On the algebraization of many-sorted logics*

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Abstract. The theory of abstract algebraic logic aims at drawing a strong bridge between logic and universal algebra, namely by generalizing the well known connection between classical propositional logic and Boolean algebras. Despite of its successfulness, the current scope of application of the theory is rather limited. Namely, logics with a many-sorted language simply fall out from its scope. Herein, we propose a way to extend the existing theory in order to deal also with many-sorted logics, by capitalizing on the theory of many-sorted equational logic. Besides showing that a number of relevant concepts and results extend to this generalized setting, we also analyze in detail the examples of first-order logic and the paraconsistent logic $\mathcal{C}_1$ of da Costa.

1 Introduction

The general theory of *abstract algebraic logic* (AAL from now on) was first introduced in [1]. It aims at providing a strong bridge between logic and universal algebra, namely by generalizing the so-called *Lindenbaum-Tarski method*, which led to the well known connection between classical propositional logic and Boolean algebras. Within AAL, one explores the relationship between a given logic and a suitable algebraic theory, in a way that enables one to use algebraic tools to study the metalogical properties of the logic being algebraized, namely with respect to axiomatizability, definability, the deduction theorem, or interpolation [9]. Nevertheless, AAL has only been developed (as happened, until recently, also with much of the research in universal algebra) for the single-sorted case. This means that the theory applies essentially only to propositional-based logics, and that logics over many-sorted languages simply fall out of its scope. It goes without saying that rich logics, with many-sorted languages, are essential to specify and reason about complex systems, as also argued and justified by the theory of combined logics [17, 19].
Herein, we propose a way to extend the scope of applicability of AAL by generalizing to the many-sorted case several of the key concepts and results of the current theory, including several alternative *characterization results*, namely those involving the *Leibniz operator* and *maps of*

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logics. The generalization we propose assumes that the language of a logic is built from a many-sorted signature, with a distinguished sort for formulas. The algebraic counterpart of such logics will then be obtained via a strong representation over a suitable many-sorted algebraic theory, thus extending the notion of single-sorted algebraization of current AAL. Terms of other sorts may exist, though, but they do not correspond to formulas. Of course, in a logic, one reasons only about formulas, and only indirectly about terms of other sorts. Hence, we consider that the sort of formulas is the only visible sort, and we will also aim at a direct application of the theory of *hidden algebra*, as developed for instance in [18], to explore possible behavioral characterizations of the algebraic counterpart of a given many-sorted algebraizable logic.
We explore the new concepts by analyzing the example of first-order logic in a many-sorted context, and comparing with its previous unsorted study [1, 2]. We will also see how to apply our many-sorted approach in order to provide a new algebraic perspective to certain logics which are not algebraizable in current AAL. Namely, we will establish the many-sorted algebraization of the paraconsistent logic $\mathcal{C}_1$ of da Costa [8], whose single-sorted non-algebraizability is well known [16, 13].
The paper is organized as follows. In section 2 we will introduce a number of necessary preliminary notions and notations. In section 3 we will introduce the essential concepts and results of current AAL, and present some of its limitations by means of examples. Then, in section 4, we will present our generalized notion of many-sorted algebraizable logic and a detailed analysis of the examples in the generalized setting. We will also show that some relevant concepts and results of AAL smoothly extend to the many-sorted setting. Finally, section 5 draws some conclusions, and points to several topics of future research.

2 Preliminaries

In this section we introduce the preliminary notions and notations that we will need in the rest of the paper, namely concerning logic and algebra.

2.1 Logics and maps

We will adopt the Tarskian notion of logic. A logic is a pair $\mathcal{L} = \langle L, \vdash \rangle$, where L is a set of formulas and $\vdash \subseteq 2^L \times L$ is a consequence relation satisfying the following conditions, for every $\Gamma \cup \Phi \cup \{\varphi\} \subseteq L$:

Reflexivity: if $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$;
Cut: if $\Gamma \vdash \varphi$ for all $\varphi \in \Phi$, and $\Phi \vdash \psi$ then $\Gamma \vdash \psi$;
Weakening: if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Phi$ then $\Phi \vdash \varphi$.

We will consider only these three conditions. However, Tarski also considered a *finitariness* condition (see [21]):

Finitariness: if $\Gamma \vdash \varphi$ then $\Gamma' \vdash \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

In the sequel if $\Gamma, \Phi \subseteq L$, we shall write $\Gamma \vdash \Phi$ whenever $\Gamma \vdash \varphi$ for all $\varphi \in \Phi$. We say that φ and ψ are *interderivable*, which is denoted by $\varphi \dashv\vdash \psi$, if $\varphi \vdash \psi$ and $\psi \vdash \varphi$. In the same way, given $\Gamma, \Phi \subseteq L$ we say that Γ and Φ are *interderivable*, if $\Gamma \vdash \Phi$ and $\Phi \vdash \Gamma$. The *theorems* of $\mathcal{L}$ are the formulas φ such that $\emptyset \vdash \varphi$. A *theory* of $\mathcal{L}$, or briefly a $\mathcal{L}$-*theory*, is a set Γ of formulas such that if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$. Given a set Γ, we can consider the set $\Gamma^{\vdash}$, the smallest theory containing Γ. The set of all theories of $\mathcal{L}$ is denoted by $Th_{\mathcal{L}}$. It is easy to see that $\langle Th_{\mathcal{L}}, \subseteq \rangle$ forms a complete partial order.

We will need to use a rather strong notion of map of logics. Let $\mathcal{L} = \langle L, \vdash \rangle$ and $\mathcal{L}' = \langle L', \vdash' \rangle$ be two logics. A map θ from $\mathcal{L}$ to $\mathcal{L}'$ is a function $\theta : L \to 2^{L'}$ such that, if $\Gamma \vdash \varphi$ then $(\bigcup_{\gamma \in \Gamma} \theta(\gamma)) \vdash' \theta(\varphi)$. The map θ is said to be *conservative* when $\Gamma \vdash \varphi$ iff $(\bigcup_{\gamma \in \Gamma} \theta(\gamma)) \vdash' \theta(\varphi)$. A strong representation of $\mathcal{L}$ in $\mathcal{L}'$ is a pair (θ, τ) of conservative maps $\theta : \mathcal{L} \to \mathcal{L}'$ and $\tau : \mathcal{L}' \to \mathcal{L}$ such that:

i) For all $\varphi \in L$ we have that $\varphi \dashv\vdash \tau[\theta(\varphi)]$;
ii) For all $\varphi' \in L'$ we have that $\varphi' \dashv\vdash' \theta[\tau(\varphi')]$.

Note that the cases where $\theta(\varphi)$ is a singleton set for every $\varphi \in L$, or is a finite set for every $\varphi \in L$, are usual particular cases of the above definition of map. For the sake of notation we will use $\theta[\Gamma] = \bigcup_{\gamma \in \Gamma} \theta(\gamma)$. Hence, a map θ is such that if $\Gamma \vdash \varphi$ then $\theta[\Gamma] \vdash \theta(\varphi)$. Analogously, it is conservative when $\Gamma \vdash \varphi$ iff $\theta[\Gamma] \vdash \theta(\varphi)$. Note also that, because of the symmetry of the definition of strong representation, we also have a strong representation (τ, θ) of $\mathcal{L}'$ in $\mathcal{L}$. The existence of a strong representation of $\mathcal{L}$ in $\mathcal{L}'$ intuitively means that the consequence relation of $\mathcal{L}$ can be represented in $\mathcal{L}'$, and vice-versa, such that they are, in some precise sense, inverse of each other. Actually, θ and τ induce an isomorphism of the complete partial orders of theories of $\mathcal{L}$ and $\mathcal{L}'$. It is not difficult to see that, if we assume the conservativeness of θ and consider any function $\tau : L' \to 2^L$ that satisfies **ii)**, then we can conclude that τ is in fact a conservative map from $\mathcal{L}'$ to $\mathcal{L}$ that also satisfies **i)**.

2.2 Algebra

Recall that a many-sorted signature is a pair $\Sigma = \langle S, O \rangle$ where S is a set (of *sorts*) and $O = \{O_{ws}\}_{w \in S^*, s \in S}$ is an indexed family of sets. For simplicity, we write $f : s_1 \dots s_n \to s$ for an element f of $O_{s_1 \dots s_n s}$. As usual, we denote by $T_\Sigma(X)$ the S-indexed family of carrier sets of the free Σ-algebra $\mathbf{T_\Sigma(X)}$ with generators taken from a sorted family $X = \{X_s\}_{s \in S}$ of variable sets. Often, we will need to write terms over a given finite set of variables $t \in T_\Sigma(x_1 : s_1, \dots, x_n : s_n)$. For simplicity, we will denote such a term by $t(x_1 : s_1, \dots, x_n : s_n)$. Moreover, if T is a set whose elements are all terms of this form, we will denote this fact by writing $T(x_1 : s_1, \dots, x_n : s_n)$. Fixed X, we will use $t_1 \approx t_2$ to represent an equation $\langle t_1, t_2 \rangle$ between Σ-terms t_1 and t_2 of the same sort. If both terms are of sort s, we will dub $t_1 \approx t_2$ an s-equation. The set of all Σ-equations will be written as Eq_Σ. Moreover, we will denote conditional equations by $t_1 \approx u_1 \wedge \dots \wedge t_n \approx u_n \to t \approx u$. A set Θ whose elements are all equations over terms of the form $t(x_1 : s_1, \dots, x_n : s_n)$, will also

be dubbed $\Theta(x_1 : s_1, \ldots, x_n : s_n)$. A substitution $\sigma = \{\sigma_s : X_s \to T_\Sigma(X)_s\}_{s\in S}$ is an indexed family of functions. As usual, $\sigma(t)$ denotes the term obtained by uniformly applying σ to each variable in t. Given $t(x_1 : s_1, \ldots, x_n : s_n)$ and terms $t_1 \in T_\Sigma(X)_{s_1}, \ldots, t_n \in T_\Sigma(X)_{s_n}$, we will write $t(t_1, \ldots, t_n)$ to denote the term $\sigma(t)$ where σ is a substitution such that $\sigma_{s_1}(x_1) = t_1, \ldots, \sigma_{s_n}(x_n) = t_n$. Extending everything to sets of terms, given $T(x_1 : s_1, \ldots, x_n : s_n)$ and $U \in T_\Sigma(X)_{s_1} \times \cdots \times T_\Sigma(X)_{s_n}$, we will use $T[U] = \bigcup_{\langle t_1,\ldots,t_n\rangle \in U} T(t_1, \ldots, t_n)$.
Given a Σ-algebra $\mathbf{A}$, we will use A_s to denote its carrier set of sort s. As usual, given an equation $t_1 \approx t_2$ of sort s, we write $\mathbf{A} \Vdash t_1 \approx t_2$ to denote the fact $\mathbf{A}$ is a model of, or satisfies, the equation. The same applies to conditional equations. Given a class K of Σ-algebras, we define the consequence relation $\vDash^K_\Sigma$ as follows: $\Theta \vDash^K_\Sigma t_1 \approx t_2$ when, for every $\mathbf{A} \in K$, if $\mathbf{A} \Vdash u_1 \approx u_2$ for every $u_1 \approx u_2 \in \Theta$ then also $\mathbf{A} \Vdash t_1 \approx t_2$. We may omit the superscript and simply write $\vDash_\Sigma$ if K is the class of all Σ-algebras. We will use Eqn^K_Σ to refer to the logic $\langle Eq_\Sigma, \vDash^K_\Sigma\rangle$.
From now on we will assume that all signatures have a distinguished sort ϕ, for formulas. Moreover, we will assume that $X_\phi = \{\xi_i : i \in \mathbb{N}\}$ and will simply write ξ_k instead of $\xi_k : \phi$. Given a specification $\langle \Sigma, \Phi\rangle$ where Φ is a set of Σ-equations, we define the induced set of formulas $L_{\Sigma,\Phi}$ to be the carrier set of sort ϕ of the initial model $\mathbf{L}_{\mathbf{\Sigma},\mathbf{\Phi}} = \mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X})_{/\mathbf{\Phi}}$ of Φ. When $\Phi = \emptyset$, we will simply write $\mathbf{L}_{\mathbf{\Sigma}}$. Moreover, we will use $BEqn^K_\Sigma$ to refer to the logic $\langle Eq_\Sigma, \vDash^K_{\Sigma,bhv}\rangle$, where $\vDash^K_{\Sigma,bhv}$ is the behavioral consequence relation defined for instance as in [18, 10], by considering ϕ to be the unique visible sort and adopting a suitable set of visible contexts.

3 Limitations of the current theory of AAL

In this section, we intend to illustrate some of the limitations of the current theory of AAL. For that purpose, we begin by briefly presenting the essential notions and results of the theory. Still, it is not our aim to survey AAL, but rather to focus on what will be relevant in the rest of the paper. A recent comprehensive survey of AAL is [9], where the proofs (or pointers to the proofs) of the results we will mention can be found.

3.1 Concepts and results of unsorted AAL

The original formulation of AAL in [1] considered only finitary logics. Currently, the finitariness condition has been dropped [9]. Still, the objects of study of current AAL are logics whose formulas have some additional algebraic structure, namely their set of formulas is freely obtained from a propositional (single-sorted) signature.

Definition 1 (Structural single-sorted logic).
A *structural single-sorted logic* is a pair $\mathcal{L} = \langle \Sigma, \vdash\rangle$, where Σ is a single-sorted signature and $\langle L_\Sigma, \vdash\rangle$ is a logic that also satisfies:

Structurality: if $\Gamma \vdash \varphi$ then $\sigma[\Gamma] \vdash \sigma(\varphi)$ for every substitution σ.

Clearly, ϕ must be the unique sort of Σ. Finally, we can introduce the main notion of AAL.

Definition 2 (Single-sorted algebraizable logic).
A structural single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *algebraizable* if there exists a class K of Σ-algebras, a set $\Theta(\xi)$ of Σ-equations, and a set $E(\xi_1, \xi_2)$ of $\mathcal{L}$-formulas such that the following conditions hold:

- for every $\Gamma \cup \{\varphi\} \subseteq L_\Sigma$, $\Gamma \vdash \varphi$ iff $\Theta[\Gamma] \vDash^K_\Sigma \Theta(\varphi)$;
- for every $\Delta \cup \{\varphi \approx \psi\} \subseteq Eq_\Sigma$, $\Delta \vDash^K_\Sigma \varphi \approx \psi$ iff $E[\Delta] \vdash E(\varphi, \psi)$;
- $\xi \dashv\vdash E[\Theta(\xi)]$ and $\xi_1 \approx \xi_2 \mathrel{=\!\mid\!\vDash^K_\Sigma} \Theta[E(\xi_1, \xi_2)]$.

The set Θ of equations is called the set of *defining equations*, E is called the set of *equivalential formulas*, and K is called an *equivalent algebraic semantics* for $\mathcal{L}$. The notion of algebraizable logic intuitively means that the consequence relation of $\mathcal{L}$ can be captured by the equational consequence relation $\vDash^\Sigma_K$, and vice-versa, in a logically inverse way. When a logic is algebraizable and both Θ and E are finite, we say the logic is *finitely algebraizable*. Other variants of the notion of algebraizability and their relationships are illustrated in Fig. 1. Note however that, in this paper, we will not explore them.
In [1] Blok and Pigozzi proved interesting results concerning the uniqueness and axiomatization of an equivalent algebraic semantics of a given finitary and finitely algebraizable logic. They proved that a class of algebras K is an equivalent algebraic semantics of a finitary finitely algebraizable logic if and only if the quasivariety generated by K is also an equivalent algebraic semantics. In terms of uniqueness they showed that there is a unique quasivariety equivalent to a given finitary finitely algebraizable logic. The axiomatization of this quasivariety can be directly built from an axiomatization of the logic being algebraized, as stated in the following result.

Theorem 1. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a (finitary) structural single-sorted logic obtained from a deductive system formed by a set of axioms Ax and a set of inference rules R. Assume that $\mathcal{L}$ is finitely algebraizable with Θ and E. Then, the equivalent quasivariety semantics is axiomatized by the following equations and conditional-equations:*

- *$\Theta(\varphi)$ for each $\varphi \in Ax$;*
- *$\Theta[E(\xi, \xi)]$;*
- *$\Theta(\psi_0) \wedge \ldots \wedge \Theta(\psi_n) \rightarrow \Theta(\psi)$ for each rule $\frac{\psi_0 \ \ldots \ \psi_n}{\psi} \in R$;*
- *$\Theta[E(\xi_1, \xi_2)] \rightarrow \xi_1 \approx \xi_2$.*

There are several interesting and useful alternative characterizations of algebraizability. The most useful, namely to prove negative results, is perhaps the characterization that explores the properties of the so-called *Leibniz operator*. A congruence $\equiv$ in a Σ-algebra $\mathbf{A}$ is said to be *compatible* with a subset F of A_ϕ if whenever $a \in F$ and $a \equiv b$ then $b \in F$. In this case, F is a union of equivalence classes of $\equiv$. We will use $\mathrm{Cong}_\mathbf{A}$ to denote the set of all congruences of a Σ-algebra $\mathbf{A}$. Recall that $\mathrm{Cong}_\mathbf{A}$ equipped with inclusion also constitutes a complete partial order.

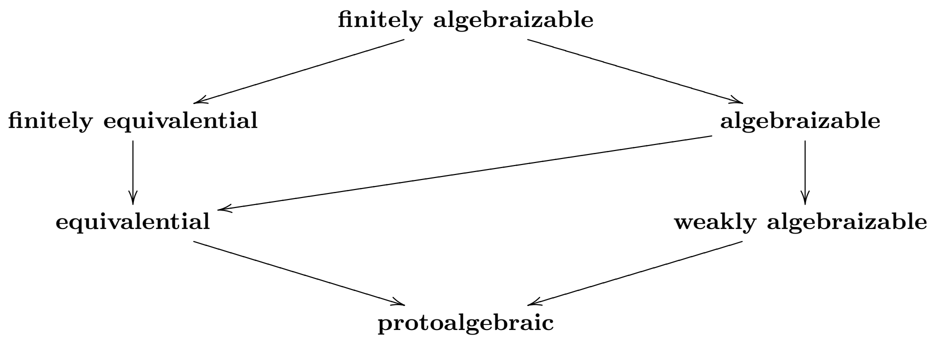

Fig. 1. A view of the Leibniz hierarchy.

Definition 3 (Leibniz operator).
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a structural single-sorted logic. The *Leibniz operator* on the formula algebra, $\Omega : Th_{\mathcal{L}} \to \mathrm{Cong}_{\mathbf{L}_{\boldsymbol{\Sigma}}}$ is such that, for each theory Γ of $\mathcal{L}$, $\Omega(\Gamma)$ is the largest congruence of $\mathbf{L}_{\boldsymbol{\Sigma}}$ compatible with Γ.

The denomination of the hierarchy considered in Fig. 1 is well justified by the fact that each of the classes of logics mentioned can be characterized by inspection of the properties of the corresponding Leibniz operator. Concerning algebraizability, we have the following result.

Theorem 2. *A structural single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable iff Ω is monotone, injective, and commutes with inverse substitutions.*

Another enlightening characterization of algebraization can be expressed using maps of logics [6].

Theorem 3. *A structural single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable iff there exists a class K of Σ-algebras and a strong representation (θ, τ) of $\mathcal{L}$ in Eqn_K^Σ such that both θ and τ commute with substitutions.*

Note that the fact that both maps commute with substitution is essential to guarantee that each can be given uniformly, respectively, by a set Θ of one-variable equations, and a set E of two-variable formulas.

3.2 Examples, good and bad

The theory of AAL is fruitful in positive and interesting examples. We will begin by introducing the two most known and simple examples.

Example 1 (Classical and Intuitionistic Propositional Logics).
The main paradigm of AAL is the well establish connection between *classical propositional logic* (CPL) and the variety of *Boolean algebras*. This was really the starting point to the idea of connecting logic with algebra, which evolved trying to generalize this connection to other logics. Other important example is *intuitionistic propositional logic* (IPL). Its

algebraization gives rise to the class of *Heyting algebras*. It is interesting to note that, in contrast to Boolean algebras, which were defined before the tools of abstract algebraic logic were first applied to generate them from CPL, Heyting algebras seem to be the first algebras of logic that were identified by applying this theory to a given axiomatization of IPL.

Even considering the enormous success of this theory, not only in the generality of its results, but also in the large amount of examples, we can point out some limitations. From our point of view, one major limitation of the existing theory is its inability to correctly deal with logics with a many-sorted language. Let us, first of all, discuss the paradigmatic example of *first-order classical logic* (FOL).

Example 2 (First-Order Classical Logic).
Research on the algebraization of FOL goes back to the seminal work initiated by Tarski in the 1940s, and published in collaboration with Henkin and Monk in [11]. In [1], Blok and Pigozzi present an single-sorted algebraization of FOL in the terms we have just introduced. Their idea was to massage the first-order language into a propositional language and then present a structural propositional deductive system **PR**, introduced by Németi, for first-order logic over this propositional language. It is then proved that **PR** is algebraizable. Moreover, it is proved that the variety equivalent to **PR** is the variety of cylindric algebras.
Despite the success of the example of FOL within AAL, we can point out some drawbacks. The first one is related with the fact that the first-order language they start from differs in several important respects from standard FOL. This is due to the fact that, since the theory only applies to single-sorted logics, the atomic formulas of FOL have to be represented, within the propositional language, as propositional variables. So, in order to preserve structurality, one has to constrain the language.
Another important drawback is the fact that, given the many-sorted character of first-order logic, where we have at least syntactic categories for terms and formulas, and possibly also for variables, it would be desirable to have an algebraic counterpart that reflects this many-sorted character. This is clearly not the case with cylindric algebras.

One of our motivations is precisely to extend the theory of AAL to cope with logics that, like first-order logic, have a many-sorted language. This will allow us to, given an algebraizable many-sorted logic, reflect its many-sorted character in its corresponding algebraic counterpart.
But it is not only at the purely many-sorted level that the limitations of current AAL arise. Even at the propositional level, there are interesting logics that fall out of the scope of the theory. It is the case of certain so-called *non-truth-functional logics*, such as the paraconsistent systems of da Costa [8]. The major problem with these logics is that they lack congruence for some connective(s). Roughly speaking, a logic is said to be paraconsistent if its consequence relation is not *explosive* [5]. We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is an *explosive logic* with respect to a *negation* connective $\neg$ if, for all formulas φ and ψ, it is true that $\{\varphi, \neg\varphi\} \vdash \psi$.

Example 3 (Paraconsistent Logic $\mathcal{C}_1$ of da Costa).
It was proved, first by Mortensen [16], and after by Lewin, Mikenberg

and Schwarze [13] that $\mathcal{C}_1$ is not algebraizable in current AAL. So, we can say that $\mathcal{C}_1$ is a example of a logic whose non-algebraizability is well studied. Nevertheless, it is rather strange that a relatively well-behaved logic fails to have an algebraic counterpart. We will briefly introduce $\mathcal{C}_1$. The language of $\mathcal{C}_1$ is generated by the unisorted signature Σ with sort ϕ and composed of the following operations:

- $\mathbf{t}, \mathbf{f} :\to \phi$, $\neg : \phi \to \phi$ and $\wedge, \vee, \supset: \phi^2 \to \phi$.

The consequence relation of $\mathcal{C}_1$ can be given by the structural deductive system composed of the following axioms:

- $\xi_1 \supset (\xi_2 \supset \xi_1)$
- $(\xi_1 \supset (\xi_2 \supset \xi_3)) \supset ((\xi_1 \supset \xi_2) \supset (\xi_1 \supset \xi_3))$
- $(\xi_1 \wedge \xi_2) \supset \xi_1$
- $(\xi_1 \wedge \xi_2) \supset \xi_2$
- $\xi_1 \supset (\xi_2 \supset (\xi_1 \wedge \xi_2))$
- $\xi_1 \supset (\xi_1 \vee \xi_2)$
- $\xi_2 \supset (\xi_1 \vee \xi_2)$
- $(\xi_1 \supset \xi_3) \supset ((\xi_2 \supset \xi_3) \supset ((\xi_1 \vee \xi_2) \supset \xi_3))$
- $\neg\neg\xi_1 \supset \xi_1$
- $\xi_1 \vee \neg\xi_1$
- $\xi_1^\circ \supset (\xi_1 \supset (\neg\xi_1 \supset \xi_2))$
- $(\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \wedge \xi_2)^\circ$
- $(\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \vee \xi_2)^\circ$
- $(\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \supset \xi_2)^\circ$
- $\mathbf{t} \equiv (\xi_1 \supset \xi_1)$
- $\mathbf{f} \equiv (\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1))$

and the rule of *modus ponens*:

- $\xi_1, \xi_1 \supset \xi_2 \vdash \xi_2$

where φ° is an abbreviation of $\neg(\varphi \wedge (\neg\varphi))$ and $\varphi \equiv \psi$ is an abbreviation of $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$.

Despite of its innocent aspect, $\mathcal{C}_1$ is a non-truth-functional logic, namely it lacks congruence for its paraconsistent negation connective. In general, it may happen that $\varphi \dashv\vdash \psi$ but $\neg\varphi \not\dashv\vdash \neg\psi$. This phenomenon leaves the non-truth-functional logics, and in particular $\mathcal{C}_1$, outside of the existing theory, since congruence is a key ingredient for the algebraization process. Still, $\mathcal{C}_1$ has other peculiarities. Although it is defined as a logic weaker that CPL, it happens that a classical negation $\sim$ can be defined in $\mathcal{C}_1$ by using the abbreviation $\sim\varphi = \varphi^\circ \wedge \neg\varphi$. Exploring this fact, da Costa himself introduced in [7] a so-called class of *Curry algebraic structures* as a possible algebraic counterpart of $\mathcal{C}_1$. In fact, nowadays, these algebraic structures are known as *da Costa algebras* [4]. However, their precise nature remains unknown, given the non-algebraizability results reported above.

One of the objectives of this paper is to point out a way to use our many-sorted approach to give a possible connection between $\mathcal{C}_1$ and the algebras of da Costa.

4 Generalizing algebraization

In this section we will propose a novel many-sorted extension of the notion of algebraization, where the major generalizations will happen at the syntactic level. The new notion is then illustrated with the help of examples, and some of its essential results are extended to the many-sorted setting.

4.1 The many-sorted generalization

Our initial aim is to extend the range of applicability of AAL. Therefore, we need to introduce a suitable notion of *structural many-sorted logic*. First of all, we will assume that the formulas of the logic are built from a many-sorted signature Σ. It is usual to assume that the syntax, namely of a logic, is defined by a free construction over the given signature Σ. In the previous section the set of formulas was precisely T_Σ, as inforced by the terminology introduced in section 2 when the formulas are built from a specification $\langle \Sigma, \Phi \rangle$ with $\Phi = \emptyset$. However, it is not unusual that certain syntactic abbreviations are assumed. For instance, in CPL one may assume that all classical connectives are primitive, or else, for instance, that negation and implication are primitive and the other connectives appear as abbreviations. In such a scenario, it makes all the sense to assume that these syntactic abbreviations correspond to equations over the syntax, thus making $\Phi \neq \emptyset$. Of course, by doing this one may be contributing to blurring the essential distinction between syntax and semantics. Still, as we will see, the development applies unrestrictedly. Hence, in general, the syntax of the logic will be specified by a pair $\langle \Sigma, \Phi \rangle$, which justifies the general definition of $L_{\Sigma,\Phi}$. Note that given a substitution σ over Σ, it is easy to see that σ is well behaved with respect to the congruence $\equiv_\Phi$ induced by Φ on $\mathbf{L}_\mathbf{\Sigma}$. Namely, given $t_1, t_2 \in L_\Sigma$ if $t_1 \equiv_\Phi t_2$ then $\sigma(t_1) \equiv_\Phi \sigma(t_2)$. Hence, it makes sense to write $\sigma(\varphi)$ for any formula $\varphi \in L_{\Sigma,\Phi}$. We will use $[t]_\Phi$ to denote the formula corresponding to the equivalence class of $t \in L_\Sigma$ under $\equiv_\Phi$.

Definition 4 (Structural many-sorted logic).
A *structural many-sorted logic* is a tuple $\mathcal{L} = \langle \Sigma, \Phi, \vdash \rangle$ where $\langle \Sigma, \Phi \rangle$ is a many-sorted specification and $\langle L_{\Sigma,\Phi}, \vdash \rangle$ is a logic that also satisfies:
Structurality: if $\Gamma \vdash \varphi$ then $\sigma[\Gamma] \vdash \sigma(\varphi)$ for every substitution σ.

It should be clear that, in the particular case of a single-sorted signature with $\Phi = \emptyset$, this notion coincides precisely with the notion of structural single-sorted logic used in the previous section. Note also that, given a structural many-sorted logic $\mathcal{L} = \langle \Sigma, \Phi, \vdash \rangle$, we can consider the following induced consequence relation $\vdash^\Phi_\mathcal{L} \subseteq \mathcal{P}(L_\Sigma) \times L_\Sigma$ defined by $T \vdash^\Phi_\mathcal{L} u$ iff $[T]_\Phi \vdash [u]_\Phi$. It is easy to see that there exists a strong representation of $\mathcal{L}$ into $\langle L_\Sigma, \vdash^\Phi_\mathcal{L} \rangle$. In particular, the theories $Th^\Phi_\mathcal{L}$ of this induced logic are isomorphic to $Th_\mathcal{L}$. When convenient, this induced consequence relation will allow us to work over L_Σ, thus avoiding the explicit reference to quotients and equivalence classes. We can now introduce our new notion of many-sorted algebraization. The key idea is to replace the role of single-sorted equational logic in current AAL by many-sorted equational logic.

Definition 5. (Many-sorted algebraizable logic)
A structural many-sorted logic $\mathcal{L} = \langle \Sigma, \Phi, \vdash \rangle$ is algebraizable if there exists a class K of $\langle \Sigma, \Phi \rangle$-models, a set $\Theta(\xi)$ of ϕ-equations and a set $E(\xi_1, \xi_2)$ of ϕ-terms such that the following conditions hold:

i) for every $T \cup \{u\} \subseteq L_\Sigma$, $T \vdash^\Phi_\mathcal{L} u$ iff $\Theta[T] \vDash^K_\Sigma \Theta(u)$;
ii) for every set $\Delta \cup \{t_1 \approx t_2\}$ of ϕ-equations, $\Delta \vDash^K_\Sigma t_1 \approx t_2$ iff $E[\Delta] \vdash^\Phi_\mathcal{L} E(t_1, t_2)$;
iii) $\xi \dashv\vdash^\Phi_\mathcal{L} E[\Theta(\xi)]$ and $\xi_1 \approx \xi_2 \dashv\vDash^K_\Sigma \Theta[E(\xi_1, \xi_2)]$.

As before, Θ is called the set of defining equations, E the set of equivalential formulas, and K is called an equivalent algebraic semantics for $\mathcal{L}$. Again, it should be clear that in the case of a single-sorted signature with $\Phi = \emptyset$ this definition coincides with the notion of algebraizable logic of current AAL.

4.2 Examples

Before we proceed, let us illustrate the new notion, namely by revisiting the examples of FOL and $\mathcal{C}_1$.

Example 4 (First-order classical logic revisited).
In example 2, we have already discussed the problems with the single-sorted algebraization of FOL developed in [1]. With our many-sorted framework we can now handle first-order logic as a two-sorted logic, with a sort for terms and a sort for formulas. This perspective seems to be much more convenient, and we no longer need to view atomic FOL formulas as propositional variables. Working out the example, whose details we omit, we manage to algebraize FOL having as an equivalent algebraic semantics the class of *two-sorted cylindric algebras*, whose restriction to the sort ϕ is a plain old cylindric algebra, but which corresponds to a regular first-order interpretation structure on the sort of terms. In the new many-sorted context it is also straightforward to algebraize many-sorted FOL.

Example 5 ($\mathcal{C}_1$ revisited).
In example 3, we made clear that the single-sorted theory of AAL has some unexpected limitations, even in the case of propositional-based logics. We will now revisit $\mathcal{C}_1$ and its algebraization in the many-sorted setting. Actually, in this new perspective, the way in which a logic is presented and, in particular, the way its language is specified, is very relevant in the algebraization process. The trick for $\mathcal{C}_1$ will be to present it as a two-sorted logic, as suggested in [3]. Namely we will consider the two-sorted syntactic specification $\langle \Sigma, \phi \rangle$ such that:

- Σ has two sorts, h and ϕ, and operations $\mathbf{t}, \mathbf{f} :\to h$, $\neg, \sim : h \to h$ and $\wedge, \vee, \supset : h^2 \to h$, as well as $o : h \to \phi$, and $\mathbf{t}, \mathbf{f} :\to \phi$, $\sim : \phi \to \phi$ and $\wedge, \vee, \supset : \phi^2 \to \phi$;
- Φ includes the following equations:

$$\sim x \approx x^\circ \wedge \neg x$$

$$o(\mathbf{t}) \approx \mathbf{t} \qquad o(\sim x) \approx \sim o(x) \qquad o(x \wedge y) \approx o(x) \wedge o(y)$$

$$o(\mathbf{f}) \approx \mathbf{f} \qquad o(x \vee y) \approx o(x) \vee o(y) \qquad o(x \supset y) \approx o(x) \supset o(y)$$

The idea is to take all the primitive syntax of $\mathcal{C}_1$ to the sort h, including the classical negation connective $\sim$ definable as an abbreviation, and to have an observation operation o into sort ϕ, where all the connectives are again available, with the exception of the non-truth-functional paraconsistent negation $\neg$. The top equation aims precisely at internalizing the definition of $\sim$. The other 6 equations simply express the truth-functional (homomorphic) nature of the corresponding connectives. It is not difficult to see that $L_{\Sigma,\Phi}$ is isomorphic to the set of $\mathcal{C}_1$-formulas. With this two-sorted perspective, it can now be shown that, taking $\Theta(\xi) = \{\xi \approx \mathbf{t}\}$ as the set of defining equations and $E(\xi_1, \xi_2) = \{\xi_1 \equiv \xi_2\}$ as the set of equivalential formulas, $\mathcal{C}_1$ is algebraizable in our generalized sense, and that the resulting algebraic counterpart is precisely the two-sorted quasivariety $K_{\mathcal{C}_1}$ proposed in [3]. We do not dwell into the details here, but we can say that the corresponding two-sorted algebras are Boolean on sort ϕ. Actually, the conditional equational specification of $K_{\mathcal{C}_1}$ only needs to use ϕ-equations, which leaves little to be said about what happens with sort h. It is certainly very interesting to understand what is the impact of the $K_{\mathcal{C}_1}$ specification over h-terms, but that is something that we can only do behaviorally, by assuming that h is a hidden-sort. If we restrict our attention to contexts that do not involve the paraconsistent negation, we can show that every algebra $\mathbf{A} \in K_{\mathcal{C}_1}$ behaviorally satisfies all the conditions in the definition of da Costa algebras. On the other hand, given any da Costa algebra, we can canonically extend it to a two-sorted algebra in $K_{\mathcal{C}_1}$. In this way, we manage to discover the connection of da Costa algebras with the algebraization of $\mathcal{C}_1$, which had never been found.

4.3 Many-sorted AAL

In order to further support our generalization of the notion of algebraizable logic, we will now show that we can also extend other notions and results of AAL. We begin by defining a many-sorted version of the Leibniz operator.

Definition 6 (Many-sorted Leibniz operator).
Let $\mathcal{L} = \langle \Sigma, \Phi, \vdash \rangle$ be a structural many-sorted logic. The *many-sorted Leibniz operator* on the term algebra, $\Omega : Th_{\mathcal{L}}^{\Phi} \to Cong_{\mathbf{L_\Sigma}}$ is such that, for each $T \in Th_{\mathcal{L}}^{\Phi}$, $\Omega(T)$ is the largest congruence of $\mathbf{L_\Sigma}$ containing Φ and compatible with T.

Note that, given $T \in Th_{\mathcal{L}}^{\Phi}$, since $\Phi \subseteq \Omega(T)$, we have that $\Omega(T)$ can be seen as a congruence on $\mathbf{L}_{\mathbf{\Sigma}/_\Phi}$. As we will see, also in the many-sorted setting, the Leibniz operator will play an important role. In fact, we are be able to generalize the characterization theorem of single-sorted algebraizable logic we gave in section 3.1.

Theorem 4. *A structural many-sorted logic $\mathcal{L} = \langle \Sigma, \Phi, \vdash \rangle$ is algebraizable iff Ω is monotone, injective, and commutes with inverse substitutions.*

Proof. This proof uses the same methodology as the proof of the single-sorted result. So, we will just give a sketch of the proof focusing on the important methodological steps. First assume that $\mathcal{L}$ is algebraizable, with K, $\Theta(\xi)$ and $E(\xi_1, \xi_2)$. Using $\Theta(\xi)$ and its properties we can define the function $\Omega_K : Th^{\Phi}_{\mathcal{L}} \to Cong_{\mathbf{L}_{\mathbf{\Sigma}}}$, such that, for every sort s, $\langle t_1, t_2\rangle \in (\Omega_K(T))_s$ iff for every ϕ-term $u(x : s)$ we have that $\Theta[T] \vDash^K_{\Sigma} u(t_1) \approx u(t_2)$. Using the properties of K, $\Theta(\xi)$ and $E(\xi_1, \xi_2)$ it is easy to prove that $\Omega_K(T)$ is the largest congruence containing Φ that is compatible with T, that is, $\Omega_K = \Omega$. The fact that Ω_K is injective, monotone and commutes with inverse substitutions also follows easily from the properties of $\Theta(\xi)$ and $E(\xi_1, \xi_2)$.
On the other direction, suppose that Ω is injective, monotone and commutes with inverse substitutions. Consider the class of algebras $K = \{\mathbf{T}_{\mathbf{\Sigma}}/_{\Omega(T)} : T \in Th^{\Phi}_{\mathcal{L}}\}$. It is clear that K is a class of $\langle \Sigma, \Phi\rangle$-models. The fact that Ω is monotone and commutes with inverse substitutions implies, according to [12], that Ω is also surjective. Hence, Ω is indeed a bijection. Our objective is to prove that $\mathcal{L}$ is algebraizable with K an equivalent algebraic semantics. We still have to find the sets of defining equations and equivalence formulas. Let $T = \{\xi_1\}^{\vdash^{\Phi}_{\mathcal{L}}}$. Let σ be the substitution over Σ such that $\sigma_\phi(\xi) = \xi_1$ for every ξ, and it is the identity in the other sorts. If we take $\Theta = \sigma[\Omega(T)_\phi]$ it can be shown, using the fact that Ω commutes with inverse substitutions, that $\Theta = \Theta(\xi_1)$ is a set of ϕ-equations and $T \vdash^{\Phi}_{\mathcal{L}} u$ iff $\Theta[T] \vDash^K_{\Sigma} \Theta(u)$. Now let us construct the set of equivalence formulas. Take now σ to be the substitution such that $\sigma_\phi(\xi_2) = \xi_2$ and $\sigma_\phi(\xi) = \xi_1$ for every $\xi \neq \xi_2$, and it is the identity in the other sorts. Take $E = \sigma[\Omega^{-1}(\{\xi_1 \approx \xi_2\}^{\vDash^K_{\Sigma}})]$. It can be proved that $E = E(\xi_1, \xi_2)$ is a set of ϕ-terms and that $\Theta[E(\xi_1, \xi_2)] \mathrel{=}\!\mid\!\vDash^K_{\Sigma} \xi_1 \approx \xi_2$. The algebraizability of $\mathcal{L}$ follows straightforwardly from these facts. □

We can also extend the characterization of algebraization using maps of logics, namely a strong representation between the given many-sorted logic and many-sorted equational logic.

Theorem 5. *A structural many-sorted logic $\mathcal{L} = \langle \Sigma, \Phi, \vdash\rangle$ is algebraizable iff there exists a class K of $\langle \Sigma, \Phi\rangle$-models and a strong representation $\langle \theta, \tau\rangle$ of $\langle L_\Sigma, \vdash^{\Phi}_{\mathcal{L}}\rangle$ in Eqn^K_Σ, such that θ is given by a set $\Theta(\xi)$ of ϕ-equations and τ by a set $E(\xi_1, \xi_2)$ of ϕ-terms.*

Proof. The result follows from the observation that conditions i), ii), and iii) of the definition of many-sorted algebraizable logic are equivalent to the fact that $\langle \theta, \tau\rangle$ is a strong representation. □

When a structural many-sorted logic $\mathcal{L}$ is algebraizable, we can sometimes provide a specification of its algebraic counterpart given a deductive system for $\mathcal{L}$.

Theorem 6. *Let $\mathcal{L} = \langle \Sigma, \Phi, \vdash\rangle$ be a structural many-sorted logic obtained from a deductive system formed by a set Ax of axioms and a set R of inference rules. Assume that $\mathcal{L}$ is finitely algebraizable with Θ and E. Then, the equivalent quasivariety semantics is axiomatized by the following equations and conditional-equations:*

i) Φ;
ii) $\Theta(\varphi)$ *for each* $[\varphi]_\Phi \in Ax$;
iii) $\Theta[E(\xi,\xi)]$;
iv) $\Theta(\psi_0) \wedge \ldots \wedge \Theta(\psi_n) \to \Theta(\psi)$ *for each* $\frac{[\psi_0]_\Phi,\ldots,[\psi_n]_\Phi}{[\psi]_\Phi} \in R$;
v) $\Theta[E(\xi_1,\xi_2)] \to \xi_1 \approx \xi_2$.

Proof. Let K be the quasivariety defined by i)-v). We will prove that K is the equivalent algebraic semantics of $\mathcal{L}$. First note that the fact that K satisfies i) is equivalent to the fact that K is a class of $\langle\Sigma,\Phi\rangle$-models. It is easy to prove that equation iii) and conditional equation v) are jointly equivalent to $\xi_1 \approx \xi_2 \dashv\models^K_\Sigma \Theta[E(\xi_1,\xi_2)]$ which is one-half of condition iii) in the definition of many-sorted algebraizable logic. It can also be verified that condition i) of the definition of algebraizable logic is equivalent to the above equations ii) and iv). It now remains to say that, as it is well known in the single-sorted case, this is enough to guarantee the algebraizability of $\mathcal{L}$. The uniqueness of the equivalent quasivariety in this case is straightforward. □

Note that, due to the inclusion of the equations in Φ, items ii) and iv) are independent of the particular choice of representatives of the equivalence classes.
In a many-sorted logic we can only reason about formulas. But, following the spirit of hidden equational logic, and considering the sort of formulas as the only visible sort, we might think of behaviorally reasoning about the other sorts. This possibility was particularly useful in example 5. It is well known that behavioral reasoning can be a complicated issue, since it involves reasoning about all possible experiments we can perform on a hidden term. There are, nevertheless, some nice approaches to tackle this problem [10, 18]. Here, we identify a very particular case that occurs when the behavioral reasoning associated with a class K of algebras is *specifiable*, in the sense that all the behaviorally valid equations can be derived, in standard equational logic, from some set of (possibly hidden) equations. We will show that, when $\mathcal{L}$ is expressive enough and it is finitely algebraizable, the behavioral reasoning (over the hole signature) associated with an equivalent algebraic semantics for $\mathcal{L}$ is specifiable. A structural many-sorted logic $\mathcal{L} = \langle\Sigma,\Phi,\vdash\rangle$ is said to be *observationally equivalential* if there exists a sorted set $E = \{E_s(x_1 : s, x_2 : s)\}_{s\in S}$ of ϕ-terms such that, for every $s \in S$ and $x,y,z \in X_s$:
- $\vdash^\Phi_\mathcal{L} E_s(x,x)$;
- $E_s(x,y) \vdash^\Phi_\mathcal{L} E_s(y,x)$;
- $E_s(x,y), E_s(y,z) \vdash^\Phi_\mathcal{L} E_s(x,z)$;
- for each operation $o \in O_{s_1\ldots s_n s}$ and $x_1,y_1 \in X_1,\ldots,x_n,y_n \in X_n$
  $\{E_{s_1}(x_1,y_1),\ldots,E_{s_n}(x_n,y_n)\} \vdash^\Phi_\mathcal{L} E_s(o(x_1,\ldots,x_n),o(y_1,\ldots,y_n))$;
- $E_\phi(\xi_1,\xi_2),\xi_1 \vdash^\Phi_\mathcal{L} \xi_2$.

Each E_s is called the set of *equivalential terms* of sort s. If, for each $s \in S$, the set E_s is finite, then $\mathcal{L}$ is called finitely observationally equivalential.

Theorem 7. *Let* $\mathcal{L} = \langle\Sigma,\Phi,\vdash\rangle$ *be a finitely algebraizable, finitary, and structural many-sorted logic, and* K *be an equivalent algebraic semantics for* $\mathcal{L}$. *If* $\mathcal{L}$ *is finitely observationally equivalential then* $\models^K_{\Sigma,bhv}$ *is specifiable.*

Proof. Since $\mathcal{L}$ is finitary and finitely algebraizable, $\vDash_{\Sigma}^{K}$ is specifiable. Let $E = \{E_s\}_{s \in S}$ the S-indexed set of equivalential formulas. Consider the sorted set $\Psi = \{\Psi_s\}_{s \in S}$ such that $\Psi_s = \Theta[E_s]$, where $\Theta(\xi)$ is the (finite) set of defining equations. Then, it is easy to prove that Ψ forms a finite set of equivalential formulas for $\vDash_{\Sigma}^{K}$. Then, using theorem 5.2.21. in [14], $\vDash_{\Sigma,bhv}^{K}$ is specifiable. □

Note that this theorem does not apply to $\mathcal{C}_1$, in examples 3 and 5, since $\mathcal{C}_1$ is not observationally equivalential. In fact, in the case of $\mathcal{C}_1$, we had to consider that the contexts were build from a subsignature to be able to behaviorally characterize its algebraic counterpart.

5 Conclusions and further work

In this paper, we have proposed a generalization of the notion of algebraizable logic that encompasses also many-sorted logics. The key ingredient of the generalization was to replace the role of single-sorted equational logic of traditional AAL by many-sorted equational logic. To support our approach we proved, in this more general setting, extended versions of several important results of AAL, including characterizations using the Leibniz operator, as well as maps of logics. We illustrated the approach by reanalyzing the examples of first-order logic and of the paraconsistent logic $\mathcal{C}_1$ in a many-sorted context. In particular, for $\mathcal{C}_1$, we managed to characterize the precise role of da Costa algebras.
Being a first attempt at this generalization, there is much to be done, and there are many more interesting results of AAL to generalize. We also aim at investigating a many-sorted version of the Leibniz hierarchy, including also protoalgebraization, weak-algebraization, and related work, such as k-deductive systems. Many further examples are also to be tried. In this front, our ultimate aim is to understand the relationship between orthomodular lattices as used in the Birkhoff and Von Neumann tradition of quantum logic and the algebraic counterpart of exogenous quantum logic [15]. An important open question is whether and how our approach can be integrated with the work on the algebraization of logics as institutions reported in [20]. Another interesting line of future work is to study the impact of our proposal with respect to the way a logic is represented within many-sorted equational logic in the context of logic combination, namely in the lines of [17].
In any case, this seems to be an area which is very fit for application of the theory many-sorted algebras, including hidden-sorts and behavioral reasoning, as developed within the formal methods community over the last decade.

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