Abstract Valuation Semantics

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Abstract

We define and study abstract valuation semantics for logics, an algebraically well-behaved version of valuation semantics. In the context of the behavioral approach to the algebraization of logics, we show, by means of meaningful bridge theorems and application examples, that abstract valuations are suited to play a role similar to the one played by logical matrices in the traditional approach to algebraization.
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1 Introduction

Logical matrices [16] can certainly be counted amongst the most widespread semantic structures used in logic. This happens for many reasons, including their convenience, but mostly because of their algebraic properties, which enable matrix semantics to fit quite naturally with the abstract approach to algebraic logic [12]. It is well-known that every structural logic can be characterized by the class of its matrix models, or even better by the class of its reduced matrix models [21]. In the case of an algebraizable logic, one even gets an equational specification of the algebraic structure underlying these models, along with a characterization of matrix congruences by means of the Leibniz operator, as well as a way of recovering the corresponding matrix filters using defining equations [3, 12].

The behavioral approach to the algebraization of logics was introduced in [8] with the aim of extending the range of applicability of the traditional tools of algebraic logic to logics with a many-sorted syntax, or including non-truth-functional connectives, and which are not algebraizable under the usual approach. There, unsorted equational logic is replaced by many-sorted behavioral equational logic (also dubbed hidden equational logic) based on the notion of behavioral equivalence [19, 13, 20], in which it is possible that one cannot distinguish between two different values if those values provide exactly the same results for all available ways of observing and experimenting with them. When used in logic, this behavioral approach allows one not only to deal algebraically with non-congruent connectives but also, in the many-sorted setting, with the intuition that all syntactic categories other than formulas (e.g., terms) can only be assessed when embedded into formulas.

Despite of its successfulness, an immediate consequence of the behavioral approach to abstract algebraic logic is that the fundamental notion of matrix semantics is no longer adequate. In particular, behavioral equivalence is in general not a congruence over the whole language of the logic. Moreover, as expected in the case of logics that are not algebraizable under the usual approach (but which may be behaviorally algebraizable), the connection between the logic and its matrix semantics may be weak and uninteresting.

Logical valuations as a general semantic tool were proposed in [10] with the aim of providing a semantic ground for logics that do not have a meaningful matrix semantics. The key idea is to drop the condition that formulas should always be interpreted homomorphically in an algebra over the same signature. Besides lacking a thorough study, namely when contrasted with the rich algebraic theory of logical matrices (see [21]), valuation semantics has also been criticized for its excessive generality (see, for instance, [11]). Still, logicians would agree that a matrix semantics is simply a clever and algebraically well-behaved way of defining a valuation semantics by collecting all possible homomorphic interpretations.

As a first step toward coping with the inadequacy of logical matrices in the behavioral setting, the work in [6, 7] explored the possibility of replacing logical matrices by an algebraically well-behaved version of valuation semantics that could serve as the semantic counterpart of the behavioral approach to the algebraization of logics, and obtained some bridge results generalizing the role played by matrix semantics in abstract algebraic logic, in the lines of [3, 9, 12, 21]. Still, some difficulties could be noticed in [7] regarding the fact that *the valuation*, as a semantic unit, has a 'local' character when contrasted with the 'global' character of a logical matrix, which gives rise to a collection of valuations, one for each possible assignment to the logical variables. This weakness reflected itself, in particular, in the asymmetric development given in [7] to the behavioral Suszko and Leibniz operators.

Herein, as already hinted in [7], we rephrase the whole process by taking as semantic units not valuations, but *abstract valuations*. We will show that they generalize logical matrices, but still retain many of their essential algebraic properties. Besides restating and proving the results obtained [7] in terms of abstract valuations, we are now also able to deal with both the behavioral Suszko and Leibniz operators in a similar way. This allows us to obtain further bridge results about the behavioral hierarchy and, at least for single-sorted logics, a characterization of behavioral equivalentiality using submodels and products of reduced abstract valuations. Two concrete illustrating examples are also analyzed.

Still, the usefulness of the line of work undertaken here will also show itself in the difficulties posed by the fully general many-sorted case, where an analog of the above mentioned result for single-sorted logics does not hold in general. This fact, indeed, points toward a defect of the formula-centric viewpoint that underlies some of the original notions of the behavioral approach. Thus, we also suggest an alternative setup where a promotion of the status of syntactic sorts other than formulas leads to stronger notions in the behavioral hierarchy, for which the envisaged algebraic characterizations are workable.

In Section 2, we introduce abstract valuations, their more basic properties, and their connection with logical matrices. Section 3 is devoted to studying the algebraic properties of abstract valuations, including its Suszko and Leibniz congruences, and the closure of abstract valuation semantics under algebraic operations. Then, in Section 4, we establish a number of bridge results with respect to the behavioral approach to the algebraization of logics and analyze a couple of illustrating examples. Section 5 draws conclusions and points towards some topics for further research.

2 Abstract valuations

In the most general case, we are interested in working with many-sorted logical languages. To start with, and also along the presentation, we will recall a few necessary notions and fix some notation.

Remark 1 *A (many-sorted) signature is a pair $\Sigma = \langle S, F\rangle$ where S is a set (of sorts) and $F = \{F_{ws}\}_{w\in S^*, s\in S}$ is an indexed family of sets (of* operations*). For simplicity, we write $f : s_1 \dots s_n \to s \in F$ for an element $f \in F_{s_1\dots s_n s}$. As usual, we denote by $T_\Sigma(X) = \{T_{\Sigma,s}(X)\}_{s\in S}$ the S-sorted family of carrier sets of the free Σ-algebra $\mathbf{T_\Sigma(X)}$ with generators taken from a sorted family $X = \{X_s\}_{s\in S}$ of variable sets. We will denote by $x{:}s$ the fact that $x \in X_s$. Often, we will need to write terms $t \in T_\Sigma(Y)$ over a given subset of variables $Y \subseteq X$. For simplicity, we will denote such a term by $t(Y)$, or even by $t(x_1{:}s_1, \dots, x_n{:}s_n)$ when $Y = \{x_1 : s_1, \dots, x_n : s_n\}$. Moreover, if T is a set whose elements are all terms of this form, we will write $T(Y)$. A substitution over Σ is a S-sorted family of functions $\sigma = \{\sigma_s : X_s \to T_{\Sigma,s}(X)\}_{s\in S}$. As usual, $\sigma(t)$ denotes the term obtained by uniformly applying σ to each variable in t. Given $t(Y)$ and $\overline{u} = \langle u_i \in T_{\Sigma,s_i}(X)\rangle_{x_i:s_i\in Y}$, we will write $t(\overline{u})$ to denote the term $\sigma(t)$ where σ is a substitution such that $\sigma_{s_i}(x_i) = u_i$ for each $x_i : s_i \in Y$. Extending everything to sets, given $T(Y)$ and $U \subseteq \prod_{x_i:s_i\in Y} T_{\Sigma,s_i}(X)$, we will use $T[U] = \bigcup_{\overline{u}\in U} T(\overline{u})$.*

In order to define logical languages, we will work with signatures $\Sigma = \langle S, F\rangle$ with a distinguished sort ϕ (the syntactic sort of formulas). We assume fixed a S-sorted family X of variables. We define the induced set of *formulas* $L_\Sigma(X)$ to be the carrier set of sort ϕ of the free algebra $\mathbf{T_\Sigma(X)}$ with generators X, that is, $L_\Sigma(X) = T_{\Sigma,\phi}(X)$. We now introduce the class of logics that is the target of our approach.

Definition 2 A *(many-sorted) logic* is a tuple $\mathcal{L} = \langle \Sigma, \vdash \rangle$ where Σ is a signature and $\vdash \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$ is a *consequence relation* satisfying, for every $\Phi \cup \Psi \cup \{\varphi\} \subseteq L_\Sigma(X)$:

- if $\varphi \in \Phi$ then $\Phi \vdash \varphi$ (***reflexivity***);
- if $\Phi \vdash \varphi$ for all $\varphi \in \Psi$, and $\Psi \vdash \psi$ then $\Phi \vdash \psi$ (***cut***);
- if $\Phi \vdash \varphi$ and $\Phi \subseteq \Psi$ then $\Psi \vdash \varphi$ (***weakening***).

$\mathcal{L}$ is further said to be ***structural*** whenever:

- if $\Phi \vdash \varphi$ then $\sigma[\Phi] \vdash \sigma(\varphi)$, for every substitution σ,

and said to be ***finitary*** whenever:

- if $\Phi \vdash \varphi$ then $\Psi \vdash \varphi$ for some finite $\Psi \subseteq \Phi$.

In this paper, unless otherwise stated, all the logics considered are assumed to be structural. Note that propositional-based single-sorted logics appear as a particular case of many-sorted logics, considering a signature $\Sigma = \langle S, F \rangle$ such that $S = \{\phi\}$.

We will use $\vdash_\mathcal{L}$ instead of just $\vdash$ to refer to the consequence relation of a given logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$. Moreover, as usual, if $\Phi, \Psi \subseteq L_\Sigma(X)$, we will write $\Psi \vdash_\mathcal{L} \Phi$ whenever $\Psi \vdash_\mathcal{L} \varphi$ for all $\varphi \in \Phi$. We say that $\varphi, \psi \in L_\Sigma(X)$ are *interderivable* in $\mathcal{L}$, which is denoted by $\varphi \dashv\vdash_\mathcal{L} \psi$, if $\varphi \vdash_\mathcal{L} \psi$ and $\psi \vdash_\mathcal{L} \varphi$. Analogously, we say that Φ and Ψ are *interderivable* in $\mathcal{L}$, which is denoted by $\Phi \dashv\vdash_\mathcal{L} \Psi$, if $\Phi \vdash_\mathcal{L} \Psi$ and $\Psi \vdash_\mathcal{L} \Phi$. The *theorems* of $\mathcal{L}$ are the formulas φ such that $\emptyset \vdash_\mathcal{L} \varphi$. A *theory* of $\mathcal{L}$ is a set of formulas Φ such that if $\Phi \vdash_\mathcal{L} \varphi$ then $\varphi \in \Phi$. As usual, $\Phi^{\vdash_\mathcal{L}}$ denotes the least theory of $\mathcal{L}$ that contains Φ. The set of theories of $\mathcal{L}$ will be denoted by $Th_\mathcal{L}$.

Valuation semantics appeared in [10] as an effort to provide a semantic ground to logics that may lack a meaningful truth-functional semantics. The underlying idea is to drop the condition that formulas should always be interpreted homomorphically, and instead accept any possible interpretation as a function from the set of formulas of the logic to a set of truth-values equipped with a subset of designated values.

Definition 3 A *valuation* over Σ is a pair $\langle v, D \rangle$ where $v : L_\Sigma(X) \to A$ is a function (where A is the set of *truth-values*), and $D \subseteq A$ is a set (of *designated* values). A *valuation semantics* over Σ is a collection $\mathcal{V}$ of valuations over Σ.

A valuation $\vartheta = \langle v, D \rangle$ over Σ *satisfies* $\varphi \in L_\Sigma(X)$, written $\vartheta \Vdash \varphi$, if $v(\varphi) \in D$. A valuation semantics $\mathcal{V}$ over Σ induces the *semantic entailment* consequence relation $\vDash_\mathcal{V}$ defined, for every $\Psi \cup \{\varphi\} \subseteq L_\Sigma(X)$, by $\Psi \vDash_\mathcal{V} \varphi$ if and only if, for every $\vartheta \in \mathcal{V}$, $\vartheta \Vdash \varphi$ whenever $\vartheta \Vdash \psi$ for each $\psi \in \Psi$.

We say that a valuation ϑ over Σ is a *model* of $\mathcal{L}$ if $\vdash_\mathcal{L} \subseteq \vDash_{\{\vartheta\}}$. The class of all valuation models of $\mathcal{L}$ is denoted by $Val(\mathcal{L})$.

It is an easy exercise to show that $\vDash_{\mathcal{V}}$ is a consequence relation, although possibly not structural nor finitary. Indeed, in such generality, valuation semantics are difficult to grasp. However, it is also clear that the usual logical matrices can be seen as defining collections of valuations.

Remark 4 *A Σ-algebra for signature $\Sigma = \langle S, F\rangle$ is a pair $\mathbf{A}$=$\langle\{A_s\}_{s\in S}, \_\mathbf{A}\rangle$, where each A_s is a non-empty set, the carrier of sort s, and $\_\mathbf{A}$ assigns to each operation $f : s_1 \dots s_n \to s$ a function $f_{\mathbf{A}} : A_{s_1} \times \dots \times A_{s_n} \to A_s$. An assignment over $\mathbf{A}$ is a S-sorted family of functions $h = \{h_s : X_s \to A_s\}_{s\in S}$. As usual, we will often overload h and use it to denote also the unique extension of the assignment to an homomorphism $h : T_\Sigma(X) \to \mathbf{A}$. Given a Σ-algebra $\mathbf{A}$, a term $t(x_1 : s_1, \dots, x_n : s_n)$ and $\langle a_1, \dots, a_n\rangle \in A_{s_1} \times \dots \times A_{s_n}$, we denote by $t_{\mathbf{A}}(a_1, \dots, a_n)$ the value $h(t)$ that t takes in $\mathbf{A}$ under an assignment h such that $h(x_1) = a_1, \dots, h(x_n) = a_n$.*

We extend, in the obvious way, the usual notion of logical matrix to the many-sorted case.

Definition 5 A *matrix* over Σ is a pair $\langle\mathbf{A}, D\rangle$ where $\mathbf{A}$ is a Σ-algebra and $D \subseteq A_\phi$ (is a *filter*). A *matrix semantics* over Σ is a collection $\mathcal{M}$ of matrices over Σ.

A matrix $m = \langle\mathbf{A}, D\rangle$ over Σ together with an assignment h over $\mathbf{A}$ *satisfy* $\varphi \in L_\Sigma(X)$, written $m, h \Vdash \varphi$, if $h(\varphi) \in D$. A matrix semantics $\mathcal{M}$ over Σ induces the *semantic entailment* consequence relation $\vDash_{\mathcal{M}}$ defined, for every $\Psi \cup \{\varphi\} \subseteq L_\Sigma(X)$, by $\Psi \vDash_{\mathcal{M}} \varphi$ if and only if, for every $m = \langle\mathbf{A}, D\rangle \in \mathcal{M}$ and every assignment h over $\mathbf{A}$, $m, h \Vdash \varphi$ whenever $m, h \Vdash \psi$ for each $\psi \in \Psi$.

As usual, we say that a matrix $m = \langle\mathbf{A}, D\rangle$ over Σ is a *model* of $\mathcal{L}$ if $\vdash_{\mathcal{L}} \subseteq \vDash_{\{m\}}$, in which case D is dubbed a $\mathcal{L}$-*filter* of $\mathbf{A}$. The class of all matrix models of $\mathcal{L}$ is denoted by $Matr(\mathcal{L})$.

In the most natural way, each matrix $m = \langle\mathbf{A}, D\rangle$ over Σ and each assignment h over $\mathbf{A}$ induce a valuation

$$\vartheta(m, h) = \langle h_\phi, D\rangle.$$

Thus, a class $\mathcal{M}$ of matrices over Σ induces a valuation semantics $\mathcal{V}(\mathcal{M})$ over Σ defined by

$$\mathcal{V}(\mathcal{M}) = \{\vartheta(m, h) \mid m = \langle\mathbf{A}, D\rangle \in \mathcal{M},\ h \text{ an assignment over } \mathbf{A}\}.$$

It is straightforward to check that $\vDash_{\mathcal{M}} = \vDash_{\mathcal{V}(\mathcal{M})}$.

Although very convenient, logical matrices are not a universal solution to the challenges posed by arbitrary logics, even if structural and finitary. Let us see one example.

Example 6 *Consider, for instance, the logic $\mathcal{K}/2$ from [2]. It is simply built over the single sorted signature $\Sigma = \langle\{\phi\}, F\rangle$, where F contains only $\neg : \phi \to \phi$ and $\Rightarrow : \phi\phi \to \phi$. Its consequence relation $\vdash_{\mathcal{K}/2}$ can be simply defined to be the* *semantic entailment associated to the set of all valuations* $\langle v, \{1\}\rangle$ *over* Σ *with* $v : L_\Sigma(X) \to \{0,1\}$ *such that, for every* $\varphi, \psi \in L_\Sigma(X)$*, the following conditions hold:*

- $v(\neg\varphi) = 0$ *if* $v(\varphi) = 1$*, and*
- $v(\varphi \Rightarrow \psi) = 0$ *iff* $v(\varphi) = 1$ *and* $v(\psi) = 0$.

Clearly, the logic would be classical if the first of the clauses above would be written with 'iff'. As it is, the logic turns out to have a classical implication but a paracomplete negation. Interestingly, however, the derived unary operation $\sim$*, with* $\sim\varphi$ *defined as an abbreviation of* $\varphi \Rightarrow (\neg\varphi)$*, still behaves as a classical negation:* $v(\sim\varphi) = 0$ *iff* $v(\varphi \Rightarrow (\neg\varphi)) = 0$ *iff* $v(\varphi) = 1$ *and* $v(\neg\varphi) = 0$ *iff* $v(\varphi) = 1$.

As we will show later, $\mathcal{K}/2$ *does not have a meaningful matrix semantics, although it is structural, finitary and very easily axiomatizable.* ▲

Besides lacking a thorough supporting theory, namely if contrasted with the rich theory of logical matrices, valuation semantics has been mostly criticized for its excessive generality, namely as it can be confused Suszko's bivalence thesis (see, for instance, [5, 11]). However, the valuation semantics used in the above example is far from being ad-hoc.

What we propose in this paper, is to adopt a suitable algebraically well-behaved version of valuation semantics that may cover such cases. The work reported in [7] has already given the first steps in this direction, by proposing so-called Γ-valuations, where Γ identifies the well-behaved operations. However, Γ-valuations are too close to valuations, in their locality, and too far away from matrices, in their globality. Indeed, just by looking at the definitions above, it is easy to understand that a valuation includes a fixed assignment, whereas we must range over all assignments when we use matrices. This crucial property, fails in [7], where it was dubbed *Laplacianism*.

Hence, as in [7], we drop the requirement that formulas must be interpreted homomorphically (in $\mathcal{K}/2$, the implication is interpreted homomorphically, but not the negation), but we still require that a certain globality is guaranteed (in $\mathcal{K}/2$, every assignment to the variables can be extended, perhaps not uniquely, to a valuation).

Remark 7 *A derived operation of type* $s_1 \dots s_n \to s$ *over* Σ *is simply a term in* $T_{\Sigma,s}(x_1 : s_1, \dots, x_n : s_n)$*. For* $w \in S^*$*, we denote by* $Der_{\Sigma,ws}$ *the set of all derived operations of type* $w \to s$ *over* Σ*. A (full) subsignature of* $\Sigma = \langle S, F\rangle$ *is a signature* $\Gamma = \langle S, F'\rangle$ *such that, for each* $w \in S^*$ *and* $s \in S$*,* $F'_{ws} \subseteq Der_{\Sigma,ws}$*. When* $\mathbf{A}$ *is a* Σ*-algebra and* Γ *a subsignature of* Σ*, we denote by* $\mathbf{A}|_\Gamma$ *the* Γ*-algebra obtained by forgetting the interpretation of all the operations not in* Γ*.*

Definition 8 Let Γ be a subsignature of Σ. An *abstract* Γ*-valuation* over Σ is a tuple $\langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ where $\mathbf{A}$ is a (concrete) Σ-algebra, $\langle \mathbf{B}, D\rangle$ is an (abstract) matrix over Γ, and $v : \mathbf{A}|_\Gamma \to \mathbf{B}$ is a surjective homomorphism (of Γ-algebras).

An *abstract* Γ*-valuation semantics* over Σ is a collection $\mathcal{AV}_\Gamma$ of abstract Γ-valuations over Σ.

An abstract Γ-valuation $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ over Σ together with a (concrete) assignment h over $\mathbf{A}$ *satisfy* $\varphi \in L_\Sigma(X)$, written $\alpha, h \Vdash \varphi$, if $v(h(\varphi)) \in D$. An abstract Γ-valuation semantics $\mathcal{AV}_\Gamma$ over Σ induces the *semantic entailment* consequence relation $\vDash_{\mathcal{AV}_\Gamma}$ defined, for every $\Psi \cup \{\varphi\} \subseteq L_\Sigma(X)$, by $\Psi \vDash_{\mathcal{AV}_\Gamma} \varphi$ if and only if, for every $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle \in \mathcal{AV}_\Gamma$ and every assignment h over $\mathbf{A}$, $\alpha, h \Vdash \varphi$ whenever $\alpha, h \Vdash \psi$ for each $\psi \in \Psi$.

We say that an abstract Γ-valuation $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ over Σ is a *model* of $\mathcal{L}$ if $\vdash_\mathcal{L} \subseteq \vdash_{\{\alpha\}}$, in which case D is dubbed a $\mathcal{L}$*-filter* of v. The class of all abstract Γ-valuation models of $\mathcal{L}$ is denoted by $AVal_\Gamma(\mathcal{L})$.

In an abstract valuation as above, note that although v is an homomorphism its kernel $\ker(v)$ is only a Γ-congruence, due to the fact that v is an homomorphism of Γ-algebras. Hence, in general, $\ker(v)$ is not a congruence on the Σ-algebra $\mathbf{A}$. In fact, an abstract Γ-valuation $\langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ over Σ can be seen (concretely) as a matrix $\langle \mathbf{A}, v^{-1}(D)\rangle$ over Σ, whereas, given an assignment h over $\mathbf{A}$, it does set up a valuation $(v \circ h)_\phi : L_\Sigma(X) \to B_\phi$. Indeed, the whole Γ-homomorphic interpretation $v \circ h$ allows us to see the role of the (abstract) matrix $\langle \mathbf{B}, D\rangle$ over Γ. Note also that the surjectivity of v provides an interesting meaning to abstract values. Clearly, for every assignment g over $\mathbf{B}$ there exist (possibly many assignments) h over $\mathbf{A}$ such that $g = v \circ h$. However, due to the fact that v is Γ-homomorphic, it is clear that $v(h(\varphi)) = g(\varphi)$ no matter the choice of h, as long as φ only uses operations in Γ.

The relationship between valuations, matrices and abstract Γ-valuations is very natural. Indeed, each matrix $m = \langle \mathbf{A}, D\rangle$ over Σ induces an abstract Γ-valuation

$$\alpha_\Gamma(m) = \langle \mathbf{A}, id, \langle \mathbf{A}|_\Gamma, D\rangle\rangle.$$

Hence, a class $\mathcal{M}$ of matrices over Σ induces an abstract Γ-valuation semantics $\mathcal{AV}_\Gamma(\mathcal{M})$ over Σ defined by

$$\mathcal{AV}_\Gamma(\mathcal{M}) = \{\alpha_\Gamma(m) \mid m \in \mathcal{M}\}.$$

On the other hand, each abstract Γ-valuation $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ over Σ and each assignment h over $\mathbf{A}$ induce a valuation

$$\vartheta(\alpha, h) = \langle (v \circ h)_\phi, D\rangle.$$

Hence, an abstract Γ-valuation semantics $\mathcal{AV}_\Gamma$ over Σ induces a valuation semantics $\mathcal{V}(\mathcal{AV}_\Gamma)$ over Σ defined by

$$\mathcal{V}(\mathcal{AV}_\Gamma) = \{\vartheta(\alpha, h) \mid \alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle \in \mathcal{AV}_\Gamma,\ h \text{ an assignment over } \mathbf{A}\}.$$

It is straightforward to check that $\vDash_{\mathcal{AV}_\Gamma} = \vDash_{\mathcal{V}(\mathcal{AV}_\Gamma)}$, and therefore that also $\vDash_\mathcal{M} = \vDash_{\mathcal{AV}_\Gamma(\mathcal{M})} = \vDash_{\mathcal{V}(\mathcal{AV}_\Gamma(\mathcal{M}))}$.

Example 9 *Recall the logic $\mathcal{K}/2$ of Example 6. Let Γ be the subsignature of Σ consisting of $\sim$: $\phi \to \phi$ and $\Rightarrow : \phi\phi \to \phi$. The consequence relation of $\mathcal{K}/2$ can be equivalently induced by the class of all abstract Γ-valuations $\langle \mathbf{A}, v, \langle \mathbf{2}, \{1\}\rangle\rangle$ over Σ, where $\mathbf{A}$ is any Σ-algebra, $\mathbf{2}$ is the $\{0,1\}$-valued Boolean algebra over Γ and v is such that, for every $a \in A_\phi$, the following condition holds:*

- $v(\neg_{\mathbf{A}} a) = 0$ *if* $v(a) = 1$.

Note that the homomorphic interpretation condition for $\Rightarrow$ (or $\sim$) is not necessary as v is necessarily a Γ-homomorphism. ▲

Contrarily to arbitrary valuation semantics, abstract valuation semantics enjoy many of the nice properties of matrix semantics. Structurality, for instance, comes for free, without the difficulties found in [7].

Proposition 10 *Let $\mathcal{AV}_\Gamma$ be an abstract Γ-valuation semantics over Σ. Then, $\vDash_{\mathcal{AV}_\Gamma}$ is structural.*

Proof: Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle \in \mathcal{AV}_\Gamma$, h an assignment over $\mathbf{A}$, and σ a substitution. It is clear that $v(h(\sigma(\varphi))) = v((h \circ \sigma)(\varphi))$ for every $\varphi \in L_\Sigma(X)$, where $(h \circ \sigma)$ is also an assignment over $\mathbf{A}$.

Hence, given $\Psi \subseteq L_\Sigma(X)$, if $\Psi \vDash_{\mathcal{AV}_\Gamma} \varphi$ and $\alpha, h \Vdash \sigma(\psi)$ for every $\psi \in \Psi$, then $\alpha, (h \circ \sigma) \Vdash \psi$ for every $\psi \in \Psi$. Therefore, $\alpha, (h \circ \sigma) \Vdash \varphi$, and also $\alpha, h \Vdash \sigma(\varphi)$. Thus, we conclude that $\sigma[\Psi] \vDash_{\mathcal{AV}_\Gamma} \sigma(\varphi)$ and $\vDash_{\mathcal{AV}_\Gamma}$ is structural. ■

One can easily bring the usual Lindenbaum-Tarski constructions to the setting of abstract valuations. For each set $\Phi \subseteq L_\Sigma(X)$, we can define the abstract Γ-valuation $\lambda_\Gamma(\Phi) = \langle \mathbf{T}_\mathbf{\Sigma}(\mathbf{X}), id, \langle \mathbf{T}_\mathbf{\Sigma}(\mathbf{X})|_\Gamma, \Phi\rangle\rangle$. The abstract valuations of this form are dubbed *Lindenbaum abstract Γ-valuations* over Σ. Given a logic $\mathcal{L}$, the family of Lindenbaum abstract Γ-valuations $\lambda_\Gamma(\Phi)$ based on theories Φ of $\mathcal{L}$ is called the *Lindenbaum abstract Γ-bundle* of $\mathcal{L}$ and denoted by $Lind_\Gamma(\mathcal{L})$.

Proposition 11 *Let $\mathcal{L}$ be a many-sorted logic over Σ, and Γ a subsignature of Σ. Then, $Lind_\Gamma(\mathcal{L}) \subseteq AVal_\Gamma(\mathcal{L})$ and $\vdash_\mathcal{L} = \vDash_{AVal_\Gamma(\mathcal{L})} = \vDash_{Lind_\Gamma(\mathcal{L})}$.*

Proof: To prove that $Lind_\Gamma(\mathcal{L}) \subseteq AVal_\Gamma(\mathcal{L})$, let $\Phi \in Th_\mathcal{L}$. We must prove that $\lambda_\Gamma(\Phi)$ is a model of $\mathcal{L}$. Note that an assignment over $\mathbf{T}_\mathbf{\Sigma}(\mathbf{X})$ is simply a substitution. Assume that $\Psi \vdash_\mathcal{L} \varphi$ and let σ be a substitution such that $\lambda_\Gamma(\Phi), \sigma \Vdash \psi$ for every $\psi \in \Psi$, that is, $id[\sigma[\Psi]] = \sigma[\Psi] \subseteq \Phi$. Using the structurality of $\mathcal{L}$, we also have that $\sigma[\Psi] \vdash_\mathcal{L} \sigma(\varphi)$. Thus, we can conclude that $\sigma(\varphi) \in \Phi$, that is, $\lambda_\Gamma(\Phi), \sigma \Vdash \varphi$, and $\lambda_\Gamma(\Phi)$ is indeed a model of $\mathcal{L}$.

Therefore, it is clear that $\vdash_\mathcal{L} \subseteq \vDash_{AVal_\Gamma(\mathcal{L})}$ and $\vdash_\mathcal{L} \subseteq \vDash_{Lind_\Gamma(\mathcal{L})}$. Thus, in order to show the converse inclusions, it is enough to prove that whenever $\Psi \nvdash_\mathcal{L} \varphi$ then there exists $\Phi \in Th_\mathcal{L}$ and a substitution σ such that $\lambda_\Gamma(\Phi)$ and σ satisfy Ψ but not φ. Easily, it suffices to take $\Phi = \Psi^{\vdash_\mathcal{L}}$ and σ the identity substitution. ■

3 Algebraic properties

We will now study a number of useful algebraic properties of the notion of abstract valuation semantics put forth in the previous section, which generalize properties of the fruitful theory of logical matrices.

Definition 12 Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ be an abstract Γ-valuation over Σ. A *congruence* of α is simply a congruence of the matrix $\langle \mathbf{B}, D\rangle$ over Γ, that is, a congruence θ on the Γ-algebra $\mathbf{B}$ that is *compatible* with D, in the sense that $b \in D$ iff $b' \in D$ for each $\langle b, b'\rangle \in \theta_\phi$.

Given a congruence θ of α, the corresponding θ*-reduced* abstract Γ-valuation is $\alpha/_\theta = \langle \mathbf{A}, [\_]_\theta \circ v, \langle \mathbf{B}/_\theta, [D]_\theta\rangle\rangle$.

Note that when reducing an abstract valuation, as above, the Σ-algebra $\mathbf{A}$ remains untouched. This is due to the fact that θ is only a Γ-congruence.

Expectedly, the reduction of an abstract valuation by a congruence yields a logically equivalent abstract valuation.

Lemma 13 *Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ be an abstract Γ-valuation over Σ, and θ a congruence of α. For every formula $\varphi \in L_\Sigma(X)$ and every assignment h over $\mathbf{A}$, we have that $\alpha, h \Vdash \varphi$ iff $\alpha/_\theta, h \Vdash \varphi$.*

Proof: It suffices to note that $\alpha/_\theta, h \Vdash \varphi$ iff $[v(h(\varphi))]_\theta \in [D]_\theta$ iff $v(h(\varphi)) \in D$ iff $\alpha, h \Vdash \varphi$. ■

As in the theory of logical matrices, there are two particularly interesting congruences that we can associate to a given abstract valuation.

Definition 14 Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ be an abstract Γ-valuation over Σ.

The *Leibniz* Γ*-congruence* $\mathbf{\Omega}_\Gamma(\alpha)$ of α is simply the Leibniz congruence $\mathbf{\Omega}(m)$ of the matrix $m = \langle \mathbf{B}, D\rangle$ over Γ, that is, the largest congruence on the Γ-algebra $\mathbf{B}$ that is compatible with D.

Analogously, if α is a model of $\mathcal{L}$, the *Suszko* Γ*-congruence* $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha)$ of α is the largest congruence on the Γ-algebra $\mathbf{B}$ that is compatible with every $\mathcal{L}$-filter $D' \supseteq D$ of v.

We denote by α^* and $\widetilde{\alpha^*}$ the reduced valuations $\alpha/_{\mathbf{\Omega}_\Gamma(\alpha)}$ and $\alpha/_{\widetilde{\mathbf{\Omega}}_\Gamma(\alpha)}$, respectively.

The abstract Γ-valuation α is said to be *Leibniz reduced*, respectively *Suszko reduced*, when $\mathbf{\Omega}_\Gamma(\alpha)$, respectively $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha)$, is the identity.

Given a logic $\mathcal{L}$, we will denote by $AVal^*_\Gamma(\mathcal{L})$, respectively $\widetilde{AVal^*_\Gamma}(\mathcal{L})$, the collection of all Leibniz reduced, respectively Suszko reduced, abstract valuation models of $\mathcal{L}$. We will also denote by $Lind^*_\Gamma(\mathcal{L})$, respectively $\widetilde{Lind^*_\Gamma}(\mathcal{L})$, the class of all Leibniz, respectively Suszko, reductions of Lindenbaum abstract Γ-valuation models of $\mathcal{L}$.

Note that in the above definition of Suszko Γ-congruence, it does not make sense to speak of the Leibniz congruence $\mathbf{\Omega}(m)$ of the matrix $m = \langle \mathbf{B}, D\rangle$ over Γ, since m cannot be seen, by itself, as a model of $\mathcal{L}$.

Below, we provide alternative, easier to work with, characterizations of the Leibniz and Suszko congruences of an abstract valuation.

Proposition 15 *Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ be an abstract Γ-valuation over Σ. The following are equivalent:*

1. $\langle b, b'\rangle \in \mathbf{\Omega}_\Gamma(\alpha)$;

2. *for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and every $b_1 \in B_{s_1}, \ldots, b_n \in B_{s_n}$, $\varphi_\mathbf{B}(b, b_1, \ldots, b_n) \in D$ iff $\varphi_\mathbf{B}(b', b_1, \ldots, b_n) \in D$;*

3. *for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$, and every $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$, $a \in v^{-1}(b)$ and $a' \in v^{-1}(b')$, $v(\varphi_\mathbf{A}(a, a_1, \ldots, a_n)) \in D$ iff $v(\varphi_\mathbf{A}(a', a_1, \ldots, a_n)) \in D$.*

Proof: Conditions 1 and 2 are well-known to be equivalent, as $\mathbf{\Omega}_\Gamma(\alpha)$ is $\mathbf{\Omega}(\langle \mathbf{B}, D\rangle)$.

To see that conditions 2 and 3 are equivalent, just recall that v is surjective and that whenever $v(a) = b$, $v(a_1) = b_1$, $\ldots$, $v(a_n) = b_n$, we have that $v(\varphi_\mathbf{A}(a, a_1, \ldots, a_n)) = \varphi_\mathbf{B}(v(a), v(a_1), \ldots, v(a_n)) = \varphi_\mathbf{B}(b, b_1, \ldots, b_n)$, as $\varphi \in L_\Gamma(X)$ and v is a Γ-homomorphism. ∎

Proposition 16 *Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ be an abstract Γ-valuation model of a logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$. The following are equivalent:*

1. $\langle b, b'\rangle \in \widetilde{\mathbf{\Omega}}_\Gamma(\alpha)$;

2. *for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$, every $b_1 \in B_{s_1}, \ldots, b_n \in B_{s_n}$, and every $\mathcal{L}$-filter $D' \supseteq D$ of v, $\varphi_\mathbf{B}(b, b_1, \ldots, b_n) \in D'$ iff $\varphi_\mathbf{B}(b', b_1, \ldots, b_n) \in D'$;*

3. *for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$, every $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$, $a \in v^{-1}(b)$ and $a' \in v^{-1}(b')$, and every $\mathcal{L}$-filter $D' \supseteq D$ of v, $v(\varphi_\mathbf{A}(b, b_1, \ldots, b_n)) \in D'$ iff $v(\varphi_\mathbf{B}(b', b_1, \ldots, b_n)) \in D'$.*

Proof: The fact that condition 1 implies condition 2 is an immediate consequence of the fact that $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha)$ is a Γ-congruence of $\mathbf{B}$ compatible with every $\mathcal{L}$-filter $D' \supseteq D$ of v.

To see that condition 2 implies condition 1, it suffices to note that the Γ-congruence θ defined on $\mathbf{B}$ precisely by $\langle b, b'\rangle \in \theta$ iff condition 2 holds, is necessarily compatible with every $\mathcal{L}$-filter $D' \supseteq D$ of v. Therefore, $\theta \subseteq \widetilde{\mathbf{\Omega}}_\Gamma(\alpha)$ as $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha)$ is by definition the largest such congruence.

Conditions 2 and 3 are equivalent for exactly the same reasons of Proposition 15 above. ∎

As a corollary, exactly as in the theory of logical matrices, for every abstract Γ-valuation model $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ of a logic $\mathcal{L}$, we have

$$\widetilde{\mathbf{\Omega}}_\Gamma(\alpha) = \bigcap_{\mathcal{L}\text{-filter } D' \supseteq D \text{ of } v} \mathbf{\Omega}_\Gamma(\langle \mathbf{A}, v, \langle \mathbf{B}, D'\rangle\rangle).$$

In particular, this implies that $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha) \subseteq \mathbf{\Omega}_\Gamma(\alpha)$, and consequently we always have that $AVal^*_\Gamma(\mathcal{L}) \subseteq \widetilde{AVal^*_\Gamma}(\mathcal{L})$.

Proposition 17 *Let $\mathcal{L}$ be a many-sorted logic over Σ, and Γ a subsignature of Σ. Then, $\vdash_\mathcal{L} = \vDash_{AVal^*_\Gamma(\mathcal{L})} = \vDash_{Lind^*_\Gamma(\mathcal{L})} = \vDash_{\widetilde{AVal^*_\Gamma}(\mathcal{L})} = \vDash_{\widetilde{Lind^*_\Gamma}(\mathcal{L})}$.*

Proof: The result is an immediate consequence of Proposition 11 and Lemma 13. ■

Still, the classes of Leibniz and Suszko reduced abstract valuation models of a logic have rich algebraic properties. Let us define some useful constructions over abstract valuations.

Definition 18 The *direct product* of a collection $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle \mid i \in I\}$ of abstract Γ-valuations over Σ is the abstract Γ-valuation $\Pi_{i\in I}\alpha_i = \langle \Pi_{i\in I}\mathbf{A}_i, \langle v_i(\_)\rangle_{i\in I}, \langle \Pi_{i\in I}\mathbf{B}_i, \Pi_{i\in I}D_i\rangle\rangle$.

Lemma 19 *Let $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle \mid i \in I\}$ be a collection of abstract Γ-valuations over Σ. For every logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, $\varphi \in L_\Sigma(X)$ and assignment h over $\Pi_{i\in I}\mathbf{A}_i$, we have that $\Pi_{i\in I}\alpha_i, h \Vdash \varphi$ iff $\alpha_i, h_i \Vdash \varphi$ for every $i \in I$.*

Hence, given a logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, $\Pi_{i\in I}\alpha_i \in AVal_\Gamma(\mathcal{L})$ iff $\{\alpha_i \mid i \in I\} \subseteq AVal_\Gamma(\mathcal{L})$.

Proof: It suffices to note that $\Pi_{i\in I}\alpha_i, h \Vdash \varphi$ iff $\langle v_i(h_i(\varphi))\rangle_{i\in I} \in \Pi_{i\in I}D_i$ iff $v_i(h_i(\varphi)) \in D_i$ for every $i \in I$ iff $\alpha_i, h_i \Vdash \varphi$ for every $i \in I$.

Moreover, it is straightforward to check that there is a one-to-one correspondence between assignments h over $\Pi_{i\in I}\mathbf{A}_i$ and collections $\{h_i\}_{i\in I}$ where each h_i is an assignment over $\mathbf{A}_i$. ■

A suitable, perhaps less obvious, notion of substructure is also possible for abstract valuations.

Definition 20 A *subvaluation* of a abstract Γ-valuation $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ over Σ is an abstract Γ-valuation $\alpha' = \langle \mathbf{A}', v', \langle \mathbf{B}', D'\rangle\rangle$ such that:

- $\langle \mathbf{B}', D'\rangle$ is a submatrix of $\langle \mathbf{B}, D\rangle$, that is, $\mathbf{B}'$ is a subalgebra of $\mathbf{B}$ and $D' = D \cap B'_\phi$, and
- there exists Σ-algebra homomorphism $f : \mathbf{A}' \to \mathbf{A}$ such that $v \circ f = \iota \circ v'$ where $\iota : \mathbf{B}' \to \mathbf{B}$ is the obvious inclusion.

Lemma 21 *Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ be an abstract Γ-valuation over Σ, and $\alpha' = \langle \mathbf{A}', v', \langle \mathbf{B}', D'\rangle\rangle$ a subvalution of α. For every logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, $\varphi \in L_\Sigma(X)$ and assignment h' over $\mathbf{A}'$, we have that $\alpha', h' \Vdash \varphi$ iff $\alpha, f \circ h' \Vdash \varphi$, where $f : \mathbf{A}' \to \mathbf{A}$ is such that $v \circ f = \iota \circ v'$.*

Hence, given a logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, if $\alpha \in AVal_\Gamma(\mathcal{L})$ then $\alpha' \in AVal_\Gamma(\mathcal{L})$.

Proof: It suffices to note that $\alpha, f \circ h' \Vdash \varphi$ iff $v((f \circ h')(\varphi)) = v(f(h'(\varphi))) = (v \circ f)(h'(\varphi)) = (\iota \circ v')(h'(\varphi)) = \iota(v'(h'(\varphi))) \in D$ iff $v'(h'(\varphi)) \in \iota^{-1}(D) = D'$ iff $\alpha', h' \Vdash \varphi$. ■

Along the same lines, we can define a suitable notion of isomorphism.

Definition 22 Abstract Γ-valuations $\alpha_1 = \langle \mathbf{A}_1, v_1, \langle \mathbf{B}_1, D_1\rangle\rangle$ and $\alpha_2 = \langle \mathbf{A}_2, v_2, \langle \mathbf{B}_2, D_2\rangle\rangle$ over Σ are *isomorphic* if the following conditions hold:

- $\langle \mathbf{B}_1, D_1\rangle$ and $\langle \mathbf{B}_2, D_2\rangle$ are isomorphic matrices over Γ (with isomorphisms $j : \mathbf{B}_1 \to \mathbf{B}_2$ and $j' : \mathbf{B}_2 \to \mathbf{B}_1$), and
- there exist Σ-homomorphisms $f : \mathbf{A}_1 \to \mathbf{A}_2$ and $f' : \mathbf{A}_2 \to \mathbf{A}_1$ such that $v_2 \circ f = j \circ v_1$ and $v_1 \circ f' = j' \circ v_2$.

Lemma 23 *Let $\alpha_1 = \langle \mathbf{A}_1, v_1, \langle \mathbf{B}_1, D_1\rangle\rangle$ and $\alpha_2 = \langle \mathbf{A}_2, v_2, \langle \mathbf{B}_2, D_2\rangle\rangle$ be isomorphic Γ-valuations over Σ. For every logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, $\varphi \in L_\Sigma(X)$ and assignments h_1 over $\mathbf{A}_1$ and h_2 over $\mathbf{A}_2$, we have that $\alpha_1, h_1 \Vdash \varphi$ iff $\alpha_2, f \circ h_1 \Vdash \varphi$ and $\alpha_2, h_2 \Vdash \varphi$ iff $\alpha_1, f' \circ h_2 \Vdash \varphi$, where $f : \mathbf{A}_1 \to \mathbf{A}_2$ and $f' : \mathbf{A}_2 \to \mathbf{A}_1$ are such that $v_2 \circ f = j \circ v_1$ and $v_1 \circ f' = j' \circ v_2$.*

Hence, given a logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, $\alpha_1 \in AVal_\Gamma(\mathcal{L})$ iff $\alpha_2 \in AVal_\Gamma(\mathcal{L})$.

Proof: As the argument is symmetric, we only prove that $\alpha_1, h_1 \Vdash \varphi$ iff $\alpha_2, f \circ h_1 \Vdash \varphi$. Just note that $\alpha_2, f \circ h_1 \Vdash \varphi$ iff $v_2((f \circ h_1)(\varphi)) = v_2(f(h_1(\varphi))) = (v_2 \circ f)(h_1(\varphi)) = (j \circ v_1)(h_1(\varphi)) = j(v_1(h_1(\varphi))) \in D_2$ iff $v_1(h_1(\varphi)) \in j^{-1}(D_2) = D_1$ iff $\alpha_1, h_1 \Vdash \varphi$. ■

Note that although unusual, these definitions of subvaluation and isomorphism are quite natural if one understands that in abstract valuations, the concrete values (over Σ) are just a passing point towards the abstract ones (over Γ). All these notions are well supported by a corresponding notion of *homomorphism* between abstract valuations. Given abstract Γ-valuations $\alpha_1 = \langle \mathbf{A}_1, v_1, \langle \mathbf{B}_1, D_1\rangle\rangle$ and $\alpha_2 = \langle \mathbf{A}_2, v_2, \langle \mathbf{B}_2, D_2\rangle\rangle$ over Σ an homomorphism $g : \alpha_1 \to \alpha_2$ is simply an homomorphism $g : \langle \mathbf{B}_1, D_1\rangle \to \langle \mathbf{B}_2, D_2\rangle$ of the matrices over Γ (that is, $g : \mathbf{B}_1 \to \mathbf{B}_2$ is an homomorphism of Γ-algebras such that $g[D_1] \subseteq D_2$), for which there exists a Σ-homomorphism $f : \mathbf{A}_1 \to \mathbf{A}_2$ such that $v_2 \circ f = g \circ v_1$.

Now, using the constructions above, we can also define subdirect products of abstract valuations.

Definition 24 A *subdirect product* of a collection $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle \mid i \in I\}$ of abstract Γ-valuations over Σ is a subvaluation $\alpha' = \langle \mathbf{A}', v', \langle \mathbf{B}', D'\rangle\rangle$ of the direct product $\Pi_{i\in I}\alpha_i$ such that the projections $\pi'_i : \mathbf{B}' \to \mathbf{B}_i$ are surjective for every $i \in I$.

Lemma 25 *Let $\alpha' = \langle \mathbf{A}', v', \langle \mathbf{B}', D'\rangle\rangle$ be a subdirect product of a collection $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle \mid i \in I\}$ of abstract Γ-valuations over Σ. For every formula $\varphi \in L_\Sigma(X)$ and every assignment h' over $\mathbf{A}'$, we have that $\alpha', h' \Vdash \varphi$ iff $\alpha_i, (f \circ h')_i \Vdash \varphi$ for every $i \in I$, where $f : \mathbf{A}' \to \Pi_{i\in I}\mathbf{A}_i$ is such that $\langle v_i(\_)\rangle_{i\in I} \circ f = \iota \circ v'$.*

Hence, given a logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, if $\{\alpha_i \mid i \in I\} \subseteq AVal_\Gamma(\mathcal{L})$ then $\alpha' \in AVal_\Gamma(\mathcal{L})$.

Proof: Immediate from Propositions 19 and 21. ■

The following is a generalization of a well-known result about logical matrices, whose formulation was only made possible by adopting the notion of abstract valuation. Namely, we show that the closure under subdirect products of the Leibniz reduced abstract valuation models of a logic yields precisely the Suszko reduced abstract valuation models, which are, on their turn, closed under subdirect products. As usual, we assume that such closures are all defined up to isomorphism.

Proposition 26 *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a logic and Γ a subsignature of Σ. The following conditions hold:*

1. *the class $\widetilde{AVal^*_\Gamma}(\mathcal{L})$ is closed under subdirect products, and*
2. *the closure under subdirect products of the class $AVal^*_\Gamma(\mathcal{L})$ coincides with $\widetilde{AVal^*_\Gamma}(\mathcal{L})$.*

Proof:

1. Let $\alpha' = \langle \mathbf{A}', v', \langle \mathbf{B}', D'\rangle\rangle$ be a subdirect product of a collection $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle\}_{i\in I} \subseteq \widetilde{AVal^*_\Gamma}(\mathcal{L})$. Clearly, α' is a model of $\mathcal{L}$. What remains to be proved is that α' is Suszko reduced.

   First we introduce some notation. Denote by $LF'(X)$ the least $\mathcal{L}$-filter of α' that contains X and, for each $i \in I$, denote by $LF_i(X)$ the least $\mathcal{L}$-filter of α_i that contains X. An easy adaptation of a well-know basic result of the theory of logical matrices allows us to conclude that $LF_i(\pi'_i[LF'(X)]) = LF_i(\pi'_i[X])$ from the fact that each projection $\pi'_i : \mathbf{B}' \to \mathbf{B}_i$ is surjective.

   Now, let us show that $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha')$ is the identity. Assume that $\langle b'_1, b'_2\rangle \in \widetilde{\mathbf{\Omega}}_{\Gamma,s}(\alpha')$ for some sort s of Σ. Using Lemma 16, equivalently, we know that for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$, every $c'_1 \in B'_{s_1}, \ldots, c'_n \in B'_{s_n}$, and every $\mathcal{L}$-filter $D'' \supseteq D'$ of v', it is the case that $\varphi_{\mathbf{B}'}(b'_1, c'_1, \ldots, c'_n) \in D''$ iff $\varphi_{\mathbf{B}'}(b'_2, c'_1, \ldots, c'_n) \in D''$. This condition is equivalent to requiring that $LF'(\{\varphi_{\mathbf{B}'}(b'_1, c'_1, \ldots, c'_n)\} \cup D') = LF'(\{\varphi_{\mathbf{B}'}(b'_2, c'_1, \ldots, c'_n)\} \cup D')$, for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and every $c'_1 \in B'_{s_1}, \ldots, c'_n \in B'_{s_n}$. Using the above result on least $\mathcal{L}$-filters we have that $LF_i(\pi'_i[\{\varphi_{\mathbf{B}'}(b'_1, c'_1, \ldots, c'_n)\} \cup D']) = LF_i(\pi'_i[\{\varphi_{\mathbf{B}'}(b'_2, c'_1, \ldots, c'_n)\} \cup D'])$, that is, $LF_i(\{\varphi_{\mathbf{B}_i}(b'_{1,i}, c'_{1,i}, \ldots, c'_{n,i})\} \cup D_i) = LF_i(\{\varphi_{\mathbf{B}_i}(b'_{2,i}, c'_{1,i}, \ldots, c'_{n,i})\} \cup D_i)$.

Now, since each π'_i is assumed to be surjective, it is clear that the values $c'_{1,i}, \ldots, c'_{n,i}$ range over the whole $\mathbf{B}_i$ as $c'_1, \ldots, c'_n$ range over $\mathbf{B}'$. Thus, tracing the steps backwards, we can conclude that $\langle b'_{1,i}, b'_{2,i}\rangle \in \widetilde{\mathbf{\Omega}}_{\Gamma,s}(\alpha_i)$ for each $i \in I$. But each α_i are Suszko reduced, and so we can conclude that $b'_{1,i} = b'_{2,i}$ for every $i \in I$, that is, $b'_1 = b'_2$.

2. As $AVal^*_\Gamma(\mathcal{L}) \subseteq \widetilde{AVal^*_\Gamma}(\mathcal{L})$, property 1 above implies that the closure under subdirect products of $AVal^*_\Gamma(\mathcal{L})$ is contained in $\widetilde{AVal^*_\Gamma}(\mathcal{L})$.

   To prove the converse inclusion let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle \in \widetilde{AVal^*_\Gamma}(\mathcal{L})$. Clearly, one can identify $\{D' \supseteq D : D' \text{ is a } \mathcal{L}\text{-filter of } v\}$ with an indexed collection $\{D_i : i \in I\}$ for some suitable set I. Obviously, $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha) = \bigcap_{i\in I} \mathbf{\Omega}_\Gamma(\alpha_i)$ is the identity, where each $\alpha_i = \langle \mathbf{A}, v, \langle \mathbf{B}, D_i\rangle\rangle$. Let us consider the collection $\{\alpha^*_i\}_{i\in I}$ of corresponding Leibniz reduced abstract valuations. Note that the Γ-homomorphism $\iota : \mathbf{B} \to \Pi_{i\in I}\mathbf{B}/_{\mathbf{\Omega}_\Gamma(\alpha_i)}$ defined by $\iota(b) = \langle [b]_{\mathbf{\Omega}_\Gamma(\alpha_i)}\rangle_{i\in I}$ is injective. Therefore, α is isomorphic to a subvaluation of $\Pi_{i\in I}\alpha^*_i$, that is, a subdirect product of $\{\alpha^*_i : i \in I\} \subseteq AVal^*_\Gamma(\mathcal{L})$, and we conclude the proof.

■

We conclude this series of results by proving a suitably adapted version of Bloom's theorem [4], characterizing finitary logics by means of ultraproducts.

Remark 27 *Recall that, given a set I, an* ultrafilter on I *is a set $\mathcal{U}$ consisting of subsets of I such that the following conditions hold:* **(1)** $\emptyset \notin \mathcal{U}$; **(2)** *if $A \in \mathcal{U}$ and $A \subseteq B$ then $B \in \mathcal{U}$;* **(3)** *if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$;* **(4)** *if $A \subseteq I$, then either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.*

Given a collection $\{\mathbf{B}_i \mid i \in I\}$ of Γ-algebras and an ultrafilter $\mathcal{U}$ on I, the relation $\sim_\mathcal{U}$ defined by $b \sim_\mathcal{U} b'$ iff $\{i \mid b_i = b'_i\} \in \mathcal{U}$ is easily seen to be a Γ-congruence on $\Pi_{i\in I}\mathbf{B}_i$.

Definition 28 Let I be a set and $\mathcal{U}$ an ultrafilter on I. The *ultraproduct modulo* $\mathcal{U}$ of a collection $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle \mid i \in I\}$ of abstract Γ-valuations over Σ is the abstract valuation $\Pi_\mathcal{U}\alpha_i = \langle \Pi_{i\in I}\mathbf{A}_i, [\langle v_i(\_)\rangle_{i\in I}]_{\sim_\mathcal{U}}, \langle (\Pi_{i\in I}\mathbf{B}_i)/_{\sim_\mathcal{U}}, \{[b]_{\sim_\mathcal{U}} \mid \{i \mid b_i \in D_i\} \in \mathcal{U}\}\rangle\rangle$.

Lemma 29 *Let $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle \mid i \in I\}$ be a collection of abstract Γ-valuations over Σ and $\mathcal{U}$ an ultrafilter on I. For every formula $\varphi \in L_\Sigma(X)$ and* *every assignment h over $\Pi_{i\in I}\mathbf{A}_i$, we have that $\Pi_{\mathcal{U}}\alpha_i, h \Vdash \varphi$ iff $\{i \mid \alpha_i, h_i \Vdash \varphi\} \in \mathcal{U}$.*

Proof: It suffices to note that $\Pi_{\mathcal{U}}\alpha_i, h \Vdash \varphi$ iff $[\langle v_i(h_i(\varphi))\rangle_{i\in I}]_{\sim_{\mathcal{U}}} \in \{[b]_{\sim_{\mathcal{U}}} \mid \{i \mid b_i \in D_i\} \in \mathcal{U}\}$ iff $\{i \mid v_i(h_i(\varphi)) \in D_i\} \in \mathcal{U}$ iff $\{i \mid \alpha_i, h_i \Vdash \varphi\} \in \mathcal{U}$. ■

Finally, we can prove the desired result.

Proposition 30 *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a logic and Γ a subsignature of Σ. Then, $\mathcal{L}$ is finitary if and only if the class $AVal_\Gamma(\mathcal{L})$ is closed under ultraproducts.*

Proof: Suppose first that $\mathcal{L}$ is finitary. Let $\{\alpha_i = \langle \mathbf{A}_i, v_i, \langle \mathbf{B}_i, D_i\rangle\rangle \mid i \in I\} \subseteq AVal_\Gamma(\mathcal{L})$ be a collection of abstract models of $\mathcal{L}$, and $\mathcal{U}$ an ultrafilter on I. To show that $\Pi_{\mathcal{U}}\alpha_i \in AVal_\Gamma(\mathcal{L})$, suppose that $\Psi \vdash_{\mathcal{L}} \varphi$ and let h be an assignment over $\Pi_{i\in I}\mathbf{A}_i$ such that $\Pi_{\mathcal{U}}\alpha_i, h \Vdash \psi$ for every $\psi \in \Psi$. Since $\mathcal{L}$ is finitary, there must exist $\{\psi_1, \ldots, \psi_n\} \subseteq \Psi$ such that $\psi_1, \ldots, \psi_n \vdash_{\mathcal{L}} \varphi$. For each $1 \leq j \leq n$, we have that $\Pi_{\mathcal{U}}\alpha_i, h \Vdash \psi_j$, and thus $I_j = \{i \mid \alpha_i, h \Vdash \psi_j\} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter we have that $I_1 \cap \ldots \cap I_n \in \mathcal{U}$. Note also that, since each α_i is an abstract valuation model of $\mathcal{L}$, $I_1 \cap \ldots \cap I_n \subseteq \{i \mid \alpha_i, h \Vdash \varphi\}$. Since $\mathcal{U}$ is an ultrafilter we have that $\{i \mid \alpha_i, h \Vdash \varphi\} \in \mathcal{U}$ and so $\Pi_{\mathcal{U}}\alpha_i, h \Vdash \varphi$, and we can conclude that $\Pi_{\mathcal{U}}\alpha_i \in AVal_\Gamma(\mathcal{L})$.

Suppose now that $AVal_\Gamma(\mathcal{L})$ is closed under ultraproducts. To prove that $\mathcal{L}$ is finitary let Ψ be infinite and assume that $\Psi' \nvdash_{\mathcal{L}} \varphi$, for every finite $\Psi' \subseteq \Psi$. Let I denote the set of all finite subsets of Ψ. For each $i \in I$, define $i^\bullet = \{j \in I \mid i \subseteq j\}$. Using well-known results on ultrafilters [21] we can conclude that there exists an ultrafilter $\mathcal{U}$ over I that contains the family $\{i^\bullet \mid i \in I\}$. For each $i \in I$, let us consider the Lindenbaum abstract Γ-valuation models $\alpha_i = \lambda_\Gamma(i^{\vdash_{\mathcal{L}}}) \in AVal_\Gamma(\mathcal{L})$. Let $\Pi_{\mathcal{U}}\alpha_i$ be the ultraproduct of the collection by the ultrafilter $\mathcal{U}$. Then, for every $\psi \in \Psi$ we have that $\{\psi\}^\bullet \subseteq \{i \mid \alpha_i, id \Vdash \psi\}$. So, $\{i \mid \alpha_i, id \Vdash \psi\} \in \mathcal{U}$, and consequently we have that $\Pi_{\mathcal{U}}\alpha_i, id \Vdash \psi$ for every $\psi \in \Psi$. However, $\{i \mid \alpha_i, id \Vdash \varphi\} = \emptyset \notin \mathcal{U}$ and so $\Pi_{\mathcal{U}}\alpha_i, id \nVdash \varphi$. Since $\Pi_{\mathcal{U}}\alpha_i \in AVal_\Gamma(\mathcal{L})$ we have that $\Psi \nvdash_{\mathcal{L}} \varphi$, and we can conclude that $\mathcal{L}$ is finitary. ■

At this point, it is clear that the theory of abstract valuations is rich and well-behaved enough to play a role that parallels that of logical matrices. In general, the class of matrix models that are canonically associated with a logic $\mathcal{L}$ is typically not the whole family $Matr(\mathcal{L})$, but rather the subclass of Suszko reduced matrix models of $\mathcal{L}$, that we will denote here by $\widetilde{Matr^*}(\mathcal{L})$. Actually, one often studies $Matr^*(\mathcal{L})$, the subclass of Leibniz reduced matrix models of $\mathcal{L}$, but it is well known that the two classes coincide for a wide range of logics, a subject to which we will come back in the next section.

For the time being, one can naturally think of extending the above rationale to the present setting. In [7], there has already been a similar proposal for the class of valuations that should be canonically associated with a logic. Herein, we will propose the abstract valuation semantics that should be canonically associated with a logic, and then show that the corresponding class of valuations coincides with the one defined in [7].

Definition 31 Let $\mathcal{L} = \langle\Sigma, \vdash\rangle$ be a logic and Γ a subsignature of Σ. The *canonical abstract* Γ*-valuation semantics associated to* $\mathcal{L}$ is $\widetilde{AVal^*_\Gamma}(\mathcal{L})$.

Although using different notation, the class of valuations canonically associated to a logic in [7] can be seen to correspond, up to isomorphism, to the valuations obtained from the Suszko reduction of the abstract Γ-valuations corresponding to the matrix models of $\mathcal{L}$. We show that this is compatible with the definition above.

Proposition 32 *Let* $\mathcal{L} = \langle\Sigma, \vdash\rangle$ *be a logic and* Γ *a subsignature of* Σ. *Then,* $\widetilde{AVal^*_\Gamma}(\mathcal{L})$ *coincides with the closure for isomorphisms of the class* $\{\widetilde{\alpha^*_\Gamma}(m) \mid m \in Matr(\mathcal{L})\}$.

Proof: It is clear that $\widetilde{\alpha^*_\Gamma(m)} \in \widetilde{AVal^*_\Gamma}(\mathcal{L})$ for every $m \in Matr(\mathcal{L})$. Thus, we are left with showing that every Suszko reduced abstract Γ-valuation $\alpha = \langle\mathbf{A}, v, \langle\mathbf{B}, D\rangle\rangle \in \widetilde{AVal^*_\Gamma}(\mathcal{L})$ is isomorphic to $\alpha^*_\Gamma(m)$ for some $m \in Matr(\mathcal{L})$.

As we have observed before, $m = \langle\mathbf{A}, v^{-1}(D)\rangle \in Matr(\mathcal{L})$. Hence, $\alpha_\Gamma(m) = \langle\mathbf{A}, id, \langle\mathbf{A}|_\Gamma, v^{-1}(D)\rangle\rangle$ and $\widetilde{\alpha^*_\Gamma(m)} = \langle\mathbf{A}, [\_]_\theta, \langle(\mathbf{A}|_\Gamma)/_\theta, [v^{-1}(D)]_\theta\rangle\rangle$, where we use $\theta = \widetilde{\boldsymbol{\Omega}}_\Gamma(\alpha_\Gamma(m))$ for ease of notation.

Given that α is reduced, it is not difficult to check that $\theta = \ker(v)$. Therefore, as v is surjective, $j : (\mathbf{A}|_\Gamma)/_\theta \to \mathbf{B}$ defined by $j([a]_\theta) = v(a)$ establishes an isomorphism of the Γ-matrices, with inverse $j' : \mathbf{B} \to (\mathbf{A}|_\Gamma)/_\theta$ defined by $j'(b) = [v^{-1}(b)]_\theta$. This concludes the proof, as by definition $j \circ [\_]_\theta = v$ and $j' \circ v = [\_]_\theta$. ■

Note that a similar proof would also hold if we considered the Leibniz reduced models, instead.

4 Applications to the behavioral algebraization of logics

In this section we explore the notion of abstract valuation semantics in the context of the behavioral approach to the algebraization of logics, as introduced in [8]. Namely, we obtain bridge results characterizing classes of logics in the behavioral Leibniz hierarchy. The following result, similar to the one obtained in [7], generalizes a well-known characterization of protoalgebraicity to the behavioral setting.

Proposition 33 *Let* $\mathcal{L} = \langle\Sigma, \vdash\rangle$ *be a many-sorted logic and* Γ *a subsignature of* Σ. *The following conditions are equivalent:*

1. $\mathcal{L}$ *is* Γ*-behaviorally protoalgebraic;*
2. $\widetilde{\boldsymbol{\Omega}}_\Gamma(\alpha) = \boldsymbol{\Omega}_\Gamma(\alpha)$ *for every* $\alpha \in AVal_\Gamma(\mathcal{L})$*;*

3. $AVal^*_\Gamma(\mathcal{L})$ is closed under subdirect products.

Proof: We first prove that conditions 1 and 2 are equivalent. Suppose first that $\mathcal{L}$ is Γ-behaviorally protoalgebraic. Using a characterization result given in [8], we can conclude that $\mathcal{L}$ has a parameterized Γ-equivalence system $\Delta(x:\phi, y:\phi, Z)$, where Z is a set of parametric variables. Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle \in AVal_\Gamma(\mathcal{L})$ and $b, b' \in B_s$ such that $\langle b, b'\rangle \in \mathbf{\Omega}_{\Gamma,s}(\alpha)$. Since $\mathbf{\Omega}_\Gamma(\alpha)$ is a Γ-congruence we have that the pair $\langle \varphi_{\mathbf{B}}(b, c_1, \ldots, c_n),\ \varphi_{\mathbf{B}}(b', c_1, \ldots, c_n)\rangle \in \mathbf{\Omega}_{\Gamma,\phi}(\alpha)$, for every $\varphi(x_0 : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and every $c_1 \in B_{s_1}, \ldots, c_n \in B_{s_n}$. Since α is a model of $\mathcal{L}$ and Δ defines the behavioral Leibniz congruence in every model we have that $\Delta_{\mathbf{B}}(\varphi_{\mathbf{B}}(b, c_1, \ldots, c_n), \varphi_{\mathbf{B}}(b', c_1, \ldots, c_n), \overline{u}) \subseteq D$ for every $\overline{u} \in \prod_{z:s\in Z} B_s$. Let $D' \supseteq D$ be a $\mathcal{L}$-filter. Suppose that $\varphi_{\mathbf{B}}(b, c_1, \ldots, c_n) \in D'$. Then, since $\Delta_{\mathbf{B}}(\varphi_{\mathbf{B}}(b, c_1, \ldots, c_n), \varphi_{\mathbf{B}}(b', c_1, \ldots, c_n), \overline{u}) \subseteq D \subseteq D'$ for every $\overline{u} \in \prod_{z:s\in Z} B_s$, we can conclude by (MP) that $\varphi_{\mathbf{B}}(b', c_1, \ldots, c_n) \in D'$. We can prove that if $\varphi_{\mathbf{B}}(b', c_1, \ldots, c_n) \in D'$ then also $\varphi_{\mathbf{B}}(b, c_1, \ldots, c_n) \in D'$ in a similar way. So, for every $\varphi(x_0 : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and every $a_1 \in B_{s_1}, \ldots, a_n \in B_{s_n}$ we have that $\varphi_{\mathbf{B}}(b, c_1, \ldots, c_n) \in D'$ iff $\varphi_{\mathbf{B}}(b', c_1, \ldots, c_n) \in D'$. Therefore, using Proposition 16 we can conclude that $\langle b, b'\rangle \in \widetilde{\mathbf{\Omega}}_{\Gamma,s}(\alpha)$. Since it is always the case that $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha) \subseteq \mathbf{\Omega}_\Gamma(\alpha)$, we can conclude that $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha) = \mathbf{\Omega}_\Gamma(\alpha)$.

For the converse implication, assume that $\widetilde{\mathbf{\Omega}}_\Gamma(\alpha) = \mathbf{\Omega}_\Gamma(\alpha)$ for every $\alpha \in AVal_\Gamma(\mathcal{L})$. Then, using the fact that $\widetilde{\mathbf{\Omega}}_\Gamma$ is always monotone, we can conclude that $\mathbf{\Omega}_\Gamma$ is also monotone. Using a characterization result given in [8] we can conclude that $\mathcal{L}$ is Γ-behaviorally protoalgebraic.

The fact that conditions 2 and 3 are equivalent is an immediate consequence of Proposition 26. ■

Thus, for behaviorally protoalgebraic logics, expectedly, we can identify its canonical abstract valuation semantics simply with the class of Leibniz reduced models.

Without getting into the details of the behavioral setting, namely of behavioral equational logic, for which we refer the reader to [8, 7], we need to introduce a corresponding notion of satisfaction of an equation, or quasi-equation, by an abstract valuation.

Remark 34 *We will use $t \approx u$ to represent an equation between terms $t, u \in T_{\Sigma,s}(X)$ of the same sort s, in which case we dub it an s-equation. The S-sorted set of all Σ-equations will be written as Eq_Σ. We will denote quasi-equations by $(t_1 \approx u_1)\ \&\ldots\&\ (t_n \approx u_n) \rightsquigarrow (t \approx u)$. A set Θ of equations with variables in $\{x_1 : s_1, \ldots, x_n : s_n\}$ will be dubbed $\Theta(x_1 : s_1, \ldots, x_n : s_n)$. Given an abstract valuation $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ we say that α with an assignment h over $\mathbf{A}$ satisfies the equation $t \approx u$, in symbols $\alpha, h \Vdash t \approx u$ if $v(h(t)) = v(h(u))$.*

Given values $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$ we say that α satisfies $\Theta(a_1, \ldots, a_n)$, in symbols $\alpha \Vdash \Theta(a_1, \ldots, a_n)$, whenever $\alpha, h \Vdash \Theta$ for an assignment h over $\mathbf{A}$ such that $h(x_1) = a_1, \ldots, h(x_n) = a_n$.

We say that α satisfies $t \approx u$, in symbols $\alpha \Vdash t \approx u$, if $\alpha, h \Vdash t \approx u$ for every assignment h over $\mathbf{A}$. Moreover, we say that α satisfies a quasi-equation $(t_1 \approx u_1) \,\&\ldots\&\, (t_n \approx u_n) \rightsquigarrow (t \approx u)$, denoted by $\alpha \Vdash (t_1 \approx u_1) \,\&\ldots\&\, (t_n \approx u_n) \rightsquigarrow (t \approx u)$, whenever, for every assignment h over $\mathbf{A}$ we have $\alpha, h \Vdash t \approx u$ whenever $\alpha, h \Vdash t_i \approx u_i$ for every $i \in \{1, \ldots, n\}$.

In the traditional theory of algebraization of logics, one associates to each algebraizable logic its largest equivalent algebraic semantics. In the case of a finitary and finitely algebraizable logic it is possible to obtain an equational specification of its largest equivalent algebraic semantics from a given axiomatization of the logic. This fact is particularly important to study concrete examples. The next result shows that, in the present setting, we can obtain a similar result. Namely, given an axiomatization of a finitary and finitely Γ-behaviorally algebraizable logic $\mathcal{L}$ we can build a (behavioral) equational specification of its canonical abstract Γ-valuation semantics.

Proposition 35 *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a finitary many-sorted logic presented by a Hilbert-style deductive system composed of a set Ax of axioms and a set Ir of inference rules, and let Γ be a subsignature of Σ. Suppose that $\mathcal{L}$ is Γ-behaviorally finitely algebraizable with $\Theta(x : \phi)$ a set of defining equations and $\Delta(x : \phi, y : \phi) \subseteq L_\Gamma(X)$ a set of equivalence formulas. Then, $\langle \mathbf{A}, v, \langle \mathbf{B}, D \rangle \rangle \in AVal^*_\Gamma(\mathcal{L})$ iff it satisfies the quasi-equations*

i) $\Theta(\varphi)$, *for every axiom* $\varphi \in Ax$;

ii) $\Theta(\psi_1) \wedge \ldots \wedge \Theta(\psi_n) \rightsquigarrow \Theta(\varphi)$ *for every* $\langle \psi_1, \ldots, \psi_n, \varphi \rangle \in Ir$;

iii) $\Theta(\Delta(x, y)) \rightsquigarrow x \approx y$,

and

iv) $D = \{v(a) \mid a \in A_\phi, \alpha \Vdash \Theta(a)\}$.

Proof: First let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D \rangle \rangle \in AVal^*_\Gamma(\mathcal{L})$. We aim to prove that α satisfies conditions i)-iv). Since $\mathcal{L}$ is Γ-behaviorally algebraizable, we have $x \dashv\vdash_\mathcal{L} \Delta[\Theta(x)]$. The fact that Δ defines $\boldsymbol{\Omega}_\Gamma(\alpha)$ and the fact that α is Leibniz reduced jointly imply that, for every formula $\varphi \in L_\Sigma(X)$ and every assignment h over $\mathbf{A}$, we have $\alpha, h \Vdash \varphi$ iff $\alpha, h \Vdash \Theta(\varphi)$. This means that condition iv) is satisfied. Therefore, it is clear that α satisfies quasi-equations i) and ii) since α is a model of $\mathcal{L}$. Condition iii) follows from the fact that α is reduced.

Now suppose that $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D \rangle \rangle$ satisfies conditions i)-iv). Condition iv) together with conditions i) and ii) imply that α is a model of $\mathcal{L}$. Therefore, Δ defines $\boldsymbol{\Omega}_\Gamma(\alpha)$ and quasi-equation iii) allows us to conclude that α is reduced. ■

We now present two examples where we illustrate the use of the result above to obtain an algebraic specification of the corresponding canonical abstract valuation semantics.

$$
\begin{array}{ll}
\textbf{(A1)} & \vdash_{\mathcal{K}/2} A \Rightarrow (B \Rightarrow A) \\
\textbf{(A2)} & \vdash_{\mathcal{K}/2} (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \\
\textbf{(A3)} & \vdash_{\mathcal{K}/2} ((A \Rightarrow B) \Rightarrow A) \Rightarrow A \\
\textbf{(A4)} & \vdash_{\mathcal{K}/2} A \Rightarrow ((\neg A) \Rightarrow B) \\
\textbf{(MP)} & A, A \Rightarrow B \vdash_{\mathcal{K}/2} B
\end{array}
$$

Example 36 *Recall the logic $\mathcal{K}/2$ from Examples 6, 9. First of all, let us remark that, as also noted in [15], the logic is very simply axiomatizable by:*

The same is to say that $\vdash_{\mathcal{K}/2}$ is the least consequence relation that contains all instances of the axioms **(A1-4)** *and is closed for* **(MP)**. *Note that* **(A1-2)** *are the usual axioms of positive implication,* **(A3)** *is Peirce's law, and* **(A4)** *is a form of ex falso. The rule* **(MP)** *is just modus ponens.*

Still, $\vdash_{\mathcal{K}/2}$ is not algebraizable, in the traditional sense. To show this, we consider the 5-valued algebra $\mathbf{A}$ with $A_\phi = \{1, a, b, 0, z\}$ where the operations are interpreted according to the tables below.

	$\neg_{\mathbf{A}}$
1	0
a	b
b	a
0	1
z	0

$\Rightarrow_{\mathbf{A}}$	1	a	b	0	z
1	1	a	b	0	z
a	1	1	b	b	b
b	1	a	1	a	a
0	1	1	1	1	1
z	1	1	1	1	1

Using a well-known result of [3], it is sufficient to find two distinct $\mathcal{K}/2$-filters of $\mathbf{A}$ whose corresponding Leibniz congruences coincide. Tedious but trivial calculations show that the axioms always evaluate to 1. Additionally, in order to accomodate modus ponens, a quick inspection of the table for implication shows that any $\mathcal{K}/2$-filter containing a and b must also contain 0 and z, and that any $\mathcal{K}/2$-filter containing 0 or z must be trivial. Thus, the algebra $\mathbf{A}$ has exactly four $\mathcal{K}/2$-filters: $\{1\}$, $\{1, a\}$, $\{1, b\}$ and the trivial filter $\{1, a, b, 0, z\}$. However, it is also not difficult to check that $\mathbf{A}$ has only two congruences: the identity and the total relations. Thus, necessarily, the Leibniz congruences of the $\mathcal{K}/2$-matrices with non-trivial filters are all the identity.

Nevertheless, the logic is obviously protoalgebraic. However, its canonical matrix semantics is quite uninteresting. In fact, it is easy to show, for instance that its Lindenbaum matrix $\langle \mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X}), \emptyset^{\vdash_{\mathcal{K}/2}} \rangle$ is reduced. Just note that identifying any two distinct formulas φ and ψ, forces the identification of $(\neg\neg\varphi) \Rightarrow (\neg\neg\varphi)$ and $(\neg\neg\varphi) \Rightarrow (\neg\neg\psi)$. However, while the first is clearly a theorem of $\mathcal{K}/2$, the second is not. To see this, just pick any bivaluation v such that $v(\neg\varphi) = 0$ and $v(\neg\neg\varphi) = 1$, but $v(\neg\psi) = v(\neg\neg\psi) = 0$.

However, $\mathcal{K}/2$ is Γ-behaviorally algebraizable with Γ consisting of $\Rightarrow$ and $\sim$, equivalence $\Delta(\varphi, \psi) = \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$ and defining set $\Theta(\varphi) = \{\varphi \approx (\varphi \Rightarrow \varphi)\}$. Thus, according to Proposition 35, its canonical abstract Γ-valuation semantics is defined precisely as the class of abstract Γ-valuations $\langle \mathbf{A}, v, \langle \mathbf{B}, \{\top\}\rangle\rangle$, where

A *is any* Σ*-algebra,* **B** *is a Boolean algebra with top element* $\top$*, bottom element* $\bot$ *and meet operation* $\sqcap$*, and* v *is such that the following condition holds:*

- $v(x) \sqcap v(\neg x) = \bot$. ▲

Example 37 *Nelson's constructive logic with strong negation* $\mathcal{N}$ *was introduced in [17]. Its language is obtained by adding a strong negation connective* $\sim$ *to the language of intuitionistic propositional logic (IPL). Concretely, the language of* $\mathcal{N}$ *is obtained from the single-sorted signature* $\Sigma = \langle\{\phi\}, F\rangle$ *where* $F_{\epsilon\phi} = \emptyset$*,* $F_{\phi\phi} = \{\neg, \sim\}$*,* $F_{\phi^2\phi} = \{\to, \vee, \wedge\}$ *and* $F_{\phi^n\phi} = \emptyset$*, for all* $n > 2$*. As usual, we can define* $\mathbf{f} = (\varphi \wedge (\neg\varphi))$ *and* $\mathbf{t} = (\varphi \to \varphi)$*, where* $\varphi \in L_\Sigma(X)$ *is some fixed but arbitrary formula. We can define the intuitionistic equivalence as usual as* $\xi_1 \leftrightarrow \xi_2 = (\xi_1 \to \xi_2) \wedge (\xi_2 \to \xi_1)$ *and we can also define a strong implication* $(\xi_1 \Rightarrow \xi_2) = (\xi_1 \to \xi_2) \wedge (\sim \xi_2 \to \sim \xi_1)$*.*

$\mathcal{N}$ *is axiomatizable by the axioms of IPL together with:*

(A1)	$\vdash_{\mathcal{N}} \sim (A \to B) \leftrightarrow (A \wedge \sim B)$
(A2)	$\vdash_{\mathcal{N}} \sim (A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
(A3)	$\vdash_{\mathcal{N}} \sim (A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
(A4)	$\vdash_{\mathcal{N}} (\sim \neg A) \leftrightarrow A$
(A5)	$\vdash_{\mathcal{N}} (\sim\sim A) \leftrightarrow A$
(A6)	$\vdash_{\mathcal{N}} (\sim A \vee \neg A) \leftrightarrow \neg A$
(MP)	$A, A \Rightarrow B \vdash_{\mathcal{N}} B$

Axioms **(A1-6)** *express the relation between strong negation and the other connectives.*

Let $\Gamma = \langle S, F'\rangle$ *be the subsignature of* Σ *such that* $F'_{\phi\phi} = \{\neg\}$ *and* $F'_{ws} = F_{ws}$ *for every* $ws \in S^*$ *such that* $ws \neq \phi\phi$*. Note that the subsignature* Γ *is nothing but the intuitionistic reduct of* Σ*.*

Although algebraizable according to the standard notion [18], the behavioral algebraization of $\mathcal{N}$ *was studied in [8], where it was proved that the logic is* Γ*-behaviorally algebraizable with* $\Delta(\varphi, \psi) = \{\varphi \leftrightarrow \psi\}$ *as equivalence formulas and* $\Theta(\varphi) = \{\varphi \approx \mathbf{t}\}$ *as defining equations.*

Our goal in this example is to use Proposition 35 to obtain an algebraic specification of $AVal^*_\Gamma(\mathcal{N})$*, the class of abstract* Γ*-valuations canonically associated with* $\mathcal{N}$*. Hence,* $AVal^*_\Gamma(\mathcal{N})$ *is precisely the class of all abstract valuations* $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ *satisfying:*

i) $\varphi \approx \mathbf{t}$ *for every axiom* φ *of* $\mathcal{N}$*;*

ii) $(x \approx \mathbf{t})$ & $((x \to y) \approx \mathbf{t}) \rightsquigarrow (y \approx \mathbf{t})$*;*

iii) $((x \leftrightarrow y) \approx \mathbf{t}) \rightsquigarrow (x \approx y)$*;*

iv) $D = \{\mathbf{t_B}\}$*.*

Since $\mathcal{N}$ *includes the axioms of IPL, conditions* i)*-*iii) *immediately imply that* **B** *must be a Heyting algebra. Condition* i) *in the case of axioms* **(A1-6)** *specifies how the* Γ*-homomorphism* v *interprets the connective* $\sim$*.*

Rephrasing, the abstract Γ-valuation semantics canonically associated with $\mathcal{N}$ is the class of abstract Γ-valuations $\alpha = \langle \mathbf{A}, v, \langle \mathbf{H}, \top \rangle \rangle$ where $\mathbf{A}$ is a Σ-algebra, $\mathbf{H}$ is a Heyting algebra with underlying order $\sqsubseteq$, top element $\top$, bottom element $\perp$, complement operation $-$, meet operation $\sqcap$ and join operation $\sqcup$, and v satisfies:

- $v(\sim (x \to y)) = v(x) \sqcap v(\sim y)$;
- $v(\sim (x \wedge y)) = v(\sim x) \sqcup v(\sim y)$;
- $v(\sim (x \vee y)) = v(\sim x) \sqcap v(\sim y)$;
- $v(\sim \neg x) = v(x)$;
- $v(\sim\sim x) = v(x)$;
- $v(\sim x) \sqsubseteq -v(x)$. ▲

We now present a bridge result characterizing behaviorally equivalential single-sorted logics.

Proposition 38 Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic and Γ a subsignature of Σ. Then, $\mathcal{L}$ is Γ-behaviorally equivalential if and only if $AVal^*_\Gamma(\mathcal{L})$ is closed under subvaluations and products.

Proof: Suppose first that $\mathcal{L}$ is Γ-behaviorally equivalential with $\Delta(x, y)$ a Γ-equivalence system. Since, in particular, $\mathcal{L}$ is Γ-behaviorally protoalgebraic, we can use Proposition 33 to conclude that $AVal^*_\Gamma(\mathcal{L})$ is closed under subdirect products, and, hence, closed under products. We now prove that $AVal^*_\Gamma(\mathcal{L})$ is also closed under subvaluations. Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D \rangle \rangle \in AVal^*_\Gamma(\mathcal{L})$ and $\alpha' = \langle \mathbf{A}', v', \langle \mathbf{B}', D' \rangle \rangle$ a subvaluation of α. We aim to prove that $\alpha' \in AVal^*_\Gamma(\mathcal{L})$. Clearly $\alpha' \in AVal_\Gamma(\mathcal{L})$, that is, α' is a model of $\mathcal{L}$. We now prove that α' is reduced. For all $a, b \in B'$ we have that $\langle a, b \rangle \in \mathbf{\Omega}_\Gamma(\alpha')$ iff $\Delta_{\mathbf{B}'}(a, b) \subseteq D'$. Since α' is a subvaluation of α we have that $\Delta_{\mathbf{B}'}(a, b) = \Delta_{\mathbf{B}}(a, b)$ and $D' = D \cap B'$. Therefore, we have that $\Delta_{\mathbf{B}'}(a, b) \subseteq D'$ iff $\Delta_{\mathbf{B}}(a, b) \subseteq D$. We can then conclude that $\langle a, b \rangle \in \mathbf{\Omega}_\Gamma(\alpha')$ iff $\langle a, b \rangle \in \mathbf{\Omega}_\Gamma(\alpha)$ iff (since $\mathbf{B}$ is reduced) $a = b$. Therefore, $\mathbf{B}'$ is reduced.

Conversely, assume that $AVal^*_\Gamma(\mathcal{L})$ is closed under subvaluations and products. In particular, we have that $AVal^*_\Gamma(\mathcal{L})$ is closed under subdirect products. Thus, using Proposition 33 we have that $\mathcal{L}$ is Γ-behaviorally protoalgebraic, and using a characterization result proved in [8] we can conclude that $\mathbf{\Omega}_\Gamma$ is monotone. We now prove that $\mathbf{\Omega}_\Gamma$ commutes with inverse substitutions, that is, for every substitution σ and every $\mathcal{L}$-theory Φ we have that $\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi)) = \mathbf{\Omega}_\Gamma(\sigma^{-1}(\Phi))$.

Fix a substitution σ and a $\mathcal{L}$-theory Φ and consider the following abstract valuations

$$\alpha = \langle \mathbf{T_\Sigma}(\mathbf{X}), [\_]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}, \langle (\mathbf{T_\Sigma}(\mathbf{X})|_\Gamma)/_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}, [\sigma^{-1}(\Phi)]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))} \rangle \rangle$$

and

$$\lambda^*_\Gamma(\Phi) = \langle \mathbf{T_\Sigma}(\mathbf{X}), [\_]_{\mathbf{\Omega}_\Gamma(\Phi)}, \langle(\mathbf{T_\Sigma}(\mathbf{X})|_\Gamma)/_{\mathbf{\Omega}_\Gamma(\Phi)}, [\Phi]_{\mathbf{\Omega}_\Gamma(\Phi)}\rangle\rangle.$$

By construction we have that $\lambda^*_\Gamma(\Phi) \in AVal^*_\Gamma(\mathcal{L})$. We now prove that α is isomorphic to a subvaluation of $\lambda^*_\Gamma(\Phi)$.

Let $\iota : (\mathbf{T_\Sigma}(\mathbf{X})|_\Gamma)/_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))} \to (\mathbf{T_\Sigma}(\mathbf{X})|_\Gamma)/_{\mathbf{\Omega}_\Gamma(\Phi)}$ be as follows

$$\iota([\varphi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}) = [\sigma(\varphi)]_{\mathbf{\Omega}_\Gamma(\Phi)}.$$

ι is well-defined:
Let $\varphi, \psi \in T_\Sigma(X)$ such that $[\varphi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))} = [\psi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}$. Then, $\langle\varphi, \psi\rangle \in \sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))$ and consequently $\langle\sigma(\varphi), \sigma(\psi)\rangle \in \mathbf{\Omega}_\Gamma(\Phi)$. Therefore, we have that $\iota([\varphi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}) = \iota([\psi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))})$.
ι is a Γ-homomorphism:
Trivial, since $[\_]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}$, $[\_]_{\mathbf{\Omega}_\Gamma(\Phi)}$ and σ are Γ-homomorphisms.
ι is injective:
Let $\varphi, \psi \in T_\Sigma(X)$ such that $\iota([\varphi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}) = \iota([\psi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))})$. Therefore, $[\sigma(\varphi)]_{\mathbf{\Omega}_\Gamma(\Phi)} = [\sigma(\psi)]_{\mathbf{\Omega}_\Gamma(\Phi)}$ and we can conclude that $\langle\sigma(\varphi), \sigma(\psi)\rangle \in \mathbf{\Omega}_\Gamma(\Phi)$. Then, $\langle\varphi, \psi\rangle \in \sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))$ and we conclude $[\varphi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))} = [\psi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}$.
Subvaluation conditions:
First let us prove that

$$\iota[[\sigma^{-1}(\Phi)]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}] = [\Phi]_{\mathbf{\Omega}_\Gamma(\Phi)} \cap \iota[(\mathbf{T_\Sigma}(\mathbf{X})|_\Gamma)/_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}].$$

Then, $[\varphi]_{\mathbf{\Omega}_\Gamma(\Phi)} \in \iota[[\sigma^{-1}(\Phi)]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}]$ iff $[\varphi]_{\mathbf{\Omega}_\Gamma(\Phi)} = [\sigma(\psi)]_{\mathbf{\Omega}_\Gamma(\Phi)}$ for some $\psi \in \sigma^{-1}(\Phi)$ iff $[\varphi]_{\mathbf{\Omega}_\Gamma(\Phi)} = [\sigma(\psi)]_{\mathbf{\Omega}_\Gamma(\Phi)}$ for some $\sigma(\psi) \in \Phi$ iff (as $\mathbf{\Omega}_\Gamma(\Phi)$ is compatible with Φ) $[\varphi]_{\mathbf{\Omega}_\Gamma(\Phi)} \in [\Phi]_{\mathbf{\Omega}_\Gamma(\Phi)} \cap \iota[(\mathbf{T_\Sigma}(\mathbf{X})|_\Gamma)/_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}]$.

Finally, we prove that there exists a homomorphism $g : \mathbf{T_\Sigma}(\mathbf{X}) \to \mathbf{T_\Sigma}(\mathbf{X})$ such that , $[\_]_{\mathbf{\Omega}_\Gamma(\Phi)} \circ g = \iota \circ [\_]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}$. We just have to take $g = \sigma$. It is clear that for every formula φ we have $\iota([\varphi]_{\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))}) = [g(\varphi)]_{\mathbf{\Omega}_\Gamma(\Phi)}$.

Since α is isomorphic to a subvaluation of $\lambda^*_\Gamma(\Phi) \in AVal^*_\Gamma(\mathcal{L})$ and since we are assuming that $AVal^*_\Gamma(\mathcal{L})$ is closed under subvaluations we can conclude that $\alpha \in AVal^*_\Gamma(\mathcal{L})$. Therefore, $\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi))$ is the largest Γ-congruence compatible with $\sigma^{-1}(\Phi)$, that is, $\sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi)) = \mathbf{\Omega}_\Gamma(\sigma^{-1}(\Phi))$.

Now that we have proved that $\mathbf{\Omega}_\Gamma$ is monotone and commutes with inverse substitutions we can use a characterization result of [8] to conclude that $\mathcal{L}$ is Γ-behaviorally equivalential. ■

It is perhaps surprising that the proof of this result does not generalize for many-sorted logics. This happens, however, not as a consequence of the behavioral approach in itself, but due to a feature (or defect) of its original development in the many-sorted case. Namely, the interested reader can confirm that several of the characterization properties obtained for the behavioral hierarchy only hold for sort ϕ, that is, for formulas. Namely, in the proof above, the characterization result from [8] used in the last step states that $\mathcal{L}$ is Γ-behaviorally equivalential if and only if $\mathbf{\Omega}_\Gamma$ is monotone and $\mathbf{\Omega}_{\Gamma,\phi}$ commutes with inverse substitutions. On the other hand, in the first step of the proof, a Γ-equivalence system is given only with respect to sort ϕ, which prevents us from completely characterizing the Leibniz operator for other sorts, if they exist.

This formula-centric feature of the approach in [8] is justified by the fact that other syntactic sorts can only be assessed in the context of formulas. However, the approach can perhaps be classified as excessively formula-centric. A more democratic treatment of other syntactic sorts, can actually be accommodated. We do not need to put aside the idea that other syntactic sorts are only assessed in the context of formulas, we just need to require that the logic is expressive enough in order to talk about the structure of sorts other than formulas.

Namely, we can put forward the following stronger notion of behavioral equivalentiality, by just requiring the existence of equivalence sets of formulas with respect to all sorts, instead of just with respect to sort ϕ, as in the original notion of behavioral equivalentiality[1].

Definition 39 Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of $\Sigma = \langle S, F \rangle$. Then, $\mathcal{L}$ is *strongly* Γ*-behaviorally equivalential* if there exists a S-sorted collection $\{\Delta_s(x : s, y : s)\}_{s \in S} \subseteq \{L_\Gamma(\{x : s, y : s\})\}_{s \in S}$ of formulas such that for every sort $s \in S$:

(R)	$\vdash \Delta_s(x, x)$
(S)	$\Delta_s(x, y) \vdash \Delta_s(y, x)$
(T)	$\Delta_s(x, y), \Delta_s(y, z) \vdash \Delta_s(x, z)$
(MP)	$\Delta_\phi(x, y), x \vdash y$
(RP$_\Gamma$)	$\Delta_{s_1}(x_1, y_1), \ldots, \Delta_{s_n}(x_n, y_n) \vdash \Delta_s(c[x_1, \ldots, x_n], c[y_1, \ldots, y_n])$ for every $c : s_1 \ldots s_n \to s \in Der_{\Gamma,s}$

In this case, $\{\Delta_s\}_{s \in S}$ is called a strong Γ-behavioral equivalence for $\mathcal{L}$.

In the next proposition we group two straightforward generalizations of interesting properties regarding behavioral equivalentiality and the behavioral Leibniz operator. The first one generalizes the intimate connection between equivalence sets and the Leibniz congruence. The second property generalizes Herrmann's criterion for equivalentiality [14].

Proposition 40 *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *a many-sorted logic and* Γ *a subsignature of* $\Sigma = \langle S, F \rangle$. *Let* $\{\Delta_s(x : s, y : s)\}_{s \in S} \subseteq \{L_\Gamma(\{x : s, y : s\})\}_{s \in S}$ *be a* S*-sorted collection of formulas. Then,*

i) if $\{\Delta_s\}_{s \in S}$ *is a strong* Γ*-behavioral equivalence for* $\mathcal{L}$ *then, for every* $\mathcal{L}$*-theory* Φ *and* $t_1, t_2 \in T_{\Sigma,s}(X)$, $\langle t_1, t_2 \rangle \in \boldsymbol{\Omega}_{\Gamma,s}(\Phi)$ *iff* $\Delta_s(t_1, t_2) \subseteq \Phi$.

[1] Despite the well known difficulties posed by presenting predicate logics in a structural way, we could understand the path we are proposing here along the intuition that it is reasonable to think of predicate logic with equality to be equivalential in a stronger sense than predicate logic without equality.

ii) Herrmann's Test: Assuming that $\mathcal{L}$ is Γ-behaviorally protoalgebraic, $\{\Delta_s\}_{s\in S}$ is a strong Γ-behavioral equivalence for $\mathcal{L}$ iff for every $s \in S$ we have $\vdash \Delta_s(x,x)$ and $\langle x,y\rangle \in \mathbf{\Omega}_{\Gamma,s}(\Delta_s(x,y)^{\vdash})$.

We can now prove the corresponding characterization of strongly behaviorally equivalential logics using the behavioral Leibniz operator. We will say that $\Sigma = \langle S,F\rangle$ is Γ-*standard* with respect to a subsignature Γ provided that $T_{\Gamma,s}(\emptyset) \neq \emptyset$ for every $s \in S$.

Proposition 41 *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted logic and Γ a subsignature of Σ. If Σ is Γ-standard, then the following conditions are equivalent:*

1. *$\mathcal{L}$ is strongly Γ-behaviorally equivalential;*
2. *$\mathbf{\Omega}_\Gamma$ is monotone and commutes with inverse substitutions;*
3. *$\mathbf{\Omega}_\Gamma$ is monotone and $\sigma[\mathbf{\Omega}_\Gamma(\Phi)] \subseteq \mathbf{\Omega}_\Gamma((\sigma[\Phi])^{\vdash})$, for every substitution σ and $\mathcal{L}$-theory Φ.*

Proof: $(1 \Rightarrow 2)$ Suppose that $\mathcal{L}$ is strongly Γ-behaviorally equivalential and let $\{\Delta_s\}_{s\in S}$ be a strong Γ-behavioral equivalence set for $\mathcal{L}$. Since, in particular, $\mathcal{L}$ is Γ-behaviorally equivalential, then it is also Γ-behaviorally protoalgebraic. Using a characterization result proved in [8] we can conclude that $\mathbf{\Omega}_\Gamma$ is monotone. To prove that $\mathbf{\Omega}_\Gamma$ commutes with inverse substitutions, take $\Phi \in Th_{\mathcal{L}}$ and a substitution σ. Let $t_1, t_2 \in T_{\Sigma,s}(X)$ and consider the following sequence of equivalent sentences: $\langle t_1,t_2\rangle \in \sigma^{-1}(\mathbf{\Omega}_{\Gamma,s}(\Phi))$ iff $\langle \sigma(t_1),\sigma(t_2)\rangle \in \mathbf{\Omega}_{\Gamma,s}(\Phi)$ iff $\Delta_s(\sigma(t_1),\sigma(t_2)) \subseteq \Phi$ iff $\sigma[\Delta_s(t_1,t_2)] \subseteq \Phi$ iff $\Delta_s(t_1,t_2) \subseteq \sigma^{-1}(\Phi)$ iff $\langle t_1,t_2\rangle \in \mathbf{\Omega}_{\Gamma,s}(\sigma^{-1}(\Phi))$.

$(2 \Rightarrow 3)$ Let $\Phi \in Th_{\mathcal{L}}$ and σ a substitution. Let $\Phi_0 = (\sigma[\Phi])^{\vdash}$. It is obvious that $\Phi \subseteq \sigma^{-1}(\Phi_0)$ and since $\mathbf{\Omega}_\Gamma$ is monotone we have that $\mathbf{\Omega}_\Gamma(\Phi) \subseteq \mathbf{\Omega}_\Gamma(\sigma^{-1}(\Phi_0))$. Since $\mathbf{\Omega}_\Gamma$ commutes with inverse substitutions we have that $\mathbf{\Omega}_\Gamma(\sigma^{-1}(\Phi_0)) = \sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi_0))$. Therefore, we have that $\mathbf{\Omega}_\Gamma(\Phi) \subseteq \sigma^{-1}(\mathbf{\Omega}_\Gamma(\Phi_0))$, which yields $\sigma[\mathbf{\Omega}_\Gamma(\Phi)] \subseteq \mathbf{\Omega}_\Gamma((\sigma[\Phi])^{\vdash})$.

$(3 \Rightarrow 1)$ Suppose that condition 3 holds. Using Proposition 40, $\mathcal{L}$ is equivalential provided some $\{\Delta_s(x:s, y:s)\}_{s\in S} \subseteq \{L_\Gamma(\{x:s, y:s\})\}_{s\in S}$ satisfies, for each $s \in S$, $\vdash \Delta_s(x,x)$ and $\langle x,y\rangle \in \mathbf{\Omega}_{\Gamma,s}(\Delta_s(x,y)^{\vdash})$. Recall that since $\mathcal{L}$ is Γ-standard there exists a closed term over Γ for each sort $s \in S$. For each $s \in S$, let σ^s be a substitution such that $\sigma^s_s(x) = x$ and $\sigma^s_s(z) = y$ for every $z \in X_s$ and $z \neq x$ and, for every $s' \in S$ such that $s' \neq s$ and every $z \in X_{s'}$, $\sigma^s_{s'}(z) = t_{s'}$ where $t_{s'}$ is a closed Γ-term of sort s'. Let $T^s = \{\varphi(x,y,Z) \in L_\Gamma(X) : \vdash \varphi(x,x,Z)\}$. Now take $\Delta_s(x,y) = \sigma^s[T^s]$. It is easy to see that $\vdash \Delta_s(x,x)$. It is easy to prove that $\langle x,y\rangle \in \mathbf{\Omega}_{\Gamma,s}((T^s)^{\vdash})$. So, $\langle \sigma^s(x),\sigma^s(y)\rangle \in \sigma^s[\mathbf{\Omega}_{\Gamma,s}((T^s)^{\vdash})]$. Using the hypothesis we have that $\langle \sigma^s(x),\sigma^s(y)\rangle \in \mathbf{\Omega}_{\Gamma,s}((\sigma^s[T^s])^{\vdash})$. Since $\sigma^s(x) = x$ and $\sigma^s(y) = y$ we have that $\langle x,y\rangle \in \mathbf{\Omega}_{\Gamma,s}(\Delta_s(x,y)^{\vdash})$. Hence, Δ is a strong Γ-behavioral equivalence. ■

We can finally prove the many-sorted analogue of Proposition 38, but now using strong behavioral equivalentiality.

Proposition 42 Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. If Σ is Γ-standard, $\mathcal{L}$ is strongly Γ-behaviorally equivalential if and only if $AVal^*_\Gamma(\mathcal{L})$ is closed under subvaluations and products.

Proof: Suppose first that $\mathcal{L}$ is strongly Γ-behaviorally equivalential with $\{\Delta_s\}_{s\in S}$ a strong Γ-behavioral equivalence system for $\mathcal{L}$. Since, in particular, $\mathcal{L}$ is Γ-behaviorally protoalgebraic we can use Proposition 33 to conclude that $AVal^*_\Gamma(\mathcal{L})$ is closed under subdirect products, and, hence, closed under products. We now prove that $AVal^*_\Gamma(\mathcal{L})$ is also closed under subvaluations. Let $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle \in AVal^*_\Gamma(\mathcal{L})$ and $\alpha' = \langle \mathbf{A}', v', \langle \mathbf{B}', D'\rangle\rangle$ a subvaluation of α. We aim to prove that $\alpha' \in AVal^*_\Gamma(\mathcal{L})$. Clearly $\alpha' \in AVal_\Gamma(\mathcal{L})$, that is, α' is a model of $\mathcal{L}$. We now prove that α' is reduced. For all $s \in S$ and $a, b \in B'_s$ we have that $\langle a, b\rangle \in \mathbf{\Omega}_{\Gamma,s}(\alpha')$ iff $(\Delta_s)_{\mathbf{B}'}(a, b) \subseteq D'$. Since α' is a subvaluation of α we have that $(\Delta_s)_{\mathbf{B}'}(a, b) = (\Delta_s)_{\mathbf{B}}(a, b)$ and $D' = D \cap B'$. Therefore, we have that $(\Delta_s)_{\mathbf{B}'}(a, b) \subseteq D'$ iff $(\Delta_s)_{\mathbf{B}}(a, b) \subseteq D$. We can then conclude that $\langle a, b\rangle \in \mathbf{\Omega}_{\Gamma,s}(\alpha')$ iff $\langle a, b\rangle \in \mathbf{\Omega}_{\Gamma,s}(\alpha)$ iff (since $\mathbf{B}$ is reduced) $a = b$. Therefore, $\mathbf{B}'$ is reduced.

The converse implication follows exactly as in the single-sorted case. The only difference is that we now have to use Proposition 41 since the characterization result used in the single-sorted case only holds for $\mathbf{\Omega}_{\Gamma,\phi}$. ■

5 Conclusion

We have proposed and studied a notion of abstract valuation semantics for structural logics, with algebraic properties similar to those of logical matrices. Our aim was to provide a consistent revision of the initial proposal in [6, 7], that would overcome its limitations and allow for a smooth connection with the behavioral approach to the algebraization of logics, where logical matrices are not suited. Namely, using abstract valuations, we were able to treat both the Suszko and Leibniz congruences in a symmetric way, thus providing a solid foundation for defining and studying the canonical semantics of a logic, in a way that can be seen as a natural extension of the reduced matrix models of a logic, as reinforced by the results obtained.

Moreover, note that the very notion of abstract valuation put forth in Definition 8, along with the subsequent comments, allows us to draw an envisaged and very simple connection with Avron's non-deterministic matrices [1]. Namely, given a Γ-valuation $\alpha = \langle \mathbf{A}, v, \langle \mathbf{B}, D\rangle\rangle$ over Σ, we can easily associate with it a (abstract) non-deterministic matrix over Σ where the operations in Γ behave as usual, by extending $\mathbf{B}$ with non-deterministic interpretations $f_\mathbf{B} : B_1 \times \cdots \times B_n \to 2^{B_s}$ defined by

$$f_\mathbf{B}(b_1, \ldots, b_n) = \{v(f_\mathbf{A}(a_1, \ldots, a_n)) \mid a_1 \in v^{-1}(b_1), \ldots, a_n \in v^{-1}(b_n)\}$$

for each Σ-operation $f : s_1 \dots s_n \to s$ not in Γ. Further exploration of this relationship will be the focus of future research.

Still, we must say that the most interesting development motivated by the work reported here is related with our observations at the end of Section 4. Indeed, the work with abstract valuations led us to conclude that the original approach to behavioral algebraization of logics proposed in [8] is defensible, but unnecessarily formula-centric. The solution found, illustrated with respect to behavioral equivalentiality, opens the door for revising the whole behavioral approach to the algebraization of logics. For instance, it is clear that a behaviorally algebraizable logic in the sense of [8] cannot be shown to be strongly behaviorally equivalential, in general, unless it is single-sorted. Clearly, this suggests a corresponding definition of *strong behavioral algebraization*. Note, in any case, that this does not mean that the original definition is meaningless. It is not, and it can be understood as a behavioral view of a many-sorted logic whose language was single-sortedly flattened into just formulas. But, we can also see a meaning behind stronger notions, where other syntactic sorts also come into play. In full generality, we can imagine a corresponding interesting notion of behavioral algebraization with respect to any subset of syntactic sorts containing ϕ. A thorough pursuit of this path is another interesting topic for further investigation.

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