On the characterization of fibred logics, with applications to conservativity and finite-valuedness *

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Abstract

Fibring is a general mechanism for combining logics that provides valuable insight on designing and understanding complex logical systems. To date, most research on fibring has focused on its model and proof-theoretic aspects, and on transference results for relevant metalogical properties. But we are still far from understanding in full the way mixed reasoning emerges from the logics being combined, which is preventing us from having a fully satisfactory semantics for fibred logics and, consequently, limiting the usability of the general results obtained.

In [16], assuming no shared connectives, we have presented an effective characterization of mixed reasoning in terms of the component logics, taking only variables as hypotheses. Despite these restrictions, the result immediately proved to have very interesting applications.

In this paper we extend our previous characterization of mixed reasoning for disjoint fibring to arbitrary non-mixed hypotheses. While still not completely satisfactory, as the characterization still cannot cover reasoning from mixed hypotheses, and even less fibred logics with shared connectives, the result again proves to be extremely useful. We illustrate its power by exploring two meaningful applications. To start with, we provide the first full characterization of conservativity for logics obtained by disjoint fibring, extending the partial results of [19]. Then, we take a semantic detour and use our characterization of mixed reasoning to show that (disjoint) fibring does not preserve finite (N)-valuedness.

Keywords: Combined logics, disjoint fibring, mixed reasoning, conservativity, finite-valuedness.

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1 Introduction

Fibring is a powerful and appealing mechanism for combining logics, a valuable tool for the construction and analysis of complex logics, and thus a key ingredient of the general theory of universal logic [3, 4]. The track of work on fibred logics has led to a substantial understanding of the semantics and proof-theory of a big class of these logics, including some general soundness and completeness preservation results (see [2, 6, 10, 17, 20, 22, 25]). However, little is known about the mechanics of the emergence of mixed reasoning in combined logics, which is keeping us far from having a satisfactory grasp on the model-theory of fibring that may have a real impact on applications.

Recently, we have presented a first effective characterization of mixed reasoning in terms of the component logics, for the case of disjoint fibring and taking only variables as hypotheses (see [16]). Of course, such a result is highly insufficient. Though important, disjoint fibring is only a special case of the combination mechanism, and variables are the simplest possible formulas we can think of. Still, the result immediately proved to have very interesting applications, and allowed us to obtain a newly found decidability preservation result for the theoremhood problem in disjointly fibred logics. Thus, extending this characterization is certainly bound to pay off.

Despite the fact that fibring becomes much more interesting (and difficult) when the logics being combined have some connectives in common, we see our results in this paper as an essential step towards more ambitious goals. Concretely, we shall not be able to drop the disjointness requirement, but instead the main result we obtain here is an extension of our previous characterization of mixed reasoning for disjoint fibring to arbitrary non-mixed hypotheses. While this may seem a small step, as the characterization still cannot cover reasoning from mixed hypotheses, the result is far from trivial, and indeed proves to be extremely useful. As an illustration of the power and usefulness of our main result, we shall consider two applications: a study of the conservativity problem for disjoint fibring, and a negative result about the preservation of finite-(N)valuedness by fibring.

As first proposed by Dov Gabbay in [12, 14], given two logics $\mathcal{L}_1$ and $\mathcal{L}_2$, fibring should combine $\mathcal{L}_1$ and $\mathcal{L}_2$ into the smallest logical system for the combined language which is a conservative extension of both $\mathcal{L}_1$ and $\mathcal{L}_2$. However, it is not hard to see that a conservative extension of two given logics may not always exist. Still, this circumstance does not necessarily imply that the construction is meaningless, as one can then aim at being "as conservative as possible". This idea has led the community to understand the fibring operation as yielding the smallest logic that extends $\mathcal{L}_1$ and $\mathcal{L}_2$, without worrying about conservativity [9], a question that was mostly put aside. For some time, the only general result on the subject was a general proof of the conservativity of fusion (fibring) of modal logics [23], which used ideas from modal semantics that could not be easily generalized. Particular forms of failure of conservativity, known as *the collapsing problem* were also studied in [13, 8, 21, 11]. However, if one considers the problem in full generality, it is immediate that a complete characterization of conservativity for fibred logics is far from obvious, even more so when the logics at hand share some of their connectives (which is nevertheless the case of fusion). More recently, a partial result about the conservativity of logics obtained by disjoint fibring was obtained in [19], covering only non-trivial component logics without quasi-theorems (*strongly determined logics*, in the author's terminology). Using our main result, we shall obtain a complete characterization of the conservativity problem for disjoint fibring, thus extending the result of [19].

As we already mentioned, there have been various attempts to provide fibring with an appropriate semantical counterpart. Despite some interesting results, like sufficient conditions for completeness preservation, these attempts are not fully satisfactory and, in particular, have reduced practical use. In another application of our main characterization result, we try to shed some light on why this is the case, using a common definition of model, based on the widely accepted notion of logical matrix [24] or, even more generally, of non-deterministic matrix [1]. We prove that even when given two logics that are each characterized by a finite model, their fibring may not be characterizable by a finite model. The fact that finite-(N)valuedness is not preserved by fibring is a good illustration of the difficulties behind fibred semantics.

In Section 2, we recall the notions and basic results needed throughout the paper, namely about fibred logics. With a few exceptions, the presentation follows closely [16], of which this paper can be seen as a natural successor. In Section 3, we introduce some useful notation to deal with mixed formulas, and then motivate and obtain our key technical result (Proposition 3.12) characterizing mixed reasoning from non-mixed hypotheses in logics obtained by disjoint fibring. Sections 4 and 5 present meaningful applications of this result. Section 4 is devoted to the conservativity problem for fibred logics, and provides a full characterization result for the case of disjoint fibring (Theorem 4.4), summarized in Figure 1. Section 5 studies the (non-)preservation of finite-(N)valuedness by fibring (Corollary 5.3). We conclude, in Section 6, with an assessment of the results obtained and paths to pursue in future work.

2 Preliminaries

In this section we recall the essential concepts that we are dealing with in this paper, and introduce some useful notions and notations.

2.1 (Trans)finite sequences

Along the paper, we will need to deal with (not necessarily finite) sequences of objects. Let A be a set (of objects). Given an ordinal η, we use $\overline{a} = \langle a_\kappa \rangle_{\kappa<\eta}$ to denote a η-long sequence of elements of A, or simply a η-sequence, understood as a function from $\{\kappa : \kappa < \eta\}$ to A. The η-sequence $\overline{a}$ will be said to be *injective* if it is injective as a function, that is, when $a_i \neq a_j$ for all $i, j < \eta$ with $i \neq j$. As usual, if $\tau \leq \eta$, the sequence $\langle a_\kappa \rangle_{\kappa<\tau}$ will be dubbed a *prefix* of $\overline{a}$.

Note that when η is a limit ordinal, a η-sequence does not have a last element. On the contrary, if η is a successor ordinal, and in particular a finite ordinal, then a η-sequence $\overline{a}$ can be understood as $a_0, a_1, \ldots, a_{\eta-1}$, and may also be represented by $\langle a_\kappa \rangle_{\kappa \leq \eta-1}$. The 0-sequence (*empty* sequence) is simply not represented.

2.2 Syntax

A *signature* is a $\mathbb{N}_0$-indexed family $\Sigma = \{\Sigma^{(n)}\}_{n\in\mathbb{N}_0}$ of sets. The elements of $\Sigma^{(n)}$ are dubbed n-place *connectives*. Being indexed families of sets, the usual set-theoretic notions can be smoothly extended to signatures. We will sometimes abuse notation, and confuse Σ with the set $(\biguplus_{n\in\mathbb{N}_0} \Sigma^{(n)})$ of all its connectives, and write $c \in \Sigma$ when c is some n-place connective and $c \in \Sigma^{(n)}$. For this reason, the *empty signature*, with no connectives at all, will be simply denoted by $\emptyset$.

Let Σ, Σ' be two signatures. We say that Σ is a *subsignature* of Σ', and write $\Sigma \subseteq \Sigma'$, whenever $\Sigma^{(n)} \subseteq \Sigma'^{(n)}$ for every $n \in \mathbb{N}_0$. Expectedly, we can also define the *intersection* $\Sigma \cap \Sigma' = \{\Sigma^{(n)} \cap \Sigma'^{(n)}\}_{n\in\mathbb{N}_0}$, *union* $\Sigma \cup \Sigma' = \{\Sigma^{(n)} \cup \Sigma'^{(n)}\}_{n\in\mathbb{N}_0}$, and *difference* $\Sigma' \setminus \Sigma = \{\Sigma'^{(n)} \setminus \Sigma^{(n)}\}_{n\in\mathbb{N}_0}$ of signatures. Clearly, $\Sigma \cap \Sigma'$ is the largest subsignature of both Σ and Σ', and contains the connectives *shared* by Σ_1 and Σ_2. When there are no shared connectives we have that $\Sigma \cap \Sigma' = \emptyset$. Analogously, $\Sigma \cup \Sigma'$ is the smallest signature that has both Σ and Σ' as subsignatures, and features all the connectives from both Σ and Σ' in a *combined signature*. Furthermore, $\Sigma' \setminus \Sigma$ is the largest subsignature of Σ' which does not share any connectives with Σ.

Given a signature Σ and a set P of *variables*, the generated set of *formulas* is the carrier set $L_\Sigma(P)$ of the free Σ-algebra generated by P. For simplicity, given $\oplus \in \Sigma^{(2)}$, $n \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$, we shall often write $\varphi_1 \oplus \varphi_2$ instead of $\oplus(\varphi_1, \varphi_2)$, and also $\bigoplus_{1\leq i\leq n} \varphi_i$ instead of $((\ldots((\varphi_1 \oplus \varphi_2) \oplus \varphi_3) \oplus \ldots) \oplus \varphi_n)$.

In the sequel, we shall assume that signatures are countable and sets of variables are denumerable. We assume fixed a denumerable set P of variables. If Σ is a countable signature then $L_\Sigma(P)$ is clearly denumerable.

If $\varphi \in L_\Sigma(P)$ then we define the *head of* φ to be either $\mathsf{head}(\varphi) = p$ when $\varphi = p \in P$, or $\mathsf{head}(\varphi) = c$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma^{(n)}$. Clearly, if $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$ then $L_\Sigma(P) \subseteq L_{\Sigma'}(P')$. Of course, given $\psi \in L_{\Sigma'}(P')$, $\mathsf{head}(\psi)$ may not be in Σ nor P.

We also define the *set of variables occurring in* φ to be either $\mathsf{var}(\varphi) = \{p\}$ when $\varphi = p \in P$, or $\mathsf{var}(\varphi) = \bigcup_{i=1}^{n} \mathsf{var}(\varphi_n)$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma^{(n)}$. We extend the notation to sets of formulas in the obvious way.

Let $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$. A Σ'-*substitution* is a function $\sigma : P \to L_{\Sigma'}(P')$, which extends freely to a function $\sigma : L_\Sigma(P) \to L_{\Sigma'}(P')$. Given a formula $\varphi \in L_\Sigma(P)$, $\sigma(\varphi)$ is the *instance* of φ by σ, sometimes denoted simply by φ^σ, and is the result of uniformly replacing each variable $p \in P$ occurring in φ by $\sigma(p)$. When $\Gamma \subseteq L_\Sigma(P)$ we use Γ^σ to denote $\{\varphi^\sigma : \varphi \in \Gamma\}$.

2.3 Logical consequence

A *logic (over signature* Σ*)* is a tuple $\mathcal{L} = \langle \Sigma, \_^\vdash \rangle$, where $\_^\vdash : 2^{L_\Sigma(P)} \to 2^{L_\Sigma(P)}$ is a consequence operator (see [24], for instance), that is, it satisfies the following properties:

$$\begin{array}{ll} \Gamma \subseteq \Gamma^\vdash & (\textit{extensiveness}) \\ \Gamma^\vdash \subseteq (\Gamma \cup \Delta)^\vdash & (\textit{monotonicity}) \\ (\Gamma^\vdash)^\vdash \subseteq \Gamma^\vdash & (\textit{idempotence}) \\ (\Gamma^\vdash)^\sigma \subseteq (\Gamma^\sigma)^\vdash & (\textit{structurality}) \end{array}$$

for every $\Gamma, \Delta \subseteq L_\Sigma(P)$ and $\sigma : P \to L_\Sigma(P)$. Note that we do not require, in general, that the logic is *finitary*, i.e., it may happen that $\Gamma^\vdash$ properly contains the union of all $\Gamma_0^\vdash$ for finite $\Gamma_0 \subseteq \Gamma$. Meaningful examples of logics that will be used throughout the paper will be presented below.

As usual, we shall confuse the consequence operator with its induced Tarskian consequence relation. Thus, given $\varphi \in L_\Sigma(P)$, we will write $\Gamma \vdash \varphi$ whenever $\varphi \in \Gamma^\vdash$. When $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ is finite we write $\varphi_1, \dots, \varphi_n \vdash \varphi$ instead of $\{\varphi_1, \dots, \varphi_n\} \vdash \varphi$. Moreover, as usual, if $\Gamma = \emptyset$ we write $\vdash \varphi$ instead of $\emptyset \vdash \varphi$, and dub φ a *theorem* of $\mathcal{L}$. A formula φ that is not a theorem of $\mathcal{L}$ but such that $\psi \vdash \varphi$ for every $\psi \in L_\Sigma(P)$ is dubbed a *quasi-theorem*, or simply a *q-theorem*. Clearly, φ is a q-theorem of $\mathcal{L}$ provided that $\not\vdash \varphi$ but $p \vdash \varphi$ for some $p \in P$ that does not occur in φ. It is immediate that a logic cannot both have theorems and q-theorems.

We shall call any $\Gamma \subseteq L_\Sigma(P)$ such that $\Gamma = \Gamma^\vdash$ a *theory* of $\mathcal{L}$, and denote the set of all theories of $\mathcal{L}$ by $\mathbf{Th}(\mathcal{L})$. It is well known that $\mathbf{Th}(\mathcal{L})$ constitutes a complete lattice under the inclusion ordering (see [24], for instance). The bottom theory of the lattice is $\emptyset^\vdash$, whereas the top theory is $L_\Sigma(P)$, also called the *inconsistent* theory. When $\Gamma^\vdash$ is inconsistent we say that Γ is $\vdash$*-explosive*.

A logic $\mathcal{L} = \langle \Sigma, \_^\vdash \rangle$ is said to be *consistent* if $\emptyset^\vdash \neq L_\Sigma(P)$. Clearly, $\mathcal{L}$ is *inconsistent* (not consistent) precisely when $\vdash p$ for some $p \in P$, or alternatively when $\mathbf{Th}(\mathcal{L}) = \{L_\Sigma(P)\}$. $\mathcal{L}$ is said to be *trivial*, if for all non-empty $\Gamma \subseteq L_\Sigma(P)$ we have $\Gamma^\vdash = L_\Sigma(P)$. Equivalently, $\mathcal{L}$ is trivial if there exist distinct variables $p, q \in P$ such that $p \vdash q$. Another equivalent characterization is that $\mathcal{L}$ is trivial if $\mathbf{Th}(\mathcal{L}) \subseteq \{\emptyset, L_\Sigma(P)\}$. Of course, all inconsistent logics are trivial. Moreover, easily, a trivial logic is consistent if and only if it has a q-theorem, if and only if all formulas are q-theorems, if and only if it has no theorems.

We say that a logic $\mathcal{L}' = \langle \Sigma', \_^{\vdash'} \rangle$ *extends* $\mathcal{L} = \langle \Sigma, \_^\vdash \rangle$ if $\Sigma \subseteq \Sigma'$, and $\vdash \subseteq \vdash'$, in the sense that $\Gamma^\vdash \subseteq \Gamma^{\vdash'}$ for every $\Gamma \subseteq L_\Sigma(P)$. We say that the extension of $\mathcal{L}$ by $\mathcal{L}'$ is *conservative* if for all $\Gamma \subseteq L_\Sigma(P)$, $\Gamma^\vdash = \Gamma^{\vdash'} \cap L_\Sigma(P)$. It is perhaps more common to express these properties in terms of the induced consequence relations. Clearly, $\mathcal{L}'$ extends $\mathcal{L}$ when $\Gamma \vdash \varphi$ implies $\Gamma \vdash' \varphi$ for all

$\Gamma \cup \{\varphi\} \subseteq L_\Sigma(P)$. Furthermore, the extension is conservative precisely when $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash' \varphi$.

Given a signature Σ, it is well known that the set of all logics over Σ, $\mathbf{Log}(\Sigma)$, constitutes a complete lattice under the extension ordering defined above (see [24], for instance).

For every $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$ there is a natural (conservative) extension of $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ to Σ' defined, for $\Delta \cup \{\psi\} \subseteq L_{\Sigma'}(P')$, by $\Delta \vdash \psi$ if there exist $\Gamma \cup \{\varphi\} \subseteq L_\Sigma(P)$ and $\sigma : P \to L_{\Sigma'}(P')$ such that $\Gamma \vdash \varphi$, $\Delta = \Gamma^\sigma$ and $\psi = \varphi^\sigma$. It is not hard to see that this extension is still a Tarksian consequence relation. The next lemma, which will be useful later on, sheds some light on what happens in proofs in such (non-trivial) extended logics when none of the hypotheses has its head in Σ.

Lemma 2.1. *Let $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ be non-trivial, $\Sigma \subseteq \Sigma'$, $\Delta \subseteq L_{\Sigma'}(P)$ such that for all $\psi \in \Delta$ we have $\mathsf{head}(\psi) \notin \Sigma$. Given $p \in P$, if $\Delta \vdash p$ then $p \in \Delta$.*

Proof. If $\Delta \vdash p$ then there exist $\Gamma \cup \{\varphi\} \subseteq L_\Sigma(P)$ and $\sigma : P \to L_{\Sigma'}(P)$ such that $\Gamma \vdash \varphi$, $\Delta = \Gamma^\sigma$ and $p = \varphi^\sigma$. Since $\mathsf{head}(\psi) \notin \Sigma$ for every $\psi \in \Delta$ we conclude that $\Gamma \cup \{\varphi\} \subseteq P$. Let $q \neq p$ and $\sigma' : P \to L_{\Sigma'}(P)$ be defined as

$$\sigma'(r) = \begin{cases} p \text{ if } r \in \Gamma, \\ q \text{ if } r \notin \Gamma. \end{cases}$$

By structurality of $\vdash$ we conclude that $\Gamma^{\sigma'} \vdash \varphi^{\sigma'}$. Assume now, for *reductio*, that $p \notin \Delta$. As a consequence we also know that $\varphi \notin \Gamma$ and thus $\Gamma^{\sigma'} = \{p\} \vdash q = \varphi^{\sigma'}$, which implies that $\mathcal{L}$ is trivial and leads to a contradiction. ☐

2.4 Hilbert calculi

A *Hilbert calculus* is a pair $\mathscr{H} = \langle \Sigma, R \rangle$ where Σ is a signature, and $R \subseteq 2^{L_\Sigma(P)} \times L_\Sigma(P)$ is a set of *inference rules*. Given $\langle \Delta, \psi \rangle \in R$, we refer to Δ as the set of *premises* and to ψ as the *conclusion* of the rule. When the set of premises is empty, ψ is dubbed an *axiom*. A rule is said to be *finitary* if it has a finite set of premises, and $\mathscr{H}$ is said to be *finitary* of all the rules in R are finitary. An inference rule $\langle \Delta, \psi \rangle \in R$ is often denoted by $\frac{\Delta}{\psi}$, or simply by $\frac{\psi_1 \; \dots \; \psi_n}{\psi}$ if $\Delta = \{\psi_1, \dots, \psi_n\}$ is finite, or even by $\frac{}{\psi}$ if $\Delta = \emptyset$.

Given $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$, a Hilbert calculus $\mathscr{H} = \langle \Sigma, R \rangle$ induces a consequence operator $\_^{\vdash_\mathscr{H}}$ on $L_{\Sigma'}(P')$ such that, for each $\Gamma \subseteq L_{\Sigma'}(P')$, $\Gamma^{\vdash_\mathscr{H}}$ is the least set that contains Γ and is closed for all applications of instances of the inference rules in R, that is, if $\frac{\Delta}{\psi} \in R$ and $\sigma : P \to L_{\Sigma'}(P')$ is such that $\Delta^\sigma \subseteq \Gamma^{\vdash_\mathscr{H}}$ then $\psi^\sigma \in \Gamma^{\vdash_\mathscr{H}}$. Of course, this definition induces a logic $\mathcal{L}_\mathscr{H} = \langle \Sigma, \_^{\vdash_\mathscr{H}} \rangle$.

The definition of $\mathcal{L}_\mathscr{H}$ above is arguably too abstract, as it does not highlight the sequence of rule applications that leads one to conclude that $\Gamma \vdash_\mathscr{H} \varphi$, when that is the case. Let us be more detailed. Given $\Sigma \subseteq \Sigma'$, $P \subseteq P'$, and

$\Gamma \subseteq L_{\Sigma'}(P')$, a $\mathscr{H}$*-derivation from* Γ is a sequence $\overline{\varphi} = \langle \varphi_\kappa \rangle_{\kappa<\eta}$ of formulas in $L_{\Sigma'}(P')$, for some ordinal η, such that, for each $\kappa < \eta$, either $\varphi_\kappa \in \Gamma$, or there is $\frac{\Delta}{\psi} \in R$ and $\sigma : P \to L_{\Sigma'}(P')$ with $\psi^\sigma = \varphi_\kappa$ and $\Delta^\sigma \subseteq \{\varphi_\tau : \tau < \kappa\}$.

The fact that $\overline{\varphi}$ is a $\mathscr{H}$-derivation from Γ is denoted by $\Gamma \vdash_{\mathscr{H}} \overline{\varphi}$. We say that such a derivation is a $\mathscr{H}$*-proof from* Γ of each of its formulas, as it is clear that any prefix of a $\mathscr{H}$-derivation from Γ is also a $\mathscr{H}$-derivation from Γ.

Clearly, $\Gamma \vdash_{\mathscr{H}} \varphi$ precisely if φ has a $\mathscr{H}$-proof from Γ, that is, there exists some $\mathscr{H}$-derivation $\langle \varphi_\kappa \rangle_{\kappa<\eta}$ from Γ such that $\varphi = \varphi_\kappa$ for some $\kappa < \eta$. Of course, in that case, $\langle \varphi_\iota \rangle_{\iota<\kappa+1}$ is a $\mathscr{H}$-proof of φ from Γ ending in φ.

When the Hilbert system is identified with a subscript $\mathscr{H} = \mathscr{H}_{\mathsf{sub}}$ we drop the $\mathscr{H}$ in $\vdash_{\mathscr{H}_{\mathsf{sub}}}$ and write just $\vdash_{\mathsf{sub}}$.

Example 2.2. Along the paper, in order to illustrate the problems at hand and the results obtained we will use the following collection of examples:

- $\mathscr{H}_{\mathsf{inc}(\Sigma)} = \langle \Sigma, R_{\mathsf{inc}} \rangle$, for each signature Σ, where R_{inc} has the unique rule

$$\frac{}{p}.$$

- $\mathscr{H}_{\mathsf{tonk}} = \langle \Sigma_{\mathsf{tonk}}, R_{\mathsf{tonk}} \rangle$, where Σ_{tonk} has a unique 2-place connective tonk, and R_{tonk} has the rules

$$\frac{p}{\mathsf{tonk}(p,q)} \qquad \frac{\mathsf{tonk}(p,q)}{q}.$$

- $\mathscr{H}_{\mathsf{cls}} = \langle \Sigma_{\mathsf{cls}}, R_{\mathsf{cls}} \rangle$, where Σ_{cls} has a unique 2-place connective $\Rightarrow$, and R_{cls} has the rules

$$\frac{}{p \Rightarrow (q \Rightarrow p)} \qquad \frac{}{(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))}$$

$$\frac{}{((p \Rightarrow q) \Rightarrow p) \Rightarrow p} \qquad \frac{p \quad p \Rightarrow q}{q}.$$

- $\mathscr{H}_{\mathsf{int}} = \langle \Sigma_{\mathsf{int}}, R_{\mathsf{int}} \rangle$, where Σ_{int} has a unique 2-place connective $\to$, and R_{int} has the rules

$$\frac{}{p \to (q \to p)} \qquad \frac{}{(p \to (q \to r)) \to ((p \to q) \to (p \to r))} \qquad \frac{p \quad p \to q}{q}.$$

- $\mathscr{H}_{\mathsf{qcls}} = \langle \Sigma_{\mathsf{qcls}}, R_{\mathsf{qcls}} \rangle$, where $\Sigma_{\mathsf{qcls}} = \Sigma_{\mathsf{cls}}$ and R_{qcls} has the rules

$$\frac{s}{p \Rightarrow (q \Rightarrow p)} \qquad \frac{s}{(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))}$$

$$\frac{s}{((p \Rightarrow q) \Rightarrow p) \Rightarrow p} \qquad \frac{p \quad p \Rightarrow q}{q}.$$

- $\mathscr{H}_{\mathsf{top}} = \langle \Sigma_{\mathsf{top}}, R_{\mathsf{top}} \rangle$, where Σ_{top} has a unique 0-place connective $\top$, and R_{top} has the unique rule

$$\frac{}{\top}.$$

- $\mathscr{H}_{\mathsf{neg}} = \langle \Sigma_{\mathsf{neg}}, R_{\mathsf{neg}} \rangle$, where Σ_{neg} has a unique 1-place connective $\neg$, and R_{neg} has the rules

$$\frac{p}{\neg\neg p} \qquad \frac{\neg\neg p}{p} \qquad \frac{p \quad \neg p}{q}.$$

- $\mathscr{H}_{\mathsf{cnj}} = \langle \Sigma_{\mathsf{cnj}}, R_{\mathsf{cnj}} \rangle$, where Σ_{cnj} has a unique 2-place connective $\wedge$, and R_{cnj} has the rules

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q} \qquad \frac{p \quad q}{p \wedge q}.$$

- $\mathscr{H}_{\mathsf{djn}} = \langle \Sigma_{\mathsf{djn}}, R_{\mathsf{djn}} \rangle$, where Σ_{djn} has a unique 2-place connective $\vee$, and R_{djn} has the rules

$$\frac{p}{p \vee q} \qquad \frac{p \vee p}{p} \qquad \frac{p \vee q}{q \vee p} \qquad \frac{p \vee (q \vee r)}{(p \vee q) \vee r}.$$

Clearly, each $\mathscr{H}_{\mathsf{inc}(\Sigma)}$ induces an inconsistent logic, whereas $\mathscr{H}_{\mathsf{tonk}}$ is Prior's infamous tonk system and induces a consistent but trivial logic. The calculi $\mathscr{H}_{\mathsf{cls}}$ and $\mathscr{H}_{\mathsf{int}}$ induce the logics of *classical implication* and *intuitionistic implication*, respectively. On the other hand, $\mathscr{H}_{\mathsf{qcls}}$ induces the logic of classical implication, but without theorems, and we dub it *quasi-classical*. Finally, $\mathscr{H}_{\mathsf{top}}$ induces the logic of (classical or intuitionistic) top (*verum*), $\mathscr{H}_{\mathsf{neg}}$ the logic of (classical or intuitionistic) *negation*, $\mathscr{H}_{\mathsf{cnj}}$ the logic of (classical or intuitionistic) *conjunction* and $\mathscr{H}_{\mathsf{djn}}$ the logic of (classical or intuitionistic) *disjunction*. Note that with the possible exception of the $\mathscr{H}_{\mathsf{inc}(\Sigma)}$ calculi, all other examples have very simple signatures with a single connective. This is a deliberate choice meant to keep the focus of attention on the relevant problems ahead, and not on the relative complexity of the syntax. △

All the systems introduced in Example 2.2 are well known, with the possible exception of $\mathscr{H}_{\mathsf{djn}}$. By reading [5] one may even be led to believe that classical disjunction does not have a finite axiomatization. However, Rautenberg proves in [18] that the axiomatization shown above is complete.

Given a logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, we can easily associate with it a Hilbert calculus $\mathscr{H}_{\mathcal{L}} = \langle \Sigma, \vdash \rangle$, where the $\_^{\vdash}$ consequence operator in the former is replaced by the induced $\vdash$ consequence relation in the latter. It is easy to check that $\mathcal{L}_{\mathscr{H}_{\mathcal{L}}} = \mathcal{L}$ (see [24], for instance). For simplicity, we will use $\mathcal{L}_{\mathsf{name}}$ to denote the logic $\mathcal{L}_{\mathscr{H}_{\mathsf{name}}}$, namely for the calculi named in Example 2.2.

2.5 Fibring

Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be two logics. The *fibring* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the smallest logic $\mathcal{L}_1 \bullet \mathcal{L}_2$ over the joint signature $\Sigma_{12} = \Sigma_1 \cup \Sigma_2$ that extends both $\mathcal{L}_1$ and $\mathcal{L}_2$. A direct characterization of this fibred logic can be most easily given by first defining the fibring of Hilbert calculi.

Given Hilbert calculi $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ let their *fibring* be the Hilbert calculus

$$\mathscr{H}_1 \bullet \mathscr{H}_2 = \langle \Sigma_{12}, R_1 \cup R_2 \rangle.$$

Clearly, besides joining the given signatures, which will allow us to build so-called *mixed formulas*, the fibring of the two calculi consists in simply putting together their rules, thus allowing a form of *mixed reasoning*.

We can now give a simple characterization of the fibring of two logics $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\mathcal{L}_1 \bullet \mathcal{L}_2 = \mathcal{L}_{\mathscr{H}_{\mathcal{L}_1} \bullet \mathscr{H}_{\mathcal{L}_1}}.$$

This mean that if $\mathcal{L}_1 \bullet \mathcal{L}_2 = \langle \Sigma_{12}, \_^{\vdash_{12}} \rangle$ then, given $\Gamma \subseteq L_{\Sigma_{12}}(P)$, $\Gamma^{\vdash_{12}}$ is obtained by a (possibly transfinite) sequence of alternate applications of $\vdash_1$ and $\vdash_2$ using substitutions $\sigma : P \to L_{\Sigma_{12}}(P)$.

Both for logics and Hilbert calculi, when there are no shared connectives, i.e. $\Sigma_1 \cap \Sigma_2 = \emptyset$, the fibring is usually said to be *disjoint*.

3 Mixed reasoning

In this section we will introduce a number of useful notions and notations that allow us to breakdown the possibly interleaved syntax of mixed formulas, and then use them to give a thorough description of consequences of non-mixed hypotheses in disjoint fibring in terms of (any)one of the component logics, by suitably encoding the necessary interaction with the other component.

3.1 Monoliths and skeletons

Let $\Sigma \subseteq \Sigma'$ be signatures. We shall call a Σ-*monolith of* $\psi \in L_{\Sigma'}(P)$ to any outermost subformula of ψ whose head is in $\Sigma' \setminus \Sigma$. The set $\mathsf{Mon}_{\Sigma}(\psi)$ of all

Σ-monoliths of ψ is defined as follows:

$$\mathsf{Mon}_\Sigma(\psi) = \begin{cases} \emptyset & \text{if } \psi \in P, \\ \bigcup_{i=1}^{n} \mathsf{Mon}_\Sigma(\psi_i) & \text{if } \psi = c(\psi_1, \ldots, \psi_n) \text{ and } c \in \Sigma^{(n)}, \\ \{\psi\} & \text{otherwise.} \end{cases}$$

We extend the notation also to sets of formulas, using $\mathsf{Mon}_\Sigma(\Delta)$ to denote $\bigcup_{\psi \in \Delta} \mathsf{Mon}_\Sigma(\psi)$, given $\Delta \subseteq L_{\Sigma'}(P)$. Clearly, if $\Gamma \subseteq L_\Sigma(P)$ then $\mathsf{Mon}_\Sigma(\Gamma) = \emptyset$.

We shall now consider a reasonable way of defining the perspective, from the point of view of Σ, that one may have of a formula in $L_{\Sigma'}(P)$. For the purpose, we use a denumerable set $X = \{x_\psi : \psi \in L_{\Sigma'}(P)\}$ of additional propositional variables, disjoint from P. We define the function $\mathsf{skel}_\Sigma : L_{\Sigma'}(P) \to L_\Sigma(P \cup X)$ as follows:

$$\mathsf{skel}_\Sigma(\psi) = \begin{cases} \psi & \text{if } \psi \in P, \\ c(\mathsf{skel}_\Sigma(\psi_1), \ldots, \mathsf{skel}_\Sigma(\psi_n)) & \text{if } \psi = c(\psi_1, \ldots, \psi_n) \text{ and } c \in \Sigma^{(n)}, \\ x_\psi, & \text{otherwise.} \end{cases}$$

We call $\mathsf{skel}_\Sigma(\psi)$ the *Σ-skeleton of ψ*. Clearly, $\mathsf{skel}_\Sigma(\psi)$ is obtained from ψ by substituting each of its Σ-monoliths ϕ by the variable x_ϕ.

Given two η-sequences $\overline{\alpha}$ and $\overline{\beta}$ of $L_{\Sigma'}(P)$ formulas, with $\overline{\alpha}$ injective, we write $\psi[\overline{\alpha}/\overline{\beta}]_\Sigma$ to denote the formula obtained by replacing each occurrence of α_i as a Σ-monolith of ψ by β_i, for all $i < \eta$. It is not difficult to check that $\psi[\overline{\alpha}/\overline{\beta}]_\Sigma = (\mathsf{skel}_\Sigma(\psi))^\sigma$ where σ is a substitution $\sigma : P \cup X \to L_{\Sigma'}(P)$ such that $\sigma(x_{\alpha_i}) = \beta_i$ for all $i < \eta$ and $\sigma(y) = y$ for $y \in P \cup (X \setminus \{x_{\alpha_i} : i < \eta\})$. This square bracket notation extends to sets of formulas in the obvious manner.

Example 3.1. Let Σ be the signature with exactly two connectives, a 0-place connective c and a 2-place connective g, that is, $\Sigma^{(0)} = \{c\}$, $\Sigma^{(2)} = \{g\}$ and $\Sigma^{(n)} = \emptyset$ for all $n \in \mathbb{N}_0 \setminus \{0, 2\}$. Let Σ' extend Σ with an additional 1-place connective f, that is, ${\Sigma'}^{(0)} = \{c\}$, ${\Sigma'}^{(1)} = \{f\}$, ${\Sigma'}^{(2)} = \{g\}$ and ${\Sigma'}^{(n)} = \emptyset$ for all $n \in \mathbb{N}_0 \setminus \{0, 1, 2\}$.

Taking the $L'_\Sigma(P)$ formula $\psi = g(f(p), g(c, f(g(f(c), f(p)))))$ we have that

$$\mathsf{Mon}_\Sigma(\psi) = \{f(p), f(g(f(c), f(p)))\}.$$

Note, in particular, that the subformula $f(c)$ is not a Σ-monolith of ψ because if occurs inside the (outermost) monolith $f(g(f(c), f(p)))$. For the same reason, $f(p)$ is only a Σ-monolith of ψ because it also occurs outside $f(g(f(c), f(p)))$.

Moreover,

$$\psi[f(p)/\beta]_\Sigma = g(\beta, g(c, f(g(f(c), f(p))))),$$

noting that β only replaces the leftmost occurrence of $f(p)$ in ψ, where it is a Σ-monolith, leaving the second untouched. △

We close this section with a simple result, that we borrow from [16]. We include also its proof since it is quite small but may help the reader to understand what is happening and hopefully work as a warm up for what comes next.

Lemma 3.2. *Let $\Sigma \subseteq \Sigma'$ and $\Gamma \subseteq L_\Sigma(P)$. Then, for every $\sigma : P \to L_{\Sigma'}(P)$, and every two η-sequences $\overline{\alpha}$ and $\overline{\beta}$ of formulas in $L_{\Sigma'}(P)$, with $\overline{\alpha}$ injective, there exists $\rho : P \to L_{\Sigma'}(P)$ such that*

$$\Gamma^\rho = \Gamma^\sigma[\overline{\alpha}/\overline{\beta}]_\Sigma.$$

Proof. One should observe, to start with, that $\mathsf{Mon}_\Sigma(\Gamma) = \emptyset$. Thus, if $\alpha_\kappa \in \mathsf{Mon}_\Sigma(\varphi^\sigma)$ for some $\varphi \in \Gamma$, then there must exist a variable $p \in P$ occurring in φ such that $\alpha_\kappa \in \mathsf{Mon}_\Sigma(\sigma(p))$. Hence, the substitution defined by $\rho(q) = \sigma(q)[\overline{\alpha}/\overline{\beta}]_\Sigma$ for every $q \in P$ satisfies the conditions of the lemma. □

Note that the lemma reflects the fact that the occurrence of Σ-monoliths in instances of $L_\Sigma(P)$ formulas is only possible if they are brought about by the substitution. As a corollary, we obtain the following result.

Corollary 3.3. *Let $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, $\Sigma \subseteq \Sigma'$ and $\Delta \cup \{\varphi\} \subseteq L_{\Sigma'}(P)$. Then,*

$$\Delta \vdash \varphi \textit{ if and only if } \mathsf{skel}_\Sigma(\Delta) \vdash \mathsf{skel}_\Sigma(\varphi).$$

Proof. If $\Delta \vdash \varphi$, by definition, there exist $\Gamma \cup \{\psi\} \subseteq L_\Sigma(P)$ and $\sigma : P \to L_{\Sigma'}(P)$ such that $\Gamma \vdash \psi$, $\Gamma^\sigma = \Delta$ and $\psi^\sigma = \varphi$. Let $M = \mathsf{Mon}_\Sigma(\Delta \cup \{\varphi\})$. Consider any injective sequence $\overline{\alpha}$ of formulas in M, where every formula in M appears (exactly once), and define $\overline{\beta}$ to be the same length sequence such that each $\beta_\iota = x_{\alpha_\iota}$. Note that $\varphi'[\overline{\alpha}/\overline{\beta}]_\Sigma = \mathsf{skel}_\Sigma(\varphi')$ for every $\varphi' \in \Delta \cup \{\varphi\}$. The statement follows simply by applying Lemma 3.2 to $\Gamma \cup \{\psi\}$, $\overline{\alpha}$ and $\overline{\beta}$, and then the structurality of $\vdash$ under the resulting substitution ρ.

The fact that $\mathsf{skel}_\Sigma(\Delta) \vdash \mathsf{skel}_\Sigma(\varphi)$ implies $\Delta \vdash \varphi$ follows easily from the structurality of $\vdash$ by considering a substitution $\sigma : P \cup X \to L_{\Sigma'}(P)$ such that $\sigma(p) = p$ for $p \in P$, and $\sigma(x_\psi) = \psi$. □

3.2 Consequences of non-mixed formulas

Let us now have a technical look at the patterns of mixed reasoning that occur in fibred logics, when starting with sets of non-mixed hypotheses. Consider the following example showing the irrelevance of certain monoliths in derivations from non-mixed formulas in logics obtained by disjoint fibring.

Example 3.4. Consider the fibred logic $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{neg}} = \langle \Sigma_{\mathsf{cls}} \cup \Sigma_{\mathsf{neg}}, \_^{\vdash} \rangle$. Let

$$\begin{aligned}
\psi_0 &= p,\\
\psi_1 &= p \to (q \to p),\\
\psi_2 &= q \to p,\\
\psi_3 &= \neg\neg(q \to p),\\
\psi_4 &= \neg\neg(q \to p) \to (\neg t \to \neg\neg(q \to p)),\\
\psi_5 &= (\neg t \to \neg\neg(q \to p)).
\end{aligned}$$

Clearly, $p \vdash \langle\psi_i\rangle_{i<6}$. We shall see that in this proof, from the point of view of Σ_{cls}, the Σ_{cls}-monoliths $\neg\neg(q \to p)$ and $\neg t$ have different roles and relevance.

It is not hard to check that if we substitute all the occurrences of $\neg t$ with any other formula β, we still obtain a proof in $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{neg}}$ from p. Letting $\psi'_i = \psi_i[\neg t/\beta]_{\Sigma_{\mathsf{cls}}}$ we get

$$\begin{aligned}
\psi'_0 &= p,\\
\psi'_1 &= p \to (q \to p),\\
\psi'_2 &= q \to p,\\
\psi'_3 &= \neg\neg(q \to p),\\
\psi'_4 &= \neg\neg(q \to p) \to (\beta \to \neg\neg(q \to p)),\\
\psi'_5 &= (\beta \to \neg\neg(q \to p)),
\end{aligned}$$

and it is still the case that $p \vdash \langle\psi'_i\rangle_{i<6}$.

The same does not happen in general with $\neg\neg(q \to p)$. For instance, taking $\beta = r \in P \setminus \{p\}$ and $\psi''_i = \psi_i[\neg\neg(q \to p)/r]_{\Sigma_{\mathsf{cls}}}$ we have

$$\begin{aligned}
\psi''_0 &= p,\\
\psi''_1 &= p \to (q \to p),\\
\psi''_2 &= q \to p,\\
\psi''_3 &= r,\\
\psi''_4 &= r \to (\neg t \to r),\\
\psi''_5 &= (\neg t \to r),
\end{aligned}$$

but $p \not\vdash \langle\psi''_i\rangle_{i<6}$ as $\psi''_3 = r$ is not an hypothesis, and also cannot be justified by any rule applied to the previous formulas.

These examples show that if a formula appears in a proof, we cannot hope in general to be able to replace its occurrences as a monolith by any other formula along the proof. They also suggest that this may be possible with formulas that do not occur in the proof sequence. Namely, note that $\neg t \notin \{\psi_i : i < 6\}$ but $\neg\neg(q \to p) = \psi_3$. △

In the next lemma we shall prove, as hinted by Example 3.4, that monoliths not appearing in a proof sequence are indeed irrelevant in that proof, and thus can be safely replaced. Note that the disjointness of the signatures is instrumental in proving this result. The lemma extends a similar result obtained in [16], where it was obtained only for the case $\Gamma \subseteq P$.

Lemma 3.5. *Let $\mathscr{H}_1 = \langle\Sigma_1, R_1\rangle$ and $\mathscr{H}_2 = \langle\Sigma_2, R_2\rangle$ be Hilbert calculi such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\Gamma \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$, and $\overline{\psi} = \langle\psi_\kappa\rangle_{\kappa<\eta}$ a sequence of $L_{\Sigma_{12}}(P)$ formulas. If $\Gamma \vdash_{12} \overline{\psi}$ and $\alpha \in L_{\Sigma_{12}}(P)$, then we have that either*

- *$\alpha = \psi_\kappa$ for some $\kappa < \eta$, or*
- *$\Gamma \vdash_{12} \langle\psi_\kappa[\alpha/\beta]_{\Sigma_i}\rangle_{\kappa<\eta}$ for every $\beta \in L_{\Sigma_{12}}(P)$ and $i \in \{1, 2\}$.*

Proof. Let us assume that $\alpha \neq \psi_\kappa$ for every $\kappa < \eta$. The proof of the second condition follows by complete transfinite induction on the size η of the derivation. For each $\iota < \tau \leq \eta$, we assume, by induction hypothesis, that $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\iota}$, and show that it implies $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$, with $i \in \{1, 2\}$.

If $\tau = 0$ the result is trivial, as the derivation is empty. If τ is a limit ordinal the result is immediate, by definition of derivation. If τ is a successor ordinal, we have to consider two cases.

(1) $\psi_{\tau-1} \in \Gamma$.

If $\psi_{\tau-1} \in \Gamma$, as $\Gamma \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$ (the hypothesis are not mixed), we have for each $i \in \{1, 2\}$ that $\mathsf{Mon}_{\Sigma_i}(\psi_{\tau-1}) = \emptyset$ if $\psi_{\tau-1} \in L_{\Sigma_i}(P)$, or $\mathsf{Mon}_{\Sigma_i}(\psi_{\tau-1}) = \{\psi_{\tau-1}\}$ if $\psi_{\tau-1} \notin L_{\Sigma_i}(P)$. Since we know that $\alpha \neq \psi_{\tau-1}$, we have for $i \in \{1, 2\}$ that $\alpha \notin \mathsf{Mon}_{\Sigma_i}(\psi_{\tau-1})$, and therefore $\psi_{\tau-1}[\alpha/\beta]_{\Sigma_i} = \psi_{\tau-1} \in \Gamma$.

By induction hypothesis we have that $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau-1}$ and so, by definition of derivation, we also have $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$.

(2) $\psi_{\tau-1} = \varphi^\sigma$, $\frac{\Delta}{\varphi} \in R_1 \cup R_2$, and $\Delta^\sigma \subseteq \{\psi_\kappa : \kappa < \tau - 1\}$.

Here we have two possibilities, given that $i \in \{1, 2\}$.

(a) $\frac{\Delta}{\varphi} \in R_i$.

Applying Lemma 3.2 to $\Delta \cup \{\varphi\} \subseteq L_{\Sigma_i}(P)$, σ, α and β, we know that there exists ρ such that $\varphi^\rho = \varphi^\sigma[\alpha/\beta]_{\Sigma_i} = \psi_{\tau-1}[\alpha/\beta]_{\Sigma_i}$, and also $\Delta^\rho = \Delta^\sigma[\alpha/\beta]_{\Sigma_i} \subseteq \{\psi_\kappa[\alpha/\beta]_{\Sigma_i} : \kappa < \tau - 1\}$.
By induction hypothesis we have that $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau-1}$ and so, by definition of derivation, we also have $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$.

(b) $\frac{\Delta}{\varphi} \in R_j$ with $j \in \{1, 2\}$ and $j \neq i$.

If $\delta \in \Delta$ and $\alpha \in \mathsf{Mon}_{\Sigma_i}(\delta^\sigma)$ then (since $\delta \in L_{\Sigma_j}(P)$) either $\alpha = \delta^\sigma$ or $\mathsf{head}(\delta^\sigma) \in \Sigma_i$. By assumption the former cannot be the case, therefore we must have $\mathsf{head}(\delta^\sigma) \in \Sigma_i$, and consequently $\delta \in P$. Consider the substitution defined by $\rho(q) = \sigma(q)[\alpha/\beta]_{\Sigma_i}$ for every $q \in P$. Clearly, as above, $\varphi^\rho = \varphi^\sigma[\alpha/\beta]_{\Sigma_i} = \psi_{\tau-1}[\alpha/\beta]_{\Sigma_i}$, and also $\Delta^\rho = \Delta^\sigma[\alpha/\beta]_{\Sigma_i} \subseteq \{\psi_\kappa[\alpha/\beta]_{\Sigma_i} : \kappa < \tau - 1\}$.
By induction hypothesis we have $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau-1}$ and so, by definition of derivation, we also have $\Gamma \vdash_{12} \langle \psi_\kappa[\alpha/\beta]_{\Sigma_i} \rangle_{\kappa<\tau}$. □

Having in mind our key result, the following definition is in hand, with the purpose of using the variables in X to represent contextual information regarding the alternation between uses of $\vdash_1$ and $\vdash_2$ in $\vdash_{12}$-derivations. For convenience, below, we work with $X_* = \{x_*\} \cup X$, where the extra variable x_* will be used to represent in $\vdash_1$ some generic provable formula in $\vdash_2$, or vice-versa.

Definition 3.6. Let $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ be Hilbert calculi, $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$ and $i \in \{1, 2\}$.
We define $\Gamma^+ \subseteq \Gamma \cup \{x_*\} \cup \mathsf{var}(\Gamma)$ by

- $x_* \in \Gamma^+$ whenever $(\Gamma^{\vdash_1} \cup \Gamma^{\vdash_2}) \neq \emptyset$,
- if $p \in \mathsf{var}(\Gamma)$ then $p \in \Gamma^+$ whenever $p \in \Gamma^\omega = \bigcup_{n \in \mathbb{N}_0} \Gamma^n$, where
  - $\Gamma^0 = \Gamma$, and
  - $\Gamma^{n+1} = \{p \in \mathsf{var}(\Gamma) : \Gamma^n \vdash_1 p \text{ or } \Gamma^n \vdash_2 p\}$.

We also define $X^i_\Gamma(\psi) \subseteq X$ such that

- $x_\phi \in X^i_\Gamma(\psi)$ whenever $\phi \in \mathsf{Mon}_{\Sigma_i}(\psi)$ and $\Gamma \vdash_{12} \phi$.

Although the definition of Γ^ω may seem involved, it is worth noting that in the non-mixed case, when $\Gamma \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$, it will follow, as we show below, that $\Gamma^\omega = \{p \in \mathsf{var}(\Gamma) : \Gamma \vdash_{12} p\}$.

The following lemma tells us that the number of iterations needed to calculate Γ^ω is bounded by the cardinality of the set $\mathsf{var}(\Gamma)$, being of course finite when Γ is finite. Furthermore, the result justifies the ω bound in the definition, since we are assuming that P is denumerable.

Lemma 3.7. *Let $p \in P$ and $\Gamma \subseteq L_{\Sigma_{12}}(P)$. If $\Gamma^{|\mathsf{var}(\Gamma)|} \vdash_1 p$ or $\Gamma^{|\mathsf{var}(\Gamma)|} \vdash_2 p$ then $p \in \Gamma^{|\mathsf{var}(\Gamma)|}$.*

Proof. The result follows easily from the fact that $|\mathsf{var}(\Gamma)| \leq \omega$ by observing, for all $n \in \mathbb{N}_0$, that $\Gamma^{n+1} \setminus \Gamma^n \subseteq \mathsf{var}(\Gamma)$, and also that if $\Gamma^{n+1} = \Gamma^n$ then $\Gamma^n = \Gamma^\omega$. $\square$

Example 3.8. If $\mathscr{H}_1 = \mathscr{H}_{\mathsf{cnj}}$, $\mathscr{H}_2 = \mathscr{H}_{\mathsf{tonk}}$ then, for $i \in \{1, 2\}$, it is easy to check that the corresponding fibred logic, induced by the Hilbert calculus $\mathscr{H}_{\mathsf{cnj}} \bullet \mathscr{H}_{\mathsf{tonk}}$, is trivial. Therefore, the following equalities hold.

$$\begin{aligned}
\emptyset^+ &= \emptyset \\
\{p\}^+ &= \{p, x_*\} \\
\{p \wedge q\}^+ &= \{p \wedge q, p, q, x_*\} \\
X^i_\emptyset(\psi) &= \emptyset \\
X^i_\Gamma(p) &= \emptyset \\
X^1_{\{p\}}(p \wedge \mathsf{tonk}(p, q)) &= \{x_{\mathsf{tonk}(p,q)}\} \\
X^2_{\{p\}}(p \wedge \mathsf{tonk}(p, q)) &= \{x_{p \wedge \mathsf{tonk}(p,q)}\}
\end{aligned}$$

If $\mathscr{H}_1 = \mathscr{H}_{\mathsf{cnj}}$, $\mathscr{H}_2 = \mathscr{H}_{\mathsf{cls}}$ then the following equality holds.

$$\{p \wedge q, p \to r\}^+ = \{p \wedge q, p \to r, p, q, r, x_*\}$$

It is worth noting that while the first three $\_^+$ examples, corresponding to the fibred logic $\mathscr{H}_{\mathsf{cnj}} \bullet \mathscr{H}_{\mathsf{tonk}}$ can be obtained in just one iteration, the last example, concerning $\mathscr{H}_{\mathsf{cnj}} \bullet \mathscr{H}_{\mathsf{cls}}$, needs two iterations. Indeed, the *modus ponens* rule of $\mathscr{H}_{\mathsf{cls}}$ can only be used after obtaining p (and also q) from $p \wedge q$ in $\mathscr{H}_{\mathsf{cnj}}$. △

The following lemma uses the newly introduced notions and provides sufficient conditions for a consequence from non-mixed hypotheses to hold in a logic obtained by (disjoint) fibring.

Lemma 3.9. *Let $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ be Hilbert calculi such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\psi \in L_{\Sigma_{12}}(P)$ and $\Gamma \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$. Given $i, j \in \{1, 2\}$ with $i \neq j$,*

$$\Gamma^+, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi) \text{ or } \Gamma^+ \text{ is } \vdash_j \text{-explosive}$$

implies

$$\Gamma \vdash_{12} \psi.$$

Proof. If $\Gamma^{\vdash_{12}} \neq \emptyset$ fix $\gamma \in \Gamma^{\vdash_{12}}$. Let $\sigma : P \cup X_* \to L_{\Sigma_{12}}(P)$ be such that $\sigma(p) = p$ if $p \in P$, $\sigma(x_\phi) = \phi$, and $\sigma(x_*) = \gamma$ if $\Gamma^{\vdash_{12}} \neq \emptyset$.

Now, on one hand, if we have that $\Gamma^+, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$ then using structurality we obtain $(\Gamma^+)^\sigma, (X^i_\Gamma(\psi))^\sigma \vdash_i (\mathsf{skel}_{\Sigma_i}(\psi))^\sigma$. However, clearly, $\Gamma^\sigma = \Gamma$, $(\Gamma^+ \cup X^i_\Gamma(\psi))^\sigma \subseteq \Gamma^{\vdash_{12}}$, $(\mathsf{skel}_{\Sigma_i}(\psi))^\sigma = \psi$, and we conclude that $\Gamma \vdash_{12} \psi$.

If, on the other hand, we know that Γ^+ is $\vdash_j$-explosive, then we have $\Gamma^+ \vdash_j \psi$. However, we also have $\Gamma^\sigma = \Gamma$, $(\Gamma^+)^\sigma \subseteq \Gamma^{\vdash_{12}}$, $\psi^\sigma = \psi$, and we conclude again that $\Gamma \vdash_{12} \psi$. □

From Lemma 3.9, we know that $\Gamma \vdash_{12} \psi$ whenever we prove that either $\Gamma^+, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$ or Γ^+ is $\vdash_j$-explosive, for $i \neq j$. Below, this implication is strengthened to an equivalence. This means that given a proof $\Gamma \vdash_{12} \psi$, there is a proof of $\mathsf{skel}_{\Sigma_i}(\psi)$ in $\mathcal{L}_i$ from $\Gamma^+ \cup X^i_\Gamma(\psi)$ provided that Γ^+ is not $\vdash_j$-explosive. In the next example we give some intuition on why this is the case.

Example 3.10. Take the fibred logic $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{djn}} = \langle \Sigma_{\mathsf{cnj}} \cup \Sigma_{\mathsf{djn}}, \_^{\vdash} \rangle$, and consider $\Gamma = \{p \wedge q\}$ and $\psi = (p \vee t_1) \wedge (q \vee t_2)$. It is clear that $\Gamma \vdash \psi$, as the following inferences hold in the component logics:

$$\begin{aligned} p \wedge q &\vdash_{\mathsf{cnj}} p \\ p \wedge q &\vdash_{\mathsf{cnj}} q \\ p &\vdash_{\mathsf{djn}} p \vee t_1 \\ q &\vdash_{\mathsf{djn}} q \vee t_2 \\ p \vee t_1, q \vee t_2 &\vdash_{\mathsf{cnj}} (p \vee t_1) \wedge (q \vee t_2). \end{aligned}$$

It is worth noting that the interaction between the component logics is guided by the subformulas of Γ and ψ that are derived in one component logic and then used in the other, and that this effect is cumulative. Indeed, it follows on the side of $\mathcal{L}_{\mathsf{cnj}}$ that p and q are derivable form Γ, which is captured in the construction of $\Gamma^+ = \{x_*, p \wedge q, p, q\}$. Then, this fact is used on the $\mathcal{L}_{\mathsf{djn}}$-side to build $p \vee t_1$ and $q \vee t_2$, which is captured by $X_\Gamma^{\mathsf{cnj}}(\psi) = \{x_{p\vee t_1}, x_{q \vee t_2}\}$. Finally, with these formulas we can build ψ on the $\mathcal{L}_{\mathsf{cnj}}$-side. In other words, $\Gamma^+, X_\Gamma^{\mathsf{cnj}}(\psi) \vdash_{\mathsf{cnj}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\psi)$.

Let now $\Gamma = \{p \vee q\}$ and $\psi = (p \wedge p) \vee (q \wedge q)$. It is not too difficult to get convinced that $\Gamma \not\vdash \psi$. But how can we justify that? The reason for this is actually simple. Starting from Γ there is no way to obtain any other subformula. Surely not by applying rules of $\mathscr{H}_{\mathsf{cnj}}$, but not even by applying rules of $\mathscr{H}_{\mathsf{djn}}$. Hence $\Gamma^+ = \{x_*, p \vee q\}$, and for the same reason $X_\Gamma^{\mathsf{djn}}(\psi) = \emptyset$. Easily then $\Gamma^+, X_\Gamma^{\mathsf{djn}}(\psi) \not\vdash_{\mathsf{djn}} \mathsf{skel}_{\Sigma_{\mathsf{djn}}}(\psi)$. △

3.3 The characterization

Before we state and prove our main characterization result, we need an additional lemma showing that one can always work under the assumption that there is a fresh variable in P.

Lemma 3.11. *Let $p_0, p_1, \ldots \in P$ be an enumeration of P and $\mathsf{nxt} : P \to L_\Sigma(P)$ be such that $\mathsf{nxt}(p_k) = p_{k+1}$. The following facts hold, for $i \in \{1,2\}$:*

1. *if $\Gamma \vdash_{12} \psi$ then $\Gamma^{\mathsf{nxt}} \vdash_{12} \psi^{\mathsf{nxt}}$,*
2. *if $(\Gamma^{\mathsf{nxt}})^+, X^i_{\Gamma^{\mathsf{nxt}}}(\psi^{\mathsf{nxt}}) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi^{\mathsf{nxt}})$ then $\Gamma^+, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$,*
3. *if $(\Gamma^{\mathsf{nxt}})^+$ is $\vdash_i$-explosive then Γ^+ is also $\vdash_i$-explosive.*

Proof. The first statement follows simply from the structurality of $\vdash_{12}$.

In order to prove the last two statements, let us consider the substitution $\mathsf{prv} : P \cup X_* \to L_\Sigma(P \cup X_*)$ such that

$$\mathsf{prv}(y) = \begin{cases} p_k & \text{if } y = p_{k+1}, \\ x_\varphi & \text{if } y = x_{\varphi^{\mathsf{nxt}}}, \\ y & \text{otherwise.} \end{cases}$$

Concerning the second statement, if $(\Gamma^{\mathsf{nxt}})^+, X^i_{\Gamma^{\mathsf{nxt}}}(\psi^{\mathsf{nxt}}) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi^{\mathsf{nxt}})$ then it follows that $((\Gamma^{\mathsf{nxt}})^+)^{\mathsf{prv}}, (X^i_{\Gamma^{\mathsf{nxt}}}(\psi^{\mathsf{nxt}}))^{\mathsf{prv}} \vdash_i (\mathsf{skel}_{\Sigma_i}(\psi^{\mathsf{nxt}}))^{\mathsf{prv}}$, by just using the structurality of $\vdash_i$. As $\mathsf{prv} \circ \mathsf{nxt}$ is the identity on P, using now the structurality of $\vdash_{12}$, we have that $((\Gamma^{\mathsf{nxt}})^+)^{\mathsf{prv}} = \Gamma^+$, $(X^i_{\Gamma^{\mathsf{nxt}}}(\psi^{\mathsf{nxt}}))^{\mathsf{prv}} = X^i_\Gamma(\psi)$, and $\mathsf{skel}_{\Sigma_i}(\psi) = (\mathsf{skel}_{\Sigma_i}(\psi^{\mathsf{nxt}}))^{\mathsf{prv}}$. Hence, it follows that $\Gamma^+, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$.

Concerning the third statement, let us further consider the substitution $\overline{\mathsf{nxt}} : P \cup X_* \to L_\Sigma(P \cup X_*)$ such that

$$\overline{\mathsf{nxt}}(y) = \begin{cases} p_{k+1} & \text{if } y = p_k, \\ x_* & \text{if } y = x_*, \\ x_{\varphi^{\mathsf{nxt}}} & \text{if } y = x_\varphi. \end{cases}$$

Clearly, $\overline{\mathsf{nxt}}$ is a completion of nxt such that $\mathsf{prv} \circ \overline{\mathsf{nxt}}$ is the identity on $P \cup X_*$. Let us assume that $(\Gamma^{\mathsf{nxt}})^+$ is $\vdash_i$-explosive. In order to show that $\Gamma^+ \vdash_i \varphi$, for any $\varphi \in L_{\Sigma_{12}}(P \cup X_*)$, it suffices to note that $(\Gamma^{\mathsf{nxt}})^+ \vdash_i \varphi^{\overline{\mathsf{nxt}}}$ and by structurality $((\Gamma^{\mathsf{nxt}})^+)^{\mathsf{prv}} \vdash_i (\varphi^{\overline{\mathsf{nxt}}})^{\mathsf{prv}}$. The proof is concluded by reusing the argument used in the proof of the second statement. □

We can finally tackle our key characterization result, relating proofs from non-mixed hypotheses in disjoint fibrings with proofs in one of the component logics, by incorporating the derivable subformulas emerging from interaction with the other component.

Proposition 3.12. *Let $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ be Hilbert calculi such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\psi \in L_{\Sigma_{12}}(P)$ and $\Gamma \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$. Then, for $i, j \in \{1, 2\}$ with $i \neq j$,*

$$\Gamma \vdash_{12} \psi$$

if and only if

$$\Gamma^+, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi) \text{ or } \Gamma^+ \text{ is } \vdash_j\text{-explosive.}$$

Proof. The bottom to top implication follows by Lemma 3.9.

Let us now consider the top to bottom implication, and assume that we have $\Gamma \vdash_{12} \psi$. We will work under the assumption that $p_0 \notin \mathsf{var}(\Gamma)$ (this is crucial in the proof of subcase (2)(b), below). Lemma 3.11 allows us to make this assumption without any loss of generality. Indeed, from $\Gamma \vdash_{12} \psi$ we know that $\Gamma^{\mathsf{nxt}} \vdash_{12} \psi^{\mathsf{nxt}}$ while being sure that $p_0 \notin \mathsf{var}(\Gamma^{\mathsf{nxt}})$. From here, our proof below will guarantee that $(\Gamma^{\mathsf{nxt}})^+, X^i_{\Gamma^{\mathsf{nxt}}}(\psi^{\mathsf{nxt}}) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi^{\mathsf{nxt}})$ or $(\Gamma^{\mathsf{nxt}})^+$ is $\vdash_j$-explosive, and the lemma allows us to conclude that $\Gamma^+, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$ or Γ^+ is $\vdash_j$-explosive, as desired.

Clearly, $\Gamma^{\vdash_{12}} \neq \emptyset$ and so $x_* \in \Gamma^+$. If Γ^+ is $\vdash_j$-explosive the statement immediately follows. Hence, we proceed assuming that we have Γ^+ is not $\vdash_j$-explosive. Let $\mathscr{H}_{12} = \mathscr{H}_1 \bullet \mathscr{H}_2$. The proof follows by complete transfinite induction on the length of $\mathscr{H}_{12}$-derivations. Given that $\Gamma \vdash_{12} \psi$, there must exist a $\mathscr{H}_{12}$-derivation $\overline{\psi} = \langle \psi_\kappa \rangle_{\kappa < \eta+1}$ from Γ such that $\psi_\eta = \psi$. We want to show that $\Gamma^+, X^i_\Gamma(\psi_\eta) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\eta)$. Thus, we will prove that $\Gamma^+, X^i_\Gamma(\psi_\tau) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\tau)$ for any $\tau \leq \eta$ assuming, by induction hypothesis, that the top to bottom implication holds for any $\mathscr{H}_{12}$-derivation with length smaller than τ, and for both $i = 1, 2$.

Note that the case when $\mathsf{head}(\psi_\tau) \in \Sigma_j$ is trivial. Indeed, in that situation, we have that $\mathsf{Mon}_{\Sigma_i}(\psi_\tau) = \{\psi_\tau\}$, and thus $\mathsf{skel}_{\Sigma_i}(\psi_\tau) = x_{\psi_\tau} \in X^i_\Gamma(\psi_\tau)$. But then, clearly, $\Gamma^+, X^i_\Gamma(\psi_\tau) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\tau)$. We assume, henceforth, that either $\psi_\tau \in P$ or $\mathsf{head}(\psi_\tau) \in \Sigma_i$, meaning that $\mathsf{skel}_{\Sigma_j}(\psi_\tau) \in P \cup X$.

We have to consider two cases.

(1) $\psi_\tau \in \Gamma$.

In this case, $\Gamma^+, X_\Gamma(\psi_\tau) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\tau)$ as $\mathsf{skel}_{\Sigma_i}(\psi_\tau) = \psi_\tau \in \Gamma \subseteq \Gamma^+$, since $\Gamma \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$, and we are working under the assumption that $\mathsf{head}(\psi_\tau) \notin \Sigma_j$.

(2) $\psi_\tau = \varphi^\sigma$, $\frac{\Delta}{\varphi} \in R_t$ for some $t \in \{1,2\}$, and $\Delta^\sigma \subseteq \{\psi_\kappa : \kappa < \tau\}$.

Since $\{\psi_\kappa : \kappa < \tau\} \vdash_t \psi_\tau$, by applying Corollary 3.3 we obtain that $\{\mathsf{skel}_{\Sigma_t}(\psi_\kappa) : \kappa < \tau\} \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\tau)$. By induction hypothesis we then have that $\Gamma^+, X^t_\Gamma(\psi_\kappa) \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\kappa)$ for each $\kappa < \tau$, and therefore, we have that $\Gamma^+, \bigcup_{\kappa<\tau} X^t_\Gamma(\psi_\kappa) \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\tau)$.

Now we consider two possibilities.

(a) $\frac{\Delta}{\varphi} \in R_i$, and so $t = i$.

Consider $\mu : P \cup X_* \to L_{\Sigma_{12}}(P \cup X_*)$ such that $\mu(x_\phi) = x_*$ if $\phi \notin \mathsf{Mon}_{\Sigma_i}(\psi_\tau)$, and $\mu(y) = y$ otherwise. We have $(\Gamma^+ \cup X^i_\Gamma(\psi_\kappa))^\mu \subseteq \Gamma^+ \cup X^i_\Gamma(\psi_\tau)$ for each $\kappa < \tau$, and $(\mathsf{skel}_{\Sigma_i}(\psi_\tau))^\mu = \mathsf{skel}_{\Sigma_i}(\psi_\tau)$. By structurality and monotonicity we get $\Gamma^+, X^i_\Gamma(\psi_\tau) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\tau)$.

(b) $\frac{\Delta}{\varphi} \in R_j$, and so $t = j$.

If $\psi_\kappa = \psi_\tau$ for some $\kappa < \tau$, by induction hypothesis we have that $\Gamma^+, X^i_\Gamma(\psi_\kappa) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\kappa)$ and so $\Gamma^+, X^i_\Gamma(\psi_\tau) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\tau)$.

If $\psi_\kappa \neq \psi_\tau$ for all $\kappa < \tau$, but $\psi_\tau \in P \cap \Gamma^+$ then $\mathsf{skel}_{\Sigma_i}(\psi_\tau) = \psi_\tau$ and therefore we also have $\Gamma^+, X^i_\Gamma(\psi_\tau) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\tau)$.

We finish the proof by showing that no other case is possible. That is, assuming that either $\psi_\tau \in P \setminus \Gamma^+$ or $\mathsf{head}(\psi_\tau) \in \Sigma_i$, along with the fact that $\psi_\kappa \neq \psi_\tau$ for all $\kappa < \tau$, we will derive a contradiction.

(i) $\psi_\tau = p \in P \setminus \Gamma^+$.

Consider $\nu : P \cup X_* \to L_{\Sigma_{12}}(P \cup X_*)$ such that $\nu(x) = x_*$ for all $x \in X_*$, and $\nu(q) = q$ for all $q \in P$. Clearly $(\Gamma^+)^\nu = \Gamma^+$, $(\bigcup_{\kappa<\tau} X_\Gamma(\psi'_\kappa))^\nu \subseteq \{x_*\} \subseteq \Gamma^+$ and $\nu(\psi_\tau) = \nu(p) = p$. Therefore, using the structurality of $\vdash_j$, we get $\Gamma^+ \vdash_j p$. Here we have to split the proof in yet another two cases.

· If $p \notin \mathsf{var}(\Gamma^+)$ then, by structurality of $\vdash_j$, we easily conclude that Γ^+ is $\vdash_j$-explosive, which is a contradiction.

· If $p \in \mathsf{var}(\Gamma^+)$ then $\Gamma \neq \emptyset$. Let $\gamma \in \Gamma$ and consider a substitution ρ such that $\tau(q) = q$ for $q \in P$ and $\rho(x_*) = \gamma$. By structurality of $\vdash_j$ we get $(\Gamma^+)^\rho = \Gamma^\omega \vdash_j p = p^\rho$. Lemma 3.7 the implies that $p \in \Gamma^+$, contradicting $p \in P \setminus \Gamma^+$.

(ii) $\mathsf{head}(\psi_\tau) \in \Sigma_i$.

Recall that we are assuming that $p_0 \notin \mathsf{var}(\Gamma)$. We define, for all $\kappa \leq \tau$, $\psi'_\kappa = \psi_\kappa[\psi_\tau/p_0]_{\Sigma_j}$. Clearly, $\psi'_\tau = \psi_\tau[\psi_\tau/p_0]_{\Sigma_j} = p_0$. Let $\mu : P \cup X_* \to L_{\Sigma_{12}}(P \cup X_*)$ defined by $\mu(x_{\psi_\tau}) = p_0$, and $\mu(y) = y$ for $y \neq x_{\psi_\tau}$. Easily, we have that $\mathsf{skel}_{\Sigma_j}(\psi'_\kappa) = (\mathsf{skel}_{\Sigma_j}(\psi_\kappa))^\mu$ for all $\kappa \leq \tau$.

Since $\{\mathsf{skel}_{\Sigma_j}(\psi_\kappa) : \kappa < \tau\} \vdash_j \mathsf{skel}_{\Sigma_j}(\psi_\tau)$, by structurality of $\vdash_j$ we obtain that $(\{\mathsf{skel}_{\Sigma_j}(\psi_\kappa) : \kappa < \tau\})^\mu \vdash_j (x_{\psi_\tau})^\mu$ and therefore, $\{\mathsf{skel}_{\Sigma_j}(\psi'_\kappa) : \kappa < \tau\} \vdash_j p_0$.

As $\Gamma \vdash_{12} \langle \psi_\kappa \rangle_{\kappa<\tau}$ and we assumed that $\psi_\kappa \neq \psi_\tau$ for all $\kappa < \tau$, we can use Lemma 3.5 to conclude that $\Gamma \vdash_{12} \langle \psi'_\kappa \rangle_{\kappa<\tau}$, and by induction hypothesis, we get that $\Gamma^+, X^j_\Gamma(\psi'_\kappa) \vdash_j \mathsf{skel}_{\Sigma_j}(\psi'_\kappa)$ for each $\kappa < \tau$. Thus, we also have $\Gamma^+, \bigcup_{\kappa<\tau} X^j_\Gamma(\psi'_\kappa) \vdash_j p_0$.
Using the substitution ν as defined in **(i)**, and arguing in the same manner, we obtain a contradiction. □

In the next example we highlight the relevance of each element of the construction in the previous proposition.

Example 3.13. Take the fibred logic $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}} = \langle \Sigma, {}_{\_}^{\vdash} \rangle$, where $\Sigma = \Sigma_{\mathsf{cnj}} \cup \Sigma_{\mathsf{neg}}$. Let $\Gamma = \{\neg\neg p, q\}$ and consider the following sequence of formulas

$$\begin{aligned}
\psi_0 &= \neg\neg p,\\
\psi_1 &= p,\\
\psi_2 &= q,\\
\psi_3 &= \neg\neg q,\\
\psi_4 &= p \wedge \neg\neg q,\\
\psi_5 &= \neg\neg(p \wedge \neg\neg q),\\
\psi_6 &= (\neg\neg p) \wedge q,\\
\psi_7 &= (\neg\neg(p \wedge \neg\neg q)) \wedge ((\neg\neg p) \wedge q).
\end{aligned}$$

We have that $\Gamma \vdash \langle \psi_k \rangle_{k<8}$, and in particular $\Gamma \vdash \psi_7$, because ψ_0, ψ_2 appear as hypothesis in Γ, ψ_1, ψ_3, ψ_5 appear by application of rules of $\mathscr{H}_{\mathsf{neg}}$, and ψ_4, ψ_6, ψ_7 appear by application of rules of $\mathscr{H}_{\mathsf{cnj}}$.

Note that $\Gamma^+ = \{\neg\neg p, q, p, x_*\}$. It is clear that Γ^+ is not $\vdash_{\mathsf{cnj}}$-explosive nor $\vdash_{\mathsf{neg}}$-explosive, as both $\mathcal{L}_{\mathsf{cnj}}$ and $\mathcal{L}_{\mathsf{neg}}$ are fragments of classical logic. We shall see that we can extract from the above derivation two derivations justifying $\Gamma^+, X^{\mathsf{cnj}}_\Gamma(\psi_7) \vdash_{\mathsf{cnj}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\psi_7)$ and $\Gamma^+, X^{\mathsf{neg}}_\Gamma(\psi_7) \vdash_{\mathsf{neg}} \mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\psi_7)$, respectively. In this case, as $\mathsf{head}(\psi_7) \in \Sigma_{\mathsf{cnj}}$, the derivation on the $\mathcal{L}_{\mathsf{cnj}}$ side will be much more informative than the derivation on the $\mathcal{L}_{\mathsf{neg}}$ side.

Indeed, if we consider $\mathcal{L}_{\mathsf{neg}}$, we get $X^{\mathsf{neg}}_\Gamma(\psi_7) = \{x_{\psi_7}\}$ and $\mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\psi_7) = x_{\psi_7}$, and we trivially have $\Gamma^+, X^{\mathsf{neg}}_\Gamma(\psi_7) \vdash_{\mathsf{neg}} \mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\psi_7)$ – the derivation only retains the ($\mathsf{skel}_{\Sigma_{\mathsf{neg}}}$ of the) last step of the original sequence.

It is more interesting to consider $\mathcal{L}_{\mathsf{cnj}}$. Easily, we have that $X^{\mathsf{cnj}}_\Gamma(\psi_7) = \{x_{\psi_0}, x_{\psi_5}\}$ and $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\psi_7) = x_{\psi_5} \wedge (x_{\psi_0} \wedge q)$. Using the same rules justifying ψ_6 and ψ_7 in the original derivation, we easily see that $q, x_{\psi_0}, x_{\psi_5} \vdash_{\mathsf{cls}} x_{\psi_5} \wedge (x_{\psi_0} \wedge q)$. Note that the other step of the original derivation that was justified on the $\mathcal{L}_{\mathsf{cnj}}$ side, ψ_4, is simply not necessary here as it is "obscured" by double-negation in ψ_5.

In the case above, $x_* \in \Gamma^+$ did not play a significant role. In order to clarify the importance of x_*, let us now consider the fibred logic $\mathcal{L}_{\mathsf{qcls}} \bullet \mathcal{L}_{\mathsf{top}} = \langle \Sigma, {}_{\_}^{\vdash} \rangle$, where $\Sigma = \Sigma_{\mathsf{qcls}} \cup \Sigma_{\mathsf{top}}$. In Example 4.1, we saw that although $\nvdash_{\mathsf{qcls}} p \Rightarrow (q \Rightarrow p)$ we have that $\vdash p \Rightarrow (q \Rightarrow p)$. Indeed, $\vdash \langle \top, p \Rightarrow (q \Rightarrow p) \rangle$ is a valid derivation:

$\top$ appears as an axiom of $\mathcal{L}_{\mathsf{top}}$, and $p \Rightarrow (q \Rightarrow p)$ as an application of the corresponding $\mathcal{L}_{\mathsf{qcls}}$ rule. In this case $\emptyset^+ = \{x_*\}$, and it is clear that $\{x_*\}$ is neither $\vdash_{\mathsf{qcls}}$-explosive, nor $\vdash_{\mathsf{top}}$-explosive.

Focusing on the $\mathcal{L}_{\mathsf{qcls}}$ side, to derive $\mathsf{skel}_{\Sigma_{\mathsf{qcls}}}(p \Rightarrow (q \Rightarrow p)) = p \Rightarrow (q \Rightarrow p)$ we need to have some initial formula. Note also that $X_\emptyset^{\mathsf{qcls}}(p \Rightarrow (q \Rightarrow p)) = \emptyset$. However, as $x_* \vdash_{\mathsf{qcls}} p \Rightarrow (q \Rightarrow p)$, we have $\emptyset^+ \vdash_{\mathsf{qcls}} p \Rightarrow (q \Rightarrow p)$. △

4 Conservativity

In this section we tackle the first of two meaningful applications of Proposition 3.12. Namely, we use it to close, in the disjoint case, a long standing question of the theory of fibring: conservativity. Namely, we prove a full conservativity characterization result for disjoint fibring, extending the partial result obtained in [19]. We start by reviewing the conservativity problem for fibring, by means of a series of examples.

4.1 The problem

Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be two logics. By definition of fibring, $\mathcal{L}_1 \bullet \mathcal{L}_2$ extends both $\mathcal{L}_1$ and $\mathcal{L}_2$, but when can we guarantee that $\mathcal{L}_1 \bullet \mathcal{L}_2$ extends $\mathcal{L}_1$ and $\mathcal{L}_2$ conservatively?

Example 4.1. Let $\mathscr{H}_1 = \langle \Sigma, R \rangle$ be any Hilbert calculus with a (useful) rule $\frac{\Gamma}{\varphi} \in R$ such that $\varphi \notin \Gamma$, and $\mathscr{H}_2 = \langle \Sigma, \emptyset \rangle$. It is easy to check that $\mathcal{L}_{\mathscr{H}_1} \bullet \mathcal{L}_{\mathscr{H}_2} = \mathcal{L}_{\mathscr{H}_1}$ and thus it does not extend $\mathcal{L}_{\mathscr{H}_2}$ conservatively. Namely, it is clear that

$$\Gamma \nvdash_2 \varphi \quad \text{but} \quad \Gamma \vdash_1 \varphi.$$

This is not surprising as one might argue that the two logics share some syntax, as they even have the same signature, but their consequences do not agree on all shared formulas. Though the argument is correct, this intuition is still misleading. In fact, the simplest way to avoid such a clash would be to require that the fibring be disjoint. However, one can easily show that even disjoint fibring can lead to situations where conservativity fails.

Let us consider the logics $\mathcal{L}_{\mathsf{qcls}} = \langle \Sigma_{\mathsf{qcls}}, \_^{\mathsf{qcls}} \rangle$ and $\mathcal{L}_{\mathsf{top}} = \langle \Sigma_{\mathsf{top}}, \_^{\mathsf{top}} \rangle$ from Example 2.2. Clearly, $\Sigma_{\mathsf{qcls}} \cap \Sigma_{\mathsf{top}} = \emptyset$. However, $\mathcal{L}_{\mathsf{qcls}} \bullet \mathcal{L}_{\mathsf{top}} = \langle \Sigma_{\mathsf{qcls}} \cup \Sigma_{\mathsf{top}}, \_^{\vdash} \rangle$ does not extend $\mathcal{L}_{\mathsf{qcls}}$ conservatively. Namely, note that

$$\nvdash_{\mathsf{qcls}} p \Rightarrow (q \Rightarrow p) \quad \text{but} \quad \vdash p \Rightarrow (q \Rightarrow p).$$

Indeed, it suffices to note that $\vdash_{\mathsf{top}} \top$ implies $\vdash \top$, $s \vdash_{\mathsf{qcls}} p \Rightarrow (q \Rightarrow p)$ implies $\top \vdash p \Rightarrow (q \Rightarrow p)$ using structurality, and thus $\vdash p \Rightarrow (q \Rightarrow p)$. △

These examples show us that although the conservativity problem for fibring is clearly more troublesome when there are shared connectives, there is still something fundamental that needs to be better understood at the simpler level of disjoint fibring. This is precisely where Proposition 3.12 may be of help. Let us analyze a few more disjoint examples.

Example 4.2. Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ be a trivial logic, and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ some other logic such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. The fibred logic $\mathcal{L}_1 \bullet \mathcal{L}_2 = \langle \Sigma_{12}, \_^{\vdash_{12}} \rangle$ is obviously trivial, as $p \vdash_1 q$ implies $p \vdash_{12} q$. Thus, $\mathcal{L}_1 \bullet \mathcal{L}_2$ can only extend $\mathcal{L}_2$ conservatively if $\mathcal{L}_2$ is also trivial, that is, $p \vdash_2 q$.

However, though necessary, the triviality of $\mathcal{L}_2$ may not be sufficient. Take, for instance, $\mathcal{L}_1 = \mathcal{L}_{\mathsf{inc}(\Sigma)}$ and $\mathcal{L}_2 = \mathcal{L}_{\mathsf{tonk}}$ for any signature Σ such that $\mathsf{tonk} \notin \Sigma$, as defined in Example 2.2. Both logics are obviously trivial, but $\mathcal{L}_{\mathsf{inc}(\Sigma)}$ is inconsistent while $\mathcal{L}_{\mathsf{tonk}}$ is consistent. Their fibring $\mathcal{L}_{\mathsf{inc}(\Sigma)} \bullet \mathcal{L}_{\mathsf{tonk}} = \langle \Sigma \cup \Sigma_{\mathsf{tonk}}, \_^{\vdash} \rangle$ does not extend $\mathcal{L}_{\mathsf{tonk}}$ conservatively, as it is clearly inconsistent. In particular, we have

$$\nvdash_{\mathsf{tonk}} p \quad \text{but} \quad \vdash p. \qquad \triangle$$

The examples above emphasize the impact that triviality and inconsistency have on the conservativity problem. However, getting rid of such pathological cases is still not completely satisfactory, as the next examples will help illustrate.

Example 4.3. Take the logics $\mathcal{L}_{\mathsf{cls}}$ and $\mathcal{L}_{\mathsf{int}}$ from Example 2.2. Their fibring $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{int}}$ does conservatively extend both $\mathcal{L}_{\mathsf{cls}}$ and $\mathcal{L}_{\mathsf{int}}$, as shown in [7], and also as a consequence of [19].

However, let us consider a small variation, and take the logic $\mathcal{L}_{\mathsf{qcls}}$ instead of $\mathcal{L}_{\mathsf{cls}}$. It turns out that the fibring $\mathcal{L}_{\mathsf{qcls}} \bullet \mathcal{L}_{\mathsf{int}} = \langle \Sigma_{\mathsf{qcls}} \cup \Sigma_{\mathsf{int}}, \_^{\vdash} \rangle$ is not conservative anymore. Namely, the extension of $\mathcal{L}_{\mathsf{qcls}}$ by $\mathcal{L}_{\mathsf{qcls}} \bullet \mathcal{L}_{\mathsf{int}}$ is not conservative, as it happens that

$$\nvdash_{\mathsf{qcls}} p \Rightarrow (q \Rightarrow p) \quad \text{but} \quad \vdash p \Rightarrow (q \Rightarrow p).$$

Indeed, similarly to Example 4.1, it suffices to note that $\vdash_{\mathsf{int}} p \to (q \to p)$ implies $\vdash p \to (q \to p)$, $s \vdash_{\mathsf{qcls}} p \Rightarrow (q \Rightarrow p)$ implies $p \to (q \to p) \vdash p \Rightarrow (q \Rightarrow p)$ using structurality, and thus $\vdash p \Rightarrow (q \Rightarrow p)$. $\triangle$

At this point one might still argue that the failure identified above has to do with the fact that the logic $\mathcal{L}_{\mathsf{qcls}}$ is somewhat artificial. We show below that such a consideration, though reasonable, is not fundamental.

Example 4.4. Recall from Example 4.1 that also $\mathcal{L}_{\mathsf{qcls}} \bullet \mathcal{L}_{\mathsf{top}}$ fails to be a conservative extension of $\mathcal{L}_{\mathsf{qcls}}$.

However, let us consider now the logic $\mathcal{L}_{\mathsf{neg}} = \langle \Sigma_{\mathsf{neg}}, \_^{\vdash_{\mathsf{neg}}} \rangle$ from Example 2.2. It turns out that $\mathcal{L}_{\mathsf{qcls}} \bullet \mathcal{L}_{\mathsf{neg}}$ is a conservative extension of both $\mathcal{L}_{\mathsf{qcls}}$ and $\mathcal{L}_{\mathsf{neg}}$, as we will show below. $\triangle$

4.2 Characterization

Our main result about conservativity of disjoint fibring follows.

Theorem 4.5. *Let $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ be two Hilbert calculi, with $\Sigma_1 \cap \Sigma_2 = \emptyset$, and $i, j \in \{1, 2\}$ such that $i \neq j$.*

Then, $\mathcal{L}_{\mathscr{H}_1} \bullet \mathcal{L}_{\mathscr{H}_2}$ is a conservative extension of $\mathcal{L}_{\mathscr{H}_i}$ if and only if the following two properties are satisfied:

- *if $\mathcal{L}_{\mathscr{H}_j}$ is trivial then $\mathcal{L}_{\mathscr{H}_i}$ is trivial, and*
- *if $\mathcal{L}_{\mathscr{H}_j}$ has theorems then $\mathcal{L}_{\mathscr{H}_i}$ does not have q-theorems.*

Proof. Let us first prove that the two conditions are necessary. Assume that indeed $\mathcal{L}_{\mathscr{H}_1} \bullet \mathcal{L}_{\mathscr{H}_2}$ is a conservative extension of $\mathcal{L}_{\mathscr{H}_i}$.

If $\mathcal{L}_{\mathscr{H}_j}$ is trivial then $p \vdash_j q$ for any distinct $p, q \in P$, which implies that $p \vdash_{12} q$, and in turn, by the conservativeness assumption, also implies that $p \vdash_i q$, and we conclude that $\mathcal{L}_{\mathscr{H}_i}$ is trivial.

If $\mathcal{L}_{\mathscr{H}_j}$ has a theorem, say $\varphi \in L_{\Sigma_j}(P)$, then $\vdash_j \varphi$, which implies that $\vdash_{12} \varphi$. Suppose, for *reductio*, that $\mathcal{L}_{\mathscr{H}_i}$ has a q-theorem $\psi \in L_{\Sigma_i}(P)$. Pick $p \in P$ such that p does not occur in ψ. As ψ is a q-theorem of $\mathcal{L}_{\mathscr{H}_i}$, we have that $p \vdash_i \psi$, which on its turn implies that $p \vdash_{12} \psi$. Consider the substitution $\sigma : P \to L_{\Sigma_{12}}(P)$ defined by $\sigma(p) = \varphi$, and $\sigma(q) = q$ for $q \neq p$. Easily, $p^\sigma = \varphi$ and $\psi^\sigma = \psi$, and by structurality we get that $\varphi \vdash_{12} \psi$. But we already know that $\vdash_{12} \varphi$, and therefore we get $\vdash_{12} \psi$. As we assumed that $\mathcal{L}_{\mathscr{H}_1} \bullet \mathcal{L}_{\mathscr{H}_2}$ is a conservative extension of $\mathcal{L}_{\mathscr{H}_i}$, we get $\vdash_i \psi$, which contradicts the fact that ψ is a q-theorem of $\mathcal{L}_{\mathscr{H}_i}$. We conclude that $\mathcal{L}_{\mathscr{H}_i}$ does not have q-theorems.

We are now left with proving that the two conditions are sufficient for guaranteeing that $\mathcal{L}_{\mathscr{H}_1} \bullet \mathcal{L}_{\mathscr{H}_2}$ is a conservative extension of $\mathcal{L}_{\mathscr{H}_i}$. Assume that both conditions hold, and let $\Gamma \cup \{\varphi\} \subseteq L_{\Sigma_i}(P)$ be such that $\Gamma \vdash_{12} \varphi$.

If $\mathcal{L}_{\mathscr{H}_j}$ is trivial then, by assumption, $\mathcal{L}_{\mathscr{H}_i}$ is also trivial. There are two possibilities for a trivial logic: either all formulas are theorems of $\mathcal{L}_{\mathscr{H}_i}$, or all formulas are q-theorems of $\mathcal{L}_{\mathscr{H}_i}$. In the former case, φ is a theorem and thus $\Gamma \vdash_i \varphi$. In the latter case, φ is a q-theorem and thus $\Gamma \vdash_i \varphi$ if $\Gamma \neq \emptyset$. We just need to show here that the case $\Gamma = \emptyset$ is impossible. Suppose, for *reductio*, that $\mathcal{L}_{\mathscr{H}_i}$ is trivial and has q-theorems, but $\Gamma = \emptyset$. As $\vdash_{12} \varphi$ but $\mathcal{L}_{\mathscr{H}_i}$ has no theorems, it must be the case that $\mathcal{L}_{\mathscr{H}_j}$ has theorems. Then, by assumption, it follows that $\mathcal{L}_{\mathscr{H}_i}$ has no q-theorems, which contradicts the hypothesis.

If $\mathcal{L}_{\mathscr{H}_j}$ is not trivial then, by Lemma 2.1, as $\Gamma \subseteq L_{\Sigma_i}(P)$, we can easily conclude that Γ^+ is not $\vdash_j$-explosive. Thus, we can apply Proposition 3.12 to $\Gamma \vdash_{12} \varphi$ and conclude that $\Gamma^+, X^i_\Gamma(\varphi) \vdash_i \mathsf{skel}_{\Sigma_i}(\varphi)$. Note, however, that $\varphi \in L_{\Sigma_i}(P)$ and therefore $\mathsf{Mon}_{\Sigma_i}(\varphi) = \emptyset$, $X^i_\Gamma(\varphi) = \emptyset$ and $\mathsf{skel}_{\Sigma_i}(\varphi) = \varphi$. Moreover, we know that $\varphi \in \Gamma^{\vdash_{12}} \neq \emptyset$ and thus $x_* \in \Gamma^+$. Hence, we have $\Gamma^+ = \Gamma^\omega \cup \{x_*\}$ and $\Gamma^+ \vdash_i \varphi$. It is also easy to see that in this case $\Gamma^\omega \subseteq \Gamma^{\vdash_i}$. Let us prove by induction on $n \in \mathbb{N}$ that $\Gamma^n \subseteq \Gamma^{\vdash_i}$. The base case is straightforward. If $n = k+1$ then by induction hypothesis, $\Gamma^n \subseteq \Gamma^{\vdash_i}$, and to finish the proof if suffices to show that $\{p \in \mathsf{var}(\Gamma) : \Gamma^n \vdash_j p\} \subseteq \Gamma^n$, which follows easily from the fact that $\Gamma^n \subseteq L_{\Sigma_i}(P)$ by using Lemma 2.1 and the non-triviality of $\vdash_j$. Hence we conclude that $\Gamma, x_* \vdash_i \varphi$.

If $\Gamma \vdash_i \psi$ for some $\psi \in L_{\Sigma_i}(P)$, just consider a substitution $\sigma : P \cup X_* \to L_{\Sigma_i}(P)$ such that $\sigma(p) = p$ if $p \in P$, and $\sigma(x_*) = \psi$. Clearly, $\Gamma^\sigma = \Gamma$, $\varphi^\sigma = \varphi$ and $x_*^\sigma = \psi$, and from $\Gamma, x_* \vdash_i \varphi$, by structurality, we get $\Gamma, \psi \vdash_i \varphi$. As we assumed that $\Gamma \vdash_i \psi$, we conclude that $\Gamma \vdash_i \varphi$.

If there is no $\psi \in L_{\Sigma_i}(P)$ such that $\Gamma \vdash_i \psi$, then we know not only that $\Gamma = \emptyset$ but also that $\mathcal{L}_{\mathscr{H}_i}$ has no theorems. In that case, as we have $\vdash_{12} \varphi$, we also know that the fibred logic has theorems, and therefore $\mathcal{L}_{\mathscr{H}_j}$ must have theorems. Thus, by assumption, we also know that $\mathcal{L}_{\mathscr{H}_i}$ does not have q-theorems. At this point, as we have that $x_* \vdash_i \varphi$, x_* does not occur in φ, and φ cannot be a q-theorem, we can conclude that φ is a theorem and thus $\vdash_i \varphi$. $\square$

		$\mathcal{L}_1$ trivial: thms	$\mathcal{L}_1$ trivial: q-thms	$\mathcal{L}_1$ non-trivial: thms	$\mathcal{L}_1$ non-trivial: q-thms	$\mathcal{L}_1$ non-trivial: none
$\mathcal{L}_2$ triv.	thms	$1, 2$	2	2	2	2
	q-thms	1	$1, 2$	–	2	2
$\mathcal{L}_2$ non-triv.	thms	1	–	$1, 2^\flat$	2	$1, 2^\flat$
	q-thms	1	1	1	$1, 2$	$1, 2$
	none	1	1	$1, 2^\flat$	$1, 2$	$1, 2^\flat$

Figure 1: Conservativity of disjoint fibring, summarized.

Our characterization of the conservativity problem for disjoint fibring is synthesized in the table of Figure 1, where we equate all the possible combinations of the relevant property of triviality with the existence of theorems (thms) and q-theorems (q-thms) for each of the component logics. The symbol $i \in \{1, 2\}$ appears in the table entries corresponding exactly to the combinations in which $\mathcal{L}_i$ is extended conservatively.

Example 4.2 illustrates the combinations in the two leftmost columns of the table, when the first logic is trivial. Example 4.3 illustrates the third line entries, in the third and fourth columns, when fibring a non-trivial logic with theorems and another non-trivial logic with theorems or q-theorems. Example 4.4 exemplifies the penultimate entry of the last line of the table, when fibring two non-trivial logics, one with q-theorems, the other with neither theorems nor q-theorems. The remaining situations, (1) when one combines two non-trivial logics both with quasi-theorems, and (2) when one combines a non-trivial logic without quasi-theorems and a non-trivial logic without theorems nor quasi-theorems, always lead to a conservative extension, and can be easily illustrated. For case (1), it would suffice to consider the fibring of $\mathcal{L}_{\mathsf{qcls}}$ with a similarly obtainable theoremless version of $\mathcal{L}_{\mathsf{int}}$. For case (2), one might consider $\mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_{\mathsf{neg}}$.

It is also worth noting that the conservativity result for disjoint fibring provided by [19] covers only four entries in our table, which we marked with $^\flat$, corresponding to the combination of two non-trivial logics without quasi-theorems.

Indeed, the author of [19] considers only what he calls *fully determined logics*: logics that are determined by a class of matrices having as designated values a non-empty proper subset of the carrier. These are shown to correspond to non-trivial logics having no *mere-followers*, or quasi-theorems in our terminology. Another advantage of our result is that it allows a finer analysis of the conservativity problem, as it also describes the cases where only one of the component logics is conservatively extended by the fibred logic.

5 A peek over the semantical side of fibring

As a further illustration of the power of Proposition 3.12, which goes well beyond conservativity, we consider a second interesting application. Namely, we show that finite-(N)valuedness is not preserved by fibring, in general.

As we already mentioned, there have been various attempts to provide fibring with an appropriate semantical counterpart. Despite some interesting results, like sufficient conditions for completeness preservation, these attempts are not fully satisfactory and, in particular, have reduced practical use. Our result helps to justify why this is the case, namely using the widely accepted notion of matrix semantics [24] or, even more generally, of non-deterministic matrix semantics [1], which we recall.

Definition 5.1. A *non-deterministic matrix (Nmatrix)* for a signature Σ is a tuple $M = \langle A, D, O\rangle$, where:

- A is a non-empty set (of *truth-values*),
- $D \subseteq A$ (the set of *designated* truth-vales), and
- for each $c \in \Sigma^{(n)}$ then $\tilde{c} \in O$ is a function $\tilde{c} : A^n \to 2^A \setminus \{\emptyset\}$ (*interpretation*).

If for every $n \in \mathbb{N}_0$, $c \in \Sigma^{(n)}$ and $a_1, \dots, a_n \in A$, we have that $\tilde{c}(a_1, \dots, a_n)$ is a singleton, then M is simply said to be a *(deterministic) matrix.*

A valuation over M is a function $v : L_\Sigma(P) \to A$ such that

$$v(c(\psi_1, \dots, \psi_n)) \in \tilde{c}(v(\psi_1), \dots, v(\psi_n))$$

for every $n \in \mathbb{N}_0$, $c \in \Sigma^{(n)}$ and $\psi_1, \dots, \psi_n \in L_\Sigma(P)$. Of course, when M is deterministic, a valuation v is determined by the values of $v(p)$ for $p \in P$.

We say that $M, v \models \varphi$ if $v(\varphi) \in D$, and $M, v \models \Gamma$ if $M, v \models \psi$ for all $\psi \in \Gamma$. Moreover, we say that $\Gamma \models_M \varphi$ if for all valuations v over M, $M, v \models \Gamma$ implies $M, v \models \varphi$. Given a class $\mathcal{M}$ of (N)matrices for Σ, we say that $\Gamma \models_{\mathcal{M}} \varphi$ if $\Gamma \models_M \varphi$ for every $M \in \mathcal{M}$. It is straightforward to check that $\models_{\mathcal{M}}$ is always a Tarskian consequence relation.

If $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ is a logic, we say that (the *semantics*) $\mathcal{M}$ is $\mathcal{L}$-*sound* if $\models_{\mathcal{M}} \subseteq \vdash$, $\mathcal{M}$ is $\mathcal{L}$-*complete* if $\vdash \subseteq \models_{\mathcal{M}}$, and $\mathcal{M}$ is $\mathcal{L}$-*adequate* if $\vdash = \models_{\mathcal{M}}$. When

$\mathcal{M} = \{M\}$ is $\mathcal{L}$-adequate, we also say that M is a *characteristic (N)matrix* for $\mathcal{L}$, and if M is finite we say that $\mathcal{L}$ is *finite-(N)valued*.

It is well known that every Tarskian consequence relation can be given a semantics based on logical matrices, and thus also on Nmatrices. Characteristic (N)matrices, however, do not exist in general. See [24, 1] for a discussion of these questions.

The main result in this section is that an important aspect of many familiar logics, the existence of a finite characteristic (N)matrix, is not preserved by (disjoint) fibring. For this purpose we present examples where the preservation of this property fails. We consider pairs of logics having simple finite characteristic matrices, and prove that their fibring is not characterizable by a single finite (N)matrix, profitting from Proposition 3.12 as follows: given logics $\mathcal{L}_i = \langle \Sigma_i, \_^{\vdash_i} \rangle$ for $i = 1, 2$, we define for each $k \in \mathbb{N}$ a set $\Gamma_k \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$ and a formula $\varphi_k \in L_{\Sigma_1 \cup \Sigma_2}(P)$; making make use of Proposition 3.12 we prove that $\Gamma_k \nvdash_{12} \varphi_k$; finally, using the pigeonhole principle, we conclude that each $\mathcal{L}_{12}$-sound Nmatrix M with less than k elements is such that $\Gamma_k \vdash_M \varphi_k$, thus concluding that $\mathcal{L}_{12} \neq \mathcal{L}_M$.

We consider the fragments of classical logic $\mathcal{L}_{\mathsf{djn}}$ and $\mathcal{L}_{\mathsf{neg}}$, each characterized by the matrix based on the corresponding restriction of the 2-valued Boolean algebra, and prove that their fibring is not characterizable by a single finite Nmatrix[1].

Proposition 5.2. *$\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{neg}}$ is not finite-Nvalued.*

Proof. Let $\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{neg}} = \langle \Sigma_{\mathsf{djn}} \cup \Sigma_{\mathsf{neg}}, \_^{\vdash} \rangle$. For each $k \in \mathbb{N}$, consider:

$$\Gamma_k = \{p_i \vee p_j : 0 \leq i < j \leq k\}, \text{ and}$$
$$\varphi_k = \bigvee_{0 \leq i \leq k} \neg\neg(p_i \vee p_{k+1}).$$

We start by proving that $\Gamma_k \nvdash \varphi_k$, for any $k \in \mathbb{N}$. For the purpose, we will take advantage of Proposition 3.12.

We first show that $(\Gamma_k)^+ = \{x_*\} \cup \Gamma_k$. Clearly $x_* \in (\Gamma_k)^+$, since $\Gamma_k \neq \emptyset$. Thus, by definition, $(\Gamma_k)^+ = \{x_*\} \cup (\Gamma_k)^\omega$. But, as we shall see next, $(\Gamma_k)^\omega = (\Gamma_k)^1 = \Gamma_k$. For that, we need to check that $\Gamma_k \nvdash_{\mathsf{djn}} p_i$ and $\Gamma_k \nvdash_{\mathsf{cnj}} p_i$, for all $0 \leq i \leq k$.

To see that $\Gamma_k \nvdash_{\mathsf{djn}} p_i$, as $\Gamma_k \subseteq L_{\Sigma_{\mathsf{djn}}}(P)$, we use the 2-valued characteristic matrix for classical disjunction, and consider a valuation $v : P \to \{0, 1\}$ such that $v(p_j) = 1$ for $j \neq i$ and $v(p_i) = 0$, which clearly satisfies Γ_k but not p_i.

To check that $\Gamma_k \nvdash_{\mathsf{neg}} p_i$, we can apply Corollary 3.3, and show instead that $\mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\Gamma_k) \nvdash_{\mathsf{neg}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(p_i)$, that is, $\{x_{p_i \vee p_j} : 0 \leq i < j \leq k\} \nvdash_{\mathsf{neg}} p_i$. As above, we use the 2-valued characteristic matrix for classical negation, and consider a valuation $v : P \cup X \to \{0, 1\}$ such that $v(p_i) = 0$ and $v(y) = 1$ for all $y \neq p_i$, which clearly does the job.

[1] *En passant*, note that $\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{neg}}$ is a conservative extension of both $\mathcal{L}_{\mathsf{djn}}$ and $\mathcal{L}_{\mathsf{neg}}$, as a result of Theorem 4.5.

Now, it follows easily from the above that $(\Gamma_k)^+$ is not $\vdash_{\mathsf{neg}}$-explosive nor $\vdash_{\mathsf{djn}}$-explosive. Using Proposition 3.12 it now suffices to show that

$$(\Gamma_k)^+, X^{\mathsf{djn}}_{\Gamma_k}(\varphi_k) \nvdash_{\mathsf{djn}} \mathsf{skel}_{\Sigma_{\mathsf{djn}}}(\varphi_k).$$

Let us show that $X^{\mathsf{djn}}_{\Gamma_k}(\varphi_k) = \emptyset$. Indeed, by definition, we have that $X^{\mathsf{djn}}_{\Gamma_k}(\varphi_k) = \{x_{\neg\neg(p_i \vee p_{k+1})} : 0 \leq i \leq k, \Gamma_k \vdash \neg\neg(p_i \vee p_{k+1})\}$. It is clear that $\Gamma_k \vdash \neg\neg(p_i \vee p_{k+1})$ if and only if $\Gamma_k \vdash p_i \vee p_{k+1}$, as $\phi \vdash_{\mathsf{neg}} \neg\neg\phi$ and $\neg\neg\phi \vdash_{\mathsf{neg}} \phi$. Thus, we need to show that $\Gamma_k \nvdash p_i \vee p_{k+1}$ for each $0 \leq i \leq k$. We use again Proposition 3.12, which, together with the fact that $(\Gamma_k)^+$ is not $\vdash_{\mathsf{neg}}$-explosive, reduces our problem to checking that $(\Gamma_k)^+, X^{\mathsf{djn}}_{\Gamma_k}(p_i \vee p_{k+1}) \nvdash_{\mathsf{djn}} \mathsf{skel}_{\Sigma_{\mathsf{djn}}}(p_i \vee p_{k+1})$. As $p_i \vee p_{k+1} \in L_{\Sigma_{\mathsf{djn}}}(P)$, we have that $X^{\mathsf{djn}}_{\Gamma_k}(p_i \vee p_{k+1}) = \emptyset$ and $\mathsf{skel}_{\Sigma_{\mathsf{djn}}}(p_i \vee p_{k+1}) = p_i \vee p_{k+1}$. Hence, we just need to prove that $(\Gamma_k)^+ \nvdash_{\mathsf{djn}} p_i \vee p_{k+1}$. Again, as above, we can just consider a valuation $v : P \cup X \to \{0,1\}$ such that $v(p_{k+1}) = v(p_i) = 0$, and $v(y) = 1$ for all $y \neq p_i$ and $y \neq p_{k+1}$, over the 2-valued characteristic matrix for classical disjunction.

We are left with showing that

$$(\Gamma_k)^+ = \{x_*\} \cup \{p_i \vee p_j : 0 \leq i < j \leq k\} \nvdash_{\mathsf{djn}} \mathsf{skel}_{\Sigma_{\mathsf{djn}}}(\varphi_k) = \bigvee_{0 \leq i \leq k} x_{\neg\neg(p_i \vee p_{k+1})}.$$

Again, we can just consider a valuation $v : P \cup X \to \{0,1\}$ such that $v(x_*) = v(p_i) = 1$ for all $0 \leq i \leq k$, and $v(x_\phi) = 0$ for all $x_\phi \in X$, over the 2-valued characteristic matrix for classical disjunction.

At last, we show that for every $(\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{neg}})$-sound Nmatrix $M = \langle A, D, O\rangle$, if $k > |A|$ (i.e., M has less than k elements), then $\Gamma_k \vDash_M \varphi_k$. Let $M = \langle A, D, O\rangle$ be $(\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{neg}})$-sound and v a valuation over M. The following 3 facts hold:

- **(a)** if $v(p \vee q) \in D$ and $v(p) = v(q) = a$ then $a \in D$, because M is sound and $q \vee q \vdash_{\mathsf{djn}} q$,
- **(b)** if $v(\psi) \in D$ then $v(\psi \vee \varphi) \in D$, since M is sound and $p \vdash_{\mathsf{djn}} p \vee q$,
- **(c)** if $v(\psi) \in D$ then $v(\neg\neg\psi) \in D$, since M is sound and $q \vdash_{\mathsf{neg}} \neg\neg q$.

Assume that $v(\Gamma_k) \subseteq D$. By the pigeonhole principle we know that $v(p_i) = v(p_j)$ for some $0 \leq i < j \leq k$. Therefore by **(a)** we get that $v(p_i) \in D$. Then, by **(b)**, we must have $v(p_i \vee p_{k+1}) \in D$. Finally, by **(c)**, we obtain that $v(\neg\neg(p_i \vee p_{k+1})) \in D$. We conclude that $v(\varphi_k) \in D$, again using **(b)**, and hence that $\Gamma_k \vDash_M \varphi_k$. □

As a corollary, we immediately get that finite-Nvaluedness is not preserved by fibring.

Corollary 5.3. *Finite-(N)valuedness is not preserved by fibring.*

Recently, in [15], Humberstone analyzed the logic $\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{cnj}}$, and proved that it is not finite-valued. In order to prove this, the author shows, with the help of a clever translation to modal logic **K4!**, that the logic has infinitely many non-equivalent formulas in a single propositional variable. As a further application, we shall also consider $\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{cnj}}$ and further prove, using the general recipe introduced and exemplified above, that this logic is also not characterizable by a finite Nmatrix. As every matrix is also an Nmatrix, our result implies that of Humberstone, but not the other way around. As a matter of fact, Humberstone's method cannot be used to prove this stronger result. Indeed, it is straightforward to define a simple finite Nmatrix generating a logic with infinitely many non-equivalent formulas in a single propositional variable.

Proposition 5.4. $\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{cnj}}$ *is not finitely-Nvalued.*

Proof. Let $\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{cnj}} = \langle \Sigma_{\mathsf{djn}} \cup \Sigma_{\mathsf{cnj}}, \_^{\vdash} \rangle$ and, for each $k \in \mathbb{N}$, consider:

$$\Gamma_k = \{p_0\} \cup \{p_i \vee p_j : 1 \le i < j \le k\}, \text{ and}$$
$$\varphi_k = \bigvee_{1 \le i \le k} (p_0 \wedge p_i).$$

We start by proving that $\Gamma_k \nvdash \varphi_k$[2], for any $k \in \mathbb{N}$, by taking advantage of Proposition 3.12.

We first show that $(\Gamma_k)^+ = \{x_*\} \cup \Gamma_k$. As before, $x_* \in (\Gamma_k)^+$ since $\Gamma_k \neq \emptyset$, and therefore $(\Gamma_k)^+ = \{x_*\} \cup (\Gamma_k)^\omega$. In order to show that $(\Gamma_k)^\omega = (\Gamma_k)^1 = \Gamma_k$, we need to check that $\Gamma_k \nvdash_{\mathsf{djn}} p_i$ and $\Gamma_k \nvdash_{\mathsf{cnj}} p_i$, for $i \neq 0$.

To see that $\Gamma_k \nvdash_{\mathsf{djn}} p_i$, as $\Gamma_k \subseteq L_{\Sigma_{\mathsf{djn}}}(P)$, we can resort to the 2-element characteristic matrix for classical disjunction, and consider a valuation $v : P \to \{0, 1\}$ such that $v(p_j) = 1$ for $j \neq i$ and $v(p_i) = 0$, which clearly satisfies Γ_k but not p_i.

To check that $\Gamma_k \nvdash_{\mathsf{cnj}} p_i$, as $\Gamma_k \nsubseteq L_{\Sigma_{\mathsf{cnj}}}(P)$, we can apply Corollary 3.3, and prove instead that $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\Gamma_k) \nvdash_{\mathsf{cnj}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(p_i)$, that is, $\{p_0\} \cup \{x_{p_i \vee p_j} : 1 \le i < j \le k\} \nvdash_{\mathsf{cnj}} p_i$. As above, we resort to the 2-element characteristic matrix for classical conjunction, and consider a valuation $v : P \cup X \to \{0, 1\}$ such that $v(p_i) = 0$, and $v(y) = 1$ for all $y \neq p_i$, which clearly does the job.

Now, it follows easily from the above that $(\Gamma_k)^+$ is not $\vdash_{\mathsf{cnj}}$-explosive nor $\vdash_{\mathsf{djn}}$-explosive. Using Proposition 3.12 it now suffices to show that

$$(\Gamma_k)^+, X^{\mathsf{djn}}_{\Gamma_k}(\varphi_k) \nvdash_{\mathsf{djn}} \mathsf{skel}_{\Sigma_{\mathsf{djn}}}(\varphi_k).$$

Let us show that $X^{\mathsf{djn}}_{\Gamma_k}(\varphi_k) = \emptyset$. Indeed, by definition, $X^{\mathsf{djn}}_{\Gamma_k}(\varphi_k) = \{x_{p_0 \wedge p_i} : 1 \le i \le k, \Gamma_k \vdash p_0 \wedge p_i\}$. In order to check that $\Gamma_k \nvdash p_0 \wedge p_i$ we will use again

[2] Note that it immediately follows that, in $\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{cnj}}$, conjunction does not distribute over disjunction as usual in classical logic, i.e., for $k > 1$,

$$p_0 \wedge (\bigvee_{1 \le i \le k} p_i) \nvdash \bigvee_{1 \le i \le k} (p_0 \wedge p_i).$$

It suffices to observe that $\Gamma_k \vdash p_0 \wedge (\bigvee_{1 \le i \le k} p_i)$ and $\bigvee_{1 \le i \le k} (p_0 \wedge p_i) = \varphi_k$.

Proposition 3.12, which, together with the fact that $(\Gamma_k)^+$ is not $\vdash_{\mathsf{djn}}$-explosive reduces our problem to checking that $\Gamma_k, X^{\mathsf{cnj}}_{\Gamma_k}(p_0 \wedge p_i) \nvdash_{\mathsf{cnj}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(p_0 \wedge p_i)$. Since $p_0 \wedge p_i \in L_{\Sigma_{\mathsf{cnj}}}(P)$, we have that $X^{\mathsf{cnj}}_{\Gamma_k}(p_0 \wedge p_i) = \emptyset$ and $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(p_0 \wedge p_i) = p_0 \wedge p_i$. As above, to check that $\Gamma_k \nvdash_{\mathsf{cnj}} p_0 \wedge p_i$, as $\Gamma_k \nsubseteq L_{\Sigma_{\mathsf{cnj}}}(P)$, we can apply Corollary 3.3, and check instead that $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\Gamma_k) \nvdash_{\mathsf{cnj}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(p_0 \wedge p_i)$, that is,

$$\{p_0\} \cup \{x_{p_i \vee p_j} : 1 \leq i < j \leq k\} \nvdash_{\mathsf{cnj}} p_0 \wedge p_i.$$

Again, as above, we can just consider a valuation $v : P \cup X \to \{0, 1\}$ such that $v(p_i) = 0$, and $v(y) = 1$ for all $y \neq p_i$, over the 2-element characteristic matrix for classical conjunction.

We are left with showing that $(\Gamma_k)^+, x_* \nvdash_{\mathsf{djn}} \bigvee_{1 \leq i \leq k} x_{p_0 \wedge p_i} = \mathsf{skel}_{\Sigma_{\mathsf{djn}}}(\varphi_k)$. Again, we use the 2-element characteristic matrix for classical disjunction, and consider a valuation $v : P \cup X_* \to \{0, 1\}$ such that $v(y) = 1$ if $y \in P \cup \{x_*\}$, and $v(y) = 0$ otherwise.

Finally, we show that for every $(\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{cnj}})$-sound Nmatrix $M = \langle A, D, O \rangle$, if $k > |A \setminus D|$ (i.e., M has less than k non-designated truth-values), then $\Gamma_k \models_M \varphi_k$. Let $M = \langle A, D, O \rangle$ be $(\mathcal{L}_{\mathsf{djn}} \bullet \mathcal{L}_{\mathsf{cnj}})$-sound and v a valuation over M. As in Proposition 5.2 we can easily conclude that:

(a) if $v(p_i \vee p_j) \in D$ and $v(p_i) = v(p_j) = a$ then $a \in D$, and

(b) if $v(p_i) \in D$ for some $1 \leq i \leq k$ then $v(\varphi_k) \in D$ as long as $v(p_0) \in D$, since M is sound and $p_0, p_i \vdash_{\mathsf{cnj}} p_0 \wedge p_i \vdash_{\mathsf{djn}} \varphi_k$.

Assume that $v(\Gamma_k) \subseteq D$, that is, $v(p_0) \in D$ and $v(p_i \vee p_j) \in D$ for all $1 \leq i < j \leq k$. If $\{v(p_1), \ldots, v(p_k)\} \cap D = \emptyset$ then, by the pigeonhole principle, we have that $v(p_i) = v(p_j)$ for some $1 \leq i < j \leq k$, and from **(a)** we conclude that $v(p_i) \in D$ for some $1 \leq i \leq k$. Therefore, using **(b)**, we get $v(\varphi_k) \in D$ and hence that $\Gamma_k \models_M \psi_n$. □

6 Conclusion

We gave a full characterization of the patterns of reasoning from non-mixed hypothesis in fibred logics without shared connectives, extending the result obtained in [16]. Though this extension may seem diminute, its interest is well justified by the new insights about fibred logics it allows and the scope of its consequences. Indeed, our new result is a great deal better in capturing the way the resulting logic emerges from its components, and its usefulness goes well beyond its previous version. Using the characterization obtained in this paper we could have easily extended the decidability preservation result of [16] from theoremhood to arbitrary reasoning from non-mixed hypothesis in logics obtained by disjoint fibring. Still, this would not be too interesting, as we discuss below. Instead, we explored other two meaningful applications which could not be tackled using the result in [16].

To start with, we provided necessary and sufficient conditions for a disjointly fibred logic to be a conservative extension of its component logics. This result is the first full account of conservativity for disjoint fibring, and completes the partial answer to this problem given in [19].

Further, we also illustrated the usefulness of our characterization of mixed reasoning in a completely different setting, namely in understanding the difficulties behind effectively describing fibred semantics in terms of the semantics of the component logics. Concretely, we have shown that the fibring of finite-(N)valued logics may result in logics that are not finitely-(N)valued.

There are two ways in which our main characterization result can possibly be strengthened.

The first possible extension would consist in admitting arbitrary sets of mixed hypotheses, instead of only non-mixed hypotheses, while keeping the disjointness requirement. Such an extension does not seem hopeless, and it should allow us to prove that disjoint fibring preserves decidability in general. The characterization result obtained in this paper already allows improving the decidability of theoremhood obtained in [16], but we refrain from working out the details as we still fall short of a general decidability preservation result. We hope to report on this line of work in a forthcoming paper.

The latter possibility would be to go the full way, and allow the logics to share connectives. Such an extension of our characterization result seems to be very far from trivial, and is currently beyond the scope of our tools. Indeed, disjointness is a key ingredient of basic results like Lemma 3.5. Still, such an extension, even if not covering all the cases, would certainly provide us with a much deeper understanding of the fibring mechanism, and could potentially have a myriad of practical applications, including characterizations of conservativity and decidability also for logics sharing connectives, not to mention the added mastery of fibred semantics.

On the conservativity front, beyond the disjoint case, it is nevertheless clear that the problem can be explored even without such a powerful tool, at least to the point of establishing general enough sufficient conditions on the component logics to guarantee that they are conservatively extended by fibring. Further investigations are necessary, but we are working on such an approach by taking advantage of suitable translations between logics.

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