Combining classical and intuitionistic implications *

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Abstract. We present a simple logic that combines, in a conservative way, the implicative fragments of both classical and intuitionistic logics, thus settling a problem posed by Dov Gabbay in [5]. We also show that the logic can be given a nice complete axiomatization by adding four simple mixed axioms to the usual axiomatizations of classical and intuitionistic implications.

1 Introduction

In [5], Dov Gabbay pointed out a difficulty with the fibring methodology for combining logics [6, 1] that became known as *the collapsing problem*. The argument given there suggested that there would be no way of combining classical and intuitionistic propositional logics, *CPL* and *IPL* respectively, in a way that would not collapse the intuitionistic connectives into classical ones. Following this spirit, Andreas Herzig and Luis Fariñas del Cerro have proposed in [4] a combined logic $C + J$ that starts from the expected combined semantic setting, that could be axiomatized, not by adding the axiomatizations of *CPL* and *IPL* together with some interaction rules, but rather by modifying these axioms along with their scope of applicability.

Actually, Gabbay's argument, which depends only on the implicative fragments of the logics, holds in fact for any logic combining classical and intuitionistic reasoning in a way that preserves the *deduction theorem* for both implications. However, a simple combination of the usual axiomatizations of classical and intuitionistic implications yields a system where none of the deduction theorems seems to hold. The question is how to prove this. A semantic argument would need to consider suitable, non-trivial, combined models. The natural place to look for such combined models would be to consider a *fibred semantics*. Still, a fibred model cannot do the job, because fibring classical and intuitionistic models together will only result in models where the intuitionistic implication becomes classical. Indeed, the semantic fibring of two logics does not always result in a conservative extension of the logics being combined. Looking for a way around this problem, we have proposed in [2, 3] an extension of the fibring methodology,

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called *cryptofibring*. In the new setting, we have obtained a more general notion of combined model and have managed to identify situations as the above mentioned collapsing problem as particular cases of non-conservative combinations. As a by-product, we have shown that a conservative extension of *CPL* and *IPL* is possible, and suggested a simple way to construct combined non-trivial models. A characterization of this (large) class of models, as well as a number of general results about cryptofibring, can be found in [2, 3].

In this paper we study in detail the implicative fragment of the combined logic characterized by a subclass of these models. We prove that the resulting logic, *CIPL*, which features both a classical and an intuitionistic implication, is a conservative extension of the implicative fragments of both classical and intuitionistic logics. We also show that *CIPL* can be given a nice complete axiomatization by adding four simple mixed axioms to the usual axiomatizations of classical and intuitionistic implications. These mixed axioms will guarantee, in particular, that the classical implication be strictly stronger than the intuitionistic one. We assume nothing but acquaintance with textbook logic. A useful source on intuitionistic logic is [12].

In Section 2, we review Gabbay's argument for the collapse when combining classical and intuitionistic logics, namely at the light of the logic $C+J$ of Herzig and del Cerro, and show how to overcome these difficulties by introducing the class of models for the combined logic *CIPL*. Then, in Section 3, we study a number of interesting properties of these models, that will eventually lead us to showing that *CIPL* is indeed a conservative extension of the implicative fragments of both *CPL* and *IPL*. Particular emphasis will be put on studying the interactions between classical and intuitionistic implications. Section 4 is devoted to proposing an axiomatization of *CIPL*, and to proving, using standard techniques, its soundness and completeness with respect to the semantics. We conclude, in Section 5, with a discussion of related and future work.

2 Combining models

Let us consider a set P of classical propositional symbols, and let us use $\Rightarrow$ to denote classical implication. It is well known that the implicative fragment of *CPL*, corresponding to the language defined by the grammar $\mathcal{L}_c ::= P \mid (\mathcal{L}_c \Rightarrow \mathcal{L}_c)$, can be characterized deductively by the axioms

(C1) $A \Rightarrow (B \Rightarrow A)$
(C2) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
(C3) $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$

and the inference rule

$$\frac{A \quad (A \Rightarrow B)}{B} \quad (\mathbf{CMP}).$$

Note that **C3** is *Peirce's law* and **CMP** the rule of *Modus Ponens*. We use $\vdash_c$ to denote the resulting deductive consequence relation, that is, we write $\Gamma \vdash_c A$

to denote the fact that there exists a derivation of A from the set of hypotheses Γ using **C1-C3** and **CMP**.

Analogously, let us consider a set Q of intuitionistic propositional symbols, and let us use $\rightarrow$ to denote classical implication. The implicative fragment of *IPL*, corresponding to the language defined by $\mathcal{L}_i ::= Q \mid (\mathcal{L}_i \rightarrow \mathcal{L}_i)$, is also well known to be characterized deductively by the axioms

$$\begin{array}{ll} \textbf{(I1)} & A \rightarrow (B \rightarrow A) \\ \textbf{(I2)} & (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \end{array}$$

and the inference rule

$$\frac{A \quad (A \rightarrow B)}{B} \qquad (\textbf{IMP}).$$

Comparing with the classical case, we have just excluded Peirce's law and rephrased the first two axioms and Modus Ponens using the intuitionistic arrow. We use now $\vdash_i$ to denote the resulting deductive consequence relation. Obviously, classical implication is stronger than intuitionistic implication. In particular, we have $\vdash_c ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ but $\nvdash_i ((A \rightarrow B) \rightarrow A) \rightarrow A$. Recall also that both implications enjoy the *deduction theorem*, that is

- $\Gamma, A \vdash_c B$ iff $\Gamma \vdash_c A \Rightarrow B$, where $\Gamma \cup \{A, B\} \subseteq \mathcal{L}_c$, and
- $\Gamma, A \vdash_i B$ iff $\Gamma \vdash_i A \rightarrow B$, where $\Gamma \cup \{A, B\} \subseteq \mathcal{L}_i$.

In order to combine the two fragments of *CPL* and *IPL*, let us consider adding together the axioms **C1-C3**, **I1-I2**, and the rules **CMP**, **IMP**, now over the combined language given by $\mathcal{L} ::= P \mid Q \mid (\mathcal{L} \Rightarrow \mathcal{L}) \mid (\mathcal{L} \rightarrow \mathcal{L})$. Henceforth, we will assume that the sets of classical and intuitionistic propositional symbols are disjoint, that is, $P \cap Q = \emptyset$. Denoting by $\vdash$ the resulting consequence relation, it will be reasonable to expect that $A \rightarrow B \vdash A \Rightarrow B$. However, the converse would be highly undesirable, not only because classical implication should be strictly stronger than intuitionistic implication, but also because the two would collapse. Gabbay's argument for the collapse of $\rightarrow$ into $\Rightarrow$ after the combination [5] was based on the assumption that the deduction theorem of each of the two implications in isolation would be transported to their combination. If that was the case, since we have $A \Rightarrow B, A \vdash B$ simply by using **CMP**, we could use the deduction theorem for $\rightarrow$ and immediately obtain $A \Rightarrow B \vdash A \rightarrow B$. Still, in this setting, it is not at all obvious that we can still use the deduction theorem over the combined language. Actually, although $\vdash A \rightarrow B$ implies $\vdash A \Rightarrow B$, it is even unclear how to obtain $A \rightarrow B \vdash A \Rightarrow B$ in the first place. If we take a little time trying to prove any meaningful interaction between $\rightarrow$ and $\Rightarrow$ using $\vdash$ we will soon be convinced that most probably the two implications do not collapse. Clearly, it would suffice to prove that $\nvdash ((A \rightarrow B) \rightarrow A) \rightarrow A$. Of course, we might try to prove this using some sort of combinatory argument over the possible deductions. But it will be much more enlightening to try and use a semantic argument, that is, to look for some sort of combined model m that falsifies $((A \rightarrow B) \rightarrow A) \rightarrow A$.

Let $\mathcal{M}_c$ stand for the class of all classical two-valued models. That is, a model in $\mathcal{M}_c$ is simply a function $v : P \to \{\bot, \top\}$ where, as usual, we define satisfaction of classical formulas inductively by:

- $v \Vdash_c p$ iff $v(p) = \top$;
- $v \Vdash_c A \Rightarrow B$ iff $v \nVdash_c A$ or $v \Vdash_c B$.

We will use $\vDash_c$ to denote the induced *entailment relation*, that is, given $\Gamma \cup \{A\} \subseteq \mathcal{L}_c$, we have $\Gamma \vDash_c A$ provided that $v \Vdash \Gamma$ implies $v \Vdash A$ for every $v \in \mathcal{M}_c$. It will be helpful to view classical models as *logical matrices* [13]. Given $v \in \mathcal{M}_c$, its associated matrix is $M(v) = \langle\{\bot, \top\}, \top, \cdot_v\rangle$, where $\{\bot, \top\}$ is the set of possible *truth-values*; $\top$ is the *designated* value; $p_v = v(p)$; and $\Rightarrow_v$ is given by the usual truth table, shown below.

$\Rightarrow_v$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$

Clearly, v satisfies the formula A if and only $A_v = \top$, the designated truth-value.

For intuitionistic logic, we shall consider $\mathcal{M}_i$ to be the usual class of *rooted Kripke models*. That is, a model $k \in \mathcal{M}_i$ is a tuple $k = \langle W, \leq, V\rangle$ where W is a nonempty set partially ordered by $\leq$ and with a least element, that we will denote by w_0, and $V : Q \to \mathcal{U}_\leq$ is a function, where $\mathcal{U}_\leq$ is the set of all *uppersets* of $\langle W, \leq\rangle$, that is, all sets $U \subseteq W$ such that if $w \in U$ and $w \leq w'$ then also $w' \in U$. Recall that $k \Vdash_i A$ iff $(k, w) \Vdash_i A$ for every $w \in W$, where the local satisfaction relation at a fixed world w is defined inductively by

- $(k, w) \Vdash_i q$ iff $w \in V(q)$;
- $(k, w) \Vdash_i A \to B$ iff for every $w' \geq w$, $(k, w') \nVdash_i A$ or $(k, w') \Vdash_i B$.

As above, we will use $\vDash_i$ to denote the induced *entailment relation*. Given $\Gamma \cup \{A\} \subseteq \mathcal{L}_i$, we have $\Gamma \vDash_i A$ provided that $k \Vdash \Gamma$ implies $k \Vdash A$ for every $k \in \mathcal{M}_i$. The logical matrix corresponding to an intuitionistic model $k = \langle W, \leq, V\rangle \in \mathcal{M}_i$, is $M(k) = \langle\mathcal{U}_\leq, W, \cdot_k\rangle$, where the truth-values are the uppersets; the designated upperset is W; $q_k = V(q)$; and $U_1 \to_k U_2 = \{w \in W :$ if $w \leq w'$ and $w' \in U_1$ then $w' \in U_2\}$. It is clear that k satisfies the formula A if and only $A_k = W$.

It is obvious that the sort of combined models for $C+J$ considered by Herzig and del Cerro in [4] does not fit our purposes, since their target was rather to stick with the obvious way of extending the intuitionistic models with an interpretation for classical implication, as is done in the usual Kripke structures for modal logic, and to axiomatize these models using adapted versions of the original axioms. Concretely, they considered an extended satisfaction relation $\Vdash$ over an intuitionistic model k such that

- $(k, w) \Vdash A \Rightarrow B$ iff $(k, w) \nVdash A$ or $(k, w) \Vdash B$.

However, as they show, these models will turn the axiomatization of intuitionistic implication unsound.

A second alternative would be to look for a convenient combined model obtained by *fibring* [6, 1] a classical and an intuitionistic model. However, this is well-known to fail too because every fibred model will collapse the two implications. In order to understand why this happens, it suffices to note that fibring $v \in \mathcal{M}_c$ and $k \in \mathcal{M}_i$ requires that both models, seen as logical matrices, share the same set of truth-values and the same designated value (modulo possible renaming). Since every possible v yields exactly two truth-values, we get stuck with intuitionistic models $k = \langle W, \leq, V \rangle$ such that $\mathcal{U}_{\leq}$ has exactly two elements: $\emptyset$ and W. This means that there is only one world, i.e., $W = \{w_0\}$. By letting $\bot = \emptyset$ and $\top = W$ we can then get a fibred model, but one where intuitionistic implication is interpreted as $\rightarrow_k$, whose classical truth-table is shown below.

$\rightarrow_k$	$\emptyset$	W
$\emptyset$	W	W
W	$\emptyset$	W

However, as explained in [3], cryptofibring offers another option. We need a combined model m that behaves like v and k on the classical and intuitionistic fragments, respectively, but which is as free as possible on combined formulas, that is, a model m that extends k with an interpretation for $\Rightarrow$ that behaves as v on $\bot = \emptyset$ and $\top = W$. The model should also guarantee the persistence requirement that is essential for the soundness of the intuitionistic axioms. Clearly, $\rightarrow_k$ as defined above always yields an upperset. It is easy to see that $U_1 \rightarrow_k U_2$ is precisely the largest upperset contained in $X = (W \setminus U_1) \cup U_2$, like the *interior* operation in a topological space. We just compute X and get rid of the worlds $w \in X$ for which there exists w' such that $w \leq w'$ but $w' \notin X$. Still, there are other ways of achieving this. Our proposal is to interpret classical implication in such a way that $U_1 \Rightarrow_m U_2$ is the least upperset containing X, mimicking now the *closure* operation in a topological space. That is, we just compute X and add w' provided that there exists $w \in X$ such that $w \leq w'$ and $w' \notin X$. To interpret the classical propositional symbols we must add a copy of the classical valuation v to each world of k. The interpretation of classical symbols will be persistent in a very strong way: either p holds at all worlds, or at none.

Definition 1 (Combined models).
A *combined model* for the language $\mathcal{L}$ is a tuple $m = \langle W, \leq, \overline{V} \rangle$ where $\langle W, \leq \rangle$ is a partial order with a least element and $\overline{V} : P \cup Q \rightarrow \wp(W)$ is such that

- $\overline{V}(p) \in \{\emptyset, W\}$ for every $p \in P$;
- $\overline{V}(q) \in \mathcal{U}_{\leq}$ for every $q \in Q$.

The satisfaction of a formula $A \in \mathcal{L}$ is defined by $m \Vdash A$ if and only if $(m, w) \Vdash A$ for every $w \in W$, where the satisfaction at a world is defined inductively by

- $(m, w) \Vdash p$ iff $w \in \overline{V}(p)$;
- $(m, w) \Vdash q$ iff $w \in \overline{V}(q)$;
- $(m, w) \Vdash A \Rightarrow B$ iff there exists $w' \leq w$ such that $(m, w') \nVdash A$ or $m, w' \Vdash B$;
- $(m, w) \Vdash A \rightarrow B$ iff for every $w' \geq w$, $(m, w') \nVdash A$ or $(m, w') \Vdash B$.

We denote by $\mathcal{M}$ the class of all combined models.

We now define the semantics of our combined logic *CIPL*. Let $\Gamma \cup \{A\} \subseteq \mathcal{L}$.

Definition 2 (*CIPL*).
The *entailment relation* $\vDash$ of *CIPL* over the language $\mathcal{L}$ is defined, as usual, from satisfaction: $\Gamma \vDash A$ if $m \Vdash \Gamma$ implies $m \Vdash A$ for every $m \in \mathcal{M}$.

Our work, in the next sections, will be to study *CIPL* in detail.

3 Interacting implications

To start our study of *CIPL*, via a semantic analysis of the class of models $\mathcal{M}$, we first note that there is a direct correspondence between combined models and pairs of classical and intuitionistic models. Given a combined model $m = \langle W, \leq, \overline{V} \rangle$, we can define classical and intuitionistic models

- $m|_c : P \to \{\bot, \top\}$ with $m|_c(p) = \begin{cases} \top \text{ if } \overline{V}(p) = W \\ \bot \text{ if } \overline{V}(p) = \emptyset \end{cases}$; and
- $m|_i = \langle W, \leq, \overline{V}|_Q \rangle$.

Conversely, given $v \in \mathcal{M}_c$ and $k = \langle W, \leq, V \rangle$ we can define a combined model

- $v \oplus k = \langle W, \leq, \overline{V} \rangle$ with $\overline{V}(p) = \begin{cases} W \text{ if } v(p) = \top \\ \emptyset \text{ if } v(p) = \bot \end{cases}$ and $\overline{V}(q) = V(q)$.

Proposition 1. *Let $m \in \mathcal{M}$, $v \in \mathcal{M}_c$ and $k \in \mathcal{M}_i$. We have:*

1. *$m|_c \in \mathcal{M}_c$ and $m|_i \in \mathcal{M}_i$;*
2. *$v \oplus k \in \mathcal{M}$;*
3. *$m|_c \oplus m|_i = m$;*
4. *$(v \oplus k)|_c = v$ and $(v \oplus k)|_i = k$.*

As a consequence, $\mathcal{M} = \{v \oplus k : v \in \mathcal{M}_c, k \in \mathcal{M}_i\}$.

Proof. The properties 1–4 are all straightforward from the definitions. The fact $\mathcal{M} = \{v \oplus k : v \in \mathcal{M}_c, k \in \mathcal{M}_i\}$ is a simple consequence of these properties. □

A simple observation is that a combined model $v \oplus k$ indeed extends v and k, in the respective fragments.

Proposition 2. *Let $m \in \mathcal{M}$, $x \in \{c, i\}$ and $A \in \mathcal{L}_x$. We have $m \Vdash A$ if and only if $m|_x \Vdash_x A$.*

Proof. We first prove the result for $x = c$. Given any $w \in W$, it suffices to show that $(m, w) \Vdash A$ if and only if $m|_c \Vdash_c A$. The proof follows by induction on $A \in \mathcal{L}_c$. For a classical propositional symbol p, we have $(m, w) \Vdash p$ iff $w \in \overline{V}(p)$ iff $\overline{V}(p) \neq \emptyset$ iff $\overline{V}(p) = W$ iff $m|_c(p) = \top$ iff $m|_c \Vdash_c p$. Consider now the case $A \Rightarrow B \in \mathcal{L}_c$. Then, $(m, w) \Vdash A \Rightarrow B$ iff there exists $w' \leq w$ such that $(m, w') \nVdash A$ or $(m, w') \Vdash B$ iff, by induction hypothesis, $m|_c \nVdash_c A$ or $m|_c \Vdash_c B$ iff $m|_c \Vdash_c A \Rightarrow B$.

The proof for $x = i$ and $A \in \mathcal{L}_i$ is similar. Given any $w \in W$, it suffices to show that $(m, w) \Vdash A$ if and only if $(m|_i, w) \Vdash_i A$. The proof follows by induction on A. Consider first the case of an intuitionistic propositional symbol q. Then, $(m, w) \Vdash q$ iff $w \in \overline{V}(q)$ iff $w \in \overline{V}|_Q(q)$ iff $(m|_i, w) \Vdash_i q$. Consider now the case $A \rightarrow B \in \mathcal{L}_i$. Then, $(m, w) \Vdash A \rightarrow B$ iff $(m, w') \nVdash A$ or $(m, w') \Vdash B$, for every $w' \geq w$, iff, by induction hypothesis, $(m|_i, w') \nVdash_i A$ or $(m|_i, w') \Vdash_i B$, for every $w' \geq w$, iff $(m|_i, w) \Vdash_i A \rightarrow B$. □

What we already know is enough to prove that *CIPL* is indeed a conservative extension of both the implicative fragments of *CPL* and *IPL*.

Theorem 1 (Conservativeness). *Let $x \in \{c, i\}$ and $\Gamma \cup \{A\} \subseteq \mathcal{L}_x$. We have $\Gamma \vDash A$ if and only if $\Gamma \vDash_x A$.*

Proof. The proof of the left-to-right implication is a direct application of Propositions 1–2, using the fact that both $\mathcal{M}_c \neq \emptyset$ and $\mathcal{M}_i \neq \emptyset$. Namely, for $x = c$, assume that $\Gamma \cup \{A\} \subseteq \mathcal{L}_c$ and $\Gamma \vDash A$. Let $v \in \mathcal{M}_c$ be such that $v \Vdash_c \Gamma$. Since $\mathcal{M}_i \neq \emptyset$ we can pick any intuitionistic model $k \in \mathcal{M}_i$ and build $v \oplus k \in \mathcal{M}$, according to Proposition 1. Hence, by using Proposition 2, we also know that $v \oplus k \Vdash \Gamma$ and by definition of entailment it follows that $v \oplus k \Vdash A$. Again using Proposition 2 we can conclude that $v \Vdash_c A$ and thus $\Gamma \vDash_c A$. The proof for $x = i$ is analogous.

The right-to-left implication is just a consequence of Proposition 2. □

Let us now investigate, in detail, the properties of our combined models. Recall that we claimed, as a justification for their definition, that even the interpretation of classical formulas would be persistent.

Theorem 2 (Persistence).
Let $m = \langle W, \leq, \overline{V} \rangle \in \mathcal{M}$, $w \in W$ and $A \in \mathcal{L}$. If $(m, w) \Vdash A$ and $w \leq w'$ then $(m, w') \Vdash A$.

Proof. The proof follows by case analysis on the structure of formulas. The cases of atomic formulas, both classical and intuitionistic, follow straightforwardly from the definition of $\overline{V}$.

If $(m, w) \Vdash A \Rightarrow B$ then there exists $w'' \leq w$ such that $(m, w'') \nVdash A$ or $(m, w'') \Vdash B$. But, $w'' \leq w'$ and so $(m, w') \Vdash A \Rightarrow B$.

If $(m, w) \Vdash A \rightarrow B$ then either $(m, w'') \nVdash A$ or $(m, w'') \Vdash B$, for all $w'' \geq w$. It is straightforward to see that for any $w' \geq w$, $(m, w') \Vdash A \rightarrow B$. □

Note that, as a corollary of persistence, the satisfaction of any formula $A \in \mathcal{L}$ by a model m can be simply checked at the root world w_0.

Corollary 1 (Satisfaction at the root).
Let $m \in \mathcal{M}$ and $A \in \mathcal{L}$. We have $m \Vdash A$ if and only if $(m, w_0) \Vdash A$.

Another direct consequence of Theorem 2 is that the satisfaction of formulas involving classical implication can be simplified.

Corollary 2 (Satisfaction of classical implication).
Let $A, B \in \mathcal{L}$*,* $m \in \mathcal{M}$ *and* $w \in W$*. We have* $(m, w) \Vdash A \Rightarrow B$ *if and only if* $(m, w_0) \nVdash A$ *or* $(m, w) \Vdash B$*.*

As should be expected, formulas in the classical fragment are even more than persistent, they are constant.

Proposition 3. *Let* $m = \langle W, \leq, \overline{V} \rangle \in \mathcal{M}$*,* $w, w' \in W$ *and* $A \in \mathcal{L}_c$*. We have* $(m, w) \Vdash A$ *if and only if* $(m, w') \Vdash A$*.*

Proof. The proof follows by induction on the structure of A. The case of atomic formulas is straightforward from the definition of $\overline{V}$. Consider $A \Rightarrow B \in \mathcal{L}_c$. Then, $(m, w) \Vdash A \Rightarrow B$ iff $(m, w_0) \nVdash A$ or $(m, w) \Vdash B$ iff, by induction hypothesis, $(m, w_0) \nVdash A$ or $(m, w') \Vdash B$ iff $(m, w') \Vdash A \Rightarrow B$. □

We can now start to inspect closely the relationship between the two implications. As desired, we will show that classical implication is strictly stronger than intuitionistic implication.

Proposition 4. *Let* $A, B \in \mathcal{L}$*,* $m = \langle W, \leq, \overline{V} \rangle \in \mathcal{M}$ *and* $w \in W$*. If* $(m, w) \Vdash A \rightarrow B$ *then* $(m, w) \Vdash A \Rightarrow B$*. As a consequence, we have* $A \rightarrow B \vDash A \Rightarrow B$*.*

Proof. Assume that $(m, w) \Vdash A \rightarrow B$. Then, for every $w' \geq w$, either $(m, w') \nVdash A$ or $(m, w') \Vdash B$. Assume now, by absurd, that $(m, w) \nVdash A \Rightarrow B$. Then, $(m, w_0) \Vdash A$ and $(m, w) \nVdash B$. Since $(m, w_0) \Vdash A$ and $w_0 \leq w$, by persistence, we must also have $(m, w') \Vdash A$. Therefore, it must be the case that $(m, w') \Vdash B$ for every $w' \geq w$. But then, if we let $w' = w$ we obtain $(m, w) \Vdash B$, a contradiction. □

The converse of the previous result does not hold, in general. Consider two intuitionistic propositional symbols q_1 and q_2, and take the model $m = \langle \{w_0, w_1\}, \leq, \overline{V} \rangle$ such that $w_0 \leq w_1$, $\overline{V}(q_1) = \{w_1\}$ and $\overline{V}(q_2) = \emptyset$. It is straightforward to see that $m \Vdash q_1 \Rightarrow q_2$ but $m \nVdash q_1 \rightarrow q_2$. Consequently, he have that $q_1 \Rightarrow q_2 \nvDash q_1 \rightarrow q_2$. Still, there are certain particular situations in which the two implications coincide. A simple sufficient condition is that A be a classical formula.

Proposition 5. *Let* $A \in \mathcal{L}_c$*,* $B \in \mathcal{L}$*,* $m = \langle W, \leq, \overline{V} \rangle \in \mathcal{M}$ *and* $w \in W$*. Then,* $(m, w) \Vdash (A \Rightarrow B) \rightarrow (A \rightarrow B)$*. As a consequence, we have* $\vDash (A \Rightarrow B) \rightarrow (A \rightarrow B)$*.*

Proof. Suppose, by absurd, that $(m, w) \nVdash (A \Rightarrow B) \rightarrow (A \rightarrow B)$, for A classic. Then, there exists $w' \geq w$ such that $(m, w') \Vdash A \Rightarrow B$ and $(m, w') \nVdash A \rightarrow B$. On one hand, there exists $w'' \geq w'$ such that $(m, w'') \Vdash A$ and $(m, w'') \nVdash B$. On the other hand, either $(m, w_0) \nVdash A$ or $(m, w') \Vdash B$. As A is classic, by Proposition 3, the first condition is not possible. By Theorem 2, the second condition is also impossible. Hence, we have a contradiction. □

Note that, as a consequence of the previous result, if A is classical then $A \Rightarrow B \vDash A \rightarrow B$. The essential ingredient of the proof is that the value of classical formulas does not change as one goes up the order on worlds. Actually, even if A is not classical, a similar situation arises as long as the value of A can be guaranteed not to change, that is, if A holds.

Proposition 6. *Let $A, B \in \mathcal{L}$, $m = \langle W, \leq, \overline{V} \rangle \in \mathcal{M}$ and $w \in W$. Then, $(m, w) \Vdash A \Rightarrow ((A \Rightarrow B) \rightarrow (A \rightarrow B))$. As a consequence, $\vDash A \Rightarrow ((A \Rightarrow B) \rightarrow (A \rightarrow B))$.*

Proof. Suppose, by absurd, that $(m, w) \nVdash A \Rightarrow ((A \Rightarrow B) \rightarrow (A \rightarrow B))$. Then, $(m, w_0) \Vdash A$ and $(m, w) \nVdash (A \Rightarrow B) \rightarrow (A \rightarrow B)$. Hence, there exists $w' \geq w$ such that $(m, w') \Vdash A \Rightarrow B$ and $(m, w') \nVdash A \rightarrow B$. On one hand, there exists $w'' \geq w'$ such that $(m, w'') \Vdash A$ and $(m, w'') \nVdash B$. On the other hand, either $(m, w_0) \nVdash A$ or $(m, w') \Vdash B$. The first condition is clearly impossible. The second condition is also impossible, by Theorem 2, since $w \leq w' \leq w''$. □

The two implication connectives further interact in a number of interesting ways. Namely, note that if A holds in a world then also does $B \Rightarrow A$.

Proposition 7. *Let $A, B \in \mathcal{L}$, $m = \langle W, \leq, \overline{V} \rangle \in \mathcal{M}$ and $w \in W$. Then, $(m, w) \Vdash A \rightarrow (B \Rightarrow A)$. As a consequence we have $\vDash A \rightarrow (B \Rightarrow A)$.*

Proof. Assume, by absurd, that $(m, w) \nVdash A \rightarrow (B \Rightarrow A)$. Then, there exists $w' \geq w$ such that $(m, w') \Vdash A$ and $(m, w') \nVdash B \Rightarrow A$. Hence, $(m, w_0) \Vdash B$ and, more importantly, $(m, w') \nVdash A$ which is a contradiction. □

Another interesting fact is that classical implication distributes over intuitionistic implication.

Proposition 8. *Given $A, B \in \mathcal{L}$ and $m \in \mathcal{M}$, then $(m, w) \Vdash (X \Rightarrow (A \rightarrow B)) \rightarrow ((X \Rightarrow A) \rightarrow (X \Rightarrow B))$. Consequently $\vDash (X \Rightarrow (A \rightarrow B)) \rightarrow ((X \Rightarrow A) \rightarrow (X \Rightarrow B))$.*

Proof. Assume, by absurd, that $(m, w) \nVdash (X \Rightarrow (A \rightarrow B)) \rightarrow ((X \Rightarrow A) \rightarrow (X \Rightarrow B))$. Then, there exists $w' \geq w$ such that $(m, w') \Vdash X \Rightarrow (A \rightarrow B)$ and $(m, w') \nVdash (X \Rightarrow A) \rightarrow (X \Rightarrow B)$. Hence, there is $w'' \geq w'$ such that $(m, w'') \Vdash X \Rightarrow A$ and $(m, w'') \nVdash X \Rightarrow B$. So, $(m, w_0) \Vdash X$ and $(m, w'') \nVdash B$. This implies that $(m, w'') \Vdash A$ and $(m, w') \Vdash A \rightarrow B$, which is impossible because $w' \leq w''$, $(m, w'') \Vdash A$ and $(m, w'') \nVdash B$. □

Finally, we should note that a semantic form of the deduction theorem for classical implication holds.

Proposition 9. *Let $\Gamma \cup \{A, B\} \subseteq \mathcal{L}$. We have $\Gamma, A \vDash B$ if and only if $\Gamma \vDash A \Rightarrow B$.*

Proof. Assume that $\Gamma, A \vDash B$ and let $m \in \mathcal{M}$ be such that $m \Vdash \Gamma$. We need to prove that $(m, w_0) \Vdash A \Rightarrow B$. If $(m, w_0) \nVdash A$ we are done. Let us suppose, otherwise, that $(m, w_0) \Vdash A$. Then, we know that $m \Vdash A$ as well and by definition of entailment it follows that $m \Vdash B$ and, in particular, that $(m, w_0) \Vdash A \Rightarrow B$. Hence, we have $\Gamma \vDash A \Rightarrow B$.

Conversely, assume that $\Gamma \vDash A \Rightarrow B$ and let $m \in \mathcal{M}$ be such that $m \Vdash \Gamma$ and $m \Vdash A$. By entailment, we also known that $m \Vdash A \Rightarrow B$. Therefore, we have $(m, w_0) \Vdash A \Rightarrow B$ and $(m, w_0) \Vdash A$, and it follows immediately that $(m, w_0) \Vdash B$, that is, $m \Vdash B$. Thus, we get $\Gamma, A \vDash B$. □

For intuitionistic implication, however, the deduction theorem does not hold in general. Given the previous result, this is not unexpected since we already know that the two implications, in general, do not coincide. Note, in particular, that $A \Rightarrow B, A \vDash B$ but $A \Rightarrow B \nvDash A \rightarrow B$.

4 Axiomatization and completeness

At this point, we are ready to propose an axiomatization for the combined logic *CIPL*.

Definition 3. The axiomatization of *CIPL* consists of the axioms

(C1)	$A \Rightarrow (B \Rightarrow A)$
(C2)	$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
(C3)	$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$
(I1)	$A \rightarrow (B \rightarrow A)$
(I2)	$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(X1)	$A \rightarrow (B \Rightarrow A)$
(X2)	$(A \Rightarrow B) \rightarrow (A \rightarrow B)$, for A classical
(X3)	$A \rightarrow ((A \Rightarrow B) \rightarrow (A \rightarrow B))$
(X4)	$(X \Rightarrow (A \rightarrow B)) \rightarrow ((X \Rightarrow A) \rightarrow (X \Rightarrow B))$

and the inferences rules

$$\frac{A \quad (A \Rightarrow B)}{B} \quad (\mathbf{CMP})$$

$$\frac{A \quad (A \rightarrow B)}{B} \quad (\mathbf{IMP}).$$

We denote by $\vdash$ the corresponding deductive consequence relation.

As we had promised, the axiomatization was obtained by adding together the axiomatizations of classical implication, **C1-C3** and **CMP**, and intuitionistic implication, **I1-I2** and **IMP**, with four interaction axioms **X1-X4**. Note that each of these interaction axioms has already been discussed in the preceding section, namely at Propositions 7, 5, 6 and 8, respectively. The soundness of these axioms and rules can be easily obtained.

Theorem 3 (Soundness).
Let $\Gamma \cup \{A\} \subseteq \mathcal{L}$. If $\Gamma \vdash A$ then $\Gamma \vDash A$.

Proof. It suffices to show that each of the axioms and rules are sound. Let m be a model.
Let us take **C1**. Assume that $m \not\Vdash A \Rightarrow (B \Rightarrow A)$, i.e. $(m, w) \not\Vdash A \Rightarrow (B \Rightarrow A)$, for some $w \in W$. Then, $(m, w_0) \Vdash A$ and $(m, w) \not\Vdash B \Rightarrow A$, i.e. $(m, w_0) \Vdash B$ and $(m, w) \not\Vdash A$. By Proposition 2, $(m, w_0) \Vdash A$ and $(m, w) \not\Vdash A$ constitute a contradiction. Hence, $m \Vdash A \Rightarrow (B \Rightarrow A)$.
The proofs for **C2** and **C3** are similar. Let us now consider the rule **CMP**. Assume that $m \Vdash A$ and $m \Vdash A \Rightarrow B$. Furthermore, assume by absurd that $m \not\Vdash B$. Then, there is w such that $(m, w) \not\Vdash B$ and by Proposition 2, $(m, w_0) \not\Vdash B$. On the other hand, $(m, w_0) \Vdash A$ and $(m, w_0) \Vdash A \Rightarrow B$ which implies that $(m, w_0) \Vdash B$, contradicting our assumption.
Take now **I1** and assume that $m \not\Vdash A \to (B \to A)$, i.e. $(m, w) \not\Vdash A \to (B \to A)$, for some $w \in W$. Then, there is $w' \geq w$ such that $(m, w') \Vdash A$ and $(m, w') \not\Vdash B \to A$. This implies that there is $w'' \geq w'$ such that $(m, w'') \Vdash B$ and $(m, w'') \not\Vdash A$. By Proposition 2, as $w'' \geq w'$, we have a contradiction.
The proof for **I2** is similar. Let us now consider the rule **IMP**. Assume that $m \Vdash A$ and $m \Vdash A \to B$. Furthermore, assume, by absurd, that $m \not\Vdash B$. Then, there exists $w \in W$ such that $(m, w) \not\Vdash B$. On the other hand, $(m, w) \Vdash A$ and $(m, w) \Vdash A \to B$ which imply that $(m, w) \Vdash B$, which, again, is a contradiction.
Finally, the soundness for the interaction axioms **X1-X4** was already established in Propositions 5-8. □

We now proceed to establishing the completeness of the proposed axiomatization. As a preliminary step, we begin by obtaining the deduction theorem for classical implication.

Theorem 4 (Classical deduction theorem (CDED)).
Let $\Gamma \cup \{A, B\} \subseteq \mathcal{L}$. We have $\Gamma, A \vdash B$ if and only if $\Gamma \vdash A \Rightarrow B$.

Proof. Assume that $\Gamma, A \vdash B$. We prove that $\Gamma \vdash A \Rightarrow B$ by induction on the length of the derivation of $\Gamma, A \vdash B$. If B is an axiom then:

1. B	**Ax**
2. $B \Rightarrow (A \Rightarrow B)$	**C1**
3. $A \Rightarrow B$	**CMP** : 1, 2

Hence, $\Gamma \vdash A \Rightarrow B$. If $B \in \Gamma$ then the proof is similar. If B is A then $\Gamma \vdash A \Rightarrow A$, which can be derived as usual in classical logic.

If B resulted from $C \Rightarrow B$ and C using **CMP** then, by induction hypothesis, there are derivations for $\Gamma \vdash A \Rightarrow (C \Rightarrow B)$ and $\Gamma \vdash A \Rightarrow C$. Then:

$\vdots$	
n. $A \Rightarrow (C \Rightarrow B)$	Hyp.
$\vdots$	
n'. $A \Rightarrow C$	Hyp.
$n'+1$. $A \Rightarrow (C \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow B))$	**C3**
$n'+2$. $(A \Rightarrow C) \Rightarrow (A \Rightarrow B)$	**CMP** : $n, n'+1$
$n'+3$. $A \Rightarrow B$	**CMP** : $n', n'+2$

Note that this derivation depends only on Γ, thus $\Gamma \vdash A \Rightarrow B$.

Assume now that B resulted from $C \to B$ and C using **IMP**. Then, by induction hypothesis, there are derivations for $\Gamma \vdash A \Rightarrow C$ and $\Gamma \vdash A \Rightarrow (C \to B)$. Hence:

$$\begin{array}{rll} & \vdots & \\ n. & A \Rightarrow (C \to B) & \text{Hyp.} \\ & \vdots & \\ n'. & A \Rightarrow C & \text{Hyp.} \\ n'+1. & (A \Rightarrow (C \to B)) \to ((A \Rightarrow C) \to (A \Rightarrow B)) & \mathbf{X4} \\ n'+2. & (A \Rightarrow C) \to (A \Rightarrow B) & \mathbf{IMP} : n, n'+1 \\ n'+3. & A \Rightarrow B & \mathbf{IMP} : n', n'+2 \end{array}$$

Once again, this derivation depends only on Γ, thus $\Gamma \vdash A \Rightarrow B$.

The converse is straightforward, using **CMP**. □

Although it fails in general, it is possible to formulate and prove a form of deduction theorem for the intuitionistic implication. For the purpose, we just need to consider derivations obtained without using the rule of classical *Modus Ponens*. Henceforth, we write $\Gamma \vdash_{\to} A$ to denote the fact that there is a derivation of A from Γ using only the axioms and the rule **IMP**. Obviously, if $\Gamma \vdash_{\to} A$ then $\Gamma \vdash A$.

Theorem 5 (Intuitionistic deduction theorem (IDED)).
Let $\Gamma \cup \{A, B\} \subseteq \mathcal{L}$. We have $\Gamma, A \vdash_{\to} B$ if and only if $\Gamma \vdash_{\to} A \to B$.

Proof. Straightforward. □

Another useful result is that, as a consequence of the corresponding deduction theorems, each of the implications enjoys a form of *hypothetical syllogism*.

Corollary 3 (Hypothetical syllogism).
The following deductions hold for every $A, B, C \in \mathcal{L}$.

$$\begin{array}{ll} (\mathbf{CHS}) & A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C \\ (\mathbf{IHS}) & A \to B, B \to C \vdash_{\to} A \to C \end{array}$$

Going now towards the completeness proof, let us use $\Gamma^{\vdash}$ to denote the *theory* $\{A \in \mathcal{L} : \Gamma \vdash A\}$. We will also use $\Gamma^{\vdash_{\to}}$ to denote the set $\{A \in \mathcal{L} : \Gamma \vdash_{\to} A\}$. As usual, given a formula A, we will say that a theory Γ is *maximal relatively to* A if $\Gamma \nvdash A$ but $\Gamma, B \vdash A$ for every $B \notin \Gamma$.

Theorem 6 (Completeness).
Let $\Gamma \cup \{A\} \subseteq \mathcal{L}$. If $\Gamma \vDash A$ then $\Gamma \vdash A$.

Proof. Assume that $\Gamma \nvdash A$. We will show how to build a model m of Γ that does not satisfy A. Let Δ_0 be a theory extending Γ, such that Δ_0 is maximal relatively to A. Note that the existence of such a Δ_0 is guaranteed by the general form of Lindenbaum's lemma [13]. Consider now the sets $\Delta_s \subseteq \mathcal{L}$ satisfying the following conditions:

- $\Delta_0 \subseteq \Delta_s$;
- $\Delta_s \cap P = \Delta_0 \cap P$;
- $\Delta_s = \Delta_s^{\vdash_\rightarrow}$.

Consider the tuple $m = \langle W, \subseteq, \overline{V} \rangle$ where W is the set of all Δ_s sets, and $\overline{V}$ is such that $\overline{V}(q) = \{\Delta_s : q \in \Delta_s\}$. We first observe that Δ_0 fulfills the above conditions on the sets Δ_s and, consequently, $\Delta_0 \in W$. Then, we also observe that $\langle W, \subseteq \rangle$ is a partial order with a least element which, by construction, is Δ_0. Furthermore, we also have that if $p \in P$ then either $p \in \Delta_0$ and $\overline{V}(p) = W$ or $p \notin \Delta_0$ and $V(p) = \emptyset$. Finally, if $q \in Q$, $\Delta_s \in \overline{V}(q)$ and $\Delta_s \subseteq \Delta_{s'}$ then $q \in \Delta_{s'}$ and so $\Delta_{s'} \in \overline{V}(q)$, i.e. q is persistent. All these conditions imply that $m \in \mathcal{M}$. We prove some auxiliary results about m.

Lemma 1. *$B \in \Delta_0$ if and only if $B \Rightarrow A \notin \Delta_0$.*

Proof. Assume that $B \in \Delta_0$. If $B \Rightarrow A \in \Delta_0$ then $A \in \Delta_0$, by **CMP**, which is a contradiction. Assume now that $B \notin \Delta_0$. As Δ_0 is maximal relatively to A, then $\Delta_0, B \vdash A$ and by **CDED** we have $B \Rightarrow A \in \Delta_0$. □

Lemma 2. *$B \Rightarrow C \in \Delta_0$ if and only if $B \notin \Delta_0$ or $C \in \Delta_0$.*

Proof. If $B \Rightarrow C \in \Delta_0$ and $B \in \Delta_0$ then by **CMP** $C \in \Delta_0$. Assume now that $C \in \Delta_0$. Then, by **C1** and **CMP**, we have $B \Rightarrow C \in \Delta_0$. If $B \notin \Delta_0$ then, by Lemma 1, $B \Rightarrow A \in \Delta_0$. On the other hand, using **C3**, $((A \Rightarrow C) \Rightarrow A) \Rightarrow A \in \Delta_0$. Hence, by Lemma 1, $(A \Rightarrow C) \Rightarrow A \notin \Delta_0$. Again by Lemma 1, $A \Rightarrow C \in \Delta_0$. If $B \Rightarrow A \in \Delta_0$ and $A \Rightarrow C \in \Delta_0$ then, by **CHS**, $B \Rightarrow C \in \Delta_0$. □

Lemma 3. *For every $\Delta_s \in W$, $A \Rightarrow B \in \Delta_s$ if and only if $A \notin \Delta_0$ or $B \in \Delta_s$.*

Proof. Assume that $A \Rightarrow B \in \Delta_s$ and that $A \in \Delta_0 \subseteq \Delta_s$. By **X3**, $A \Rightarrow ((A \Rightarrow B) \rightarrow (A \rightarrow B)) \in \Delta_0$ and using **CMP**, $(A \Rightarrow B) \rightarrow (A \rightarrow B) \in \Delta_0 \subseteq \Delta_s$. As Δ_s is closed for **IMP** it follows that $A \rightarrow B \in \Delta_s$. Again by **IMP** we have $B \in \Delta_s$.

If $A \notin \Delta_0$ then, by Lemma 2, $A \Rightarrow B \in \Delta_0 \subseteq \Delta_s$. If $B \in \Delta_s$ then, using **X1** and the fact that Δ_s is closed for **IMP**, it follows that $A \Rightarrow B \in \Delta_s$. □

Lemma 4. *For every $\Delta_s \in W$, $B \rightarrow C \in \Delta_s$ if and only if for every Δ'_s such that $\Delta_s \subseteq \Delta'_s$, if $B \in \Delta'_s$ then $C \in \Delta'_s$.*

Proof. The left-to-right implication is straightforward by **IMP**. For the converse, consider the set $\Delta'_s = (\Delta_s \cup \{B\})^{\vdash_\rightarrow}$ and assume that $C \in \Delta'_s$. Then, either Δ'_s has the same classical propositional symbols as Δ_0 or not. Let us assume first that Δ'_s has the same classical propositional symbols as Δ_0. Then $\Delta_s, B \vdash_\rightarrow C$ and by **IDED**, it follows that $B \rightarrow C \in \Delta_s$. Assume now that Δ'_s does not have the same classical propositional symbols as Δ_0. Then, $\Delta_s, B \vdash_\rightarrow p$, for some $p \in P$ such that $p \notin \Delta_0$. By **IDED**, $\Delta_s \vdash B \rightarrow p$. On the other hand, by Lemma 1, $p \Rightarrow A \in \Delta_0$, and by Lemma 2, $A \Rightarrow C \in \Delta_0$. So, by **CHS**, $p \Rightarrow C \in \Delta_0 \subseteq \Delta_s$. As p is classic, by **X2**, $(p \Rightarrow C) \rightarrow (p \rightarrow C) \in \Delta_s$ and by **IMP** it follows that $p \rightarrow C \in \Delta_s$. So, by **IHS**, we can conclude that $B \rightarrow C \in \Delta_s$. □

Finally, we prove that, for $\varDelta_s \in W$, $(m, \varDelta_s) \Vdash B$ if and only if $B \in \varDelta_s$. We proceed by induction on B. If B is a propositional symbol the result follows from the construction of $\overline{V}$. Consider now the case $B \Rightarrow C$. Then $(m, \varDelta_s) \Vdash B \Rightarrow C$ iff either $(m, \varDelta_0) \nVdash B$ or $(m, \varDelta_s) \Vdash C$ iff, by induction hypothesis, either $B \notin \varDelta_0$ or $C \in \varDelta_s$ iff, by Lemma 3, $B \Rightarrow C \in \varDelta_s$. Finally, consider the case $B \rightarrow C$. In this case, $(m, \varDelta_s) \Vdash B \rightarrow C$ iff for every $\varDelta'_s \supseteq \varDelta_s$, either $(m, \varDelta'_s) \nVdash B$ or $(m, \varDelta'_s) \Vdash C$ iff for every $\varDelta'_s \supseteq \varDelta_s$, either $B \notin \varDelta'_s$ or $C \in \varDelta'_s$, by induction hypothesis, iff $B \rightarrow C \in \varDelta_s$, by Lemma 4.

Hence, we conclude that $m \Vdash \varGamma$ and $(m, \varDelta_0) \nVdash A$, and so $\varGamma \nvDash A$. □

5 Concluding remarks

In this paper we have introduced an extended Kripke semantics for both *CPL* and *IPL*, where classical and intuitionistic implications are shown not to coincide. We have also shown that the implicative fragment of the resulting combined logic *CIPL* is a conservative extension of the implicative fragments of both *CPL* and *IPL*, thus settling a problem raised by Dov Gabbay in [5]. In addition, we have provided a simple complete axiomatization for the combined logic, which adds four simple interaction axioms to the usual axiomatizations of classical and intuitionistic implications.

The logic *CIPL* is, to the best of our knowledge, the first logic that extends classical and intuitionistic logics in a conservative way. Note that the completeness theorem was obtained using fairly classical tools. It is interesting to note that our axiomatization works only for rooted models. Actually, if we consider arbitrary Kripke models we will get a logic that is similar in most respects to the one obtained here, with the exception of axiom **X3**. However, we were unable to find a property that would replace the role played by **X3** in the completeness proof. This is an open question for future research. Other topics for further investigation concern studying the combination of the full languages of *CPL* and *IPL*, as well as the decidability and algebraizability of the logics obtained. Note, however, that the *amalgamation*-like flavour of our class of models as expressed by Proposition 1 will still guarantee conservativeness, as well as a few other interesting independence properties, in a way similar to *Robinson's consistency theorem* for first-order theories. It is also worth mentioning that the assumption that $P \cap Q = \emptyset$ plays an important role here: the combined logic will only be a conservative extension of the classical and intuitionistic logics built over non-shared symbols.

It is interesting to note that our extended Kripke semantics, namely where classical implication is concerned, is very closely related to other extensions of intuitionistic Kripke semantics that can be found in the literature. Most notably, we should mention the *coimplication* of [11, 14], but also Humberstone's *anticipation* operator [9], or the logic of *bunched implications* of [10]. A thorougher comparison with our *CIPL*, namely with respect to their topological semantic aspects is certainly needed. Of course, from the point of view of Grzegorczyk's logic of scientific reasoning [8], there is a simple meaning for our interpretation of classical implication. However, it is hard to say whether there is any connection to constructive proofs. In any case, developments in type theories have found that classical implication plays an essential role in typing certain control structures occurring in functional programs [7]. This is a line of research that is certainly worth exploring.

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