Towards a behavioral algebraic theory of logical valuations[*]

Carlos Caleiro

SQIG-Instituto de Telecomunicações and

Dept. Mathematics, IST, TU Lisbon

Portugal

Ricardo Gonçalves[†]

CENTRIA, FCT-UNL and

SQIG-Instituto de Telecomunicações

Portugal

Abstract. Logical matrices are widely accepted as the semantic structures that most naturally fit the traditional approach to algebraic logic. The behavioral approach to the algebraization of logics extends the applicability of the traditional methods of algebraic logic to a wider range of logical systems, possibly encompassing many-sorted languages and non-truth-functional phenomena. However, as one needs to work with behavioral congruences, matrix semantics are unsuited to the behavioral setting. In [5], a promising version of algebraic valuation semantics was proposed in order to fill in this gap. Herein, we define the class of valuations that should be canonically associated to a logic, and we show, by means of new meaningful bridge results, how it is related to the behaviorally equivalent algebraic semantics of a behaviorally algebraizable logic.

Keywords: algebraic logic, logical matrix, behavioral algebraization, valuation semantics.

1. Introduction

The behavioral approach to the algebraization of logics was introduced in [7] with the aim of extending the range of applicability of the traditional tools of algebraic logic to logics with a many-sorted syntax, or including non-truth-functional connectives, and which are not algebraizable with the usual approach.

The key idea underlying the behavioral approach to algebraic logic is to shift one's attention from the notion of congruence that is central to the traditional algebraization process, to the weaker notion of behavioral equivalence. Behavioral equivalence has its roots in computer science, namely in the field of algebraic specifications of data-types, where it is often necessary to reason about data which cannot be directly accessed [16]. In such a situation, it is perfectly possible that one cannot distinguish between two different values if those values provide exactly the same results for all available ways of observing and experimenting with them. Hence, unsorted equational logic is replaced by many-sorted behavioral equational logic (sometimes called hidden equational logic) based on the notion of behavioral equivalence, given a set of available experiments. Behavioral reasoning in equational logic has been consistently developed, see for instance [14, 19].

Logical matrices [15] are widely accepted as the semantic structures that naturally fit the traditional approach to algebraic logic [13]. It is well-known that every structural logic is fully characterized by the class of its matrix models, or even better by the class of its reduced matrix models [20]. In the case of a logic algebraizable according to the traditional methods, one even gets an equational specification of the algebras underlying these matrix models, a neat characterization of matrix congruences by means of the Leibniz operator, and a way of recovering the corresponding matrix filters by using the defining equations of the algebraization [2, 13]. However, as a consequence of the behavioral generalization of the process of algebraizing logics, one finds that the fundamental notion of matrix semantics is no longer adequate. Namely due to the additional freedom provided by the notion of behavioral equivalence, the behavioral version of the Leibniz operator is in general not a congruence over the whole language of the logic. Moreover, as expected in the case of logics that are not algebraizable under the usual approach (but which may be behaviorally algebraizable), the connection between the logic and its matrix semantics may be weak and uninteresting. A paradigmatic example of this situation can be found in da Costa's system of paraconsistent logic $\mathcal{C}_1$ [8]. In fact, $\mathcal{C}_1$ is well-known not to be algebraizable using traditional means, and additionally all its Lindenbaum matrices are reduced. Still, the logic $\mathcal{C}_1$ is behaviorally algebraizable, and its resulting behaviorally equivalent algebraic semantics is quite interesting [6]. Namely, with little effort, it allows us to recover the non-truth-functional bivaluation semantics of [9].

Logical valuations as a general semantic tool were proposed in [10] precisely with the aim of providing a semantic ground for logics that, like $\mathcal{C}_1$, lack a meaningful truth-functional semantics. The key idea is, in the extreme case, to drop the condition that formulas should always be interpreted homomorphically in an algebra over the same signature. Besides lacking a thorough study, namely if contrasted to the myriad of interesting and valuable algebraic theory underlying logical matrices (see [20]), valuation semantics has been criticized for its excessive generality (see, for instance, [12]). Still, every logician would agree that a matrix semantics is simply a clever and algebraically well-behaved way of defining a valuation semantics by simply collecting all possible homomorphic interpretations. A promising algebraically well-behaved version of valuation semantics was proposed in [5] as the natural generalization of logical matrices to the behavioral setting. Namely, it drops the requirement that formulas be interpreted homomorphically, while still requiring that any exceptions must have an algebraic specification.

In this paper we review the nice properties of algebraic valuations, paralleling them with the properties of logical matrices. As our main contribution, we propose a fine-tuned version of the process leading to the envisaged semantic counterpart of behaviorally algebraizable logics that allows us to obtain new meaningful bridge results, namely generalizing to the behavioral setting the role played by reduced matrix models in the traditional approach, thus opening the way to the development of a behavioral algebraic

theory of logical valuations. The paper is organized as follows. In Section 2, we will recall the essential ingredients of the behavioral approach to the algebraization of logics, including those of a behaviorally protoalgebraic logic, a behaviorally algebraizable logic, and the behavioral Leibniz operator, along with a few relevant characterizations. We will also introduce for the first time the behavioral Suszko operator and prove some of its properties. In Section 3, we review the notion of valuation semantics proposed in [5] and some of its properties, and introduce the class of reduced valuations that should be canonically associated with a logic. Section 4 is dedicated to establishing a few bridge results that parallel for valuations, in the behavioral setting, well known bridging results between logical matrices and traditional algebraization. Finally, in Section 5, we draw some conclusions and point to some topics of future work.

2. Behavioral algebraization of logics

We will focus our attention on a wide class of logics: those whose language can be built from a rich many-sorted signature. Below, we first recall the necessary notions of universal algebra. We will also recall behavioral equational reasoning, contrasting it with traditional equational reasoning. Along the way we also fix some notation and terminology. Then, we will recall from [7] the necessary elements of the behavioral approach to algebraizing logics. We will also prove a few new characterization results.

2.1. Algebraic preliminaries

A *(many-sorted) signature* is a pair $\Sigma = \langle S, F \rangle$ where S is a set (of *sorts*) and $F = \{F_{ws}\}_{w \in S^*, s \in S}$ is an indexed family of sets (of *operations*). For simplicity, we write $f : s_1 \ldots s_n \to s \in F$ for an element $f \in F_{s_1 \ldots s_n s}$. As usual, we denote by $T_\Sigma(X) = \{T_{\Sigma,s}(X)\}_{s \in S}$ the S-sorted family of carrier sets of the free Σ-algebra $\mathbf{T}_\Sigma(\mathbf{X})$ with generators taken from a sorted family $X = \{X_s\}_{s \in S}$ of variable sets. We will denote by $x : s$ the fact that $x \in X_s$. Often, we will need to write terms $t \in T_\Sigma(Y)$ over a given subset of variables $Y \subseteq X$. For simplicity, we will denote such a term by $t(Y)$, or even by $t(x_1 : s_1, \ldots, x_n : s_n)$ when $Y = \{x_1 : s_1, \ldots, x_n : s_n\}$. Moreover, if T is a set whose elements are all terms of this form, we will write $T(Y)$. A *substitution* over Σ is a S-sorted family of functions $\sigma = \{\sigma_s : X_s \to T_{\Sigma,s}(X)\}_{s \in S}$. As usual, $\sigma(t)$ denotes the term obtained by uniformly applying σ to each variable in t. Given $t(Y)$ and $\overline{u} = \langle u_i \in T_{\Sigma,s_i}(X) \rangle_{x_i : s_i \in Y}$, we will write $t(\overline{u})$ to denote the term $\sigma(t)$ where σ is a substitution such that $\sigma_{s_i}(x_i) = u_i$ for each $x_i : s_i \in Y$. Extending everything to sets, given $T(Y)$ and $U \subseteq \prod_{x_i : s_i \in Y} T_{\Sigma,s_i}(X)$, we will use $T[U] = \bigcup_{\overline{u} \in U} T(\overline{u})$. A *derived operation* of type $s_1 \ldots s_n \to s$ over Σ is simply a term in $T_{\Sigma,s}(x_1 : s_1, \ldots, x_n : s_n)$. For $w \in S^*$, we denote by $Der_{\Sigma,ws}$ the set of all derived operations of type $w \to s$ over Σ. A *(full) subsignature* of Σ is a signature $\Gamma = \langle S, F' \rangle$ such that, for each $w \in S^*$ and $s \in S$, $F'_{ws} \subseteq Der_{\Sigma,ws}$.

Given a signature $\Sigma = \langle S, F \rangle$, a Σ-*algebra* is a pair $\mathbf{A} = \langle \{A_s\}_{s \in S}, {}_{-\mathbf{A}} \rangle$, where each A_s is a non-empty set, the *carrier of sort* s, and ${}_{-\mathbf{A}}$ assigns to each operation $f : s_1 \ldots s_n \to s$ a function $f_\mathbf{A} : A_{s_1} \times \ldots \times A_{s_n} \to A_s$. An assignment over $\mathbf{A}$ is a S-sorted family of functions $h = \{h_s : X_s \to A_s\}_{s \in S}$. As usual, we will often overload h and use it to denote also the unique extension of the assignment to an homomorphism $h : T_\Sigma(X) \to \mathbf{A}$. Given a Σ-algebra $\mathbf{A}$, a term $t(x_1 : s_1, \ldots, x_n : s_n)$ and $\langle a_1, \ldots, a_n \rangle \in A_{s_1} \times \ldots \times A_{s_n}$, then we denote by $t_\mathbf{A}(a_1, \ldots, a_n)$ the value $h(t)$ that t takes in $\mathbf{A}$ under an assignment h such that $h(x_1) = a_1, \ldots, h(x_n) = a_n$. When $\mathbf{A}$ is a Σ-algebra and Γ a subsignature of Σ, we denote by $\mathbf{A}|_\Gamma$ the Γ-algebra obtained by forgetting the interpretation of all operations not in Γ.

We will use $t \approx u$ to represent an equation between terms $t, u \in T_{\Sigma,s}(X)$ of the same sort s, in which case we dub it an s-equation. The S-sorted set of all Σ-equations will be written as Eq_Σ. We will denote quasi-equations by $(t_1 \approx u_1) \& \ldots \& (t_n \approx u_n) \to (t \approx u)$. A set Θ of equations with variables in $\{x_1 : s_1, \ldots, x_n : s_n\}$ will be dubbed $\Theta(x_1 : s_1, \ldots, x_n : s_n)$. As usual, we say that an assignment h over $\mathbf{A}$ *satisfies* the equation $t \approx u$, in symbols $\mathbf{A}, h \Vdash t \approx u$ if $h(t) = h(u)$. We say that $\mathbf{A}$ satisfies $t \approx u$, in symbols $\mathbf{A} \Vdash t \approx u$, if $\mathbf{A}, h \Vdash t \approx u$ for every assignment h over $\mathbf{A}$. Given a class $\mathbb{K}$ of Σ-algebras, the *equational consequence over Σ associated with* $\mathbb{K}$, $\vDash_\mathbb{K} \subseteq \mathcal{P}(Eq_\Sigma) \times Eq_\Sigma$, is such that $\Theta \vDash_\mathbb{K} t \approx u$ if for every $\mathbf{A} \in \mathbb{K}$ and assignment h over $\mathbf{A}$ we have that $\mathbf{A}, h \Vdash t \approx u$ whenever $\mathbf{A}, h \Vdash \Theta$. Moreover, we say that $\mathbf{A}$ satisfies a quasi-equation $(t_1 \approx u_1) \& \ldots \& (t_n \approx u_n) \to (t \approx u)$, denoted by $\mathbf{A} \Vdash (t_1 \approx u_1)$ $\& \ldots \& (t_n \approx u_n) \to (t \approx u)$, whenever $\{t_1 \approx u_1, \ldots, t_n \approx u_n\} \vDash_{\{\mathbf{A}\}} t \approx u$.

As mentioned above, the key ingredient of the behavioral approach to algebraizing logics is to use *behavioral equational logic* in the role usually played by plain *equational logic*. The distinctive feature of behavioral equational logic is the fact that the sorts are split in two disjoint sets, of *visible* and *hidden* sorts, and only certain operations of visible sort are allowed as *experiments*. In the visible sorts we can perform simple equational reasoning, but we can only reason indirectly about hidden sorts, using *behavioral indistinguishability* under the available experiments. Intuitively, we must evaluate equations involving hidden values using only their visible properties. It may happen that under all available experiments two certain hidden terms always coincide, which makes them behaviorally equivalent, even though they might actually have distinct values. We will now put forward the rigorous definitions, contrasting them with the ones for plain equational logic. A *hidden (many-sorted) signature* is a tuple $\langle \Sigma, V, \mathcal{E} \rangle$ where $\Sigma = \langle S, F \rangle$ is a many sorted-signature, $V \subseteq S$ is the set of visible sorts, and $\mathcal{E}$ is the set of available *experiments*, that is, a set terms of visible sort of the form $t(x : s, x_1 : s_1, \ldots, x_n : s_n)$ where x is a distinguished variable of hidden sort $s \in H = S \setminus V$.

Definition 2.1. Consider a hidden signature $\langle \Sigma, V, \mathcal{E} \rangle$ and a Σ-algebra $\mathbf{A}$. Given a hidden sort $s \in H$, two values $a, b \in A_s$ *are $\mathcal{E}$-behaviorally equivalent*, in symbols $a \equiv_\mathcal{E} b$, if for every experiment $t(x : s, x_1 : s_1, \ldots, x_n : s_n) \in \mathcal{E}$ and every $\langle c_1, \ldots, c_n \rangle \in A_{s_1} \times \ldots \times A_{s_n}$, we have that

$$t_\mathbf{A}(a, c_1, \ldots, c_n) = t_\mathbf{A}(b, c_1, \ldots, c_n).$$

Now that we have defined behavioral equivalence, we can talk about behavioral satisfaction of an equation by a Σ-algebra $\mathbf{A}$. We say that an assignment h over $\mathbf{A}$ $\mathcal{E}$-*behaviorally satisfies* an equation $t \approx u$ of hidden sort $s \in H$, in symbols $\mathbf{A}, h \Vdash^\mathcal{E} t \approx u$ if $h(t) \equiv_\mathcal{E} h(u)$. Expectedly, equations of visible sort are satisfied as usual, that is, if $t \approx u$ is an equation of sort $s \in V$ then we write $\mathbf{A}, h \Vdash^\mathcal{E} t \approx u$ if $h(t) = h(u)$. These notion can now be smoothly extended. We say that $\mathbf{A}$ $\mathcal{E}$-behaviorally satisfies $t \approx u$, in symbols $\mathbf{A} \Vdash^\mathcal{E} t \approx u$, if $\mathbf{A}, h \Vdash^\mathcal{E} t \approx u$ for every assignment h over $\mathbf{A}$. Given a class $\mathbb{K}$ of Σ-algebras, the *behavioral consequence over Σ associated with $\mathbb{K}$ and $\mathcal{E}$*, $\vDash_\mathbb{K}^\mathcal{E} \subseteq \mathcal{P}(Eq_\Sigma) \times Eq_\Sigma$, is such that $\Theta \vDash_\mathbb{K}^\mathcal{E} t \approx u$ if for every $\mathbf{A} \in \mathbb{K}$ and every assignment h over $\mathbf{A}$ we have that $\mathbf{A}, h \Vdash^\mathcal{E} t \approx u$ whenever $\mathbf{A}, h \Vdash^\mathcal{E} t' \approx u'$ for every $t' \approx u' \in \Theta$. Moreover, we say that $\mathbf{A}$ $\mathcal{E}$-behaviorally satisfies a quasi-equation $(t_1 \approx u_1) \& \ldots \& (t_n \approx u_n) \to (t \approx u)$, denoted by $\mathbf{A} \Vdash^\mathcal{E} (t_1 \approx u_1) \& \ldots \&$ $(t_n \approx u_n) \to (t \approx u)$, whenever $\{t_1 \approx u_1, \ldots, t_n \approx u_n\} \vDash_{\{\mathbf{A}\}}^\mathcal{E} t \approx u$. We refer the reader to [18] for more details on the subject of behavioral equational reasoning.

2.2. The behavioral approach

From now on, we will work only with signatures $\Sigma = \langle S, F \rangle$ with a distinguished sort ϕ (the syntactic sort of formulas). We assume fixed a S-sorted family X of variables. We define the induced set of *formulas* $L_\Sigma(X)$ to be the carrier set of sort ϕ of the free algebra $\mathbf{T}_\Sigma(\mathbf{X})$ with generators X, that is, $L_\Sigma(X) = T_{\Sigma,\phi}(X)$. We now introduce the class of logics that is the target of our approach.

Definition 2.2. A *(many-sorted) logic* is a tuple $\mathcal{L} = \langle \Sigma, \vdash \rangle$ where Σ is a signature and $\vdash\,\subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$ is a *consequence relation* satisfying, for every $\Phi \cup \Psi \cup \{\varphi\} \subseteq L_\Sigma(X)$:

- if $\varphi \in \Phi$ then $\Phi \vdash \varphi$ (***reflexivity***);

- if $\Phi \vdash \varphi$ for all $\varphi \in \Psi$, and $\Psi \vdash \psi$ then $\Phi \vdash \psi$ (***cut***);

- if $\Phi \vdash \varphi$ and $\Phi \subseteq \Psi$ then $\Psi \vdash \varphi$ (***weakening***).

$\mathcal{L}$ is further said to be ***structural*** whenever:

- if $\Phi \vdash \varphi$ then $\sigma[\Phi] \vdash \sigma(\varphi)$, for every substitution σ,

and said to be ***finitary*** whenever

- if $\Phi \vdash \varphi$ then $\Psi \vdash \varphi$ for some finite $\Psi \subseteq \Phi$.

In this paper, unless otherwise stated, all the logics considered are assumed to be structural.

Note that propositional-based logics appear as a particular case of many-sorted logics, considering a signature $\Sigma = \langle S, F \rangle$ such that $S = \{\phi\}$.

We will use $\vdash_\mathcal{L}$ instead of just $\vdash$ to refer to the consequence relation of a given logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$. Moreover, as usual, if $\Phi, \Psi \subseteq L_\Sigma(X)$, we will write $\Psi \vdash \Phi$ whenever $\Psi \vdash \varphi$ for all $\varphi \in \Phi$. We say that $\varphi, \psi \in L_\Sigma(X)$ are *interderivable* in $\mathcal{L}$, which is denoted by $\varphi \dashv\vdash_\mathcal{L} \psi$, if $\varphi \vdash_\mathcal{L} \psi$ and $\psi \vdash_\mathcal{L} \varphi$. Analogously, we say that Φ and Ψ are *interderivable* in $\mathcal{L}$, which is denoted by $\Phi \dashv\vdash_\mathcal{L} \Psi$, if $\Phi \vdash_\mathcal{L} \Psi$ and $\Psi \vdash_\mathcal{L} \Phi$. The *theorems* of $\mathcal{L}$ are the formulas φ such that $\emptyset \vdash_\mathcal{L} \varphi$. A *theory* of $\mathcal{L}$ is a set of formulas Φ such that if $\Phi \vdash_\mathcal{L} \varphi$ then $\varphi \in \Phi$. As usual, $\Phi^{\vdash_\mathcal{L}}$ denotes the least theory of $\mathcal{L}$ that contains Φ. The set of theories of $\mathcal{L}$ will be denoted by $Th_\mathcal{L}$.

Extending the single-sorted case, a logical matrix over a many-sorted signature Σ is a tuple $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a Σ-algebra and $D \subseteq A_\phi$. A matrix $\langle \mathbf{A}, D \rangle$ over Σ is a model of a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ if for every homomorphism $h : \mathbf{T}_\Sigma(\mathbf{X}) \rightarrow \mathbf{A}$ we have that if $T \vdash_\mathcal{L} \varphi$ then $h(\varphi) \in D$ whenever $h(\psi) \in D$ for every $\psi \in T$, in which case D is dubbed a $\mathcal{L}$-*filter* of $\mathbf{A}$. As usual, the class of all matrix models of a given logic $\mathcal{L}$ will be denoted by Matr$(\mathcal{L})$[1].

In the present setting, given the signature $\Sigma = \langle S, F \rangle$ of a logic, the corresponding free algebra will have a set of terms of each sort, but only those terms of sort ϕ will correspond to formulas of the logic. Therefore, in the logic itself, one can only observe the behavior of terms of other sorts by their indirect influence on the formulas where they appear. The behavioral approach to the algebraization of logics is

---

[1]In this paper we will borrow the terminology used, for instance, in [20]. In the modern terminology of algebraic logic [13], the classes Matr$(\mathcal{L})$ and Matr$^*(\mathcal{L})$ are instead denoted by Mod$(\mathcal{L})$ and Mod$^*(\mathcal{L})$, respectively. We dropped this terminology here, as our main point is precisely that matrices do not provide the most natural notion of model in the behavioral algebraic setting.

built over the idea of taking this situation a step further. Namely, we will hide all the sorts of Σ, including ϕ, and introduce a new unique visible sort for observing the behavior of formulas. Experiments must be carefully chosen among the well-behaved connectives of the logic, determined by a given subsignature Γ of Σ, thus possibly allowing the remaining connectives to behave in a non-congruent way. This can be achieved by considering behavioral equational logic over an extended signature. We define the extended signature $\Sigma^o = \langle S^o, F^o \rangle$ such that $S^o = S \uplus \{v\}$, where v is the newly introduced sort of *observations* of formulas. The indexed set of operations $F^o = \{F^o{}_{ws}\}_{w \in (S^o)^*, s \in S^o}$ is such that $F^o{}_{ws} = F_{ws}$ if $w \in S^*$ and $s \in S$, $F^o_{\phi v} = \{o\}$, and $F^o_{ws} = \emptyset$ otherwise. Intuitively, we are just extending the signature with a new sort v for the observations that we can perform on formulas using the observation operation o. Finally, the extended hidden signature is $\langle \Sigma^o, \{v\}, \mathcal{E}_\Gamma \rangle$ where $\mathcal{E}_\Gamma = \{o(t(x\!:\!s, x_1\!:\!s_1, \ldots, x_m\!:\!s_m)) : t \in T_{\Gamma,\phi}(X)\}$. Henceforth, we will use Γ instead of $\mathcal{E}_\Gamma$ to qualify the corresponding notions of behavioral reasoning. Before we recall the notion of a behaviorally algebraizable logic, we need a further concept. Let $\Theta(x : \phi)$ be a set of ϕ-equations. Θ is said to be Γ-*compatible* with a class $\mathbb{K}$ of Σ^o algebras if, given any variable $y : \phi$, it is the case that $x \approx y, \Theta(x) \models^\Gamma_\mathbb{K} \Theta(y)$.

Definition 2.3. A logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *behaviorally algebraizable* if there exists a subsignature Γ of Σ, a class $\mathbb{K}$ of Σ^o-algebras, a set $\Theta(x : \phi)$ of ϕ-equations Γ-compatible with $\mathbb{K}$, and a set $\Delta(x : \phi, y : \phi) \subseteq L_{\Gamma,\phi}$ of formulas such that, for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$ and for every set $\Psi \cup \{t \approx u\}$ of ϕ-equations, we have:

- $\Phi \vdash_\mathcal{L} \varphi$ iff $\Theta[\Phi] \models^\Gamma_\mathbb{K} \Theta(\varphi)$;

- $\Psi \models^\Gamma_\mathbb{K} t \approx u$ iff $\Delta[\Psi] \vdash_\mathcal{L} \Delta(t, u)$;

- $x \dashv\vdash_\mathcal{L} \Delta[\Theta(x)]$ and $x \approx y \equiv\!\models^\Gamma_\mathbb{K} \Theta[\Delta(x, y)]$.

The set Θ is called the set of *defining equations*, Δ the set of *equivalence formulas*, and $\mathbb{K}$ a *behaviorally equivalent algebraic semantics* for $\mathcal{L}$.

This definition is parameterized by the choice of the subsignature Γ of Σ. Hence, in what follows, we will say that a logic is Γ-*behaviorally algebraizable* if we want to stress the choice of Γ.

It is no accident that the definition of behaviorally algebraizable logic follows very closely the usual definition of an algebraizable logic, but with behavioral reasoning replacing the usual equational reasoning. As shown in [7], behavioral algebraization indeed enlarges the scope of the traditional theory of algebraization, but maintains many of its nice properties. Namely, given a Σ-algebra $\mathbf{A}$, there is a very natural way of defining a corresponding Γ-behavioral Leibniz operator $\mathbf{\Omega}^\mathbf{A}_\Gamma$ that maps each filter $D \subseteq A_\phi$ of $\mathbf{A}$ to the largest congruence $\mathbf{\Omega}^\mathbf{A}_\Gamma(D)$ of $\mathbf{A}|_\Gamma$ that is compatible with D. Note that $\mathbf{\Omega}^\mathbf{A}_\Gamma(D)$ is in general not a congruence over $\mathbf{A}$ if Γ is a proper subsignature of Σ. In particular, if we consider the free algebra $\mathbf{T}_\Sigma(X)$, we will write $\mathbf{\Omega}_\Gamma$ instead of $\mathbf{\Omega}^{\mathbf{T}_\Sigma(\mathbf{X})}_\Gamma$. We will use $\mathbf{\Omega}_{\Gamma,\phi}$ to denote the restriction of $\mathbf{\Omega}_\Gamma$ to the sort ϕ. The following simple result gives us a simple view of the meaning of the behavioral Leibniz congruence.

Proposition 2.1. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a logic, $M = \langle \mathbf{A}, D \rangle$ a matrix over Σ, and Γ a subsignature of Σ. Then, $\langle a, b \rangle \in \mathbf{\Omega}^\mathbf{A}_{\Gamma,s}(D)$ if and only if for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and every $c_1 \in A_{s_1}, \ldots, c_n \in A_{s_n}$ we have that:*

$$\varphi_\mathbf{A}(a, c_1, \ldots, c_n) \in D \text{ iff } \varphi_\mathbf{A}(b, c_1, \ldots, c_n) \in D.$$

Proof:

Consider the binary relation θ_M over $\mathbf{A}$ such that, for every $s \in S$, $\langle a, b \rangle \in \theta_{M,s}$ if for every formula $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and every $c_1 \in A_{s_1}, \ldots, c_n \in A_{s_n}$ we have that $\varphi_\mathbf{A}(a, c_1, \ldots, c_n) \in D$ iff $\varphi_\mathbf{A}(b, c_1, \ldots, c_n) \in D$. It is easy to conclude that θ_M is a congruence over $\mathbf{A}|_\Gamma$ that is compatible with D. We now prove that it is indeed the largest one. Let α be a congruence over $\mathbf{A}|_\Gamma$ that is compatible with D. We aim to prove that $\alpha \subseteq \theta_M$. Consider $\langle a, b \rangle \in \alpha_s$, for some $s \in S$. Since α is a congruence over $\mathbf{A}|_\Gamma$ we can conclude that, for every $\varphi(x : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and every $c_1 \in A_{s_1}, \ldots, c_n \in A_{s_n}$ we have that $\langle \varphi_\mathbf{A}(a, c_1, \ldots, c_n), \varphi_\mathbf{A}(b, c_1, \ldots, c_n) \rangle \in \alpha_\phi$. Using the fact that α is compatible with D, we can conclude that $\varphi_\mathbf{A}(a, c_1, \ldots, c_n) \in D$ iff $\varphi_\mathbf{A}(b, c_1, \ldots, c_n) \in D$. Thus, $\langle a, b \rangle \in \theta_{M,s}$.   □

Note that the behavioral Leibniz operator has some interesting features. Namely, given a sort $s \in S$, if the set $L_\Gamma(X) =$ then $\mathbf{\Omega}^\mathbf{A}_{\mathbf{\Gamma},s}(D)$ becomes trivial, that is, $\mathbf{\Omega}^\mathbf{A}_{\mathbf{\Gamma},s}(D) = A_s \times A_s$. Moreover, when $L_\Gamma(X) \neq$, even if $\mathbf{\Omega}^\mathbf{A}_{\mathbf{\Gamma},\phi}(D)$ is the identity, it may happen that $\mathbf{\Omega}^\mathbf{A}_{\mathbf{\Gamma}}(D)$ is not the identity. In general, when $\mathbf{\Omega}^\mathbf{A}_{\mathbf{\Gamma},\phi}(D)$ is the identity, we have that $\mathbf{\Omega}^\mathbf{A}_{\mathbf{\Gamma},s}(D)$ is the kernel of the (possibly non-injective) interpretation of the operators in Γ.

Nevertheless, as in the traditional approach, the behavioral Leibniz operator can be used to characterize important classes of logics with respect to their algebraic properties. Namely, a logic $\mathcal{L}$ is Γ-behaviorally algebraizable exactly when, on the theories of $\mathcal{L}$, $\mathbf{\Omega}_{\mathbf{\Gamma}}$ is injective, monotone, and commutes with inverse substitutions [7]. The behavioral Leibniz operator can also be used to generalize to the behavioral setting the class of protoalgebraic logic, which is considered as the largest class of logic amenable to meaningful algebraic treatment.

Definition 2.4. A logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *behaviorally protoalgebraic* if there exists a subsignature Γ of Σ such that, for every $\Phi \in Th_\mathcal{L}$ and $\varphi, \psi \in L_\Sigma(X)$, we have:

- if $\langle \varphi, \psi \rangle \in \mathbf{\Omega}_{\mathbf{\Gamma},\phi}(\Phi)$ then $\Phi, \varphi \vdash \psi$ and $\Phi, \psi \vdash \varphi$.

This definition is again parameterized by the choice of the subsignature Γ. We will say that a logic is Γ-*behaviorally protoalgebraic* when we want to stress this choice.

We will now present two equivalent characterizations of behaviorally protoalgebraic logics that will be useful later on. One is based on properties of the behavioral Leibniz operator, and the other on the existence of a parameterized equivalence for the logic. Both generalize to the behavioral setting well known results about protoalgebraic logics, in the standard sense, and their proofs can be found in [7].

Definition 2.5. Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a logic, Γ a subsignature of Σ. Given a set Z of variables, called *parametric variables*, a set $\Delta(x : \phi, y : \phi, Z) \subseteq L_\Gamma(X)$ is said to be a *parameterized Γ-equivalence system for $\mathcal{L}$* if it satisfies the following conditions:

- $\vdash \Delta(x, x, Z)$;

- $x, \Delta(x, y, U) \vdash y$;

- $\Delta(x, y, U) \vdash \Delta(\varphi(x), \varphi(y), U)$, for each $\varphi \in T_{\Gamma,\phi}(x : \phi)$,

where $U = \prod_{z_i : s_i \in Z} T_{\Sigma,s_i}(X)$, thus allowing the joint instantiation of the parametric variables Z in every possible way.

We can now present the characterization result.

Theorem 2.1. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a logic and Γ a subsignature of Σ. Then, the following conditions are equivalent:*

(i) $\mathcal{L}$ is Γ-behaviorally protoalgebraic;

(ii) $\mathbf{\Omega_\Gamma}$ is monotone on $Th_{\mathcal{L}}$;

(iii) there exists a parameterized Γ-equivalence system for $\mathcal{L}$.

A straightforward generalization of the standard setting allows us to prove an interesting but simple new result, which will be useful below, showing that a parameterized Γ-equivalence system can be used to define the Γ-behavioral Leibniz congruence in every matrix model of $\mathcal{L}$.

Lemma 2.1. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a logic and $\Delta(x : \phi, y : \phi, Z)$ be a parameterized Γ-equivalence system for $\mathcal{L}$. Then, for every $M = \langle \mathbf{A}, D \rangle \in Matr(\mathcal{L})$, we have that*

$$\langle a, b \rangle \in \mathbf{\Omega}_{\mathbf{\Gamma}, \phi}^{\mathbf{A}}(D) \quad iff \quad \Delta_{\mathbf{A}}(a, b, \overline{c}) \subseteq D \text{ for every } \overline{c} \in \prod_{z_i : s_i \in Z} A_{s_i}.$$

Proof:
Consider the binary relation $\theta_\Delta = \{ \langle a, b \rangle : \Delta_{\mathbf{A}}(a, b, \overline{c}) \subseteq D \text{ for every } \overline{c} \in \prod_{z_i : s_i \in Z} A_{s_i} \}$ over A_ϕ. It is easy to prove that θ_Δ is compatible with D, and that it respects a congruence property for all the formulas $\phi(x_1 : \phi, \ldots, x_n : \phi) \in L_\Gamma(X)$, that is, if $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \theta_\Delta$ then it must also be the case that $\langle \phi_{\mathbf{A}}(a_1, \ldots, a_n), \phi_{\mathbf{A}}(b_1, \ldots, b_n) \rangle \in \theta_\Delta$. Therefore, we can conclude that $\theta_\Delta \subseteq \mathbf{\Omega}_{\mathbf{\Gamma}, \phi}^{\mathbf{A}}(D)$.

To prove the converse, consider $\langle a, b \rangle \in \mathbf{\Omega}_{\mathbf{\Gamma}, \phi}^{\mathbf{A}}(D)$. Then $\langle \Delta_{\mathbf{A}}(a, a, \overline{c}), \Delta_{\mathbf{A}}(a, b, \overline{c}) \rangle \in \mathbf{\Omega}_{\mathbf{\Gamma}, \phi}^{\mathbf{A}}(D)$ for every $\overline{c} \in \prod_{z_i : s_i \in Z} A_{s_i}$. Since $M \in \text{Matr}(\mathcal{L})$ we have that $\Delta_{\mathbf{A}}(a, a, \overline{c}) \subseteq D$ for every $\overline{c} \in \prod_{z_i : s_i \in Z} A_{s_i}$. Therefore, by compatibility we can conclude that $\Delta_{\mathbf{A}}(a, b, \overline{c}) \subseteq D$. $\square$

Despite the fact that the Leibniz congruence is a key tool of algebraic logic, it is well-known that for non-protoalgebraic logics the Suszko congruence is more suitable. Nevertheless, the Suszko congruence is often neglected because it coincides with the Leibniz congruence within the class of protoalgebraic logics, and most of the work in algebraic logic focuses on the class of protoalgebraic logics. Herein, as we want to be as general as possible, we will introduce for the first time a behavioral version of the Suszko congruence. Given a Σ-algebra $\mathbf{A}$ and a filter D of $\mathbf{A}$ we can define the behavioral Suszko congruence as $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{A}}(D) = \bigcap \{ \mathbf{\Omega}_{\mathbf{\Gamma}}^{\mathbf{A}}(D') : D' \text{ is a } \mathcal{L}\text{-filter of } \mathbf{A} \text{ and } D \subseteq D' \}$. Note that, as in the traditional approach and contrarily to the definition of the Leibniz congruence, the definition of the Suszko congruence $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{A}}(D)$ does not depend exclusively on $\mathbf{A}$ and D, but also on the logic $\mathcal{L}$ to define all the $\mathcal{L}$-filters that contain D. The notation we are using does not emphasize this fact, but we should keep it in mind at all times. An immediate consequence of the definition is that $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{A}}(D) \subseteq \mathbf{\Omega}_{\mathbf{\Gamma}}^{\mathbf{A}}(D)$. As for the behavioral Leibniz congruence, we should also remark here that $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{A}}(D)$ is not a congruence over $\mathbf{A}$ if Γ is a proper subsignature of Σ. When we consider the free algebra $\mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X})$ we will write $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}$ instead of $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X})}$. The following result presents an alternative, perhaps simpler, characterization of the behavioral Suszko congruence.

Proposition 2.2. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a logic, $M = \langle \mathbf{A}, D \rangle$ a matrix over Σ, and Γ a subsignature of Σ. Then, $\langle a, b \rangle \in \widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma},s}^{\mathbf{A}}(D)$ if and only if for every formula $\varphi(x : s, x_1 : s_1, \dots, x_n : s_n) \in L_\Gamma(X)$, every $c_1 \in A_{s_1}, \dots, c_n \in A_{s_n}$, and every $\mathcal{L}$-filter D' of $\mathbf{A}$ such that $D \subseteq D'$, we have that:*

$$\varphi_{\mathbf{A}}(a, c_1, \dots, c_n) \in D' \text{ iff } \varphi_{\mathbf{A}}(b, c_1, \dots, c_n) \in D'.$$

Proof:
This result follows easily from Proposition 2.1. Just note that $\langle a, b \rangle \in \widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma},s}^{\mathbf{A}}(D)$ iff $\langle a, b \rangle \in \mathbf{\Omega}_{\mathbf{\Gamma},s}^{\mathbf{A}}(D')$ for every $\mathcal{L}$-filter D' of $\mathbf{A}$ such that $D \subseteq D'$ iff for every formula $\varphi(x : s, x_1 : s_1, \dots, x_n : s_n) \in L_\Gamma(X)$, every $c_1 \in A_{s_1}, \dots, c_n \in A_{s_n}$, and every $\mathcal{L}$-filter D' of $\mathbf{A}$ such that $D \subseteq D'$, we have that $\varphi_{\mathbf{A}}(a, c_1, \dots, c_n) \in D'$ iff $\varphi_{\mathbf{A}}(b, c_1, \dots, c_n) \in D'$. □

We can now obtain a generalization to the behavioral setting of a very nice result of the traditional setting. First we prove a useful lemma.

Lemma 2.2. *The behavioral Suszko operator $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{A}}$ is monotone, for every logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$, every Σ-algebra $\mathbf{A}$, and every subsignature Γ of Σ.*

Proof:
Assume that D_1, D_2 are two filters of $\mathbf{A}$ such that $D_1 \subseteq D_2$. Then, it is clear that $F_1 = \{D : D \text{ is a } \mathcal{L}\text{-filter of } \mathbf{A} \text{ and } D_1 \subseteq D\} \supseteq \{D : D \text{ is a } \mathcal{L}\text{-filter of } \mathbf{A} \text{ and } D_2 \subseteq D\} = F_2$. Therefore, $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{A}}(D_1) = \bigcap F_1 \subseteq \bigcap F_2 = \widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}^{\mathbf{A}}(D_2)$. □

We can now prove that the Leibniz and Suszko behavioral congruences coincide precisely when the logic at hand is behaviorally protoalgebraic.

Theorem 2.2. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a logic and Γ a subsignature of Σ. Then, $\mathcal{L}$ is Γ-behaviorally protoalgebraic if and only if $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}(\Phi) = \mathbf{\Omega}_{\mathbf{\Gamma}}(\Phi)$ for every $\Phi \in Th_{\mathcal{L}}$.*

Proof:
Suppose first that $\mathcal{L}$ is Γ-behaviorally protoalgebraic. Then, using Theorem 2.1 we know that there exists a parameterized Γ-equivalence system $\Delta(x : \phi, y : \phi, Z)$ for $\mathcal{L}$. Let $\Phi \in Th_{\mathcal{L}}$ and $t_1, t_2 \in T_{\Sigma,s}(X)$ such that $\langle t_1, t_2 \rangle \in \mathbf{\Omega}_{\mathbf{\Gamma},s}(\Phi)$. Since $\mathbf{\Omega}_{\mathbf{\Gamma}}(\Phi)$ is a Γ-congruence we have that the pair $\langle \varphi(t_1, u_1, \dots, u_n), \varphi(t_2, u_1, \dots, u_n) \rangle \in \mathbf{\Omega}_{\mathbf{\Gamma},\phi}(\Phi)$, for every $\varphi(x_0 : s, x_1 : s_1, \dots, x_n : s_n) \in L_\Gamma(X)$ and every $u_1 \in T_{\Sigma,s_1}(X), \dots, u_n \in T_{\Sigma,s_n}(X)$. Thus, using Lemma 2.1, $\Delta(\varphi(t_1, u_1, \dots, u_n), \varphi(t_2, u_1, \dots, u_n), U) \subseteq \Phi$, and using the properties of a parameterized Γ-equivalence system it is straightforward to conclude that $\Phi, \varphi(t_1, u_1, \dots, u_n) \dashv\vdash \Phi, \varphi(t_2, u_1, \dots, u_n)$. Therefore, we have that $\varphi(t_1, u_1, \dots, u_n) \in \Phi'$ if and only if $\varphi(t_2, u_1, \dots, u_n) \in \Phi'$ for every $\Phi' \in Th_{\mathcal{L}}$ with $\Phi \subseteq \Phi'$, and Proposition 2.2 allows us to conclude that $\langle t_1, t_2 \rangle \in \widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma},s}(\Phi)$. Since it is always the case that $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}} \subseteq \mathbf{\Omega}_{\mathbf{\Gamma}}$, we can conclude that $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}} = \mathbf{\Omega}_{\mathbf{\Gamma}}$.

For the converse implication, assume that $\widetilde{\mathbf{\Omega}}_{\mathbf{\Gamma}}(\Phi) = \mathbf{\Omega}_{\mathbf{\Gamma}}(\Phi)$ for every $\Phi \in Th_{\mathcal{L}}$. Then, Lemma 2.2 allow us to conclude that $\mathbf{\Omega}_{\mathbf{\Gamma}}$ is monotone on $Th_{\mathcal{L}}$. Using Theorem 2.1 we have that $\mathcal{L}$ is Γ-behaviorally protoalgebraic. □

In the classical theory of algebraization, the class of models that are canonically associated with a logic $\mathcal{L}$ is typically not the whole family $\mathrm{Matr}(\mathcal{L})$, but rather the subclasses of Leibniz or Suszko reduced matrix models. In general, a matrix for $\mathcal{L}$ can be reduced by simply factoring it with the corresponding congruence. This is however a challenge in the behavioral case, as typically the behavioral Leibniz and Suszko are congruences for just the operations in Γ.

3. Algebraic valuations

In this section we will first review the definition and properties of the notion of valuation semantics proposed in [5] as the suitable semantic companion of a logic in the behavioral setting. Then, we propose and discuss the class of logical valuations that, from our point of view, should be canonically associated with a given logic.

3.1. Logical valuations

Valuation semantics appeared in [10] as an effort to provide a semantic ground to logics that lack a meaningful matrix semantics. The essential idea is thus to drop the requirement that formulas must be interpreted homomorphically in an algebra over the same signature, and instead accept any possible interpretation as a function from the set of formulas of the logic to a set of truth-values equipped with a subset of designated values. Besides lacking a thorough supporting theory, namely if contrasted to the rich theory of logical matrices, valuation semantics has been mostly criticized for its excessive generality, namely as it can be (mis)understood at the light of Suszko's bivalence thesis (see, for instance, [4, 12]). The notion of valuation semantics proposed in [5] that we review here is intended precisely to maintain as much regularity as possible, thus allowing for a smooth algebraic treatment.

Below, we will consider fixed a signature $\Sigma = \langle S, F \rangle$ and a subsignature Γ of Σ.

Definition 3.1. A Γ-*valuation* is a triple $\vartheta = \langle \mathbf{A}, D, h \rangle$ such that:

- $\langle \mathbf{A}, D \rangle$ is a Γ-matrix, and

- h is a sorted function $h : T_\Sigma(X) \to A$ such that $h(f(t_1, \ldots, t_n)) = f_{\mathbf{A}}(h(t_1), \ldots, h(t_n))$ for every $f : s_1 \ldots s_n \to s \in \Gamma$ and $t_i \in T_{\Sigma, s_i}(X)$ with $i \in \{1, \ldots, n\}$.

A Γ-*valuation semantics* over Σ is a collection $\mathcal{V}$ of Γ-valuations.

A Γ-valuation is a matrix over the subsignature Γ of Σ together with a function that satisfies the homomorphism condition for every connective in Γ. In other words, $h : \mathbf{T}_{\boldsymbol\Sigma}(\mathbf{X})|_{\boldsymbol\Gamma} \to \mathbf{A}$ must be an homomorphism between Γ-algebras. In this way we are allowing valuations that do not necessarily satisfy the homomorphism condition with respect to connectives outside Γ. Note that the notion of matrix semantics is a particular cases of this definition, as it can be obtained by taking $\Gamma = \Sigma$ and requiring that, for each relevant Σ-algebra $\mathbf{A}$, every homomorphism $h : \mathbf{T}_{\boldsymbol\Sigma}(\mathbf{X}) \to \mathbf{A}$ is considered.

Expectedly, given a Γ-valuation $\vartheta = \langle \mathbf{A}, D, h \rangle$ and a formula $\varphi \in L_\Sigma(X)$, we say that ϑ *satisfies* φ, denoted by $\vartheta \Vdash \varphi$, if $h(\varphi) \in D$. As usual, given $\Phi \subseteq L_\Sigma(X)$, we write $\vartheta \Vdash \Phi$ whenever $\vartheta \Vdash \varphi$ for every $\varphi \in \Phi$. A Γ-valuation ϑ is said to be a model of $\mathcal{L}$ when it happens that if $\vartheta \Vdash \Phi$ and $\Phi \vdash_{\mathcal{L}} \varphi$ then

$\vartheta \Vdash \varphi$. In this case, D is called a $\mathcal{L}$-*filter* of h. The class of all Γ-valuations that are models of $\mathcal{L}$ will be denoted by $\mathrm{Val}_\Gamma(\mathcal{L})$.

Given a Γ-valuation semantics $\mathcal{V} = \{\langle \mathbf{A}_i, D_i, h_i \rangle : i \in I\}$ over Σ, we define the consequence relation associated with $\mathcal{V}$, $\vdash_\mathcal{V} \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$, by letting $\Phi \vdash_\mathcal{V} \varphi$ if for every Γ-valuation $\vartheta \in \mathcal{V}$ we have that $\vartheta \Vdash \varphi$ whenever $\vartheta \Vdash \Phi$. Note that, in general, $\vdash_\mathcal{V}$ is not structural, an important question to which we will return. In any case, let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a logic over Σ, not necessarily structural. A Γ-valuation semantics $\mathcal{V}$ is *sound* for $\mathcal{L}$ if $\vdash_\mathcal{L} \subseteq \vdash_\mathcal{V}$. Symmetrically, $\mathcal{V}$ is *adequate* for $\mathcal{L}$ if $\vdash_\mathcal{V} \subseteq \vdash_\mathcal{L}$. The Γ-valuation semantics $\mathcal{V}$ is *complete* for $\mathcal{L}$ if it is both sound and adequate, that is $\vdash_\mathcal{L} = \vdash_\mathcal{V}$.

One can easily bring the usual Lindenbaum-Tarski constructions to the setting of valuations. For each set $\Phi \subseteq L_\Sigma(X)$ of formulas, we can define the Γ-valuation $\vartheta_\Gamma^\Phi = \langle \mathbf{L_\Sigma(X)}|_\Gamma, \Phi, id \rangle$ where $id : L_\Sigma(X) \to L_\Sigma(X)$ is the identity function. The Γ-valuations of the form ϑ_Γ^Φ are dubbed *Lindenbaum Γ-valuations* for Σ. The family $\mathcal{V}_\Gamma(\mathcal{L}) = \{\vartheta_\Gamma^\Phi : \Phi \text{ is a theory of } \mathcal{L}\}$ is called the *Lindenbaum Γ-bundle* of $\mathcal{L}$.

Proposition 3.1. *For every logic $\mathcal{L}$,*

- *$\mathcal{L}$ is complete with respect to its Lindenbaum Γ-bundle $\mathcal{V}_\Gamma(\mathcal{L})$;*

- *$\mathcal{L}$ is complete with respect to $\mathrm{Val}_\Gamma(\mathcal{L})$.*

Proof:
Clearly, $\mathcal{V}_\Gamma(\mathcal{L}) \subseteq \mathrm{Val}_\Gamma(\mathcal{L})$. To see that $\mathcal{V}_\Gamma(\mathcal{L})$ is an adequate Γ-valuation semantics for $\mathcal{L}$, just suppose that $\Phi \nvdash_\mathcal{L} \varphi$ for some $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$. Then $\vartheta_\Gamma^{\Phi^{\vdash_\mathcal{L}}} \Vdash \Phi$ but $\vartheta_\Gamma^{\Phi^\vdash} \nVdash \varphi$, and hence $\Phi \nvdash_{\mathcal{V}_\Gamma(\mathcal{L})} \varphi$. As a consequence, also $\mathrm{Val}_\Gamma(\mathcal{L})$ is adequate for $\mathcal{L}$. $\square$

As the class of all matrix models of $\mathcal{L}$ in the usual approach, the class $\mathrm{Val}_\Gamma(\mathcal{L})$ is very important since it allows us to study the metalogical properties of $\mathcal{L}$. As we will show below, when a logic is behaviorally algebraizable, we are able to algebraically specify not only the class of algebras associated with the logic, but also the admissible ways that formulas can be interpreted in these algebras, as the valuations are now incorporated in the algebraic models. Note that it is precisely the extended signature Σ^o that gives the algebraic handle that allows us to specify these. There are, however, other desirable properties that a valuation semantics might enjoy.

One semantical property which is very characteristic of the algebraic setting, and which holds for a matrix semantics, is *representativity*. A Γ-valuation semantics $\mathcal{V}$ over Σ is is said to be *representative* if

- $\vartheta = \langle \mathbf{A}, D, h \rangle \in \mathcal{V}$ implies that $\vartheta \circ \sigma = \langle \mathbf{A}, D, h \circ \sigma \rangle \in \mathcal{V}$ for every substitution σ.

This last property is well known to be closely connected with structurality [20].

Theorem 3.1. *Let $\mathcal{L}$ be a logic, not necessarily structural, over signature Σ, and Γ a subsignature of Σ. Then, $\mathcal{L}$ is structural if and only if the class $\mathrm{Val}_\Gamma(\mathcal{L})$ is representative.*

Proof:
Suppose that $\mathcal{L}$ is structural, let $\vartheta \in \mathrm{Val}_\Gamma(\mathcal{L})$ and take any substitution σ. Assume that $\Phi \vdash_\mathcal{L} \varphi$ and $\vartheta \circ \sigma \Vdash \Phi$. Clearly, this is equivalent to having $\vartheta \Vdash \sigma[\Phi]$. But, by structurality, it is also the case that

$\sigma[\Phi] \vdash_{\mathcal{L}} \sigma(\varphi)$ and, as $\vartheta \in \mathrm{Val}_\Gamma(\mathcal{L})$, it follows that $\vartheta \Vdash \sigma(\varphi)$. Equivalently, then, $\vartheta \circ \sigma \Vdash \varphi$, and hence $\vartheta \circ \sigma \in \mathrm{Val}_\Gamma(\mathcal{L})$, and $\mathrm{Val}_\Gamma(\mathcal{L})$ is representative.

To prove the converse implication, given that according to Proposition 3.1 $\mathcal{L}$ is complete with respect to $\mathrm{Val}_\Gamma(\mathcal{L})$, it suffices to show that the consequence associated with an arbitrary representative valuation semantics $\mathcal{V}$ is necessarily structural. Assume that $\Phi \vdash_{\mathcal{V}} \varphi$ and take any substitution σ. Given $\vartheta \in \mathcal{V}$, if $\vartheta \Vdash \sigma[\Phi]$ then, equivalently, $\vartheta \circ \sigma \Vdash \Phi$. But we know that $\vartheta \circ \sigma \in \mathcal{V}$, and therefore $\vartheta \circ \sigma \Vdash \varphi$, or equivalently, $\vartheta \Vdash \sigma(\varphi)$. Hence, $\sigma[\Phi] \vdash_{\mathcal{V}} \sigma(\varphi)$ and $\vdash_{\mathcal{V}}$ is structural. $\qquad\square$

To see that some further important results of the fruitful theory of logical matrices generalize to valuation semantics, we end this section by recalling an example of such a result, namely an adaptation of Bloom's theorem [3]. Given a Γ-valuation $\vartheta = \langle \mathbf{A}, D, h \rangle$ a *subvaluation of* ϑ is a Γ-valuation $\vartheta' = \langle \mathbf{A}', D', h' \rangle$ such that $\mathbf{A}'$ is a Γ-subalgebra of $\mathbf{A}$, $D \cap A'_\phi = D'$, and $h' = h$. Two Γ-valuations $\vartheta = \langle \mathbf{A}, D, h \rangle$ and $\vartheta' = \langle \mathbf{A}', D', h' \rangle$ are said to be *isomorphic* if there exists an isomorphism $\iota : \mathbf{A} \to \mathbf{A}'$ of Γ-algebras such that $\iota(D) = D'$ and $\iota \circ h = h'$. Given a set $\Lambda = \{\vartheta_i = \langle \mathbf{A}_i, D_i, h_i \rangle : i \in I\}$ of Γ-valuations the *direct product of* Λ is the Γ-valuation $\Pi_{i \in I} \vartheta_i = \langle \Pi_{i \in I} \mathbf{A}_i, \Pi_{i \in I} D_i, (h_i(\_))_{i \in I} \rangle$. A *subdirect product* of Λ is a Γ-valuation $\vartheta = \langle \mathbf{A}, D, h \rangle$ such that ϑ is a subvaluation of $\Pi_{i \in I} \vartheta_i$ and each of the projections $\pi_i : \mathbf{A} \to \mathbf{A}_i$ is onto. Recall that, given a set I, an *ultrafilter on I* is a set $\mathcal{U}$ consisting of subsets of I such that: 1) $\emptyset \notin \mathcal{U}$; 2) if $A \in \mathcal{U}$ and $A \subseteq B$ then $B \in \mathcal{U}$; 3) if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$; 4) if $A \subseteq I$, then either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$. Note that, axioms 1) and 3), imply that A and $I \setminus A$ cannot both be elements of $\mathcal{U}$. Given $\Lambda = \{\vartheta_i : i \in I\}$ and an ultrafilter $\mathcal{U}$ on I we can define a (sorted) equivalence relation $\sim_\mathcal{U}$ on the direct product $\Pi_{i \in I} \vartheta_i$ as follows: $a \sim_\mathcal{U} b$ iff $\{i \in I : a_i = b_i\} \in \mathcal{U}$. The *ultraproduct* of the Γ-valuations Λ modulo an ultrafilter $\mathcal{U}$, denoted by $\Pi_\mathcal{U} \vartheta_i$, is the quotient of $\Pi_{i \in I} \vartheta_i$ by the equivalence $\sim_\mathcal{U}$ (that is indeed a congruence of the underlying Γ-algebra). Concretely, let $\Pi_\mathcal{U} \vartheta_i = \langle (\Pi_{i \in I} \mathbf{A}_i)/\mathcal{U}, [\{(a_i)_{i \in I} \in \Pi_{i \in I} A_{i,\phi} : \{i \in I : a_i \in D_i\} \in \mathcal{U}\}]_\mathcal{U}, [(h_i(\_))_{i \in I}]_\mathcal{U} \rangle$.

Theorem 3.2. *Let $\mathcal{L}$ be a logic over signature Σ, and Γ a subsignature of Σ. Then, $\mathcal{L}$ is finitary if and only if the class $Val_\Gamma(\mathcal{L})$ is closed under ultraproducts.*

Proof:
Suppose first that $\mathcal{L}$ is finitary. Let $\{\vartheta_i : i \in I\} \subseteq \mathrm{Val}_\Gamma(\mathcal{L})$ be a family of Γ-models of $\mathcal{L}$ and $\mathcal{U}$ an ultrafilter on I. We aim to prove that $\Pi_\mathcal{U} v_i \in \mathrm{Val}_\Gamma(\mathcal{L})$. So, suppose that $\Phi \vdash_{\mathcal{L}} \varphi$ and that $\Pi_\mathcal{U} \vartheta_i \Vdash \Phi$. Since $\mathcal{L}$ is finitary, there must exist $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Phi$ such that $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{L}} \varphi$. For each $1 \leq j \leq n$, we have that $\Pi_\mathcal{U} \vartheta_i \Vdash \varphi_j$, and thus $I_j = \{i \in I : \vartheta_i \Vdash \varphi_j\} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter we have that $I_1 \cap \ldots \cap I_n = \{i \in I : \vartheta_i \Vdash \{\varphi_1, \ldots, \varphi_n\}\} \in \mathcal{U}$. Note also that, since each ϑ_i is a Γ-model of $\mathcal{L}$, $I_1 \cap \ldots \cap I_n \subseteq \{i \in I : \vartheta_i \Vdash \varphi\}$. Since $\mathcal{U}$ is an ultrafilter we have that $\{i \in I : \vartheta_i \Vdash \varphi\} \in \mathcal{U}$ and so $\Pi_\mathcal{U} \vartheta_i \Vdash \varphi$.

Suppose now that $\mathrm{Val}_\Gamma(\mathcal{L})$ is closed under ultraproducts. To prove that $\mathcal{L}$ is finitary let Φ be infinite and assume that $\Phi' \nvdash_{\mathcal{L}} \psi$, for every finite $\Phi' \subseteq \Phi$. Let I denote the set of all finite subsets of Φ. For each $i \in I$, define $i^* = \{j \in I : i \subseteq j\}$. Using well-known results on ultrafilters [20] we can conclude that there exists an ultrafilter $\mathcal{U}$ over I that contains the family $\{i^* : i \in I\}$. For every $i \in I$, consider the theory $i^{\vdash_{\mathcal{L}}}$ and let $\vartheta_i = \vartheta_\Gamma^{i^{\vdash_{\mathcal{L}}}} \in \mathrm{Val}_\Gamma(\mathcal{L})$ be the corresponding Lindenbaum Γ-valuation. Let $\Pi_\mathcal{U} \vartheta_i$ be the ultraproduct of the family by the ultrafilter $\mathcal{U}$. Then, for every $\varphi \in \Phi$ we have that $\{\varphi\}^* \subseteq \{i \in I : \vartheta_i \Vdash \varphi\}$. So, $\{i \in I : \vartheta_i \Vdash \varphi\} \in \mathcal{U}$ for every $\varphi \in \Phi$, and consequently we have that $\Pi_\mathcal{U} \vartheta_i \Vdash \Phi$. But $\{i \in I : \vartheta_i \Vdash \psi\} = \emptyset$ and so $\Pi_\mathcal{U} \vartheta_i \nVdash \psi$. Since $\Pi_\mathcal{U} \vartheta_i \in \mathrm{Val}_\Gamma(\mathcal{L})$ we have that $\Phi \nvdash \varphi$, and we can conclude that $\mathcal{L}$ is finitary. $\qquad\square$

As we remarked before, the class of logical matrices canonically associated with a logic $\mathcal{L}$ is typically not the whole of $\mathrm{Matr}(\mathcal{L})$, but rather the subclass $\mathrm{Matr}^*(\mathcal{L})$ of Leibniz reduced matrices. In the behavioral setting, we can define an analogous class of reduced Γ-valuation models, by setting $\mathrm{Val}_\Gamma^*(\mathcal{L}) = \{\langle \mathbf{A}, D, h \rangle \in \mathrm{Val}_\Gamma(\mathcal{L}) : \boldsymbol{\Omega}_\Gamma^{\mathbf{A}}(D)$ is the identity$\}$. Expectedly, given $\Phi \subseteq L_\Sigma(X)$, we can also define the Γ-valuation

$$\vartheta_\Gamma^{*\Phi} = \langle (\mathbf{L}_\Sigma(\mathbf{X})|_\Gamma)/\boldsymbol{\Omega}_\Gamma(\Phi), [\Phi]_{\boldsymbol{\Omega}_\Gamma(\Phi)}, [\_]_{\boldsymbol{\Omega}_\Gamma(\Phi)} \rangle \in \mathrm{Val}_\Gamma^*(\mathcal{L}).$$

The Γ-valuations of this form are dubbed *reduced Lindenbaum Γ-valuations* for Σ. The family $\mathcal{V}_\Gamma^*(\mathcal{L}) = \{\vartheta_\Gamma^{*\Phi} : \Phi$ is a theory of $\mathcal{L}\}$ is called the *reduced Lindenbaum Γ-bundle* of $\mathcal{L}$.

Proposition 3.2. *For every logic $\mathcal{L}$,*

- *$\mathcal{L}$ is complete with respect to its reduced Lindenbaum Γ-bundle $\mathcal{V}_\Gamma^*(\mathcal{L})$;*

- *$\mathcal{L}$ is complete with respect to $\mathrm{Val}_\Gamma^*(\mathcal{L})$.*

Proof:
Noting that $\mathcal{V}_\Gamma^*(\mathcal{L}) \subseteq \mathrm{Val}_\Gamma^*(\mathcal{L})$, the results follows easily from Proposition 3.1, once we observe that, for every theory Φ of $\mathcal{L}$ and every $\varphi \in L_\Sigma(X)$, we have that $\vartheta_\Gamma^{*\Phi} \Vdash \varphi$ if and only if $\vartheta_\Gamma^{\Phi} \Vdash \varphi$. This equivalence follows easily from the fact that $\boldsymbol{\Omega}_\Gamma(\Phi)$ is compatible with Φ. □

Still, the class $\mathrm{Val}_\Gamma^*(\mathcal{L})$ is not fully satisfactory, as it fails to comply with an important global property of matrix semantics: the fact that we can consider all possible assignments over a given algebra. A Γ-valuation semantics $\mathcal{V}$ over Σ is said to be *Laplacian* if

- whenever $\vartheta = \langle \mathbf{A}, D, h \rangle \in \mathcal{V}$ then for every assignment ρ over $\mathbf{A}$ there exists a Γ-valuation $\vartheta_\rho = \langle \mathbf{A}, D, h_\rho \rangle \in \mathcal{V}$ such that $h_\rho|_X = \rho$.

In the general case, there seems to be no clear way of establishing that $\mathrm{Val}_\Gamma^*(\mathcal{L})$ is Laplacian.

3.2. Valuations canonically associated with a logic

In this section we advance our proposal of the most suitable valuation semantics that should, from an algebraic perspective, be associated with a logic. To start with, it is useful to recall what happens in the traditional approach, using matrix models. Indeed, a lot of effort was invested by algebraic logicians in finding a congruence that could be used to extend in a smooth way the Lindenbaum-Tarski construction to every logic. It became clear that the Suszko and the Leibniz congruences were the most natural choices. In the class of protoalgebraic logics these two congruences coincide, and therefore it became clear what the natural generalization of the Lindenbaum-Tarski construction for protoalgebraic logics should be. For non-protoalgebraic logics, the range of studied examples confirmed that the Suszko congruence is the most appropriate choice. Therefore, the standard class of algebras canonically associated with a logic $\mathcal{L}$ is the class $\mathrm{Alg}(\mathcal{L})$, obtained as the closure under isomorphisms of

$$\{\mathbf{A}/_{\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}(D)} : \langle \mathbf{A}, D \rangle \in \mathrm{Matr}(\mathcal{L})\}.$$

As argued in [13], this class of algebras is less important than the closure under isomorphisms of the class of Suszko reduced matrices

$$\{\langle \mathbf{A}/_{\widetilde{\mathbf{\Omega}}^{\mathbf{A}}(D)}, [D]_{\widetilde{\mathbf{\Omega}}^{\mathbf{A}}(D)}\rangle : \langle \mathbf{A}, D\rangle \in \mathrm{Matr}(\mathcal{L})\},$$

as, it is not possible, in general, to canonically associate to each $\mathbf{A} \in \mathrm{Alg}(\mathcal{L})$ a filter $D_{\mathbf{A}}$ such that $\mathcal{L}$ is complete with respect to $\{\langle \mathbf{A}, D_{\mathbf{A}}\rangle : \mathbf{A} \in \mathrm{Alg}(\mathcal{L})\}$.

Using the Leibniz congruence one can introduce another important class of reduced matrices associated with a logic $\mathcal{L}$. This is $\mathrm{Matr}^*(\mathcal{L})$, the closure under ismorphisms of the class of its Leibniz reduced matrix models

$$\{\langle \mathbf{A}/_{\mathbf{\Omega}^{\mathbf{A}}(D)}, [D]_{\mathbf{\Omega}^{\mathbf{A}}(D)}\rangle : \langle \mathbf{A}, D\rangle \in \mathrm{Matr}(\mathcal{L})\}.$$

Taking the algebraic reducts of the class of $\mathrm{Matr}^*(\mathcal{L})$ we obtain the class of algebras

$$\mathrm{Alg}^*(\mathcal{L}) = \{\mathbf{A} : \langle \mathbf{A}, D\rangle \in \mathrm{Matr}^*(\mathcal{L})\}.$$

Since, in general, $\widetilde{\mathbf{\Omega}}^{\mathbf{A}} \subseteq \mathbf{\Omega}^{\mathbf{A}}$ we have that $\mathrm{Alg}^*(\mathcal{L}) \subseteq \mathrm{Alg}(\mathcal{L})$. Moreover, in the protoalgebraic case, the classes $\mathrm{Alg}^*(\mathcal{L})$ and $\mathrm{Alg}(\mathcal{L})$ coincide. Therefore, within the class of protoalgebraic logics it is very usual to work just with the Leibniz congruence and the classes $\mathrm{Alg}^*(\mathcal{L})$ and $\mathrm{Matr}^*(\mathcal{L})$.

To pave the ground for a meaningful behavioral algebraic theory of valuations, it seems that the wisest choice will be to consider classes of reduced valuations that mimic as much as possible the case of logical matrices. However, as we have seen above, simply taking the class of Leibniz reduced valuations does not seem to be a good choice. We propose instead a rationale that is perhaps even closer to the lessons learned from logical matrices, and which departs from the simple intuition that a matrix semantics can be understood as defining the class of all valuations obtained by homomorphic interpretation. Given a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ and a subsignature Γ of Σ, let $M = \langle \mathbf{A}, D\rangle \in \mathrm{Matr}(\mathcal{L})$. As we have discussed, the Γ-behavioral Leibniz congruence, not being a congruence over the whole Σ, cannot help us to obtain a corresponding reduced matrix. Still, for each homomorphism $h : \mathbf{L}_{\mathbf{\Sigma}}(\mathbf{X}) \to \mathbf{A}$ over M, we can take the corresponding Γ-valuation $\vartheta_{M,h} = \langle \mathbf{A}|_{\Gamma}, D, h\rangle \in \mathrm{Val}_{\Gamma}(\mathcal{L})$, and obtain the reduced Γ-valuation $\vartheta^*_{M,h} \in \mathrm{Val}^*_{\Gamma}(\mathcal{L})$. We need just quotient $\vartheta_{M,h}$ by the behavioral Leibniz congruence $\mathbf{\Omega} = \mathbf{\Omega}^{\mathbf{A}|_{\Gamma}}_{\mathbf{\Gamma}}(D)$, that is, $\vartheta^*_{M,h} = \langle (\mathbf{A}|_{\Gamma})/_{\mathbf{\Omega}}, [D]_{\mathbf{\Omega}}, [\_]_{\mathbf{\Omega}} \circ h\rangle$. If we proceed like this, systematically, we obtain the classes of Γ-valuations:

$$\mathrm{MVal}_{\Gamma}(\mathcal{L}) = \{\vartheta_{M,h} : M \in \mathrm{Matr}(\mathcal{L}) \text{ and } h \text{ is a homomorphism over } M\},$$

$$\text{and}$$

$$\mathrm{MVal}^*_{\Gamma}(\mathcal{L}) = \{\vartheta^*_{M,h} : M \in \mathrm{Matr}(\mathcal{L}) \text{ and } h \text{ is a homomorphism over } M\}.$$

As we will show below, these valuation semantics are meaningful and very well-behaved. One important characteristic of $\mathrm{MVal}^*_{\Gamma}(\mathcal{L})$ which is not shared by $\mathrm{Val}^*_{\Gamma}(\mathcal{L})$ is that it is Laplacian.

Proposition 3.3. *Let $\mathcal{L}$ be a many-sorted logic. Then $\mathrm{MVal}^*_{\Gamma}(\mathcal{L})$ is both Laplacian and representative.*

Proof:
Suppose that $\vartheta = \langle \mathbf{A}, D, h\rangle \in \mathrm{MVal}^*_{\Gamma}(\mathcal{L})$. Then there exists $M = \langle \mathbf{B}, D'\rangle \in \mathrm{Matr}(\mathcal{L})$ and an homomorphism h' over M such that $\vartheta = \vartheta^*_{M,h'}$. Consider given an assignment ρ over $\mathbf{A}$. Recall that by

construction $\mathbf{A}$ results of a quotient construction from $\mathbf{B}|_\Gamma$. Therefore, it is always possible to choose an assignment h^ρ over $\mathbf{B}$ such that, for every $s \in S$ and every $x \in X_s$, we have that $h_s^\rho(x) \in \rho_s(x)$. We can then conclude that $\vartheta^*_{M,h^\rho} = \langle \mathbf{A}, D, [\_]_{\mathbf{\Omega}_{\mathbf{F}}^{\mathbf{B}}(D')} \circ h^\rho \rangle \in \mathrm{MVal}^*_\Gamma(\mathcal{L})$ and $([\_]_{\mathbf{\Omega}_{\mathbf{F}}^{\mathbf{B}}(D')} \circ h^\rho)|_X = \rho$. Hence, $\mathrm{MVal}^*_\Gamma(\mathcal{L})$ is Laplacian.

To show that $\mathrm{MVal}^*_\Gamma(\mathcal{L})$ is representative, take any substitution σ. Clearly, $h' \circ \sigma$ is also an assignment over $\mathbf{B}$. Hence, $\vartheta^*_{M,h'\circ\sigma} = \vartheta^*_{M,h'} \circ \sigma = \vartheta \circ \sigma \in \mathrm{MVal}^*_\Gamma(\mathcal{L})$. $\qquad\square$

Lemma 3.1. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Then:*

(i) *$MVal^*_\Gamma(\mathcal{L}) \subseteq Val^*_\Gamma(\mathcal{L})$;*

(ii) *for every $\vartheta \in Val^*_\Gamma(\mathcal{L})$ there exists $\vartheta' \in MVal^*_\Gamma(\mathcal{L})$ such that, for every $\varphi \in L_\Sigma(X)$, we have that $\vartheta \Vdash \varphi$ iff $\vartheta' \Vdash \varphi$;*

(iii) *for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$, $\Phi \vdash_{MVal^*_\Gamma(\mathcal{L})} \varphi$ iff $\Phi \vdash_\mathcal{L} \varphi$ iff $\Phi \vdash_{Val^*_\Gamma(\mathcal{L})} \varphi$.*

Proof:

Condition *(i)* is trivial since by construction every Γ-valuation in $\mathrm{MVal}^*_\Gamma(\mathcal{L})$ is reduced and a model of $\mathcal{L}$.

To prove condition *(ii)*, let $\vartheta = \langle \mathbf{A}, D, h \rangle \in \mathrm{Val}^*_\Gamma(\mathcal{L})$. Consider the set $\Phi_\vartheta = \{\varphi \in L_\Sigma(X) : h(\varphi) \in D\}$. It is easy to see that $\Phi_\vartheta \in Th_\mathcal{L}$. Therefore, we have that $M = \langle \mathbf{T}_\Sigma(\mathbf{X}), \Phi_\vartheta \rangle \in Matr(\mathcal{L})$, and taking the identity function $id : T_\Sigma(X) \to T_\Sigma(X)$ we have that $\vartheta^*_{M,id} \in \mathrm{MVal}^*_\Gamma(\mathcal{L})$. We just need to check that $\vartheta \Vdash \varphi$ iff $\vartheta^*_{M,id} \Vdash \varphi$, for every $\varphi \in L_\Sigma(X)$. Clearly, $\vartheta^*_{M,id} \Vdash \varphi$ iff $[\varphi]_{\mathbf{\Omega}_\Gamma(\Phi_\vartheta)} \in [\Phi_\vartheta]_{\mathbf{\Omega}_\Gamma(\Phi_\vartheta)}$ iff $\varphi \in \Phi_\vartheta$ iff $h(\varphi) \in D$ iff $\vartheta \Vdash \varphi$.

Condition *(iii)* is an immediate consequence of condition *(ii)* and Proposition 3.2. $\qquad\square$

Given these properties, it is not surprising that we propose $\mathrm{MVal}^*_\Gamma(\mathcal{L})$ as the canonical valuation semantics that should be associated with a given Γ-behaviorally protoalgebraic logic $\mathcal{L}$. Of course, if behavioral protoalgebraicity fails then it might be wise to consider instead the class of Suszko reduced valuations obtained from $\mathcal{L}$-matrices. For each $M = \langle \mathbf{A}, D \rangle \in \mathrm{Matr}(\mathcal{L})$, we need just quotient $\vartheta_{M,h}$ by the behavioral Suszko congruence $\widetilde{\mathbf{\Omega}} = \widetilde{\mathbf{\Omega}}_\Gamma^{\mathbf{A}|\Gamma}(D)$, that is, $\widetilde{\vartheta}^*_{M,h} = \langle (\mathbf{A}|_\Gamma)/_{\widetilde{\mathbf{\Omega}}}, [D]_{\widetilde{\mathbf{\Omega}}}, [\_]_{\widetilde{\mathbf{\Omega}}} \circ h \rangle$.

4.   Bridge results

In this section we will present some bridge results similar to those that connect traditional algebraization with matrix semantics, thus setting the path towards a behavioral algebraic theory of logical valuations. We also discuss our proposal at the light of the behaviorally equivalent algebraic semantics associated with a behaviorally algebraizable logic, thus reinforcing the idea that the class $\mathrm{MVal}^*_\Gamma(\mathcal{L})$ of Γ-valuations is the right behavioral companion of $\mathcal{L}$. The following theorem is a generalization of a bridge result of traditional algebraic logic.

Theorem 4.1. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. Then $\mathcal{L}$ is Γ-behaviorally protoalgebraic if and only if $MVal^*_\Gamma(\mathcal{L})$ is closed under subdirect products.*

Proof:

Let us first suppose that $\mathcal{L}$ is Γ-behaviorally protoalgebraic. Then, by Theorem 2.1, $\mathcal{L}$ has a parameterized Γ-equivalence system $\Delta(x, y, Z)$, where Z is the set of parametric variables. Let us consider a family $\Lambda = \{\vartheta_i = \langle \mathbf{A}_i, D_i, h_i \rangle\}_{i \in I}$ of Γ-valuations such that, for each $i \in I$, $\vartheta_i \in \mathrm{MVal}_\Gamma^*(\mathcal{L})$ and let $\vartheta = \langle \mathbf{A}, D, h \rangle$ be a subdirect product of Λ. It is easy to see that ϑ is a model of $\mathcal{L}$. What remains to be proved is that ϑ is reduced. For the purpose, let $a, b \in A_s$ and suppose that $\langle a, b \rangle \in \mathbf{\Omega}_{\Gamma,s}^{\mathbf{A}}(D)$. Take any $\varphi(x_0 : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$ and $d_1 \in A_{s_1}, \ldots, d_n \in A_{s_n}$. Then, we have that $\langle \varphi_{\mathbf{A}}(a, d_1, \ldots, d_n), \varphi_{\mathbf{A}}(b, d_1, \ldots, d_n) \rangle \in \mathbf{\Omega}_{\Gamma,\phi}^{\mathbf{A}}(D)$. Since ϑ is a model of $\mathcal{L}$ and $\mathrm{MVal}_\Gamma^*(\mathcal{L})$ is Laplacian we know that, for every $\overline{c} \in \prod_{z:r \in Z} A_r$ there exists $h_{\overline{c}} : T_\Sigma(X) \to A$ such that $\langle \mathbf{A}, D, h_{\overline{c}} \rangle \in \mathrm{MVal}_\Gamma^*(\mathcal{L})$, with $h_{\overline{c}}(x) = \varphi_{\mathbf{A}}(a, d_1, \ldots, d_n)$, $h_{\overline{c}}(y) = \varphi_{\mathbf{A}}(b, d_1, \ldots, d_n)$ and $h_{\overline{c},r}(z) = \overline{c}_{z:r}$ for every $z : r \in Z$. Thus, using Lemma 2.1, we have that $\Delta_{\mathbf{A}}(\varphi_{\mathbf{A}}(a, d_1, \ldots, d_n), \varphi_{\mathbf{A}}(b, d_1, \ldots, d_n), \overline{c}) \subseteq D$ for every $\overline{c} \in \prod_{z:r \in Z} A_r$. Hence, by definition of subdirect product, we also know that for each $i \in I$ it must be the case that $\Delta_{\mathbf{A}_i}(\varphi_{\mathbf{A}}(a, d_1, \ldots, d_n)_i, \varphi_{\mathbf{A}}(b, d_1, \ldots, d_n)_i, \overline{c}_i) \subseteq D_i$. Note that $\varphi_{\mathbf{A}}(a, d_1, \ldots, d_n)_i = \varphi_{\mathbf{A}_i}(a_i, d_{1,i}, \ldots, d_{n,i})$, and similarly for $\varphi_{\mathbf{A}}(b, d_1, \ldots, d_n)_i$. Moreover, as π_i is onto, we have that, for every $i \in I$, $\overline{c}_i$ ranges over $\prod_{z:r \in Z} A_{i,r}$, as $\overline{c}$ ranges over $\prod_{z:r \in Z} A_r$. Thus, for each $i \in I$, we have that $\Delta_{\mathbf{A}_i}(\varphi_{\mathbf{A}_i}(a_i, d_{1,i}, \ldots, d_{n,i}), \varphi_{\mathbf{A}_i}(b_i, d_{1,i}, \ldots, d_{n,i}), \overline{c}_i) \subseteq D_i$ for every $\overline{c}_i \in \prod_{z:r \in Z} A_{i,r}$. So, using Lemma 2.1 again, we have that $\langle \varphi_{\mathbf{A}_i}(a_i, d_{1,i}, \ldots, d_{n,i}), \varphi_{\mathbf{A}_i}(b_i, d_{1,i}, \ldots, d_{n,i}) \rangle \in \mathbf{\Omega}_{\Gamma,\phi}^{\mathbf{A}_i}(D_i)$. As this is true for every $\varphi(x_0 : s, x_1 : s_1, \ldots, x_n : s_n) \in L_\Gamma(X)$, and $d_{1,i} \in A_{i,s_1}, \ldots, d_{n,i} \in A_{i,s_n}$ range over all possible values as $d_1, \ldots, d_n$ vary due to the fact that π_i is onto, we can conclude using Theorem 2.1 that $\langle a_i, b_i \rangle \in \mathbf{\Omega}_{\Gamma,s}^{\mathbf{A}_i}(D_i)$. Since each ϑ_i is reduced we have that $a_i = b_i$ for every $i \in I$, and we can conclude that $a = b$.

To prove the converse, let $\Phi_1, \Phi_2 \in \mathit{Th}_{\mathcal{L}}$ be two theories such that $\Phi_1 \subseteq \Phi_2$. Consider the reduced Lindenbaum Γ-valuations $\vartheta_\Gamma^{*\Phi_1}$ and $\vartheta_\Gamma^{*\Phi_2}$, and let $\theta = \mathbf{\Omega}_\Gamma(\Phi_1) \cap \mathbf{\Omega}_\Gamma(\Phi_2)$. We can define a Γ-valuation $\vartheta_\theta = \langle (\mathbf{T}_\Sigma(\mathbf{X})|_\Gamma)/_\theta, [\Phi_1]_\theta, [\_]_\theta \rangle$. We have that ϑ_θ is isomorphic to a subdirect product of $\vartheta_\Gamma^{*\Phi_1}$ and $\vartheta_\Gamma^{*\Phi_2}$ by the mapping $[t]_\theta \mapsto \langle [t]_{\mathbf{\Omega}_\Gamma(\Phi_1)}, [t]_{\mathbf{\Omega}_\Gamma(\Phi_2)} \rangle$. Since $\mathrm{MVal}_\Gamma^*(\mathcal{L})$ is closed under subdirect products and $\vartheta_\Gamma^{*\Phi_1}, \vartheta_\Gamma^{*\Phi_2} \in \mathrm{MVal}_\Gamma^*(\mathcal{L})$ we can conclude that $\vartheta_\theta \in \mathrm{MVal}_\Gamma^*(\mathcal{L})$. So, $\theta = \mathbf{\Omega}_\Gamma(\Phi_1) \cap \mathbf{\Omega}_\Gamma(\Phi_2) = \mathbf{\Omega}_\Gamma(\Phi_1)$ and therefore, we can conclude that $\mathbf{\Omega}_{\Gamma,\phi}(\Phi_1) \subseteq \mathbf{\Omega}_{\Gamma,\phi}(\Phi_2)$. Using Theorem 2.1 we can conclude that $\mathcal{L}$ is Γ-behaviorally protoalgebraic. $\qquad\square$

Let us now consider that $\mathcal{L}$ is a Γ-behaviorally algebraizable with behaviorally equivalent algebraic semantics $\mathbb{K}$ and defining equations $\Theta = \{\delta^i \approx \epsilon^i : i \in I\}$. As advanced in [7], one important consequence of this assumption is that given a Σ^o-algebra $\mathbf{A} \in \mathbb{K}$, and by setting $D_{\mathbf{A}} = \{a \in A_\phi : \delta_{\mathbf{A}}^i(a) \equiv_\Gamma \epsilon_{\mathbf{A}}^i(a)$ for every $i \in I\}$, we can recover a filter such that $M_{\mathbf{A}} = \langle \mathbf{A}|_\Sigma, D_{\mathbf{A}} \rangle \in \mathrm{Matr}(\mathcal{L})$. We define

$$\mathrm{MVal}_{\Gamma,\mathbb{K}} = \{\vartheta_{M_{\mathbf{A}},h} : \mathbf{A} \in \mathbb{K} \text{ and } h \text{ is an assignment over } M_{\mathbf{A}}\}$$

and

$$\mathrm{MVal}_{\Gamma,\mathbb{K}}^* = \{\vartheta_{M_{\mathbf{A}},h}^* : \mathbf{A} \in \mathbb{K} \text{ and } h \text{ is an assignment over } M_{\mathbf{A}}\}.$$

We can now prove a sequence of results relating the behavioral consequence associated with $\mathbb{K}$ and the corresponding valuation semantics. The results generalize well-known bridge results linking matrix semantics with traditional algebraization [13].

Theorem 4.2. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a Γ-behaviorally algebraizable logic with behaviorally equivalent algebraic semantics $\mathbb{K}$. Then, $\mathit{MVal}_\Gamma^*(\mathcal{L}) = \mathit{MVal}_{\Gamma,\mathbb{K}}^*(\mathcal{L})$.*

Proof:
The fact that $\mathrm{MVal}^*_{\Gamma,\mathbb{K}}(\mathcal{L}) \subseteq \mathrm{MVal}^*_{\Gamma}(\mathcal{L})$ follows easily from the definition of $\mathrm{MVal}^*_{\Gamma,\mathbb{K}}(\mathcal{L})$.

Let us now prove the other inclusion. Let $\vartheta = \langle A, D, h \rangle \in \mathrm{MVal}^*_{\Gamma}(\mathcal{L})$. Then, there exists $M = \langle \mathbf{B}, D' \rangle \in \mathrm{Matr}(\mathcal{L})$ and an homomorphism h' over M such that $\vartheta = \vartheta^*_{M,h'}$. Consider the Σ^o-algebra $\mathbf{B}^o$ obtained from M by setting $\mathbf{B}^o|_{\Sigma} = \mathbf{B}$, and $B^o_v = \{[a]_{\mathbf{\Omega}^{\mathbf{B}}_{\Gamma}(D')} : a \in B_\phi\}$ and $o_{\mathbf{B}^o} = \lfloor\_\rfloor_{\mathbf{\Omega}^{\mathbf{B}}_{\Gamma}(D')}$. As an easy consequence of the construction of $\mathbf{B}^o$ from M we have that $M, h' \Vdash \varphi$ iff $\mathbf{B}^o, h', \Vdash^{\Gamma} \Theta(\varphi)$, for every $\varphi \in L_\Sigma(X)$. From this fact, since $\mathcal{L}$ is Γ-behaviorally algebraizable, it easily follows that $\mathbf{B}^o \in \mathbb{K}$. We aim to prove that $\vartheta = \vartheta^*_{M_{\mathbf{B}^o},h'}$, thus showing that $\vartheta \in \mathrm{MVal}^*_{\Gamma,\mathbb{K}}(\mathcal{L})$. This follows easily after we prove that $M_{\mathbf{B}^o} = M$. To prove this we start by recalling that $M_{\mathbf{B}^o} = \langle \mathbf{B}^o|_{\Sigma}, D_{\mathbf{B}^o}\rangle$, where $D_{\mathbf{B}^o} = \{a \in B_\phi : \delta_{\mathbf{B}^o}(a) \equiv_\Gamma \epsilon_{\mathbf{B}^o}(a)$ for every $\delta \approx \epsilon \in \Theta(x : \phi)\}$. It is clear that, by construction, $\mathbf{B}^o|_{\Sigma} = \mathbf{B}$. What remains to be proved is that $D_{\mathbf{B}^o} = D'$. Assuming that $\mathcal{L}$ is Γ-behaviorally algebraizable with defining equations $\Theta(x : \phi)$ and equivalence formulas $\Delta(x : \phi, y : \phi)$, we have that $x \dashv\vdash \Delta[\Theta(x)]$. Therefore, since $M \in \mathrm{Matr}(\mathcal{L})$ we have that $a \in D'$ iff $\Delta_{\mathbf{B}}(\delta_{\mathbf{B}}(a), \epsilon_{\mathbf{B}}(a)) \subseteq D'$ for every $\delta \approx \epsilon \in \Theta(x : \phi)$ iff $\langle \delta_{\mathbf{B}}(a), \epsilon_{\mathbf{B}}(a)\rangle \in \mathbf{\Omega}^{\mathbf{B}}_{\Gamma,\phi}(D')$ for every $\delta \approx \epsilon \in \Theta(x : \phi)$ iff $a \in D_{\mathbf{B}^o}$. $\qquad\square$

5. Conclusion

We have explored an algebraic based notion of valuation semantics arising naturally in semantical considerations in the behavioral approach to the algebraization of logics, and proposed the class $\mathrm{MVal}^*_{\Gamma}(\mathcal{L})$ of valuations that should be canonically associated with a logic $\mathcal{L}$. The results obtained, which generalize well known properties of matrix semantics and of reduced matrix models in traditional algebraic logic, are intended as a trigger towards the development of a consistent algebraic theory of logical valuations. That is to say that further meaningful results relating properties of the class $\mathrm{MVal}^*_{\Gamma}(\mathcal{L})$ with properties of the logic $\mathcal{L}$ are expected. The references [2, 13, 20], among others, will provide the essential guidelines for future work, in this respect. In particular, as hinted by the example of the logic $\mathcal{C}_1$ in [6], we expect to be able to obtain a general result that will allow us to construct an algebraic specification of $\mathrm{MVal}^*_{\Gamma}(\mathcal{L})$ from an axiomatization of $\mathcal{L}$. The recovery of the non-truth-functional bivaluation semantics of $\mathcal{C}_1$ from the Boolean-based algebraic valuations obtained by the behavioral algebraization process also suggest that a systematic study of the Birkhoff-like operations over valuations is crucial.

Of course, we could also associate a meaningful class of algebras to each logic, by taking advantage of its valuation semantics, as is done in the traditional setting. However, these would necessarily have to be algebras over the extended signature. Thus, we might speculate, here, that the difficulties raised by the fact that the model-theory of behavioral equational logic remains relatively unexplored (see [17]), are perhaps a hint that meaningful bridge results might be simpler to obtain using the more intuitive valuation semantics proposed in this paper.

Alternatives to the proposed valuation semantics should also be carefully inspected. In particular, we would like to have characterization results for the classes of valuations resulting from non-deterministic matrices [1], both in the static and dynamic versions, or gaggles [11], as well as bridges to the classes of logics that they characterize. We hope to report on these and related questions in forthcoming papers.

References

[1] Avron, A.: Non-deterministic matrices and modular semantics of rules, in: *Logica universalis* (J.-Y. Béziau, Ed.), Birkhäuser Verlag, 2005, 149–167.

[2] Blok, W., Pigozzi, D.: Algebraizable Logics, *Memoirs of the AMS*, **77**(396), 1989.

[3] Bloom, S. L.: Some theorems on structural consequence operations., *Studia Logica*, **34**, 1975, 1–9.

[4] Caleiro, C., Carnielli, W. A., Coniglio, M. E., Marcos, J.: Two's company: The humbug of many logical values, in: *Logica Universalis* (J.-Y. Béziau, Ed.), Birkhäuser Verlag, 2005, 169–189.

[5] Caleiro, C., Gonçalves, R.: Algebraic valuations as behavioral logical matrices, in: *WoLLIC 2009, Selected Papers* (M. Kanazawa, H. Ono, R. de Queiroz, Eds.), vol. 5514 of *Lecture Notes in Artificial Intelligence*, Springer-Verlag, 2009, 13–25.

[6] Caleiro, C., Gonçalves, R.: Behavioral algebraization of da Costa's $\mathcal{C}$-Systems, *Journal of Applied Non-Classical Logics*, **19**(2), 2009, 127–148.

[7] Caleiro, C., Gonçalves, R., Martins, M.: Behavioral algebraization of logics, *Studia Logica*, **91**(1), 2009, 63–111.

[8] da Costa, N.: On the theory of inconsistent formal systems, *Notre Dame Journal of Formal Logic*, **15**, 1974, 497–510.

[9] da Costa, N., Alves, E. H.: A semantical analysis of the calculi $\mathcal{C}_n$., *Notre Dame Journal of Formal Logic*, **18**, 1977, 621–630.

[10] da Costa, N., Béziau, J.-Y.: Théorie de la valuation, *Logique et Analyse*, **37**(146), 1994, 95–117.

[11] Dunn, J. M.: Gaggle theory: an abstraction of Galois connections and residuation, with applications to negation, implication, and various logical operators, in: *Logics in AI*, vol. 478 of *Lecture Notes in Computer Science*, Springer-Verlag, 1991, 31–51.

[12] Font, J.: Taking degrees of truth seriously, *Studia Logica*, in print.

[13] Font, J., Jansana, R., Pigozzi, D.: A Survey of Abstract Algebraic Logic, *Studia Logica*, **74**(1–2), 2003, 13–97.

[14] Goguen, J., Malcolm, G.: A hidden agenda, *Theoretical Computer Science*, **245**(1), 2000, 55–101.

[15] Łoś, J., Suszko, R.: Remarks on sentential logics, *Indagationes Mathematicae*, **20**, 1958, 177–183.

[16] Reichel, H.: Behavioural validity of conditional equations in abstract data types, in: *Contributions to general algebra 3, Proc. Conf., Vienna 1984*, 1985, 301–324.

[17] Roşu, G.: A Birkhoff-like Axiomatizability Result for Hidden Algebra and Coalgebra, *Electronic Notes in Theoretical Computer Science*, **11**, 1998, 179–196.

[18] Roşu, G.: *Hidden Logic*, Ph.D. Thesis, University of California at San Diego, 2000.

[19] Roşu, G.: Behavioral abstraction is hiding information, *Theoretical Computer Science*, **327**(1-2), 2004, 197–221.

[20] Wójcicki, R.: *Theory of Logical Calculi*, Synthese Library, Kluwer Academic Publishers, 1988.