sets of non-mixed hypotheses would certainly help shedding new light over the fibring construction, with possible applications going beyond decidability. Namely, such a result may be fundamental for solving the long standing problem of conservativity of fibring, at least in the unconstrained case. We are working towards such a result.

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