Hierarchical Logical Consequence

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Abstract

The modern view of logical reasoning as modeled by a consequence operator (instead of simply by a set of theorems) has allowed for huge developments in the study of logic as an abstract discipline. Still, it is unable to explain why it is often the case that the same designation is used, in an ambiguous way, to describe several distinct modes of reasoning over the same logical language. A paradigmatic example of such a situation is 'modal logic', a designation which can encompass reasoning over Kripke frames, but also over Kripke models, and in any case either locally (at a fixed world) or globally (at all worlds). Herein, we adopt a novel abstract notion of logic presented as a lattice-structured hierarchy of consequence operators, and explore some common proof-theoretic and model-theoretic ways of presenting such hierarchies through a collection of meaningful examples. In order to illustrate the usefulness of the notion of hierarchical consequence operators we address a few questions in the theory of combined logics, where a suitable abstract presentation of the logics being combined is absolutely essential, and we show how to define and achieve a number of interesting preservation results for fibring, in the context of **2**-hierarchies.

Keywords: Consequence operator, hierarchical lattice, deduction, semantics, fibred logics.

1 Introduction

Science is full of historical misunderstandings. Logic, as a discipline, is no exception. The fact that logic, as an independent discipline, originated with the study of classical and intuitionistic logics is probably the source of the problem that we are about to address. Indeed, for these logics, it is quite sufficient to regard a logic, in abstract, as a set of theorems. Why? Well, because both classical and intuitionistic logics are compact (or finitary), and both have a well-behaved implication connective for which the deduction theorem holds. However, nowadays, people have come up with many interesting and useful examples of logics for which, one way or the other, this idealistic state of affairs does not apply. Most notably, there is the case of modal logics, that come in various brands and flavors, and which have found applications in areas as diverse as software verification, knowledge representation, linguistics, or metaphysics. As a designation, 'modal logic' is quite ambiguous, and can encompass reasoning over Kripke frames, but also over Kripke models, and in any case either locally (at a fixed world) or globally (at all worlds). This confusion is dealt with in a relatively careless way in many textbooks, but even modern textbooks such as [2] that recognize these distinctions, end up defining a modal logic as a set of theorems!

Of course, the situation is not totally hopeless for modal logics, as long as one fully understands that in general compactness does not hold, and only a restricted form of the deduction theorem can be used, but still the situation is quite distressing. And what is more, there are of course logics where not even a decent implication connective is available. These facts are well known but often simply overlooked by most logicians, these days. A rare paper raising the awareness for this fact is [10], but it does not attempt to provide an abstract systematic treatment of the subject.

Herein, we aim at clarifying this manifold nature of logical consequence by explicitly incorporating this feature into logic presentations. Hence, in order to be able to describe in abstract all the relevant aspects of a given logic, we propose to work with a notion of logic modeled as a lattice-structured hierarchy of consequence operators. We should note from the very beginning that, without surprise, all the consequences in a given hierarchy are required to share the same set of theorems. A brief analysis of a number of meaningful examples, namely modal propositional logic, first order logic, and modal first order logic, will help to justify the definition proposed and its details. In particular, the lattice structure will allow us to evaluate the relative dependence between concepts in the logic, namely for atomic lattices. The examples will also allow us to present and explore a few typical proof-theoretic and model-theoretic ways of presenting such hierarchies. In particular, we will focus on hierarchical versions of Hilbert-style deductive calculi and, semantically, on satisfaction-based structures.

Still, our interest in hierarchical consequence operators is not merely abstract. The fact is that dealing with logics and their presentations at an adequate level of abstraction is well recognized to be an essential ingredient of the algebraic theory of combined logics [4, 18, 14, 5, 8]. Rigorously defining and studying mechanisms for combining logics has become increasingly important, given the complexity that many meaningful and useful logics have reached, in some cases incorporating features which are well understood in isolation but whose interaction is far from being clear. These features may include modalities of various kinds, but also, for instance, probabilistic aspects of complex dynamic concurrent and distributed systems. Hence, for the sake of illustration of the usefulness of the notion of hierarchical consequence, we will take the powerful mechanism for combining logics known as fibring [12, 11] and we will show how it can be naturally defined and studied in this context, once *structurality* is properly introduced. Namely, we will establish a number of interesting preservation results for fibred **2**-hierarchies, that is, hierarchies of consequence operators based on the lattice **2**. While a systematic study of fibred hierarchies is beyond the scope of this paper, we will also hint at how such a rich environment could be used to obtain different versions of modal first order logic, by suitably combining modal propositional logic with first order logic.

The paper is organized as follows. In Section 2 we introduce and illustrate the notion of hierarchical consequence operator. Then, in Section 3 we introduce and explore model-theoretic and proof-theoretic presentations of hierarchies of consequence operators. Section 4 shows how to apply the novel notion of hierarchical consequence to the systematic characterization and study of fibring as a general mechanism for combining logics. We conclude, in Section 5, with a discussion of future work.

2 Hierarchical consequence operators

In order to introduce the notion of hierarchical consequence operators, we first need to recall the notion of consequence operator à la Tarski (see [17]).

DEFINITION 2.1
Let L be a set (of formulas). A *consequence operator* over L is a map $\_^{\vdash} : \wp(L) \to \wp(L)$ such that

- $\Phi \subseteq \Phi^{\vdash}$ (extensiveness)
- $\Phi \subseteq (\Phi \cup \Psi)^{\vdash}$ (monotonicity)
- $(\Phi^{\vdash})^{\vdash} \subseteq \Phi^{\vdash}$ (idempotence)

for every $\Phi, \Psi \subseteq L$.

We will often use the relational version of the consequence operator, and write $\Phi \vdash \varphi$ instead of $\varphi \in \Phi^{\vdash}$. As usual, we say that $\varphi \in L$ is a *theorem* provided that $\emptyset \vdash \varphi$. We denote by $Cns(L)$ the set of all consequence operators over L. We define the order relation $\sqsubseteq$ on $Cns(L)$ as follows: $\vdash' \sqsubseteq \vdash''$ provided that $\Phi^{\vdash'} \subseteq \Phi^{\vdash''}$ for every $\Phi \subseteq L$. Clearly, $\sqsubseteq$ is a partial order.

We need to recall also a number of useful fact about lattices (see [1]). A partially ordered set $\langle X, \leq \rangle$ is a *lattice* whenever the join $x \vee y$ (or supremum) and the meet $x \wedge y$ (or infimum) exist for every $x, y \in X$. The lattice is *finite* if X is a finite set. The partially ordered set $\langle X, \leq \rangle$ is a *complete lattice* if the join $\bigvee S$ and the meet $\bigwedge S$ exist for every $S \subseteq X$. When it exists, the *top* (*bottom*) element of a lattice $\langle X, \leq \rangle$ is $\bigvee X$ ($\bigwedge X$) and it is denoted by $\top$ ($\bot$). A lattice $\langle X, \leq \rangle$ with $\bot$ is *atomic* if each element in $X \backslash \{\bot\}$ is a join of atoms (a is an *atom* whenever $\bot < a$ and there is no y such that $\bot < y < a$). Recall also that every finite lattice is complete.

As usual, for each $n \in \mathbb{N}$, we use $\mathbf{n}$ to denote the finite lattice $\langle \{1, 2, \ldots, n\}, \leq \rangle$, where $\leq$ is the usual order relation in $\mathbb{N}$; $\mathbf{2^2}$ denotes the lattice $\langle \{\bot, a, b, \top\}, \leq_{\mathbf{2^2}} \rangle$ where $\bot$ and $\top$ are as expected and the other two elements are not related to each other. To illustrate the structure of the lattices we will often use Hasse diagrams. The Hasse diagram of $\mathbf{2^2}$ is depicted in Figure 1.

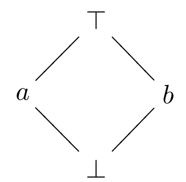

FIG. 1. The lattice $\mathbf{2^2}$.

The following is a well-known result [17].

PROPOSITION 2.2
The partially ordered set $\langle Cns(L), \sqsubseteq_L \rangle$ is a complete lattice.

We now introduce the notion of hierarchical consequence operator.

Definition 2.3
A *hierarchical consequence operator* over L is a map

$$h : \langle X, \leq \rangle \to \langle Cns(L), \sqsubseteq \rangle$$

where

- $\langle X, \leq \rangle$ is a finite lattice;
- h preserves joins;
- $\emptyset^{h(x)} = \emptyset^{h(\bot)}$ for all $x \in X$.

A hierarchical consequence operator is simply a finite sublattice of $\langle Cns(L), \sqsubseteq \rangle$, reflecting the various modes of reasoning allowed over the same logical language, which are all required to have the same set of theorems. In order to illustrate this notion, let us discuss several examples of hierarchical consequences. The first example is about classical propositional logic.

Example 2.4
Let L_p^Π be the usual classical propositional language over a set of propositional symbols Π with connectives $\neg$ and $\Rightarrow$. Let $p, q, r \in \Pi$ and consider the deductive system with the axioms

- $p \Rightarrow (q \Rightarrow p) \quad$ (A1)
- $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \quad$ (A2)
- $((\neg q) \Rightarrow (\neg p)) \Rightarrow (p \Rightarrow q) \quad$ (A3)

and the schematic rules

- $\dfrac{\varphi \qquad \varphi \Rightarrow \psi}{\psi}$ (MP)
- $\dfrac{\varphi}{\theta(\varphi)}$ (Sub)

where $\theta(\varphi)$ is the instantiation of φ by a substitution $\theta : \Pi \to L_p^\Pi$, that is, the result of uniformly replacing in φ each $s \in \Pi$ by $\theta(s)$.

These axioms and rules come as no surprise, with the possible exception of rule (Sub). In fact, usually, substitution instances are obtained implicitly by describing the axioms in a schematic way. Still, sometimes, propositional symbols tend to be confused with metavariables, in which case axiomatizations such as this one pop up naturally. Similar rules are suggested, for instance, in [17]. Although the axioms here are not schematic, the usual notion of deduction for classical propositional logic can still be obtained by restricting the use of the rule (Sub) in such a way that it can only be applied to generate theorems, that is, the premise of (Sub) must always be a formula that does not depend on the hypotheses. Let us denote the resulting consequence operator by $\vdash_u$. If the application of the rule (Sub) is not restricted we obtain a strictly stronger consequence operator $\vdash_s$. In fact, for instance, $p \vdash_s q$ but $p \not\vdash_u q$ for $p, q \in \Pi$. However, both $\vdash_u$ and $\vdash_s$ share the same set of theorems, which would make the presentations equally suitable if we were to regard the logic simply as a set of theorems. Hence, the map

$$h_p : \mathbf{2} \to Cns(L_p^\Pi)$$

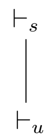

FIG. 2. A **2**-hierarchy of consequence operators for classical propositional logic.

where $h_p(1) = \vdash_u$ and $h_p(2) = \vdash_s$ is a hierarchical consequence operator (see Figure 2).

Next we discuss modal propositional logic.

EXAMPLE 2.5

Let L_m^Π be the usual (classical) propositional modal language over a set of propositional symbols Π, with connectives $\neg$ and $\Rightarrow$, and modal operator $\Box$. Abbreviations, namely $\Diamond$, are as expected. Assume the usual Kripke semantics: a frame is a pair $\mathbb{F} = \langle W, R \rangle$ with W a nonempty set (of worlds) and R a binary relation on W, and a Kripke model, based on the frame $\langle W, R \rangle$, is a triple $M = \langle W, R, \vartheta \rangle$ where $\vartheta : W \to 2^\Pi$ is a map, assigning to each world the set of propositional symbols that are true in that world. Given a Kripke model $M = \langle W, R, \vartheta \rangle$, the satisfaction of a formula φ at a world $w \in M$ is defined as expected and we write as usual $M \Vdash_w \varphi$. Moreover, $M \Vdash \varphi$ whenever $M \Vdash_w \varphi$ for every $w \in W$, and, for each frame $\mathbb{F}$, $\mathbb{F} \Vdash \varphi$ whenever $M \Vdash \varphi$ for every Kripke model M based on $\mathbb{F}$. These definitions extend to sets of formulas in the usual way.

Several notions of modal entailment can be considered, namely, the *model global* and the *model local* entailments, and the *frame global* and the *frame local* entailments. Let $\Phi \cup \{\varphi\} \subseteq L_m^\Pi$. The model global entailment, denoted by $\vDash_{mg}$, is defined as follows: $\Phi \vDash_{mg} \varphi$ if $M \Vdash \varphi$ whenever $M \Vdash \Phi$, for every Kripke model M. The model local entailment, denoted by $\vDash_{ml}$, is defined as follows: $\Phi \vDash_{ml} \varphi$ if $M \Vdash_w \varphi$ whenever $M \Vdash_w \Phi$, for every Kripke model M and world w of M. It is easy to conclude that $\vDash_{mg}$ and $\vDash_{ml}$ are consequence operators over L_m^Π. Moreover, $\vDash_{ml} \sqsubseteq \vDash_{mg}$. The frame global entailment, denoted by $\vDash_{fg}$, is as follows: $\Phi \vDash_{fg} \varphi$ if $\mathbb{F} \Vdash \varphi$ whenever $\mathbb{F} \Vdash \Phi$, for every frame $\mathbb{F}$. The frame local entailment, denoted by $\vDash_{fl}$, is as follows: $\Phi \vDash_{fl} \varphi$ if, for every every frame $\mathbb{F}$ and world w of $\mathbb{F}$, we have $M \Vdash_w \varphi$ for every M based on $\mathbb{F}$ whenever $M \Vdash_w \Phi$ for every M based on $\mathbb{F}$. Clearly, $\vDash_{fg}$ and $\vDash_{fl}$ are consequence operators over L_m^Π. It is easy to see that not only $\vDash_{fl} \sqsubseteq \vDash_{fg}$, but also $\vDash_{ml} \sqsubseteq \vDash_{fl}$ and $\vDash_{mg} \sqsubseteq \vDash_{fg}$. Furthermore, $\emptyset^{\vDash_\alpha} = \emptyset^{\vDash_\beta}$ for every $\alpha, \beta \in \{ml, mg, fl, fg\}$.

We can then consider the hierarchical consequence operator

$$h_m : \mathbf{2^2} \to Cns(L_m^\Pi)$$

where $h_m(\bot) = \vDash_{ml}$, $h_m(\top) = \vDash_{fg}$, $h_m(a) = \vDash_{mg}$ and $h_m(b) = \vDash_{fl}$ (see Figure 3).

All the order relations between the above entailments are strict. To begin with, we have $\vDash_{ml} \sqsubset \vDash_{mg}$ because $\varphi \vDash_{mg} \Box\varphi$ but, in general, $\varphi \nvDash_{ml} \Box\varphi$. This fact is related to the necessitation inference rule (from φ infer $\Box\varphi$). Every set $\Phi^{\vDash_{mg}}$ is closed for necessitation, that is, if $\varphi \in \Phi^{\vDash_{mg}}$ then $\Box\varphi \in \Phi^{\vDash_{mg}}$. But this is not the case for $\vDash_{ml}$. We also have that $\vDash_{fl} \sqsubset \vDash_{fg}$ since $(\Box\varphi) \Rightarrow \varphi \vDash_{fg} \Box((\Box\varphi) \Rightarrow \varphi)$ but $(\Box\varphi) \Rightarrow \varphi \nvDash_{fl} \Box((\Box\varphi) \Rightarrow \varphi)$, in general. This fact is also related to the necessitation

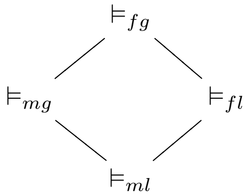

FIG. 3. A $\mathbf{2^2}$-hierarchy of consequence operators for propositional modal logic.

inference rule, but in this case we cannot just pick a propositional formula to prove that the two entailments are different. Given $p, q \in \Pi$, we have $p \vDash_{fg} q$ but, clearly, $p \not\vDash_{mg} q$. Hence, $\vDash_{mg} \sqsubset \vDash_{fg}$. In this case (uniform) substitution is involved. Every set $\Phi^{\vDash_{fg}}$ is closed for (uniform) substitution: if $\varphi \in \Phi^{\vDash_{fg}}$ then $\psi \in \Phi^{\vDash_{fg}}$ where ψ is the instance of φ by a substitution $\theta : \Pi \to L_m^{\Pi}$. But this is not the case for $\vDash_{mg}$. The same example also shows that $\vDash_{ml} \sqsubset \vDash_{fl}$ holds and that $\vDash_{fl}$ is closed for substitution whereas $\vDash_{ml}$ is not.

The next example focuses on classical first order logic.

EXAMPLE 2.6
Let $\Sigma = \langle F, P\rangle$ be a first order signature with $F = \{F_i\}_{i \in \mathbb{N}_0}$ the family of sets of function symbols and $P = \{P_i\}_{i \in \mathbb{N}}$ the family of sets of predicate symbols. Let V be a set of variables and $T(\Sigma, V)$ the usual set of terms. We denote by $L_{fol}^{(\Sigma,V)}$ the set of first order formulas defined as expected from $T(\Sigma, V)$, P, $\neg$, $\Rightarrow$, and the quantifier $\forall$. Abbreviations are as usual, namely $\exists$. Recall that an interpretation structure over $\Sigma = \langle F, P\rangle$ is a pair $I = \langle D, [\![.]\!]\rangle$, with D an nonempty set, $[\![f_i]\!] : D^i \to D$ a map and $[\![Q_i]\!] \subseteq D^i$, for $f_i \in F_i$ and $Q_i \in P_i$. A map $\rho : V \to D$ is an assignment over I. Given an interpretation structure $I = \langle D, [\![.]\!]\rangle$ and an assignment ρ, the interpretation of terms is the map $[\![.]\!]_{I,\rho} : T(\Sigma, V) \to D$ defined in the usual way. We write $I \Vdash_\rho \varphi$ for the satisfaction of a formula φ by I and ρ, defined as expected. Furthermore, $I \Vdash \varphi$ whenever $I \Vdash_\rho \varphi$ for every assignment ρ over I. The extension to sets of formulas is straightforward.

Again several notions of entailment can be considered, namely, the *validity entailment* and the *truth entailment*. Given $\Phi \cup \{\varphi\} \subseteq L_{fol}^{(\Sigma,V)}$, the validity entailment, denoted by $\vDash_v$, is defined as follows: $\Phi \vDash_v \varphi$ if $I \Vdash \varphi$ whenever $I \Vdash \Phi$, for every interpretation structure I. The truth entailment, denoted by $\vDash_t$, is defined as follows: $\Phi \vDash_t \varphi$ if $I \Vdash_\rho \varphi$ whenever $I \Vdash_\rho \Phi$, for every I and assignment ρ over I.

It can be easily checked that the entailments $\vDash_v$ and $\vDash_t$ are consequence operators over $L_{fol}^{(\Sigma,V)}$. We also have that $\emptyset^{\vDash_v} = \emptyset^{\vDash_t}$ and $\vDash_t \sqsubseteq \vDash_v$. Therefore, the map

$$h_{fol} : \mathbf{2} \to Cns(L_{fol}^{(\Sigma,V)})$$

where $h_{fol}(1) = \vDash_t$ and $h_{fol}(2) = \vDash_v$ is a hierarchical consequence operator (see Figure 4).

Note that $\varphi \vDash_v \forall x \varphi$ but, in general, $\varphi \not\vDash_t \forall x \varphi$. Hence, we have $\vDash_t \sqsubset \vDash_v$. This fact is related to the generalization inference rule (from φ infer $\forall x \varphi$). Every set $\Phi^{\vDash_v}$ is closed for generalization that is, if $\varphi \in \Phi^{\vDash_v}$ then $\forall x \varphi \in \Phi^{\vDash_v}$.

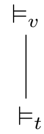

FIG. 4. A **2**-hierarchy of consequence operators for first order logic.

The last example, more complex, concerns first order modal logic.

EXAMPLE 2.7
Let $\Sigma = \langle F, P\rangle$ be a first order signature and $T(\Sigma, V)$ the set of terms over Σ and a set V of variables as in Example 2.6. We denote by $L^{(\Sigma,V)}_{mfol}$ the set of first order modal formulas defined as usual using $\neg$, $\Rightarrow$, $\forall$ and $\Box$. Let $IntS(\Sigma)$ be the class of every (first order) interpretation structures over Σ. A Kripke model is now a triple $M = \langle W, R, \upsilon\rangle$ where $\langle W, R\rangle$ is a frame as in Example 2.5 and $\upsilon : W \to IntS(\Sigma)$ is a map assigning to each $w \in W$ the interpretation structure $\upsilon(w) = \langle D_w, [\![.]\!]_w\rangle$. Let $D = \bigcup_{w\in W} D_w$. An assignment is a map $\rho : V \to D$. A Kripke model M is a model with *increasing domains* whenever $D_w \subseteq D_{w'}$ for every $w, w' \in W$ such that wRw'. An assignment ρ is said to be *compatible* with $w \in W$ if $\rho(x) \in D_w$ for every $x \in V$. The interpretation of terms in M is *rigid* if $[\![t]\!]_w = [\![t]\!]_{w'}$ for every $w, w' \in W$ and ground term t; otherwise it is *flexible*. In the sequel, we consider only first order signatures with no function symbols and Kripke models with increasing domains and rigid term interpretation. The satisfaction of φ by M and ρ at world w, denoted by $M \Vdash_{\rho,w} \varphi$, is defined in the usual way. In particular, recall that $M \Vdash_{\rho,w} \forall x\varphi$ whenever $M \Vdash_{\rho',w} \varphi$ for every assignment ρ' compatible with w and such that $\rho(y) = \rho'(y)$ for all $y \in V\backslash\{x\}$. We write $M \Vdash \varphi$ whenever $M \Vdash_{\rho,w} \varphi$ for every $w \in W$ and assignment ρ compatible with w.

Once more several notions of entailment can be defined. We consider *validity global* and *validity local* entailments as well as *truth local* entailment. Let $\Phi \cup \{\varphi\} \subseteq L^{(\Sigma,V)}_{mfol}$. The validity global entailment, $\models_{vg}$, is as follows: $\Phi \models_{vg} \varphi$ if $M \Vdash \varphi$ whenever $M \Vdash \Phi$, for every Kripke model M. The validity local entailment, $\models_{vl}$, is as follows: $\Phi \models_{vl} \varphi$ if, for every M and world w of M, $M \Vdash_{\rho,w} \varphi$ for every assignment ρ compatible with w whenever $M \Vdash_{\rho,w} \Phi$ for every assignment ρ compatible with w. These entailments are consequence operators over $L^{(\Sigma,V)}_{mfol}$ and, clearly, $\models_{vl} \sqsubseteq \models_{vg}$. Finally, the truth local entailment, $\models_{tl}$, is as follows: $\Phi \models_{tl} \varphi$ if $M \Vdash_{\rho,w} \varphi$ whenever $M \Vdash_{\rho,w} \Phi$, for every M, world w of M and assignment ρ compatible with w. This entailment is also a consequence operator over $L^{(\Sigma,V)}_{mfol}$ and $\models_{tl} \sqsubseteq \models_{vl}$. Furthermore, $\emptyset^{\models_{vg}} = \emptyset^{\models_{vl}} = \emptyset^{\models_{tl}}$. Therefore, the map

$$h_{mfol} : \mathbf{3} \to Cns(L^{(\Sigma,V)}_{mfol})$$

where $h_{mfol}(1) = \models_{tl}$, $h_{mfol}(2) = \models_{vl}$ and $h_{mfol}(3) = \models_{vg}$ is a hierarchical consequence operator (see Figure 5).

All the order relations between the above three entailments are strict. Firstly, $Q(x) \models_{vl} \forall xQ(x)$ but $Q(x) \not\models_{tl} \forall xQ(x)$. Hence, we get $\models_{tl} \sqsubset \models_{vl}$. Secondly, $Q(x) \models_{vg} \Box Q(x)$ but $Q(x) \not\models_{vl} \Box Q(x)$ and therefore $\models_{vl} \sqsubset \models_{vg}$.

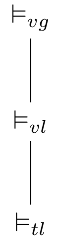

FIG. 5: A **3**-hierarchy of consequence operators for first-order modal logic with increasing domains.

A $\mathbf{2^2}$-hierarchy for first-order modal logic with constant domains could also be defined, in a straightforward manner. Note that in the case of constant domains, assignments and worlds become completely independent.

3 Deduction and semantics

Usually, in practice, consequence operators are presented either in a proof-theoretic or in a model-theoretic way. In this section we propose hierarchical versions of some of the most usual tools for presenting logical consequence.

3.1 Deductive systems

In the proof-theoretic case, we consider Hilbert-style deductive systems. Although we have used them in the examples of the previous section, a rigorous definition is due. An *inference rule* over L is a pair $\langle \Delta, \gamma \rangle$ where $\Delta \cup \{\gamma\} \subseteq L$. The elements in Δ are dubbed the *premises* and γ the *conclusion* of the rule. When $\Delta = \emptyset$ the rule is said to be an *axiom*. We will represent a rule $\langle \{\delta_1, \ldots, \delta_n\}, \gamma \rangle$ by

$$\frac{\delta_1 \quad \ldots \quad \delta_n}{\gamma}$$

It is not unusual that axiomatizations distinguish between inference rules that can be freely applied to hypotheses, and others that can only be used to generate theorems. Some of the examples of the previous section took advantage of that possibility. The following is a general abstract definition of such calculi.

DEFINITION 3.1
A *deductive system* over L is a triple $\mathcal{D} = \langle L, D, T \rangle$ where:

- D is a set of *derivation rules*;
- T is a set of *theoremhood rules*.

Given $\Phi \cup \{\varphi\} \subseteq L$ then φ is said to be *derived* from Φ, in symbols $\varphi \in \Phi^{\vdash_{\mathcal{D}}}$ or $\Phi \vdash_{\mathcal{D}} \varphi$, if φ can be obtained from Φ by application of the rules in D and T, with the proviso that the rules in T may only by applied to formulas that do not depend on the hypotheses Φ, i.e., may only be applied to *theorems*. Although not necessary, it is natural to require that $D \cap T = \emptyset$ and also that T does not contain axioms. Given a deductive system $\mathcal{D}$ we denote by $D_{\mathcal{D}}$ and $T_{\mathcal{D}}$ its sets of derivation and theoremhood rules, respectively. We denote by $Ded(L)$ the set of all deductive systems over L. We can define an order relation $\preceq$ on the elements of $Ded(L)$ as follows: $\mathcal{D} \preceq \mathcal{D}'$ provided that $D_{\mathcal{D}} \subseteq D_{\mathcal{D}'}$ and $T_{\mathcal{D}} \subseteq D_{\mathcal{D}'} \cup T_{\mathcal{D}'}$. The following is a straightforward fact.

Proposition 3.2
The partially ordered set $\langle Ded(L), \preceq \rangle$ is a complete lattice.

We can now introduce the notion of *hierarchical deductive system.*

Definition 3.3
A *hierarchical deductive system* is a map

$$h : \langle X, \leq \rangle \to \langle Ded(L), \preceq \rangle$$

where

- $\langle X, \leq \rangle$ is a finite lattice;
- h preserves joins;
- $D_{h(x)} \cup T_{h(x)} = D_{h(\bot)} \cup T_{h(\bot)}$ for all $x \in X$.

Expectedly, a hierarchical deductive system is just a finite sublattice of $\langle Ded(L), \preceq \rangle$, but we require that the set of all rules (deduction and theoremhood) be the same for all the deductive systems of the hierarchy. As a consequence of this definition, it immediately follows that, for any $x, y \in X$, $\emptyset \vdash_{h(x)} \varphi$ iff $\emptyset \vdash_{h(y)} \varphi$. Hence, clearly, a hierarchical deductive system presents a hierarchical consequence operator.

Proposition 3.4
Every hierarchical deductive system induces a hierarchical consequence operator with the same lattice structure.

Some intuitions on hierarchical deductive systems are due. Let $h : \langle X, \leq \rangle \to \langle Ded(L), \preceq \rangle$ be a hierarchical deductive system and assume that $\langle X, \leq \rangle$ is an atomic lattice. Given the bottom deductive system, the deductive system associated to each atom of X is obtained by moving some theoremhood rules to the set of derivation rules. This gives us an idea of how to proceed with the construction of whole hierarchical deductive system. Consider two theoremhood rules r_1 and r_2 in the bottom system and consider two atomic elements x and y obtained by moving r_1 and r_2, respectively, from the set of theoremhood rules to the set of derivation rules. Then, the join of these two systems will be the deductive system with both $r1$ and $r2$ as derivation rules. This construction is depicted in Figure 6.

In some cases, rules may not be added in ad-hoc manner. Consider for instance the hierarchical consequence operator presented in Example 2.7. We start by observing that, in this case, the underlying lattice is not atomic. So, if we want to define a hierarchical deductive system that induces this hierarchical consequence operator we must be careful with the order in which the rules are moved from the set of theoremhood rules to the set of derivation rules. In this case, in the first level, we move the generalization rule. In the next level, we move the necessitation rule. But there is no level corresponding to moving the necessitation rule without the generalization rule. Details on this construction are further discussed in Example 3.7 below. Let us start with some simpler examples.

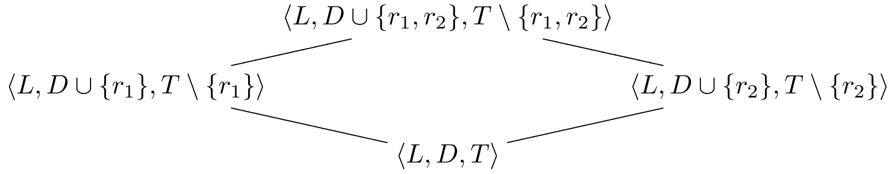

FIG. 6. Example of an (atomic) $\mathbf{2^2}$-deductive system

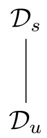

FIG. 7. A **2**-hierarchical deductive system for classical propositional logic.

EXAMPLE 3.5
Consider the following deductive systems over the language of propositional logic:

- $\mathcal{D}_u = \langle L_p^\Pi, \{A1, A2, A3, MP\}, \{Sub\}\rangle$;
- $\mathcal{D}_s = \langle L_p^\Pi, \{A1, A2, A3, MP, Sub\}, \{\}\rangle$.

As was said in Example 2.4, $\vdash_u \sqsubseteq \vdash_s$, simply because $\mathcal{D}_u \sqsubseteq \mathcal{D}_s$. It is straightforward to see that:

$$h_p : \mathbf{2} \to \mathit{Ded}(L_p^\Pi)$$

where $h_p(1) = \mathcal{D}_u$ and $h_p(1) = \mathcal{D}_s$ constitutes a hierarchical deductive system.

EXAMPLE 3.6
Recall Example 2.5 on classical modal propositional logic. Consider the following axioms and rules:

- $(\Box(p \Rightarrow q)) \Rightarrow ((\Box p) \Rightarrow (\Box q))$ (K)
- $\dfrac{\varphi}{\Box\varphi}$ (Nec)

and the following deductive systems:

- $\mathcal{D}_{ml} = \langle L_m^\Pi, \{\text{A1,A2,A3,K,MP}\}, \{\text{Nec,Sub}\}\rangle$;
- $\mathcal{D}_{mg} = \langle L_m^\Pi, \{\text{A1,A2,A3,K,MP,Nec}\}, \{\text{Sub}\}\rangle$;
- $\mathcal{D}_{fl} = \langle L_m^\Pi, \{\text{A1,A2,A3,K,MP,Sub}\}, \{\text{Nec}\}\rangle$;
- $\mathcal{D}_{fg} = \langle L_m^\Pi, \{\text{A1,A2,A3,K,MP,Nec,Sub})\}, \{\}\rangle$.

We can define the corresponding hierarchical deductive system

$$h_m : \mathbf{2^2} \to \mathit{Ded}(L_m^\Pi)$$

such that

- $h_m(\bot) = \mathcal{D}_{ml}$;
- $h_m(a) = \mathcal{D}_{mg}$;
- $h_m(b) = \mathcal{D}_{fl}$;
- $h_m(\top) = \mathcal{D}_{fg}$.

As expected, this hierarchical deductive system induces the hierarchical consequence operator presented in Example 2.5, which amounts to proving the soundness and completeness of the deductive systems. Once again, this is an atomic lattice.

Example 3.7
Recall Example 2.7 on modal first order logic. Consider the following axioms and rules:

- φ, where φ is an instance of a propositional tautology (Taut)
- $(\forall x\, \varphi) \Rightarrow \varphi^x_y$, where x and y are variables ($\forall$)
- $\dfrac{\varphi \Rightarrow \psi}{\varphi \Rightarrow \forall x\, \psi}$ provided that x is not free in φ (Gen)

and the following deductive systems:

- $\mathcal{D}_{tl} = \langle L^{(\Sigma,V)}_{mfol}, \{(Taut), (\forall), \textit{(MP)}\}, \{\textit{(Gen)},\textit{(Nec)}\}\rangle$;
- $\mathcal{D}_{vl} = \langle L^{(\Sigma,V)}_{mfol}, \{(Taut), (\forall), \textit{(MP)},\textit{(Gen)}\}, \{\textit{(Nec)}\}\rangle$;
- $\mathcal{D}_{vg} = \langle L^{(\Sigma,V)}_{mfol}, \{(Taut), (\forall), \textit{(MP)},\textit{(Gen)},\textit{(Nec)}\}, \{\}\rangle$.

With these, we can define the corresponding hierarchical deductive system

$$h_{mfol} : \mathbf{3} \rightarrow \mathit{Ded}(L^{(\Sigma,V)}_{mfol})$$

such that

- $h_{mfol}(1) = \mathcal{D}_{tl}$;
- $h_{mfol}(2) = \mathcal{D}_{vl}$;
- $h_{mfol}(3) = \mathcal{D}_{vg}$.

As was said above, in this case, the order in which the rules were moved from the set of theoremhood rules to the set of derivation rules was not arbitrary: from $h(1)$ to $h(2)$ we moved (Gen), and from $h(2)$ to $h(3)$ we moved (Nec). An observation is due: we are not able to derive the *Barcan formula* $(\forall x\, \Box\varphi) \Rightarrow (\Box\forall x\, \varphi)$, but it is well known that this formula does not hold, in general, for the case of expanding domains. However, the *converse Barcan formula* $(\Box\forall x\, \varphi) \Rightarrow (\forall x\, \Box\varphi)$ is derivable, given the special shape of the rule (Gen), as explained in [13].

When addressing combined logics, later on, we shall pay special attention to **2**-hierarchies. It is not difficult to see, anyhow, that deductive systems as introduced above induce **2**-hierarchical deductive systems in a very natural way. Given a deductive system $\mathcal{D} = \langle L, D, T\rangle$ we just need to consider $h(1) = \mathcal{D}$ and $h(2) = \langle C, D\cup T, \emptyset\rangle$. We will use $\vdash_d$ and $\vdash_t$ (for derivation and theoremhood) to refer to the consequence induced by each of these deductive systems.

3.2 Satisfaction systems

We now focus on the model-theoretic presentations. To this end, we consider semantic systems based on the notion of *satisfaction*.

Definition 3.8
A *satisfaction system* over L is a triple $\mathcal{S} = \langle L, M, \Vdash \rangle$ where:

- M is a class (of *models*);
- $\Vdash \subseteq M \times L$ is a (*satisfaction*) relation.

Given $\Phi \cup \{\varphi\} \subseteq L$ then φ is said to be *entailed* by Φ, in symbols $\varphi \in \Phi^{\models_\mathcal{S}}$ or $\Phi \models_\mathcal{S} \varphi$, if $m \Vdash \Gamma$ implies $m \Vdash \varphi$ for every $m \in M$. Given a satisfaction system $\mathcal{S}$ we denote by $M_\mathcal{S}$ and $\Vdash_\mathcal{S}$ its class of models and satisfaction relation, respectively. We denote by $Sat(L)$ the collection of all satisfaction systems over L. We can also define an order relation $\unlhd$ on satisfaction systems as follows: $\mathcal{S} \unlhd \mathcal{S}'$ provided that every $m' \in M_{\mathcal{S}'}$ is associated to a subclass $\mathcal{S}(m') \subseteq M_\mathcal{S}$ in such a way that $m' \Vdash_{\mathcal{S}'} \varphi$ iff $m \Vdash_\mathcal{S} \varphi$ for every $m \in \mathcal{S}(m')$. The idea is that a model of $\mathcal{S}'$ can be seen a class of models of $\mathcal{S}$ that share some common property, as will be illustrated in the examples. The following is a simple fact, where joins are taken by selecting as models compatible families of models from each satisfaction system (compatibility here means that the formulas jointly satisfied by the models in each family are exactly the same).

Proposition 3.9
The partially ordered set $\langle Sat(L), \unlhd \rangle$ is a complete lattice.

We can now introduce the notion of *hierarchical satisfaction system*.

Definition 3.10
A *hierarchical satisfaction system* is a map

$$h : \langle X, \leq \rangle \to \langle Sat(L), \unlhd \rangle$$

where

- $\langle X, \leq \rangle$ is a finite lattice;
- h preserves joins;
- $M_{h(x)}$ is associated to a partition of $M_{h(\bot)}$.

Again, a hierarchical satisfaction system is just a sublattice of $\langle Sat(L), \unlhd \rangle$, but we require that the class of models of each of the systems be associated to a different partition of the models at the base. As a consequence of this definition, it immediately follows that, for any $x, y \in X$, $\emptyset \models_{h(x)} \varphi$ iff $\emptyset \models_{h(y)} \varphi$. Hence, clearly, a hierarchical deductive system presents a hierarchical consequence operator.

Proposition 3.11
Every hierarchical satisfaction system induces a hierarchical consequence operator with the same lattice structure.

Some intuitions on hierarchical satisfaction systems are in order. Let $h : \langle X, \leq \rangle \to \langle Sat(L), \unlhd \rangle$ be a hierarchical satisfaction system and assume that $\langle X, \leq \rangle$ is an atomic lattice. Given the bottom satisfaction system, the satisfaction system associated to each atom of X is obtained by choosing a partition of its models. This gives us an idea of how to proceed with the construction of the whole hierarchy. When going up in the order, the partitions will be less and less refined. Of course, a join must be built by taking jointly compatible partitions.

Example 3.12
Recall Example 2.5 on propositional modal logic. Consider the satisfaction systems defined as follows:

- $\mathcal{S}_{ml} = \langle L_m^\Pi, M_{ml}, \Vdash_{ml}\rangle$ is such that:
  - $M_{ml} = \{\langle W, R, \vartheta, w\rangle : \langle W, R\rangle$ is a frame, $\vartheta : W \to 2^\Pi, w \in W\}$,
  - $\langle W, R, \vartheta, w\rangle \Vdash_{ml} \varphi$ if $\langle W, R, \vartheta\rangle \Vdash_w \varphi$ as in Example 2.5;
- $\mathcal{S}_{mg} = \langle L_m^\Pi, M_{mg}, \Vdash_{mg}\rangle$ is such that:
  - $M_{mg} = \{m_{\langle W,R,\vartheta\rangle} : \langle W, R\rangle$ is a frame and $\vartheta : W \to 2^\Pi\}$ where $m_{\langle W,R,\vartheta\rangle} = \{\langle W, R, \vartheta, w\rangle \in M_{ml} : w \in W\}$;
  - $m_{\langle W,R,\vartheta\rangle} \Vdash_{mg} \varphi$ if $\langle W, R, \vartheta\rangle \Vdash_w \varphi$ for every $w \in W$;
- $\mathcal{S}_{fl} = \langle L_m^\Pi, M_{fl}, \Vdash_{fl}\rangle$ is such that:
  - $M_{fl} = \{m_{\langle W,R,w\rangle} : \langle W, R\rangle$ is a frame and $w \in W\}$ where $m_{\langle W,R,w\rangle} = \{\langle W, R, \vartheta, w\rangle \in M_{ml} : \vartheta : W \to 2^\Pi\}$;
  - $m_{\langle W,R,w\rangle} \Vdash_{fl} \varphi$ if $\langle W, R, \vartheta\rangle \Vdash_w \varphi$ for every ϑ;
- $\mathcal{S}_{fg} = \langle L_m^\Pi, M_{fg}, \Vdash_{fg}\rangle$ is such that:
  - $M_{fg} = \{m_{\langle W,R\rangle} : \langle W, R\rangle$ is a frame$\}$ where $m_{\langle W,R\rangle} = \{\langle W, R, \vartheta, w\rangle \in M_{ml}$: for every $\vartheta : W \to 2^\Pi$ and $w \in W\}$;
  - $m_{\langle W,R\rangle} \Vdash_{fl} \varphi$ if $\langle W, R, \vartheta\rangle \Vdash_w \varphi$ for every ϑ and every $w \in W$.

With these, we can establish a hierarchical satisfaction system as follows:

$$h_m : \mathbf{2^2} \to Sat(L_m^\Pi)$$

where $h_m(\bot) = \mathcal{S}_{ml}$, $h_m(\top) = \mathcal{S}_{fg}$, $h_m(a) = \mathcal{S}_{mg}$ and $h_m(b) = \mathcal{S}_{fl}$. Observe that each M_{mg}, M_{fl} and M_{fg} is in fact a partition of M_{ml}. In fact, each model in M_{mg} can be seen as the set all models in M_{ml} that have the same frame and the same valuation, i.e., only the designated world can vary. Similarly, each model in M_{fl} is the set of all models in M_{ml} that have the same frame and the same world, and only the valuation varies. Finally, each model in M_{fg} is the set of all models in M_{ml} that have the same frame, that is, both the valuation and the world may vary. It is straightforward to conclude that this hierarchical satisfaction system induces precisely the hierarchical consequence operator presented in Example 2.5.

Observe that similar constructions could be made for the other examples. For instance, in the case of classical first order logic of Example 2.6, we could take as models for the truth satisfaction system all pairs composed of a first-order interpretation structure and of an assignment. Then, for the *validity satisfaction system* we would partition these models in a way that each validity model would be the set of truth models that share the same interpretation structure but might have different assignments.

As explained before, we will be particularly interested in **2**-hierarchies of satisfaction systems. A common way of defining the semantics of logics is to use *logical matrices* (see [17]). For **2**-hierarchies we will use instead *generalized matrices*. These are obtained by endowing the envisaged sets of truth-values with more than just a set of designated values. Namely, we require the set of truth-values to be structured also according to a consequence operation as in [4], thus recovering an early proposal of Smiley [16]. Recall that a *general matrix* over L is a triple $\langle A, \cdot_A, \mathbf{c}\rangle$ where A is a set and $\cdot_A : L \to A$ is a map and $\mathbf{c} : \wp(A) \to \wp(A)$ is a consequence operator over A. We denote by $GM(L)$ the class of all general matrices over L.

Definition 3.13
An *interpretation system* over L is a triple $\mathcal{I} = \langle L, M, \alpha\rangle$ where:

- M is a class (of *models*);
- $\alpha : M \to GM(L)$ is a map.

Given an interpretation system $\mathcal{I} = \langle L, M, \alpha\rangle$ we can easily define a **2**-hierarchical satisfaction system. Just consider:

- $h(1) = \langle L, M', \Vdash'\rangle$ where $M' = \{\langle m, T\rangle : m \in \mathcal{M}, \alpha(m) = \langle A, \cdot_A, \mathbf{c}\rangle, T \subseteq A$ and $T^{\mathbf{c}} = T\}$ and $\langle m, T\rangle \Vdash' \varphi$ iff $\varphi_A \in T$;
- $h(2) = \langle L, M, \Vdash\rangle$ where $m \in M$ with $\alpha(m) = \langle A, \cdot_A, \mathbf{c}\rangle$ is such that $m \Vdash \varphi$ iff $\varphi_A \in \emptyset^{\mathbf{c}}$.

Observe that M is indeed an obvious partition of M', each model $m \in M$ can be seen as grouping all the pairs of the form $\langle m, T\rangle$. It is straightforward to see that this constitutes a **2**-satisfaction system. Note that in $\langle L, M', \Vdash'\rangle$ we get an entailment $\vDash'$ defined by $\Phi \vDash' \varphi$ if $\varphi_A \in \Phi_A^{\mathbf{c}}$ for every $m \in M$, assuming $\alpha(m) = \langle A, \cdot_A, \mathbf{c}\rangle$. In $\langle L, M, \Vdash\rangle$ we get a stronger entailment $\vDash$ defined by $\Phi \vDash \varphi$ if $\varphi_A \in \emptyset^{\mathbf{c}}$ when $\Phi_A \subseteq \emptyset^{\mathbf{c}}$ for every $m \in M$, assuming $\alpha(m) = \langle A, \cdot_A, \mathbf{c}\rangle$.

In analogy with the deductive case, we will use $\vDash_d$ and $\vDash_t$ to refer to the consequence induced by each of these deductive systems.

4 Combining logics

After introducing the essential ingredients of hierarchical logical consequence, we now turn our attention to the rich application field of combining logics. In particular, we will show how to develop a bit of the theory of fibring as an operation within **2**-hierarchies. The mechanism of fibring logics depends in an essential way on requiring that the syntax of the logics be generated from a given *signature* of constructors and on the subsequent structurality of the consequence operators. For simplicity, here, we will deal only with propositional-based logical languages generated from simple unsorted signatures. Still, a smooth extension towards richer languages can be achieved, in the lines of [7].

4.1 Structurality

A *signature* Σ is a $\mathbb{N}$-indexed family $\{\Sigma_n\}_{n\in\mathbb{N}}$ of sets. The elements of Σ_n are known as *connectives* of arity n. Propositional symbols are a subset of Σ_0. Given a signature Σ, the generated set of *formulas* is the carrier set $L(\Sigma)$ of the free Σ-algebra. Given a Σ-algebra $\mathcal{A}$ we will denote its carrier set by A and its interpretation map by $\cdot_{\mathcal{A}}$. The *denotation* $[\![\varphi]\!]_{\mathcal{A}}$ of a formula $\varphi \in L(\Sigma)$ in $\mathcal{A}$ is inductively defined as usual: $[\![c(\varphi_1, \ldots, \varphi_n)]\!]_{\mathcal{A}} = c_{\mathcal{A}}([\![\varphi_1]\!]_{\mathcal{A}}, \ldots, [\![\varphi_n]\!]_{\mathcal{A}})$, for every $c \in \Sigma_n$ and $\varphi_1, \ldots, \varphi_n \in L(\Sigma)$.

EXAMPLE 4.1
Given a set of propositional symbols Π, we may consider the signature Σ such that $\Sigma_0 = \Pi$; $\Sigma_1 = \{\neg, \Box\}$ and $\Sigma_2 = \{\Rightarrow\}$. It is easy to see that $L(\Sigma)$ is exactly L_m^Π of Example 2.5.

Another feature that we need to consider, in order to be able to fiber logics at the deductive level, is that the rules of a deductive system must be *schematic*. For this purpose, we assume given once and for all a set Ξ of *schema variables*. Given a signature Σ, the generated set of *schema formulas* is the carrier set $SL(\Sigma)$ of the free Σ-algebra with generators Ξ. A *(ground) schema Σ-substitution* is a function $\sigma : \Xi \to L(\Sigma)$. Given a schema formula $\delta \in SL(\Sigma)$, the *instance* of δ by the schema substitution σ, denoted by $\sigma(\delta)$, is the result of uniformly replacing each schema variable ξ in δ by $\sigma(\xi)$. Clearly $\sigma(\delta) \in L(\Sigma)$.

A *schema consequence operator* over Σ is a consequence operator $\_^\vdash$ over $SL(\Sigma)$ with the additional *structurality* condition [17]:

- $\sigma(\Phi^\vdash) \subseteq \sigma(\Phi)^\vdash$, for any schema substitution σ. (structurality)

We denote by $SCns(\Sigma)$ the set of all structural consequence operators over Σ, with the order relation $\sqsubseteq$ imported from $Cns(SL(\Sigma))$. Clearly, $\langle SCns(\Sigma), \sqsubseteq\rangle$ is still a complete lattice [17]. The definition of a *hierarchical structural consequence operator* over Σ is exactly as would be expected.

A *structural deductive system* over Σ is a triple $\mathcal{SD} = \langle \Sigma, D, T\rangle$ such that $\mathcal{D} = \langle SL(\Sigma), D, T\rangle$ is a deductive system. With this definition, the underlying language of a structural deductive system is the schema language $SL(\Sigma)$. Also, inference rules are defined over this schema language and can be regarded as *schema inference rules*. We denote by $SDed(\Sigma)$ the set of all structural deductive systems over Σ, with the order relation $\preceq$ imported from $Ded(SL(\Sigma))$. As in the case of deductive systems, $\langle SDed(\Sigma), \preceq\rangle$ is a complete lattice. The notion of *hierarchical structural deductive system* follows from the definition of *hierarchical deductive system* as expected.

EXAMPLE 4.2
Let Σ be the signature defined in Example 4.1. A structural deductive system for propositional modal logic is similar to the ones presented in 3.6, but in this case written with schematic inference rules. For instance, axiom (K) should be written as $(\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2))$. Similarly, rule (MP) would be written as

$$\frac{\xi_1 \qquad \xi_1 \Rightarrow \xi_2}{\xi_2}$$

Finally, we address structurality at the semantic level. Of course, we need to work with *structural general matrices* over Σ, built over a suitable Σ-algebra of truth-values. A *structural interpretation system* is a triple $\langle \Sigma, M, \alpha\rangle$, as before, but now, for each $m \in M$, $\alpha(m) = \langle \mathcal{A}_m, \mathbf{c}_m\rangle$ where $\mathcal{A}_m$ is a Σ-algebra and $\mathbf{c}_m$ is a consequence on A_m. A structural generalized matrix over Σ induces two satisfaction systems over $L(\Sigma)$, as was defined above, but now considering all possible assignments $\mu : \Xi \to A_m$.

EXAMPLE 4.3
Consider signature Σ of Example 4.1. The corresponding structural interpretation system over Σ is $\mathcal{SI} = \langle \Sigma, M, \alpha\rangle$ where:

- M is the class of all pairs $\langle \mathcal{B}, v\rangle$ where $\mathcal{B} = \langle B, \leq\rangle$ is a Boolean algebra with an additional operation $\Box : B \to B$ such that:
  – $\Box(\top) = \top$;
  – $\Box(x \wedge y) = \Box(x) \wedge \Box(y)$,
  and $v : \Pi \to B$ is a valuation function;
- $\alpha(\langle \mathcal{B}, v\rangle) = \langle \mathcal{B}, \mathbf{c}_{\leq}\rangle$ where the denotation $[\![\varphi]\!]_{\mathcal{B}}$ is as expected and $\mathbf{c}_{\leq}$ is the closure operator on B induced by $\leq$ (see [1]) defined as follows:
  – $X^{\mathbf{c}_{\leq}} = \mathcal{U}_{\leq}(\mathcal{L}_{\leq}(X))$;
  where $\mathcal{U}_{\leq}(X)$ is the set of *upper-bounds* of X, i.e., is the set $\{b \in B : x \leq b \text{ for every } x \in X\}$ and $\mathcal{L}_{\leq}(X)$ is the set of *lower-bounds* of X, i.e., is the set $\{b \in B : b \leq x \text{ for every } x \in X\}$.

4.2 Fibring

From the experiences learnt in previous work, it is clear that a rigorous and smooth algebraic characterization of *fibring* requires a notion of logic that promotes logical consequence as a whole, rather than just theoremhood or validity. Moreover, an explicit hierarchical distinction between different modes of reasoning within the same language is essential if one wants fibring to explain, at least, the naturalness of modal logics and their fusion [12]. Thus, working at the level of hierarchical consequence operators and their presentations seems to be quite appropriate. Our aim here is not to develop a fully fledged account of fibred hierarchies. Still, we will show how to develop the theory of fibring within the setting of **2**-hierarchies, including soundness and completeness preservation results about fibred structural deductive systems and interpretation systems, by adapting work reported in [4, 7].

Now, quite simply, within the setting of **2**-hierarchical consequence operators, fibring can be very simply defined, in a pointwise manner, as the join of structural consequence operators over the combined signature. Such an abstract construction could be easily obtained along the lines of [4]. Our aim, here, is to go directly towards the deductive and semantic fibring of presentations of **2**-hierarchies, namely the fibring of structural deductive and interpretation systems.

Given that the right amount of structurality [3] is embodied in the rules of deductive systems, their *fibring* is well understood [4] and meaningful. In the sequel we consider two arbitrary structural deductive systems $\mathcal{SD}_1 = \langle \Sigma_1, D_1, T_1\rangle$ and $\mathcal{SD}_2 = \langle \Sigma_2, D_2, T_2\rangle$. For simplicity, when fibring two structural deductive systems we assume that the shared connectives are always the connectives that are present in both signatures, that is, we are sharing the signature $\Sigma_0 = \{\Sigma_{0,n}\}_{n\in\mathbb{N}}$ where $\Sigma_{0,n} = \Sigma_{1,n} \cap \Sigma_{2,n}$ for every $n \in \mathbb{N}$. We just write $c \in \Sigma_0$ whenever $c \in \Sigma_{1,n} \cap \Sigma_{2,n}$ for some $n \in \mathbb{N}$.

Definition 4.4
The *fibring* of $\mathcal{SD}_1$ and $\mathcal{SD}_2$ (sharing Σ_0) is the structural deductive system $\mathcal{SD}_1 \circledast \mathcal{SD}_2 = \langle \Sigma, D, T\rangle$ such that:

- $\Sigma_n = \Sigma_{1,n} \cup \Sigma_{2,n}$, for every $n \in \mathbb{N}$;
- $D = D_1 \cup D_2$;
- $T = T_1 \cup T_2$.

The deduction rules in the fibring are just the deduction rules of each of the deductive systems, and similarly for theoremhood rules.

Next we introduce the fibring of structural interpretation systems. Let $\mathcal{SI}_1 = \langle \Sigma_1, M_1, \alpha_1 \rangle$ and $\mathcal{SI}_2 = \langle \Sigma_2, M_2, \alpha_2 \rangle$ be two arbitrary structural interpretation systems. The shared signature is again Σ_0.

Definition 4.5
The *fibring* of $\mathcal{SI}_1$ and $\mathcal{SI}_2$ (sharing Σ_0) is the structural interpretation system $\mathcal{SI}_1 \circledast \mathcal{SI}_2 = \langle \Sigma, M, \alpha \rangle$ such that:

- $\Sigma_n = \Sigma_{1,n} \cup \Sigma_{2,n}$, for every $n \in \mathbb{N}$;
- $M = \{(m_1, m_2) \in M_1 \times M_2 : A_{m_1} = A_{m_2}, \mathbf{c}_{m_1} = \mathbf{c}_{m_2}, \text{ and } c_{\mathcal{A}_{m_1}} = c_{\mathcal{A}_{m_2}} \text{ for } c \in \Sigma_0\}$;
- $\alpha(m_1, m_2) = \langle \mathcal{A}, \mathbf{c} \rangle$ is such that
  - $A = A_{m_1} = A_{m_2}$ and $c_{\mathcal{A}} = c_{\mathcal{A}_{m_i}}$ for all $c \in \Sigma_{i,n}$, $i \in \{1, 2\}$ and $n \in \mathbb{N}$;
  - $\mathbf{c} = \mathbf{c}_{m_1} = \mathbf{c}_{m_2}$.

In the fibring $\mathcal{SI}_1 \circledast \mathcal{SI}_2 = \langle \Sigma, M, \alpha \rangle$, the class M of models consists of all pairs of models from $\mathcal{SI}_1$ and $\mathcal{SI}_2$ whose corresponding algebras have the same carrier set plus the same interpretation for shared connectives, and whose corresponding consequence operators are equal. Then, in a obvious way, the map α associates to each such pair a Σ-algebra (where the interpretations of the non shared connectives from $\mathcal{SI}_i$ are as in $\mathcal{SI}_i$, for $i = 1, 2$) and a consequence operator.

In this context, we can now state and prove a collection of soundness and completeness transference results for *fibring*. Preservation of soundness is easily just a consequence of the construction underlying *fibring*, as shown in [4].

Since we are working in a **2**-structured setting, there are two notions of soundness and two notions of completeness involved: d-soundness and t-soundness, d-completeness and t-completeness. We say that a structural interpretation system $\mathcal{SI}$ is equipped with a a structural deductive system $\mathcal{SD}$ whenever they have the same signature. Then,

- $\mathcal{SD}$ is *d-sound* for $\mathcal{SI}$ if $\vdash_d \sqsubseteq \vDash_d$;
- $\mathcal{SD}$ is *t-sound* for $\mathcal{SI}$ if $\vdash_t \sqsubseteq \vDash_t$;
- $\mathcal{SD}$ is *d-complete* for $\mathcal{SI}$ if $\vDash_d \sqsubseteq \vdash_d$;
- $\mathcal{SD}$ is *t-complete* for $\mathcal{SI}$ if $\vDash_t \sqsubseteq \vdash_t$.

When $\mathcal{SD}$ is both *d-sound* and *t-sound* for $\mathcal{SI}$ we will simply say that it is sound. Similarly, if $\mathcal{SD}$ is both *d-complete* and *t-complete* for $\mathcal{SI}$ we will say that it is complete. If we just require that the conditions hold for consequences of finite sets of hypotheses, then we will refer to finite (d/t-)soundness/completeness. If we just require that the conditions hold for consequences of empty sets of hypotheses, that is theoremhood, then we will refer to weak (d/t-)soundness/completeness.

As usual, to prove the soundness of a deductive system one has to establish the soundness of each of its rules. Let $\langle \Delta, \gamma \rangle$ be an inference rule. As expected, the rule is said to be *d-sound* for $\mathcal{SI}$ if $\Delta \vDash_d \gamma$, and it is said to be *t-sound* for $\mathcal{SI}$ if $\Delta \vDash_t \gamma$. We say that all the rules of $\mathcal{SD}$ are sound whenever all rules in D are d-sound and the rules in T are t-sound.

THEOREM 4.6
Let $\mathcal{SI}_1$ and $\mathcal{SI}_2$ be structural interpretation systems equipped with the structural deductive systems $\mathcal{SD}_1$ and $\mathcal{SD}_2$, respectively. If all the rules of $\mathcal{SD}_1$ are sound for $\mathcal{SI}_1$ and all the rules of $\mathcal{SD}_2$ are sound for $\mathcal{SI}_2$, then $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ is sound for $\mathcal{SI}_1 \circledast \mathcal{SI}_2$.

PROOF. It is immediate, by definition of fibring, that in the conditions of the theorem all the rules of $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ are sound for $\mathcal{SI}_1 \circledast \mathcal{SI}_2$. ∎

On the contrary, as should be expected, completeness preservation results are in general not so easy to obtain. The completeness transference results that we shall present are based on the fundamental notion of *fullness*, as a means of guaranteeing that we always have enough models, extending original ideas from [18], further worked out in [4]. Let $\mathcal{P}$ be a class of consequence operators. A structural interpretation system $\mathcal{SI} = \langle \Sigma, M, \alpha \rangle$ equipped with a structural deductive system $\mathcal{SD}$ is said to be *full with respect to* $\mathcal{P}$ when there exists a model $m \in M$ such that $\alpha(m) = \langle \mathcal{A}, \mathbf{c} \rangle$ for every structural generalized matrix $\langle \mathcal{A}, \mathbf{c} \rangle$ with $\langle A, \mathbf{c} \rangle \in \mathcal{P}$ that makes all the rules in $\mathcal{SD}$ sound.

Based on fullness requirements, we provide completeness proofs for several classes of structural interpretation systems, using standard techniques in logic and algebra, such as congruences, Lindenbaum-Tarski algebras and Henkin-style techniques. The crucial feature of fullness is that it is always preserved by fibring.

PROPOSITION 4.7
Let $\mathcal{SI}_1$ and $\mathcal{SI}_2$ be structural interpretation systems equipped with the structural deductive systems $\mathcal{SD}_1$ and $\mathcal{SD}_2$, respectively, both full with respect to a class $\mathcal{P}$ of consequence operators. Then, the fibred structural interpretation system $\mathcal{SI}_1 \circledast \mathcal{SI}_2$ equipped with the fibred structural deductive system $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ is full with respect to $\mathcal{P}$.

PROOF. Suppose that a structural generalized matrix $\langle \mathcal{A}, \mathbf{c} \rangle$ that makes all the rules in $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ sound is such that $\langle A, \mathbf{c} \rangle \in \mathcal{P}$. Easily, then, the restrictions of $\mathcal{A}$ to Σ_1 and Σ_2 give rise to general matrices that make the rules in each $\mathcal{SD}_i$ sound, and thus are represented in each M_i, by fullness. Hence, their pair, whose associated generalized matrix is precisely $\langle \mathcal{A}, \mathbf{c} \rangle$, is a model of the fibred system. ∎

We shall now present completeness preservation results for different choices of $\mathcal{P}$.

THEOREM 4.8
Let $\mathcal{SI}_1$ and $\mathcal{SI}_2$ be structural interpretation systems equipped with structural deductive systems $\mathcal{SD}_1$ and $\mathcal{SD}_2$ and full with respect to the class of all consequence operators. Then, the fibred structural deductive system $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ is complete for the fibred structural interpretation structure $\mathcal{SI}_1 \circledast \mathcal{SI}_2$.

PROOF. Let Σ be the signature of the combined systems, $\vdash_d$ and $\vdash_t$ the consequences induced by $\mathcal{SD}_1 \circledast \mathcal{SD}_2$. Given $\Gamma \subseteq SL(\Sigma)$ with $\Gamma^{\vdash_t} = \Gamma$, let $\mathbf{c}_\Gamma$ be the consequence operator over $SL(\Sigma)$ defined by $\Delta^{\mathbf{c}_\Gamma} = (\Gamma \cup \Delta)^{\vdash_d}$. It is straightforward to conclude that $\langle SL(\Sigma), \mathbf{c}_\Gamma \rangle$ makes all the rules of $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ sound, and thus, by Proposition 4.7, it is the structure of some fibred model. If we fix the assignment such that $\mu(\xi) = \xi$ for all schema variables, then note that $[\![\varphi]\!]^{\mu}_{SL(\Sigma)} = \varphi$ for every schema formula φ.

For t-completeness, assume that $\Delta \not\vdash_t \theta$. Let $\Gamma = \Delta^{\vdash_t}$ and consider the structure $\langle SL(\Sigma), \mathbf{c}_\Gamma \rangle$. To prove that $\Delta \not\vDash_t \theta$ it suffices to show that $\Delta \subseteq \emptyset^{\mathbf{c}_\Gamma}$ but $\theta \notin \emptyset^{\mathbf{c}_\Gamma}$. Easily, then, $\Delta \subseteq \Delta^{\vdash_t} = \Gamma \subseteq (\Gamma \cup \emptyset)^{\vdash_d} = \emptyset_{\mathbf{c}_\Gamma}$. However, $\emptyset^{\mathbf{c}_\Gamma} = \Gamma^{\vdash_d} = \Gamma = \Delta^{\vdash_t}$, since $\vdash_d \sqsubseteq \vdash_t$, and $\theta \notin \Delta^{\vdash_t}$.

For d-completeness, assume that $\Delta \not\vdash_d \theta$. Let $\Gamma = \emptyset^{\vdash_t}$ and consider the structure $\langle SL(\Sigma), \mathbf{c}_\Gamma \rangle$. To prove that $\Delta \not\vDash_d \theta$ it suffices to show that $\theta \notin \Delta^{\mathbf{c}_\Gamma}$. Easily, then, $\Delta^{\mathbf{c}_\Gamma} = (\Gamma \cup \Delta)^{\vdash_d} = (\emptyset^{\vdash_t} \cup \Delta)^{\vdash_d} = \Delta^{\vdash_d}$, since $\emptyset^{\vdash_d} = \emptyset^{\vdash_t}$. However, $\theta \notin \Delta^{\vdash_d}$. ∎

This result may seem to be a little too syntactic and artificial. Let us focus on a much more interesting class of consequence operators. Recall from example 4.3 that every partial-order $\langle A, \leq \rangle$ easily induces two polarities $\mathcal{U}_\leq$ and $\mathcal{L}_\leq$, and a cut closure $\mathbf{c}$ on A defined by $B^{\mathbf{c}} = \mathcal{U}_\leq(\mathcal{L}_\leq(B))$, where $B \subseteq A$, as in [1]. In such case, $\langle A, \mathbf{c} \rangle$ is said to be a *partial order* consequence operator.

A structural deductive system $\mathcal{SD} = \langle \Sigma, D, T \rangle$ is said to be *congruent* if for every connective $c \in \Sigma_n$, every set Γ of schema formulas closed for theorem generation, and all schema formulas $\gamma_1 \ldots, \gamma_n, \delta_1, \ldots, \delta_n$, it is the case that

$$\Gamma \cup \{c(\gamma_1 \ldots, \gamma_n)\} \vdash_d c(\delta_1, \ldots, \delta_n)$$

whenever

$$\Gamma \cup \{\gamma_i\} \vdash_d \delta_i \text{ and } \Gamma \cup \{\delta_i\} \vdash_d \gamma_i \text{ for } i = 1, \ldots, n.$$

Many deductive systems, including the ones presented above, are congruent. But there are exceptions, as for instance the paraconsistent systems of [9].

Furthermore, $\mathcal{SD}$ is said to have an *implication* connective if there exists $\Rightarrow \in \Sigma_2$ such that

$$\Gamma \vdash_d (\gamma \Rightarrow \delta) \text{ if and only if } \Gamma \cup \{\gamma\} \vdash_d \delta$$

for every set Γ of schema formulas and all schema formulas γ, δ.

Actually, the fibring of two congruent structural deductive systems sharing an implication connective is also a congruent structural deductive system with implication. The proof of this fact can be found in [18, 4].

THEOREM 4.9
Let $\mathcal{SI}_1$ and $\mathcal{SI}_2$ be structural interpretation systems equipped with the structural deductive systems $\mathcal{SD}_1$ and $\mathcal{SD}_2$ and full with respect to the class of all partial-order consequence operators. If both $\mathcal{SD}_1$ and $\mathcal{SD}_2$ are congruent and they share an implication connective, then $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ is t-complete and weak d-complete for $\mathcal{SI}_1 \circledast \mathcal{SI}_2$.

PROOF. Let Σ be the signature of the combined systems, $\vdash_d$ and $\vdash_t$ the consequences induced by $\mathcal{SD}_1 \circledast \mathcal{SD}_2$. Given $\Gamma \subseteq SL(\Sigma)$ with $\Gamma^{\vdash_t} = \Gamma$, the relation defined by $\varphi \equiv \psi$ if both $\Gamma \cup \{\varphi\} \vdash_d \psi$ and $\Gamma \cup \{\psi\} \vdash_d \varphi$ is a congruence of the free algebra $SL(\Sigma)$. Let $\mathbf{c}_{\leq_\Gamma}$ be the cut consequence operator over $SL(\Sigma)$ defined from the partial order $\leq_\Gamma$ on the quotient algebra $SL(\Sigma)_{/\equiv}$ defined by $[\varphi]_\equiv \leq_\Gamma [\psi]_\equiv$ if $\varphi \Rightarrow \psi \in \Gamma$. It is easy to conclude that $\langle SL(\Sigma)_{/\equiv}, \mathbf{c}_{\leq_\Gamma} \rangle$ is a partial order consequence that makes all the rules of $\mathcal{SD}_1 \circledast \mathcal{SD}_2$ sound, and thus, by Proposition 4.7, it is the structure of some fibred model. If we fix the assignment such that $\mu(\xi) = [\xi]_\equiv$ for all schema variables, then note that $[\![\varphi]\!]^\mu_{SL(\Sigma)} = [\varphi]_\equiv$ for every schema formula φ.

For t-completeness, assume that $\Delta \not\vdash_t \theta$. Let $\Gamma = \Delta^{\vdash_t}$ and consider the structure $\langle SL(\Sigma)_{/\equiv}, \mathbf{c}_{\leq_\Gamma}\rangle$. To prove that $\Delta \not\models_t \theta$ it suffices to show that $[\Delta]_\equiv \subseteq \emptyset^{\mathbf{c}_{\leq_\Gamma}}$ but $[\theta]_\equiv \notin \emptyset^{\mathbf{c}_{\leq_\Gamma}}$. It can be easily shown that $\emptyset^{\mathbf{c}_{\leq_\Gamma}} = \{[\psi]_\equiv : \varphi \Rightarrow \psi \in \Gamma$ for every $\varphi \in SL(\Sigma)\}$. Now, if $\delta \in \Delta \subseteq \Gamma$, since Γ is $\vdash_d$ closed, it follows from the properties of implication that $\varphi \Rightarrow \delta \in \Gamma$. However, $\xi \Rightarrow \xi \in \Gamma$ and thus $(\xi \Rightarrow \xi) \Rightarrow \theta \notin \Gamma$, or we would also have $\theta \in \Gamma$.

Weak d-completeness is just a corollary of t-completeness, since the two have the same theorems. ∎

This result can be slightly improved, obtaining finite d-completeness, as long as we require also that at least one of the systems being fibred has a well-behaved conjunction connective. Still, since it is not possible to require an infinitary conjunction, in order to obtain d-completeness, we will now dwell on an alternative approach. The following result, whose proof we omit, is an instance of the one obtained in [18] concerning algebras of sets in the style of general frames for modal logic (see [13]), and uses a Henkin-style construction. For a powerset lattice $\langle \wp(U), \subseteq\rangle$, the cut consequence $\mathbf{c}_\subseteq$ induced by the polarities as explained above is such that $B^{\mathbf{c}_\subseteq} = \{b \subseteq \wp(U) : (\bigcap B) \subseteq b\}$. A consequence operator $\langle A, \hat{\mathbf{c}}\rangle$ is said to be a *general powerset* consequence operator if $A \subseteq \wp(U)$, $U \in A$ and $\hat{\mathbf{c}}$ is the consequence induced by $\mathbf{c}_\subseteq$ on A, that is, $B^{\hat{\mathbf{c}}} = B^{\mathbf{c}_\subseteq} \cap A$ for each $B \subseteq A$.

Theorem 4.10
Let $\mathcal{SI}_1$ and $\mathcal{SI}_2$ be structural interpretation systems, full with respect to the class of all general powerset consequence operators, and equipped with the structural deductive systems $\mathcal{SD}_1$ and $\mathcal{SD}_2$. If both $\mathcal{SD}_1$ and $\mathcal{SD}_2$ are congruent and there is a shared implication connective, then $\mathcal{SD}_1 \circledast \mathcal{SD}_1$ is d-complete for $\mathcal{SI}_1 \circledast \mathcal{SI}_2$.

5 Concluding remarks

We have proposed a novel look on the notion of logic, and modeled it as a lattice-based hierarchy of consequence operators. The approach can be seen as a step further along the path that led logicians to shift the focus of their attention from mere theoremhood to the fundamental notion of logical consequence, and reflects the different but strongly related modes of reasoning that are often available over the same logical language. We have illustrated the concept with the aid of a collection of familiar examples, where the hierarchical phenomena are well represented. The choice of a lattice-based structure and the requirement of a common set of theorems was not arbitrary, as was also illustrated by the worked examples. Finiteness, on the contrary, is just a working option. Logical hierarchies based on infinite lattices (or perhaps on complete partial orders) seem to be plausible and workable. We have also defined and explored hierarchical versions of two of the most widespread vehicles for presenting logics. Namely, on the proof-theoretic side, we have studied hierarchical Hilbert-style deductive systems, and on the model-theoretic side, a suitable notion of hierarchical satisfaction system.

It turns out that our interest in regarding logics as hierarchies of consequence operators is not merely abstract. In fact, it was motivated by studying combined logics and, in particular, the mechanism of fibring. We have not even attempted to develop a fully fledged theory of fibred hierarchies, but we have shown how fibring and a number of meaningful results about fibred logics arise naturally in this context. We have looked in detail into **2**-hierarchies, not just for simplicity, but essentially because it corresponds to the setting where most of the previous work on the theory of fibring was developed. In particular, we have studied the fibred semantics of structural interpretation systems based on generalized logical matrices and a number of soundness and completeness preservation results. Still, in this regard, the paper raises many more questions than the answers it proposes. This is the case even for the fibring of **2**-hierarchies. In fact, it seems clear that combining logics seen as hierarchies should first of all include a suitable way of combining the lattices involved. As we did it, and as it has always been the case in previous work on fibring [4, 18, 14, 5, 8, 6, 7], the fibring of **2**-hierarchies yields another **2**-hierarchy as a result. However, this seems to be rather tight. If we adopt a free perspective, the fibring of two **2**-hierarchies would rather be a $\mathbf{2}^{\mathbf{2}}$-hierarchy reflecting the relative independence of the two. In the general case, the fibring of two hierarchies should be guided by a suitable lattice combination diagram, and the resulting hierarchy be structured according to a pullback lattice. Indeed, structural deductive systems and structural interpretation systems induce **2**-hierarchies of deductive systems and satisfaction systems, respectively, but the converse is unclear. The fibring of hierarchical deductive systems and satisfaction systems (actually of their structural versions) does not seem to be difficult, using known facts about their fibred non-hierarchical versions, but it certainly deserves a detailed study. An adequate category theoretic development of these ideas is envisaged, in the lines of [4]. Such a setting can also be expected to allow for a much simpler and natural solution to the problems arising, for instance, when combining modal logic with first order logic [15]. Under a suitable lattice composition diagram, it is even plausible to obtain directly first-order logic over increasing domains structured as a **3**-hierarchy. Although a little speculative, one might even explore new possibilities for completeness preservation results, taking into account the lattice structures. In particular, it does not seem to be an impossible task to infer both the Barcan formula and its converse by means of a careful systematic analysis of the paths along the lattice resulting from the combination of modal and first order logics.

Acknowledgments

This work was partially supported by FCT and EU FEDER, namely via the projects QuantLog POCI/MAT/55796/2004 and KLog PTDC/MAT/68723/2006 of SQIG-IT.

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Received submission date