Algebraic valuations as behavioral logical matrices*

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Abstract. The newly developed behavioral approach to the algebraization of logics extends the applicability of the methods of algebraic logic to a wider range of logical systems, namely encompassing many-sorted languages and non-truth-functionality. However, where a logician adopting the traditional approach to algebraic logic finds in the notion of a logical matrix the most natural semantic companion, a correspondingly suitable tool is still lacking in the behavioral setting. Herein, we analyze this question and set the ground towards adopting an algebraic formulation of valuation semantics as the natural generalization of logical matrices to the behavioral setting, by establishing a few simple but promising results. For illustration, we will use da Costa's paraconsistent logic $\mathcal{C}_1$.

Key words: algebraic logic, behavioral algebraization, logical matrix, valuation semantics.

1 Introduction

A novel behavioral approach to the algebraization of logics was introduced in [7] with the aim of extending the range of applicability of the traditional tools of algebraic logic. The extended theory is able to provide a meaningful algebraic counterpart also to logics with a many-sorted syntax, or including non-truth-functional connectives, and which are not algebraizable with the usual approach. Intuitively, while the algebraization process is usually centered around the notion of congruence, behavioral algebraization is centered around the weaker notion of behavioral equivalence. Behavioral equivalence has its roots in computer science, namely in the field of algebraic specifications of data-types, where it is often necessary to reason about data which cannot be directly accessed [16]. In such situations, it is perfectly possible that one cannot distinguish between two different values if those values provide exactly the same results for all available ways of observing and experimenting with them. Hence, unsorted equational logic is replaced by many-sorted behavioral equational logic (sometimes called hidden equational logic) based on the

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notion of behavioral equivalence, given a set of available experiments. Behavioral reasoning in equational logic has been consistently developed, see for instance [14, 19].

As a consequence of the generalization of the process of algebraizing logics to the behavioral setting, however, one finds that the central notion of matrix semantics [15] is no longer adequate. Of course, it is well-known that every structural logic is fully characterized by the class of its matrix models, or even better by the class of its reduced matrix models [20]. In the case of a logic algebraizable according to the traditional methods, one even gets an equational characterization of the algebras underlying these matrix models, a neat characterization of matrix congruences by means of the Leibniz operator, and a way of recovering the corresponding matrix filters by using the defining equations of the algebraization [2, 13]. In contrast, in the behavioral approach, this is not such an easy task. First of all, due to the additional freedom provided by the notion of behavioral equivalence, the corresponding behavioral version of the Leibniz operator is in general not a congruence over the whole language of the logic. Moreover, as expected in the case of logics that are not algebraizable under the usual approach (but which may be behaviorally algebraizable), the connection between the logic and its matrix semantics may be weak and uninteresting. A paradigmatic example of this situation can be found in da Costa's system of paraconsistent logic $\mathcal{C}_1$ [8]. In fact, $\mathcal{C}_1$ is well-known not to be algebraizable using traditional means, and additionally all its Lindenbaum matrices are reduced.

Still, the logic $\mathcal{C}_1$ is behaviorally algebraizable, and its resulting behaviorally equivalent algebraic semantics is quite interesting [6]. Namely, with little effort, it allows us to recover the non-truth-functional bivaluation semantics of [9]. Valuations as a general semantic tool were proposed in [10] precisely with the aim of providing a semantic ground for logics that, like $\mathcal{C}_1$, lack a meaningful truth-functional semantics. The key idea is, in the extreme case, to drop the condition that formulas should always be interpreted homomorphically in an algebra over the same signature. Besides lacking a thorough study, namely if contrasted to the myriad of interesting and valuable algebraic theory underlying logical matrices (see [20]), valuation semantics has been criticized for its excessive generality (see, for instance, [12]). Still, everyone would agree that matrix semantics is simply a clever and algebraically well-behaved way of defining a valuation semantics. What we propose in this paper, taking into account the experience with $\mathcal{C}_1$, and as already suggested in [7, 5], is to adopt a suitable algebraic version of valuation semantics as the natural generalization of logical matrices to the behavioral setting. Namely, we will drop the requirement that formulas must be interpreted homomorphically, but we will require that there is an algebraic way of specifying these exceptions. This is just preliminary work, in the sense that our proposal will lack a deep body of results, namely as those available about logical matrices in the usual theory of abstract algebraic logic. What we hope, herein, is to set the ground for developing such results about algebraic valuations and the behavioral approach in the near future.

The paper is organized as follows. In Section 2, we fix notation and concepts that will be necessary for the remainder of the paper, most notably behavioral equational reasoning. Section 3 briefly overviews the notion of behavioral algebraizable logic, as well as the behavioral version of the Leibniz operator. Then, in Section 4, we motivate and present the notion of valuation semantics, and some of its properties. Section 5 is dedicated to establishing a few promising results that parallel, for valuations and in the behavioral setting, well known bridging results between logical matrices and traditional algebraization. Finally, in Section 6, we draw some conclusions and point to some topics of future work.

2 Preliminaries

In this work we will focus our attention on a wide class of logics: those whose language can be built from a rich many-sorted signature. A *many-sorted signature* is a pair $\Sigma = \langle S, F\rangle$ where S is a set (of *sorts*) and $F = \{F_{ws}\}_{w\in S^*,s\in S}$ is an indexed family of sets (of *operations*). For simplicity, we write $f : s_1 \ldots s_n \to s \in F$ for an element $f \in F_{s_1 \ldots s_n s}$. As usual, we denote by $T_\Sigma(X) = \{T_{\Sigma,s}(X)\}_{s\in S}$ the S-sorted family of carrier sets of the free Σ-algebra $\mathbf{T_\Sigma(X)}$ with generators taken from a sorted family $X = \{X_s\}_{s\in S}$ of variable sets. We will denote by $x{:}s$ the fact that $x \in X_s$. Often, we will need to write terms over a given finite set of variables $t \in T_\Sigma(x_1 : s_1, \ldots, x_n : s_n)$. For simplicity, we will denote such a term by $t(x_1 : s_1, \ldots, x_n : s_n)$. Moreover, if T is a set whose elements are all terms of this form, we will write $T(x_1 : s_1, \ldots, x_n : s_n)$. A *substitution* over Σ is a S-sorted family of functions $\sigma = \{\sigma_s : X_s \to T_{\Sigma,s}(X)\}_{s\in S}$. As usual, $\sigma(t)$ denotes the term obtained by uniformly applying σ to each variable in t. Given $t(x_1 : s_1, \ldots, x_n : s_n)$ and terms $t_1 \in T_{\Sigma,s_1}(X), \ldots, t_n \in T_{\Sigma,s_n}(X)$, we will write $t(t_1, \ldots, t_n)$ to denote the term $\sigma(t)$ where σ is a substitution such that $\sigma_{s_1}(x_1) = t_1, \ldots, \sigma_{s_n}(x_n) = t_n$. Extending everything to sets, given $T(x_1 : s_1, \ldots, x_n : s_n)$ and $U \in T_{\Sigma,s_1}(X) \times \cdots \times T_{\Sigma,s_n}(X)$, we will use $T[U] = \bigcup_{\langle t_1,\ldots,t_n\rangle\in U} T(t_1, \ldots, t_n)$. A *derived operation* of type $s_1 \ldots s_n \to s$ over Σ is simply a term in $T_{\Sigma,s}(x_1 : s_1, \ldots, x_n : s_n)$. For $w \in S^*$, we denote by $Der_{\Sigma,ws}$ the set of all derived operations of type $w \to s$ over Σ. A *(full) subsignature* of Σ is a many-sorted signature $\Gamma = \langle S, F'\rangle$ such that, for each $w \in S^*$ and $s \in S$, $F'_{ws} \subseteq Der_{\Sigma,ws}$. Given a many-sorted signature $\Sigma = \langle S, F\rangle$, a *$\Sigma$-algebra* is a pair $\mathbf{A}{=}\langle\{A_s\}_{s\in S}, \__{\mathbf{A}}\rangle$, where each A_s is a non-empty set, the *carrier of sort* s, and $\__{\mathbf{A}}$ assigns to each operation $f : s_1 \ldots s_n \to s$ a function $f_{\mathbf{A}} : A_{s_1} \times \ldots \times A_{s_n} \to A_s$. An assignment over $\mathbf{A}$ is a S-sorted family of functions $h = \{h_s : X_s \to A_s\}_{s\in S}$. As usual, we will often overload h and use it to denote also the unique extension of the assignment to an homomorphism $h : T_\Sigma(X) \to \mathbf{A}$. Given a Σ-algebra $\mathbf{A}$, a term $t(x_1 : s_1, \ldots, x_n : s_n)$ and $\langle a_1, \ldots, a_n\rangle \in A_{s_1} \times \ldots \times A_{s_n}$, then we denote by $t_{\mathbf{A}}(a_1, \ldots, a_n)$ the value $h(t)$ that t takes in $\mathbf{A}$ under an assignment h such that $h(x_1) = a_1, \ldots, h(x_n) = a_n$. We denote by $\mathbf{A}|_\Gamma$ the Γ-algebra obtained by forgetting in a given Σ-algebra $\mathbf{A}$ the interpretation of all the operations not in the subsignature Γ. We will use $t \approx u$ to represent an equation between terms $t, u \in T_{\Sigma,s}(X)$ of the same sort s, in which case we dub it an s-equation. The S-sorted set of all Σ-equations will be written as Eq_Σ. We will denote quasi-equations by $(t_1 \approx u_1)$ & ... & $(t_n \approx u_n) \rightarrow (t \approx u)$. A set Θ of equations with variables in $\{x_1 : s_1, \ldots, x_n : s_n\}$ will be dubbed $\Theta(x_1 : s_1, \ldots, x_n : s_n)$. As usual, we say that an assignment h over $\mathbf{A}$ *satisfies* the equation $t \approx u$, in symbols $\mathbf{A}, h \Vdash t \approx u$ if $h(t) = h(u)$. We say that $\mathbf{A}$ satisfies $t \approx u$, in symbols $\mathbf{A} \Vdash t \approx u$, if $\mathbf{A}, h \Vdash t \approx u$ for every assignment h over $\mathbf{A}$. Given a class $\mathbb{K}$ of Σ-algebras, the *equational consequence over* Σ *associated with* $\mathbb{K}$, $\vDash_{\mathbb{K}} \subseteq \mathcal{P}(Eq_\Sigma) \times Eq_\Sigma$, is such that $\Theta \vDash_{\mathbb{K}} t \approx u$ if for every $\mathbf{A} \in \mathbb{K}$ and assignment h over $\mathbf{A}$ we have that $\mathbf{A}, h \Vdash t \approx u$ whenever $\mathbf{A}, h \Vdash \Theta$. Moreover, we say that $\mathbf{A}$ satisfies a quasi-equation $(t_1 \approx u_1)$ & ... & $(t_n \approx u_n) \rightarrow (t \approx u)$, denoted by $\mathbf{A} \Vdash (t_1 \approx u_1)$ & ... & $(t_n \approx u_n) \rightarrow (t \approx u)$, whenever $\{t_1 \approx u_1, \ldots, t_n \approx u_n\} \vDash_{\{\mathbf{A}\}} t \approx u$.

As mentioned above, the key ingredient of the behavioral approach to algebraizing logics is to use *behavioral equational logic* in the role usually played by plain *equational logic*. The distinctive feature of behavioral equational logic is the fact that the sorts are split in two disjoint sets, of *visible* and *hidden* sorts, and only certain operations of visible sort are allowed as *experiments*. In the visible sorts we can perform simple equational reasoning, but we can only reason indirectly about hidden sorts, using *behavioral indistinguishability* under the available experiments. Intuitively, we must evaluate equations involving hidden values using only their visible properties. It may happen that under all available experiments two certain hidden terms always coincide, which makes them behaviorally equivalent, even though they might actually have distinct values. We will now put forward the rigorous definitions, contrasting them with the ones for plain equational logic. A *hidden many-sorted signature* is a tuple $\langle \Sigma, V, \mathcal{E} \rangle$ where $\Sigma = \langle S, F \rangle$ is a many sorted-signature, $V \subseteq S$ is the set of visible sorts, and $\mathcal{E}$ is the set of available *experiments*, that is, a set terms of visible sort of the form $t(x : s, x_1 : s_1, \ldots, x_n : s_n)$ where x is a distinguished variable of hidden sort $s \in H = S \setminus V$.

Definition 1. Consider a hidden many-sorted signature $\langle \Sigma, V, \mathcal{E} \rangle$ and a Σ-algebra $\mathbf{A}$. Given a hidden sort $s \in H$, two values $a, b \in A_s$ *are* $\mathcal{E}$*-behaviorally equivalent*, in symbols $a \equiv_{\mathcal{E}} b$, if for every experiment $t(x{:}s, x_1{:}s_1, \ldots, x_n{:}s_n) \in \mathcal{E}$ and every $\langle a_1, \ldots, a_n \rangle \in A_{s_1} \times \ldots \times A_{s_n}$, we have that $t_{\mathbf{A}}(a, a_1, \ldots, a_n) = t_{\mathbf{A}}(b, a_1, \ldots, a_n)$.

Now that we have defined behavioral equivalence, we can talk about behavioral satisfaction of an equation by a Σ-algebra $\mathbf{A}$. We say that an assignment h over $\mathbf{A}$ $\mathcal{E}$*-behaviorally satisfies* an equation $t \approx u$ of hidden sort $s \in H$, in symbols $\mathbf{A}, h \Vvdash^{\mathcal{E}} t \approx u$ if $h(t) \equiv_{\mathcal{E}} h(u)$. Expectedly, equations of visible sort are satisfied as usual, that is, if $t \approx u$ is an equation of sort $s \in V$ then we write $\mathbf{A}, h \Vvdash^{\mathcal{E}} t \approx u$ if $h(t) = h(u)$. These notion can now be smoothly extended. We say that $\mathbf{A}$ $\mathcal{E}$-behaviorally satisfies $t \approx u$, in symbols $\mathbf{A} \Vvdash^{\mathcal{E}} t \approx u$, if $\mathbf{A}, h \Vvdash^{\mathcal{E}} t \approx u$ for every assignment h over $\mathbf{A}$. Given a class $\mathbb{K}$ of Σ-algebras, the *behavioral consequence over* Σ *associated with* $\mathbb{K}$ *and* $\mathcal{E}$, $\vDash\!\!\!\equiv^{\mathcal{E}}_{\mathbb{K}} \subseteq \mathcal{P}(Eq_\Sigma) \times Eq_\Sigma$, is such that $\Theta \vDash\!\!\!\equiv^{\mathcal{E}}_{\mathbb{K}} t \approx u$ if for every $\mathbf{A} \in \mathbb{K}$ and every assignment h over $\mathbf{A}$ we have that $\mathbf{A}, h \Vvdash^{\mathcal{E}} t \approx u$ whenever $\mathbf{A}, h \Vvdash^{\mathcal{E}} t' \approx u'$ for every $t' \approx u' \in \Theta$. Moreover, we say that $\mathbf{A}$ $\mathcal{E}$-behaviorally satisfies a quasi-equation $(t_1 \approx u_1) \,\&\ldots\&\, (t_n \approx u_n) \rightarrow (t \approx u)$, denoted by $\mathbf{A} \Vdash^{\mathcal{E}} (t_1 \approx u_1) \,\&\ldots\&\, (t_n \approx u_n) \rightarrow (t \approx u)$, whenever $\{t_1 \approx u_1, \ldots, t_n \approx u_n\} \models^{\mathcal{E}}_{\{\mathbf{A}\}} t \approx u$. We refer the reader to [18] for more details on the subject of behavioral equational reasoning.

3 Behavioral algebraization

From now on, we will work only with many-sorted signatures $\Sigma = \langle S, F\rangle$ with a distinguished sort ϕ (the syntactic sort of formulas). We assume fixed a S-sorted family X of variables. We define the induced set of *formulas* $L_\Sigma(X)$ to be the carrier set of sort ϕ of the free algebra $\mathbf{T_\Sigma(X)}$ with generators X, that is, $L_\Sigma(X) = T_{\Sigma,\phi}(X)$. We now introduce the class of logics that is the target of our approach.

Definition 2. A *many-sorted logic* is a tuple $\mathcal{L} = \langle \Sigma, \vdash\rangle$ where Σ is a many-sorted signature and $\vdash \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$ is a *consequence relation* satisfying, for every $\Phi_1 \cup \Phi_2 \cup \{\varphi\} \subseteq L_\Sigma(X)$: if $\varphi \in \Phi_1$ then $\Phi_1 \vdash \varphi$ (***reflexivity***); if $\Phi_1 \vdash \varphi$ for all $\varphi \in \Phi_2$, and $\Phi_2 \vdash \psi$ then $\Phi_1 \vdash \psi$ (***cut***); if $\Phi_1 \vdash \varphi$ and $\Phi_1 \subseteq \Phi_2$ then $\Phi_2 \vdash \varphi$ (***weakening***). A logic $\mathcal{L}$ is further said to be ***structural*** if whenever $\Phi_1 \vdash \varphi$ then, for every substitution σ, $\sigma[\Phi_1] \vdash \sigma(\varphi)$; and said to be ***finitary*** if whenever $\Phi_1 \vdash \varphi$ then $\Phi_2 \vdash \varphi$ for some finite $\Phi_2 \subseteq \Phi_1$. In this paper, unless otherwise stated, all the logics considered are assumed to be structural.

Note that propositional-like logics appear as a particular case of many-sorted logics. They can be obtained by taking ϕ to be the only sort of the signature, that is, considering a signature $\Sigma = \langle S, F\rangle$ such that $S = \{\phi\}$. In the sequel, we will use $\vdash_\mathcal{L}$ instead of just $\vdash$ to refer to the consequence relation of a given logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$. Moreover, as usual, if $\Phi_1, \Phi_2 \subseteq L_\Sigma(X)$, we will write $\Phi_1 \vdash_\mathcal{L} \Phi_2$ whenever $\Phi_1 \vdash_\mathcal{L} \varphi$ for all $\varphi \in \Phi_2$. We say that $\varphi, \psi \in L_\Sigma(X)$ are *interderivable* in $\mathcal{L}$, which is denoted by $\varphi \dashv\vdash_\mathcal{L} \psi$, if $\varphi \vdash_\mathcal{L} \psi$ and $\psi \vdash_\mathcal{L} \varphi$. Analogously, we say that Φ_1 and Φ_2 are *interderivable* in $\mathcal{L}$, which is denoted by $\Phi_1 \dashv\vdash_\mathcal{L} \Phi_2$, if $\Phi_1 \vdash_\mathcal{L} \Phi_2$ and $\Phi_2 \vdash_\mathcal{L} \Phi_1$. The *theorems* of $\mathcal{L}$ are the formulas φ such that $\emptyset \vdash_\mathcal{L} \varphi$. A *theory* of $\mathcal{L}$ is a set of formulas Φ such that if $\Phi \vdash_\mathcal{L} \varphi$ then $\varphi \in \Phi$. As usual, $\Phi^{\vdash_\mathcal{L}}$ denotes the least theory of $\mathcal{L}$ that contains Φ.
In the present setting, given the signature $\Sigma = \langle S, F\rangle$ of a logic, the corresponding free many-sorted algebra will have a set of terms of each sort, but only those terms of sort ϕ will correspond to formulas of the logic. Therefore, in the logic itself, one can only observe the behavior of terms of other sorts by their indirect influence on the formulas where they appear. The behavioral approach to the algebraization of logics is built over the idea of taking this situation a step further. Namely, we will hide all the sorts of Σ, including ϕ, and introduce a new unique visible sort for observing the behavior of formulas. Experiments must be carefully chosen among the well-behaved connectives of the logic, determined by a given subsignature Γ of Σ, thus possibly allowing the remaining connectives to behave in a non-congruently. This can be achieved by considering behavioral equational logic over an extended many-sorted

signature. We define the extended signature $\Sigma^o = \langle S^o, F^o\rangle$ such that $S^o = S \biguplus \{v\}$, where v is the newly introduced sort of *observations* of formulas. The indexed set of operations $F^o = \{{F^o}_{ws}\}_{w\in(S^o)^*, s\in S^o}$ is such that ${F^o}_{ws} = F_{ws}$ if $w \in S^*$ and $s \in S$, $F^o_{\phi v} = \{o\}$, and $F^o_{ws} = \emptyset$ otherwise. Intuitively, we are just extending the signature with a new sort v for the observations that we can perform on formulas using the observation operation o. Finally, the extended hidden signature is $\langle \Sigma^o, \{v\}, \mathcal{E}_\Gamma\rangle$ where $\mathcal{E}_\Gamma = \{o(t(x:s, x_1:s_1, \ldots, x_m:s_m)) : t \in T_{\Gamma,\phi}(X)\}$. Henceforth, we will use Γ instead of $\mathcal{E}_\Gamma$ to qualify the corresponding notions of behavioral reasoning. Before we recall the notion of a behaviorally algebraizable logic, we need a further concept. Let $\Theta(x : \phi)$ be a set of ϕ-equations. Θ is said to be *Γ-compatible* with a class $\mathbb{K}$ of Σ^o algebras if, given any variable $y : \phi$, it is the case that $x \approx y, \Theta(x) \models^{\Gamma}_{\mathbb{K}} \Theta(y)$.

Definition 3. A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$ is *behaviorally algebraizable* if there exists a subsignature Γ of Σ, a class $\mathbb{K}$ of Σ^o-algebras, a set $\Theta(x : \phi)$ of ϕ-equations Γ-compatible with $\mathbb{K}$, and a set $\Delta(x_1 : \phi, x_2 : \phi) \subseteq T_{\Gamma,\phi}(\{x_1 : \phi, x_2 : \phi\})$ of formulas such that, for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$ and for every set $\Psi \cup \{t \approx u\}$ of ϕ-equations, we have:

i) $\Phi \vdash_{\mathcal{L}} \varphi$ iff $\Theta[\Phi] \models^{\Gamma}_{\mathbb{K}} \Theta(\varphi)$;
ii) $\Psi \models^{\Gamma}_{\mathbb{K}} t \approx u$ iff $\Delta[\Psi] \vdash_{\mathcal{L}} \Delta(t, u)$;
iii) $x \dashv\vdash_{\mathcal{L}} \Delta[\Theta(x)]$ and $x_1 \approx x_2 \mathrel{=\!\!\mid\models^{\Gamma}_{\mathbb{K}}} \Theta[\Delta(x_1, x_2)]$.

The set Θ is called the set of *defining equations*, Δ the set of *equivalence formulas*, and $\mathbb{K}$ a *behaviorally equivalent algebraic semantics* for $\mathcal{L}$. This definition is parameterized by the choice of the subsignature Γ of Σ. Hence, in what follows, we will say that a logic is Γ-behaviorally algebraizable if we want to stress the choice of Γ.

The attentive reader will notice that the statement above follows very closely the usual definition of an algebraizable logic, but with behavioral reasoning replacing the usual equational reasoning. As shown in [7], behavioral algebraization indeed enlarges the scope of the traditional theory of algebraization, but maintains many of its nice properties. Namely, given a Σ-algebra $\mathbf{A}$, there is a very natural way of defining a corresponding Γ-behavioral Leibniz operator $\mathbf{\Omega}^{\mathbf{A}}_{\mathbf{\Gamma}}$ that maps each filter $D \subseteq A_\phi$ to the largest congruence $\mathbf{\Omega}^{\mathbf{A}}_{\mathbf{\Gamma}}(D)$ of $\mathbf{A}|_\Gamma$ that is compatible with D. Note that $\mathbf{\Omega}^{\mathbf{A}}_{\mathbf{\Gamma}}(D)$ is in general not a congruence over $\mathbf{A}$ if Γ is a proper subsignature of Σ. In particular, if we consider the free algebra $\mathbf{T_\Sigma(X)}$, we will write $\mathbf{\Omega_\Gamma}$ instead of $\mathbf{\Omega}^{\mathbf{T_\Sigma(X)}}_{\mathbf{\Gamma}}$. As in the traditional approach, the behavioral Leibniz operator can be used to characterize important classes of logics with respect to their algebraic properties. Namely, a logic $\mathcal{L}$ is Γ-behaviorally algebraizable exactly when, on the theories of $\mathcal{L}$, $\mathbf{\Omega_\Gamma}$ is injective, monotone, and commutes with inverse substitutions [7].

Example 1. As an application of the behavioral approach, let us consider the example of da Costa's paraconsistent logic $\mathcal{C}_1$ of [8]. As shown in detail in [7, 6], the logic is behaviorally algebraizable despite the well-known fact that it is not algebraizable according to the usual theory, and *that it further lacks a meaningful matrix semantics. We just need to let Γ include all the classical connectives of the logic, but not its non-congruent paraconsistent negation. Moreover, the resulting behaviorally equivalent algebraic semantics for the logic, $\mathbb{K}_{\mathcal{C}_1}$, is very rich, allowing not only to fully explain the relational semantics provided by the so-called da Costa algebras, but also to recapture its best known semantics based on non-truth-functional bivaluations [9].*

In the classical theory of algebraization, the class of models that are canonically associated with a logic $\mathcal{L}$ is typically not the whole family $\mathrm{Matr}(\mathcal{L})$ of matrix models of $\mathcal{L}$, but rather the subclass $\mathrm{Matr}^*(\mathcal{L})$ of $\mathcal{L}$'s reduced matrix models[3]. In general, a matrix for $\mathcal{L}$ can be reduced by simply factoring it with the corresponding Leibniz congruence. This is however a problem in the behavioral case, as typically the behavioral Leibniz congruence is not a congruence over the whole matrix.

4 Algebraic valuations

Valuation semantics appeared in [10] as an effort to provide a semantic ground to logics that, like $\mathcal{C}_1$, lack a meaningful truth-functional semantics. The underlying idea is to drop the condition that formulas should always be interpreted homomorphically in an algebra over the same signature, and instead accept any possible interpretation as a function from the set of formulas of the logic to a set of truth-values equipped with a subset of designated values. Besides lacking a thorough supporting theory, namely if contrasted to the rich theory of logical matrices, valuation semantics has been mostly criticized for its excessive generality, namely as it can be (mis)understood at the light of Suszko's bivalence thesis (see, for instance, [4, 12]). What we propose in this paper, is to adopt a suitable algebraic version of valuation semantics as the natural generalization of logical matrices to the behavioral setting. Namely we will drop the requirement that formulas must always be interpreted homomorphically, but we will still require that a certain regularity is maintained.
Given the signature Σ of a many-sorted logic, recall that a Σ-matrix is a pair $m = \langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a Σ-algebra and $D \subseteq A_\phi$ is the set of designated elements. In the matrix semantics setting formulas are interpreted by means of the homomorphic extension of any assignment h over $\mathbf{A}$. The key idea of the valuation semantics is to drop this assumption. There can be operations that are always interpreted homomorphically, but also some that are not. Below, we will consider fixed a many-sorted signature $\Sigma = \langle S, O \rangle$ and a subsignature Γ of Σ.

Definition 4. A *Γ-valuation* is a triple $\vartheta = \langle \mathbf{A}, D, h \rangle$ such that $\langle \mathbf{A}, D \rangle$ is a Γ-matrix, and h is a sorted function $h : T_\Sigma(X) \to A$ such that $h(f(t_1, \ldots, t_n)) = f_{\mathbf{A}}(h(t_1), \ldots, h(t_n))$ for every $f : s_1 \ldots s_n \to s \in \Gamma$ and $t_i \in T_{\Sigma, s_i}(X)$ with $i \in \{1, \ldots, n\}$. A *Γ-valuation semantics* over Σ is a collection $\mathcal{V}$ of Γ-valuations.

[3] We have borrowed here the terminology used, for instance, in [20]. In the modern terminology of algebraic logic [13], the classes $\mathrm{Matr}(\mathcal{L})$ and $\mathrm{Matr}^*(\mathcal{L})$ are instead denoted by $\mathrm{Mod}(\mathcal{L})$ and $\mathrm{Mod}^*(\mathcal{L})$, respectively. We dropped this terminology here, as our main point is precisely that matrices do not provide the most natural semantical approach in the behavioral algebraic setting.

A Γ-valuation is a matrix over the subsignature Γ of Σ together with a function that satisfies the homomorphism condition for every connective in Γ. In other words, $h : \mathbf{T_\Sigma}(\mathbf{X})|_{\mathbf{\Gamma}} \to \mathbf{A}$ must be an homomorphism between Γ-algebras. In this way we are allowing valuations that do not necessarily satisfy the homomorphism condition with respect to connectives outside Γ. Both the notion of matrix semantics and the original notion of valuation semantics are particular cases of this definition. The former can be obtained by taking $\Gamma = \Sigma$ and requiring that, for each relevant Σ-algebra $\mathbf{A}$, every homomorphism $h : \mathbf{T_\Sigma}(\mathbf{X}) \to \mathbf{A}$ is considered. The latter can be obtained by observing that the original valuation semantics assume fixed sets of truth and designated values. So, by letting $\Gamma = \langle S, \emptyset \rangle$ and requiring that all Γ-valuations share the same algebraic reduct, we have an original valuation semantics.
Expectedly, given a Γ-valuation $\vartheta = \langle \mathbf{A}, D, h \rangle$ and a formula $\varphi \in L_\Sigma(X)$, we say that ϑ *satisfies* φ, denoted by $\vartheta \Vdash \varphi$, if $h(\varphi) \in D$. As usual, given $\Phi \subseteq L_\Sigma(X)$, we write $\vartheta \Vdash \Phi$ whenever $\vartheta \Vdash \varphi$ for every $\varphi \in \Phi$. Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic over Σ, not necessarily structural. The Γ-valuation ϑ is said to be a *Γ-model of $\mathcal{L}$* when it happens that if $\vartheta \Vdash \Phi$ and $\Phi \vdash_\mathcal{L} \varphi$ then $\vartheta \Vdash \varphi$. In this case, D is called a *Γ-deductive filter for $\mathcal{L}$*, or just a *$\mathcal{L}$-Γ-filter*, over $\mathbf{A}$. The set of all Γ-models of $\mathcal{L}$ will be denoted by $\mathrm{Val}_\Gamma(\mathcal{L})$. Given a Γ-valuation semantics $\mathcal{V} = \{\langle \mathbf{A}_i, D_i, h_i \rangle : i \in I\}$ over Σ, we define the consequence relation associated with $\mathcal{V}$, $\vdash_\mathcal{V} \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$, by letting $\Phi \vdash_\mathcal{V} \varphi$ if for every Γ-valuation $\vartheta \in \mathcal{V}$ we have that $\vartheta \Vdash \varphi$ whenever $\vartheta \Vdash \Phi$. A Γ-valuation semantics $\mathcal{V}$ is *sound* for $\mathcal{L}$ if $\vdash_\mathcal{L} \subseteq \vdash_\mathcal{V}$. Symmetrically, $\mathcal{V}$ is *adequate* for $\mathcal{L}$ if $\vdash_\mathcal{V} \subseteq \vdash_\mathcal{L}$. The Γ-valuation semantics $\mathcal{V}$ is *complete* for $\mathcal{L}$ if it is both sound and adequate, that is $\vdash_\mathcal{L} = \vdash_\mathcal{V}$.
One can also bring the usual Lindenbaum-Tarski constructions to the setting of valuations. For each set $\Phi \subseteq L_\Sigma(X)$ of formulas, we can define the Γ-valuation $\vartheta_\Gamma^\Phi = \langle \mathbf{L_\Sigma}(\mathbf{X})|_{\mathbf{\Gamma}}, \Phi, id \rangle$ where $id : L_\Sigma(X) \to L_\Sigma(X)$ is the identity function. The Γ-valuations of the form ϑ_Γ^Φ are dubbed *Lindenbaum Γ-valuations* for Σ. The family $\mathcal{V}_\Gamma(\mathcal{L}) = \{\vartheta_\Gamma^\Phi : \Phi \text{ is a theory of } \mathcal{L}\}$ is called the *Lindenbaum Γ-bundle* of $\mathcal{L}$.

Proposition 1. *For every many-sorted logic $\mathcal{L}$,*
- *$\mathcal{L}$ is complete with respect to its Lindenbaum Γ-bundle;*
- *$\mathcal{L}$ is complete with respect to the class of its Γ-models.*

Proof. Clearly, $\mathcal{V}_\Gamma(\mathcal{L}) \subseteq \mathrm{Val}_\Gamma(\mathcal{L})$. To see that $\mathcal{V}_\Gamma(\mathcal{L})$ is an adequate Γ-valuation semantics for $\mathcal{L}$, just suppose that $\Phi \nvdash_\mathcal{L} \varphi$ for some $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$. Then $\vartheta_\Gamma^{\Phi^{\vdash_\mathcal{L}}} \Vdash \Phi$ but $\vartheta_\Gamma^{\Phi^{\vdash}} \nVdash \varphi$, and hence $\Phi \nvdash_{\mathcal{V}_\Gamma(\mathcal{L})} \varphi$. As a consequence, also $\mathrm{Val}_\Gamma(\mathcal{L})$ is adequate for $\mathcal{L}$. □

As the class of all matrix models of $\mathcal{L}$ in the usual approach, the class $\mathrm{Val}_\Gamma(\mathcal{L})$ is very important since it captures algebraically some of the metalogical properties of $\mathcal{L}$. As we will show below, when a logic is behaviorally algebraizable, we are able to algebraically specify not only the class of algebras associated with the logic, but also the admissible ways that formulas can be interpreted in these algebras, as the valuations are now incorporated in the algebraic models. Note that it is precisely the extended signature Σ^o that gives the algebraic handle that allows us to specify these. There are, however, other desirable properties that a valuation semantics might enjoy. One semantical property which is very characteristic of the algebraic setting, and which holds for a matrix semantics, is that we can consider all possible assignments over a given algebra. A Γ-valuation semantics $\mathcal{V}$ over Σ is said to be *Laplacian* if whenever $\vartheta = \langle \mathbf{A}, D, h \rangle \in \mathcal{V}$ then for every assignment ρ over $\mathbf{A}$ there exists a Γ-valuation $\vartheta_\rho = \langle \mathbf{A}, D, h' \rangle \in \mathcal{V}$ such that $h'|_X = \rho$. Another typical property of a matrix semantics is *representativity.* A Γ-valuation semantics $\mathcal{V}$ over Σ is is said to be *representative* if $\vartheta = \langle \mathbf{A}, D, h \rangle \in \mathcal{V}$ implies that $\vartheta \circ \sigma = \langle \mathbf{A}, D, h \circ \sigma \rangle \in \mathcal{V}$ for every substitution σ. This last property is well-known to be closely connected with structurality [20].

Theorem 1. *Let $\mathcal{L}$ be a many-sorted logic over signature Σ, not necessarily structural, and Γ a subsignature of Σ. Then, $\mathcal{L}$ is structural iff the class $Val_\Gamma(\mathcal{L})$ is representative.*

Proof. Suppose that $\mathcal{L}$ is structural, let $\vartheta \in \mathrm{Val}_\Gamma(\mathcal{L})$ and take any substitution σ. Assume that $\Phi \vdash_\mathcal{L} \varphi$ and $\vartheta \circ \sigma \Vdash \Phi$. Clearly, this is equivalent to having $\vartheta \Vdash \sigma[\Phi]$. But, by structurality, it is also the case that $\sigma[\Phi] \vdash_\mathcal{L} \sigma(\varphi)$ and, as $\vartheta \in \mathrm{Val}_\Gamma(\mathcal{L})$, it follows that $\vartheta \Vdash \sigma(\varphi)$. Equivalently, then, $\vartheta \circ \sigma \Vdash \varphi$, and hence $\vartheta \circ \sigma \in \mathrm{Val}_\Gamma(\mathcal{L})$, and $\mathrm{Val}_\Gamma(\mathcal{L})$ is representative. To prove the converse implication, given that according to Proposition 1 $\mathcal{L}$ is complete with respect to $\mathrm{Val}_\Gamma(\mathcal{L})$, it suffices to show that the consequence associated with an arbitrary representative valuation semantics $\mathcal{V}$ is necessarily structural. Assume that $\Phi \vdash_\mathcal{V} \varphi$ and take any substitution σ. Given $\vartheta \in \mathcal{V}$, if $\vartheta \Vdash \sigma[\Phi]$ then, equivalently, $\vartheta \circ \sigma \Vdash \Phi$. But we know that $\vartheta \circ \sigma \in \mathcal{V}$, and therefore $\vartheta \circ \sigma \Vdash \varphi$, or equivalently, $\vartheta \Vdash \sigma(\varphi)$. Hence, $\sigma[\Phi] \vdash_\mathcal{V} \sigma(\varphi)$ and $\vdash_\mathcal{V}$ is structural. □

To see that some further important results of the fruitful theory of logical matrices generalize to valuation semantics, we will end this section by presenting an example of such a result, namely an adaptation of Bloom's theorem [3] that characterizes finitary logics in our setting. We start by introducing some necessary extended operations on valuations. Given a set $\Lambda = \{\vartheta_i = \langle \mathbf{A}_i, D_i, h_i \rangle : i \in I\}$ of Γ-valuations the *direct product of Λ* is the Γ-valuation $\Pi_{i \in I} \vartheta_i = \langle \Pi_{i \in I} \mathbf{A}_i, \Pi_{i \in I} D_i, (h_i(\_))_{i \in I} \rangle$. Recall that, given a set I, an *ultrafilter on I* is a set $\mathcal{U}$ consisting of subsets of I such that: 1) $\emptyset \notin \mathcal{U}$; 2) if $A \in \mathcal{U}$ and $A \subseteq B$ then $B \in \mathcal{U}$; 3) if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$; 4) if $A \subseteq I$, then either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$. Note that, axioms 1) and 3), imply that A and $I \setminus A$ cannot both be elements of $\mathcal{U}$. Given $\Lambda = \{\vartheta_i : i \in I\}$ and an ultrafilter $\mathcal{U}$ on I we can define a (sorted) equivalence relation $\sim_\mathcal{U}$ on the direct product $\Pi_{i \in I} \vartheta_i$ as follows: $a \sim_\mathcal{U} b$ iff $\{i \in I : a_i = b_i\} \in \mathcal{U}$. The *ultraproduct* of the Γ-valuations Λ modulo an ultrafilter $\mathcal{U}$, denoted by $\Pi_\mathcal{U} \vartheta_i$, is the quotient of $\Pi_{i \in I} \vartheta_i$ by the equivalence $\sim_\mathcal{U}$ (that is indeed a congruence of the underlying Γ-algebra). Concretely, let $\Pi_\mathcal{U} \vartheta_i = \langle (\Pi_{i \in I} \mathbf{A}_i)/\mathcal{U}, [\{(a_i)_{i \in I} \in \Pi_{i \in I} A_{i,\phi} : \{i \in I : a_i \in D_i\} \in \mathcal{U}\}]_\mathcal{U}, [(h_i(\_))_{i \in I}]_\mathcal{U} \rangle$.

Theorem 2. *Let $\mathcal{L}$ be a many-sorted logic over signature Σ, and Γ a subsignature of Σ. Then, $\mathcal{L}$ is finitary iff the class $Val_\Gamma(\mathcal{L})$ is closed under ultraproducts.*

Proof. Suppose first that $\mathcal{L}$ is finitary. Let $\{\vartheta_i : i \in I\} \subseteq \text{Val}_\Gamma(\mathcal{L})$ be a family of Γ-models of $\mathcal{L}$ and $\mathcal{U}$ an ultrafilter on I. We aim to prove that $\Pi_\mathcal{U} v_i \in \text{Val}_\Gamma(\mathcal{L})$. So, suppose that $\Phi \vdash_\mathcal{L} \varphi$ and that $\Pi_\mathcal{U}\vartheta_i \Vdash \Phi$. Since $\mathcal{L}$ is finitary, there must exist $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Phi$ such that $\varphi_1, \ldots, \varphi_n \vdash_\mathcal{L} \varphi$. For each $1 \leq j \leq n$, we have that $\Pi_\mathcal{U}\vartheta_i \Vdash \varphi_j$, and thus $I_j = \{i \in I : \vartheta_i \Vdash \varphi_j\} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter we have that $I_1 \cap \ldots \cap I_n = \{i \in I : \vartheta_i \Vdash \{\varphi_1, \ldots, \varphi_n\}\} \in \mathcal{U}$. Note also that, since each ϑ_i is a Γ-model of $\mathcal{L}$, $I_1 \cap \ldots \cap I_n \subseteq \{i \in I : \vartheta_i \Vdash \varphi\}$. Since $\mathcal{U}$ is an ultrafilter we have that $\{i \in I : \vartheta_i \Vdash \varphi\} \in \mathcal{U}$ and so $\Pi_\mathcal{U}\vartheta_i \Vdash \varphi$.
Suppose now that $\text{Val}_\Gamma(\mathcal{L})$ is closed under ultraproducts. To prove that $\mathcal{L}$ is finitary let Φ be infinite and assume that $\Phi' \nvdash_\mathcal{L} \psi$, for every finite $\Phi' \subseteq \Phi$. Let I denote the set of all finite subsets of Φ. For each $i \in I$, define $i^* = \{j \in I : i \subseteq j\}$. Using well-known results on ultrafilters [20] we can conclude that there exists an ultrafilter $\mathcal{U}$ over I that contains the family $\{i^* : i \in I\}$. For every $i \in I$, consider the theory $i^{\vdash_\mathcal{L}}$ and let $\vartheta_i = \vartheta_\Gamma^{i^{\vdash_\mathcal{L}}} \in \text{Val}_\Gamma(\mathcal{L})$ be the corresponding Lindenbaum Γ-valuation. Let $\Pi_\mathcal{U}\vartheta_i$ be the ultraproduct of the family by the ultrafilter $\mathcal{U}$. Then, for every $\varphi \in \Phi$ we have that $\{\varphi\}^* \subseteq \{i \in I : \vartheta_i \Vdash \varphi\}$. So, $\{i \in I : \vartheta_i \Vdash \varphi\} \in \mathcal{U}$ for every $\varphi \in \Phi$, and consequently we have that $\Pi_\mathcal{U}\vartheta_i \Vdash \Phi$. But $\{i \in I : \vartheta_i \Vdash \psi\} = \emptyset$ and so $\Pi_\mathcal{U}\vartheta_i \nVdash \psi$. Since $\Pi_\mathcal{U}\vartheta_i \in \text{Val}_\Gamma(\mathcal{L})$ we have that $\Phi \nvdash \varphi$, and we can conclude that $\mathcal{L}$ is finitary. □

The class of models canonically associated with a logic $\mathcal{L}$ is typically not the whole of $\text{Matr}(\mathcal{L})$, but rather the subclass $\text{Matr}^*(\mathcal{L})$ of reduced matrices. In the behavioral setting, we can define an analogous class of reduced Γ-valuation models, by setting $\text{Val}^*_\Gamma(\mathcal{L}) = \{\langle \mathbf{A}, D, h\rangle \in \text{Val}_\Gamma(\mathcal{L}) : \mathbf{\Omega}^\mathbf{A}_\mathbf{\Gamma}(D) \text{ is the identity}\}$. Expectedly, given $\Phi \subseteq L_\Sigma(X)$, we can also define the Γ-valuation $\vartheta_\Gamma^{*\Phi} = \langle (\mathbf{L_\Sigma(X)}|_\mathbf{\Gamma})/\mathbf{\Omega_\Gamma}(\Phi), [\Phi]_{\mathbf{\Omega_\Gamma}(\Phi)}, [\_]_{\mathbf{\Omega_\Gamma}(\Phi)}\rangle \in \text{Val}^*_\Gamma(\mathcal{L})$. The Γ-valuations of this form are dubbed *reduced Lindenbaum Γ-valuations* for Σ. The family $\mathcal{V}^*_\Gamma(\mathcal{L}) = \{\vartheta_\Gamma^{*\Phi} : \Phi \text{ is a theory of } \mathcal{L}\}$ is called the *reduced Lindenbaum Γ-bundle* of $\mathcal{L}$.

Proposition 2. *For every many-sorted logic $\mathcal{L}$,*
- *$\mathcal{L}$ is complete with respect to its reduced Lindenbaum Γ-bundle;*
- *$\mathcal{L}$ is complete with respect to the class of its reduced Γ-models.*

Proof. Noting that $\mathcal{V}^*_\Gamma(\mathcal{L}) \subseteq \text{Val}^*_\Gamma(\mathcal{L})$, the results follows easily from Proposition 1, once we observe that, for every theory Φ of $\mathcal{L}$ and every $\varphi \in L_\Sigma(X)$, we have that $\vartheta_\Gamma^{*\Phi} \Vdash \varphi$ iff $\vartheta_\Gamma^\Phi \Vdash \varphi$. This equivalence follows easily from the fact that $\mathbf{\Omega_\Gamma}(\Phi)$ is compatible with Φ. □

5 Some bridge results

In this section we aim at giving a first step into studying the connection between algebraic valuation semantics and behavioral algebraization. We consider fixed a Γ-behaviorally algebraizable many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$ with behaviorally equivalent algebraic semantics $\mathbb{K}$ and defining equations Θ. One important consequence of assuming that $\mathcal{L}$ is Γ-behaviorally algebraizable is that given a Σ^o-algebra $\mathbf{A} \in \mathbb{K}$, and by setting $D_{\mathbf{A}} = \{a \in A_\phi : \delta^i_{\mathbf{A}}(a) \equiv_\Gamma \epsilon^i_{\mathbf{A}}(a) \text{ for every } i \in I\}$, we can recover a filter such that $\langle \mathbf{A}|_\Sigma, D_{\mathbf{A}}\rangle \in \mathrm{Matr}(\mathcal{L})$. However, as we have discussed, the behavioral Leibniz congruence cannot help us to obtain a corresponding reduced matrix. Still, if we consider instead an assignment h on this matrix and take the corresponding Γ-valuation $\vartheta_{\mathbf{A},h} = \langle \mathbf{A}|_\Gamma, D_{\mathbf{A}}, h\rangle \in \mathrm{Val}(\mathcal{L})$, we can now obtain the reduced Γ-valuation $\vartheta^*_{\mathbf{A},h} \in \mathrm{Val}^*(\mathcal{L})$. We need just quotient $\vartheta_{\mathbf{A},h}$ by the behavioral Leibniz congruence $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\mathbf{A}|_\Gamma}_{\boldsymbol{\Gamma}}(D_{\mathbf{A}})$, that is, $\vartheta^*_{\mathbf{A},h} = \langle (\mathbf{A}|_\Gamma)/\boldsymbol{\Omega}, [D_{\mathbf{A}}]_{\boldsymbol{\Omega}}, [\_]_{\boldsymbol{\Omega}} \circ h\rangle$. We define $\mathcal{V}_{\mathbb{K}} = \{\vartheta_{\mathbf{A},h} : \mathbf{A} \in \mathbb{K} \text{ and } h \text{ is an assignment over } \mathbf{A}\}$, and $\mathcal{V}^*_{\mathbb{K}} = \{\vartheta^*_{\mathbf{A},h} : \vartheta_{\mathbf{A},h} \in \mathcal{V}_{\mathbb{K}}\}$. We can now establish a sequence of results relating the behavioral consequence associated with $\mathbb{K}$ and the corresponding valuation semantics. The results generalize well-known bridge results linking matrix semantics with traditional algebraization [13].

Proposition 3. *Given $\Phi \cup \{\psi\} \subseteq L_\Sigma(X)$, we have that*

$$\Phi \vdash_{\mathcal{V}^*_{\mathbb{K}}} \varphi \;\; \textit{iff} \;\; \Phi \vdash_{\mathcal{V}_{\mathbb{K}}} \varphi \;\; \textit{iff} \;\; \Theta[\Phi] \models^{\Gamma}_{\mathbb{K}} \Theta(\varphi).$$

Proof. The property follows straightforwardly from the fact that, given an algebra $\mathbf{A} \in \mathbb{K}$, an assignment h over $\mathbf{A}$ and a formula $\varphi \in L_\Sigma(X)$, we have that $\vartheta^*_{\mathbf{A},h} \Vdash \varphi$ iff $\vartheta_{\mathbf{A},h} \Vdash \varphi$ iff $\mathbf{A}, h \Vvdash^{\Gamma} \Theta(\varphi)$. Proving this equivalence is an easy exercise and its proof will be omitted. □

Corollary 1. *If $\mathcal{L} = \langle \Sigma, \vdash\rangle$ is a Γ-behaviorally algebraizable logic with behaviorally equivalent algebraic semantics $\mathbb{K}$ then,*
- *$\mathcal{L}$ is complete with respect to the class $\mathcal{V}^*_{\mathbb{K}}$;*
- *$\mathcal{L}$ is complete with respect to the class $\mathcal{V}_{\mathbb{K}}$.*

We can further show that $\mathcal{V}^*_{\mathbb{K}}$ enjoys other properties that are typical of matrix semantics.

Proposition 4. *If $\mathcal{L} = \langle \Sigma, \vdash\rangle$ is a Γ-behaviorally algebraizable logic with behaviorally equivalent algebraic semantics $\mathbb{K}$ then $\mathcal{V}^*_{\mathbb{K}}$ is both Laplacian and representative.*

Proof. Suppose that $\vartheta = \langle \mathbf{A}, D, h\rangle \in \mathcal{V}_{\mathbb{K}}$. Then there exists $\mathbf{B} \in \mathbb{K}$ and an assignment h' over $\mathbf{B}$ such that $\vartheta = \vartheta_{\mathbf{B},h'}$.
Consider given an assignment ρ over $\mathbf{A}$. Recall that by construction $\mathbf{A}$ results of a quotient construction from $\mathbf{B}|_\Gamma$. Therefore, it is always possible to choose an assignment h^ρ over $\mathbf{B}$ such that, for every $s \in S$ and every $x \in X_s$, we have that $h^\rho_s(x) \in \rho_s(x)$. We can then conclude that $\vartheta_{\mathbf{B},h^\rho} = \langle \mathbf{A}, D, h^\rho\rangle \in \mathcal{V}^*_{\mathbb{K}}$ and $h^\rho|_X = \rho$. Hence, $\mathcal{V}^*_{\mathbb{K}}$ is Laplacian.
To show that $\mathcal{V}^*_{\mathbb{K}}$ is representative, take any substitution σ. Clearly, $h' \circ \sigma$ is also an assignment over $\mathbf{B}$. Hence, $\vartheta_{\mathbf{B},h'\circ\sigma} = \vartheta_{\mathbf{B},h} \circ \sigma \in \mathcal{V}^*_{\mathbb{K}}$. □

In [7] the authors prove that when a logic is Γ-behaviorally algebraizable we can obtain a behavioral equational specification (even if infinite) of its equivalent algebraic semantics $\mathbb{K}$. Therefore, by construction, we can also obtain a specification of the complete Γ-valuation semantics $\mathcal{V}^*_{\mathbb{K}}$. Moreover, when the logic is finitary and finitely Γ-behaviorally algebraizable the specification mentioned above is also finite.

Example 2. As shown in [7, 6], the complete reduced valuation semantics for da Costa's paraconsistent logic $\mathcal{C}_1$ resulting from its behavioral algebraization is the class $\mathcal{V}^*_{\mathbb{K}_{\mathcal{C}_1}}$ of valuations $\langle \mathbf{A}, D, h\rangle$ such that $\mathbf{A}$ is a Boolean algebra, $D = \{\top_{\mathbf{A}}\}$, and $h : L_{\mathcal{C}_1}(X) \to A$ satisfies:

- $h(\boldsymbol{t}) = \top_{\mathbf{A}}$ *and* $h(\boldsymbol{f}) = \bot_{\mathbf{A}}$;
- $h(x \wedge y) = h(x) \sqcap_{\mathbf{A}} h(y)$;
- $h(x \vee y) = h(x) \sqcup_{\mathbf{A}} h(y)$;
- $h(x \supset y) = h(x) \Rightarrow_{\mathbf{A}} h(y)$;
- $-_{\mathbf{A}} h(x) \leq_{\mathbf{A}} h(\neg x)$;
- $h(\neg\neg x) \leq_{\mathbf{A}} h(x)$;
- $h(x^\circ) \sqcap_{\mathbf{A}} h(x) \sqcap_{\mathbf{A}} h(\neg x) = \bot_{\mathbf{A}}$;
- $h(x^\circ) \sqcap_{\mathbf{A}} h(y^\circ) \leq_{\mathbf{A}} h((x \wedge y)^\circ)$;
- $h(x^\circ) \sqcap_{\mathbf{A}} h(y^\circ) \leq_{\mathbf{A}} h((x \vee y)^\circ)$;
- $h(x^\circ) \sqcap_{\mathbf{A}} h(y^\circ) \leq_{\mathbf{A}} h((x \supset y)^\circ)$.

The well-known bivaluation semantics of $\mathcal{C}_1$ from [9] can be easily recovered from these valuations by looking at the underlying irreducible Boolean algebras, as explained in [6].

6 Conclusion

We have introduced an algebraic based notion of valuation semantics that arose naturally in semantical considerations in the novel behavioral approach to the algebraization of logics. We have started to pave the way towards the development of a consistent algebraic theory of valuations, that may mirror in the behavioral setting the role played by matrix semantics in the traditional approach to algebraic logic. The results and characterizations obtained are promising. Of course, this paper raises more questions than it gives answers, but the path seems to be clear despite the difficulties raised by the fact that the model-theory of behavioral equational logic remains relatively unexplored. We refer the reader to [17] for a glimpse of what lays ahead. In any case, the references [2, 13, 20], among others, will provide the essential guidelines for future work. The experience of da Costa's logic $\mathcal{C}_1$ and the recovery of its non-truth-functional bivaluation semantics from the Boolean-based algebraic valuations obtained by the behavioral algebraization process also suggest that a systematic study of the Birkhoff-like operations over valuations is crucial. A related direction in the development of the behavioral theory of algebraization that will need attention is related with Suszko's reduction. Also alternatives to valuation semantics, as proposed here, should be carefully inspected. In particular, we would like to have characterization results for the classes of valuations resulting from non-deterministic matrices [1], both in the static and dynamic versions, or gaggles [11], as well as bridges to the classes of logics that they characterize. We hope to report on these and related questions in forthcoming papers.

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