From Fibring to Cryptofibring, A Solution to The Collapsing Problem

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Abstract. The semantic collapse problem is perhaps the main difficulty associated to the very powerful mechanism for combining logics known as fibring. In this paper we propose cryptofibred semantics as a generalization of fibred semantics, and show that it provides a solution to the collapsing problem. In particular, given that the collapsing problem is a special case of failure of conservativeness, we formulate and prove a sufficient condition for cryptofibring to yield a conservative extension of the logics being combined. For illustration, we revisit the example of combining intuitionistic and classical propositional logics.

Keywords. Combining logics, fibring, the collapsing problem, cryptofibring, conservativeness.

1. Introduction

The study of combined logics and of their relationship to the logics being combined is certainly a key issue of the general theory of universal logic [1]. Fibring is a very powerful and appealing mechanism for combining logics. As proposed by Gabbay in [15], fibring should "combine $\mathcal{L}_1$ and $\mathcal{L}_2$ into a system which is the smallest logical system for the combined language which is a conservative extension of both $\mathcal{L}_1$ and $\mathcal{L}_2$". Of course, if the languages of $\mathcal{L}_1$ and $\mathcal{L}_2$ share some common constructors, then they will be identified in the combined language. In deductive terms, fibring is very well understood. Given deductive systems for $\mathcal{L}_1$ and $\mathcal{L}_2$, one just needs to add them together in order to obtain a system for the combined logic. However, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are given in semantic terms, setting up exactly the

The authors are deeply grateful to Amilcar Sernadas and Cristina Sernadas for their essential contribution to the ideas underlying this work. This work was partially supported by FCT and EU FEDER, through POCI, namely via the Project POCI/MAT/55796/2004 QuantLog of CLC, and the recent KLog initiative of SQIG-IT. The authors also acknowledge the anonymous referees whose comments helped improving an earlier version of this paper.

semantic presentation of the combined logic is not a trivial task. Gabbay's original idea revolved around the notion of *fibring function*, an operational characterization of the meaning of formulas over the combined language that is based on switching between models of the two logics. Still, the first meaningful characterization of fibred model was proposed in [20]. Afterwards, fibring has been extensively studied and generalized, with an emphasis on the relationship between fibred deduction and fibred semantics, including very general soundness preservation results, as well as sufficient conditions for completeness preservation. See, for instance, [22, 3, 4, 11, 18, 19, 6, 7]. An up-to-date overview of fibring, its properties, applications and problems can be found in [5], which we will follow closely.

One of the most notorious problems of fibring is well identified. It is usually called *the collapsing problem*. It was recognized in [13, 14], in the context of a very simple example: the combination of intuitionistic and classical propositional logics. Indeed, even if one does not want to identify the intuitionistic and classical implications in the combined language, it turns out that all fibred models for the combination will give a classical interpretation to the intuitionistic implication, and therefore the two collapse into classical implication. In [12], a very interesting logical system combining intuitionistic and classical logics was introduced and studied, but it departs a lot from the spirit of fibring, namely because its deductive characterization does not include all the axioms and rules of both intuitionistic and classical logic (perhaps with some additional mixed axioms). Instead, it uses mixed axioms and incorporates syntactic restrictions on their instantiation. A first general solution to the collapsing problem, *modulated fibring*, was proposed in [21] using similar ideas.

Still, we argue that the last word about this problem has not yet been said. Indeed, it should be clear that, in abstract, collapsing situations are particular cases of failure of conservativeness. That is, the fibred logical system fails to be a conservative extension of at least one of the original logics. Moreover, the collapsing phenomenon should appear only in fibred semantics. This does not mean that collapses, or failures of conservativeness, cannot happen when fibring deductive systems. But if they happen, then they are unavoidable. It is well-known that, in some cases, there is no logical system that extends both given logical systems in a conservative way. It is easy to come up with such an example if one just remembers that the logical systems being combined may have shared constructors. If one fibers intuitionistic and classical logics by identifying the two implications, the collapse that one obtains does not come as a surprise. These facts, and the intrinsic difficulties associated to the characterization of models of the combined logic, are perhaps the main reasons that explain why the study of fibred semantics has been concentrating on finding an extension of the given logics, but not necessarily a conservative one. Although quite rich, the partial answers to the question of completeness preservation by fibring are certainly also related to this fact. Specially, if we contrast them with the ubiquity of soundness preservation.

In any case, it is obvious that the collapsing problem is only a problem because it is somehow unexpected. Without identifying the intuitionistic and the classical implications, it should be the case that a conservative extension of intuitionistic and classical logics exists, and we should know what its semantics looks like. With this aim, and following the initial ideas reported in [8, 9], we propose a generalization of fibred semantics, to which we call *cryptofibring*[1]. The key idea of the extension is to allow a more relaxed relationship between combined models and models of the logical systems being combined. We show that cryptofibring extends fibring in the sense that all fibred models are also cryptofibred models, but in general cryptofibring allows many more models. After contending with soundness requirements, we use this fact to obtain a combined model for intuitionistic and classical logics that shows that their cryptofibring does not suffer from the collapsing problem. Moreover, we then study the question in general, obtaining a sufficient condition for cryptofibring to be a conservative extension of the given logical systems.

We proceed as follows. In Section 2 we introduce our working universe of logical systems, and establish some relevant notions and notation. In Section 3 we overview the mechanism of fibring, some of its good properties, and we illustrate the collapsing problem. Section 4 introduces and studies cryptofibring, and uses it to provide a solution to the collapsing problem. Finally, Section 5 provides a detailed study and a sufficient condition for conservativeness in the context of cryptofibring. We conclude, in Section 6, with a summary of the results and an outline of further work.

2. Logical systems, semantics, deduction

We will be interested only in Tarskian logics. Given a set L of *formulas*, we will use lower-case greek letters to denote members of L, and upper-case greek letters to denote subsets of L.

Definition 2.1. A *logical system* is a pair $\mathcal{L} = \langle L, \vdash \rangle$ where L is a set of formulas and $\vdash : 2^L \to 2^L$ satisfies:

Extensiveness: $\Gamma \subseteq \Gamma^{\vdash}$;
Monotonicity: $\Gamma^{\vdash} \subseteq (\Gamma \cup \Phi)^{\vdash}$; and
Idempotence: $(\Gamma^{\vdash})^{\vdash} \subseteq \Gamma^{\vdash}$.

To keep as general as possible we will not require:

Finitariness: $\Gamma^{\vdash} \subseteq \bigcup_{\text{finite } \Gamma_0 \subseteq \Gamma} \Gamma_0^{\vdash}$.

We will dub a logical system *finitary* whenever finitariness holds. As usual, we will write $\Gamma \vdash \varphi$ instead of $\varphi \in \Gamma^{\vdash}$. A *theory* of $\mathcal{L}$ is a set $\Gamma \subseteq L$ such that $\Gamma = \Gamma^{\vdash}$. If $\Gamma^{\vdash} = L$ we will say that the theory is *trivial.* If necessary, in context, we will write $\vdash_{\mathcal{L}}$ instead of just $\vdash$.

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[1] *Cryptofibring* borrows its name from the use of *cryptomorphisms*, to be introduced in Section 4, as such homomorphisms between heterogeneous algebras mediated by a signature morphism were baptized by Tarlecki, Burstall and Goguen, namely in [23, page 6].

In order to combine logical systems in any meaningful way it is essential to work with logical languages that are freely generated from a collection of constructors. Although more general multi-sorted notions could be considered, e.g. as in [9], in order to avoid unnecessary complexity, we will consider herein a *signature* C to be a $\mathbb{N}$-indexed family $\{C_n\}_{n\in\mathbb{N}}$. Note that this notion of signature is sufficient to cover propositional based languages. The elements of each C_n are known as *connectives* of arity n. Propositional *symbols* appear as a subset of C_0. Given a signature C, the generated set of *formulas* is the carrier $L(C)$ of the free C-algebra. The *denotation* $[\![\varphi]\!]_{\mathbf{A}}$ of a formula $\varphi \in L(C)$ in a given C-algebra $\mathbf{A} = \langle A, \cdot_{\mathbf{A}}\rangle$, is inductively defined as usual: $[\![c(\varphi_1, \ldots, \varphi_k)]\!]_{\mathbf{A}} = c_{\mathbf{A}}([\![\varphi_1]\!]_{\mathbf{A}}, \ldots, [\![\varphi_k]\!]_{\mathbf{A}})$ for every $c \in C_k$ and formulas $\varphi_1, \ldots, \varphi_k \in L(C)$. The denotation map extends canonically to sets of formulas.

Most interesting logical systems are presented either by semantic or deductive means. Let us introduce our working notions of *interpretation system* and *deductive system*, and settle the way in which they can be understood as semantic, respectively deductive, presentations of logical systems. We assume fixed a signature C.

Our semantic presentations are based on *logical matrices*.

Definition 2.2. A *C-structure* is a pair $\mathbb{A} = \langle \mathbf{A}, T_{\mathbb{A}}\rangle$ where $\mathbf{A} = \langle A, \cdot_{\mathbf{A}}\rangle$ is a C-algebra and $T_{\mathbb{A}} \subseteq A$.

The elements of A are called *truth-values* and those in $T_{\mathbb{A}}$ are known as *designated* truth-values. In the sequel, we write $[\![\varphi]\!]_{\mathbb{A}}$ for the denotation of φ in the underlying algebra. We denote the class of all C-structures by $Str(C)$.

Definition 2.3. An *interpretation system* is a tuple $\mathcal{I} = \langle C, \mathcal{M}, \alpha\rangle$ where C is a signature, $\mathcal{M}$ is a class and $\alpha : \mathcal{M} \to Str(C)$.

The elements of $\mathcal{M}$ are called *models*. The map α associates a C-structure to each model of the interpretation system. In the sequel, we may write $\mathbb{A}_m = \langle \mathbf{A}_m, T_m\rangle$ for $\alpha(m)$, $\cdot_m$ for $\cdot_{\mathbf{A}_m}$, and $[\![\varphi]\!]_m$ for $[\![\varphi]\!]_{\alpha(m)}$.

Definition 2.4. We say that a model $m \in \mathcal{M}$ *satisfies* a formula $\varphi \in L(C)$ in $\mathcal{I}$, written $m \Vdash_{\mathcal{I}} \varphi$, if $[\![\varphi]\!]_m \in T_m$. And, given $\Gamma \subseteq L(C)$ and $\varphi \in L(C)$, we say that Γ *entails* φ in $\mathcal{I}$, written $\Gamma \vDash_{\mathcal{I}} \varphi$, if for every $m \in \mathcal{M}$, $[\![\varphi]\!]_m \in T_m$ whenever $[\![\Gamma]\!]_m \subseteq T_m$.

It is well-known that $\vDash_{\mathcal{I}}$ constitutes a Tarskian consequence on $L(C)$, possibly not finitary. The associated logical system is $\langle L(C), \vDash_{\mathcal{I}}\rangle$.

For illustration, we present interpretation systems for the implicative fragments of intuitionistic propositional logic (*IPL*) and classical propositional logic (*CPL*). We will adopt the usual Kripke-style semantics for intuitionistic logic.

Example. Let P be a given set of propositional symbols. The interpretation system for *the implicative fragment of intuitionistic propositional logic over P* is

$$IPL = \langle C^i, \mathcal{M}^i, \alpha^i\rangle$$

where:

- $C_0^i = P$ and $C_2^i = \{\to\}$, $C_1^i = C_n^i = \emptyset$ for every $n > 2$;
- $\mathcal{M}^i$ is the class of all *Kripke-models for intuitionistic logic over* P, that is, the class of all triples $m = \langle W, \leq, V\rangle$ such that $W \neq \emptyset$, $\langle W, \leq\rangle$ is a partial order and $V : P \to \mathcal{U}pp_{\leq}$, where $\mathcal{U}pp_{\leq} = \{X \subseteq W|\ \text{if } w_1 \in X \text{ and } w_1 \leq w_2 \text{ then } w_2 \in X\}$ is the set of all upper closed subsets of W;
- for each $m = \langle W, \leq, V\rangle$ in $\mathcal{M}^i$, $\mathbb{A}_m = \langle\langle \mathcal{U}pp_{\leq}, \cdot_m\rangle, \{W\}\rangle$ where:
  - $p_m = V(p)$, for every $p \in P$;
  - $(X \to_m Y) = \{w \in W|\ \{w'|\ w \leq w'\} \subseteq (W \setminus X) \cup Y\}$.

In the classical case, we will use bivaluations.

Example. Let Q be a given set of propositional symbols. The interpretation system for *the implicative fragment of classical propositional logic over* Q is

$$CPL = \langle C^c, \mathcal{M}^c, \alpha^c\rangle$$

where:

- $C_0^c = Q$ and $C_2^c = \{\Rightarrow\}$, $C_1^c = C_n^c = \emptyset$ for every $n > 2$;
- $\mathcal{M}^c$ is the class of bivaluations to the symbols in Q, that is, the class of all triples $\langle\bot, \top, v\rangle$ where $\bot \neq \top$ and $v : Q \to \{\bot, \top\}$ is a function;
- for each $m \in \mathcal{M}^c$, $\mathbb{A}_m = \langle\langle\{\bot, \top\}, \cdot_m\rangle, \{\top\}\rangle$ where:
  - $q_m = v(q)$, for every $q \in Q$;
  - $(a \Rightarrow_m b) = \bot$ iff $a = \top$ and $b = \bot$.

We now focus on the deductive counterparts of logical systems. We will adopt Hilbert-style deduction systems with schematic axioms and inference rules. For the purpose, we assume given once and for all a set Ξ of *schema variables.* Given a signature C, the generated set of *schema formulas* is the carrier $SL(C)$ of the free C-algebra with generators Ξ. A *(ground) schema C-substitution* is a function $\sigma : \Xi \to L(C)$. Given a schema formula δ, the *instance* of δ by the schema substitution σ is denoted by $\sigma(\delta)$ and is the result of simultaneously replacing each schema variable ξ in δ by $\sigma(\xi)$. Clearly, $\sigma(\delta) \in L(C)$.

Definition 2.5. A *schema C-rule* is a pair $\langle\Phi, \psi\rangle$, where $\Phi \cup \{\psi\} \subseteq SL(C)$. A rule is said to be *finite* when Φ is finite, and is said to be a *schema axiom* when Φ is empty.

In the sequel we will sometimes denote a rule $\langle\{\varphi_1, \ldots, \varphi_k\}, \psi\rangle$ by

$$\frac{\varphi_1 \ \cdots \ \varphi_k}{\psi}.$$

Definition 2.6. A *deductive system* is a pair $\mathcal{D} = \langle C, R\rangle$ where C is a signature and R is a set of finitary C-rules.

Definition 2.7. A *proof* within a deductive system $\mathcal{D}$ of $\varphi \in L(C)$ from $\Gamma \subseteq L(C)$ is a sequence $\delta_1, \ldots, \delta_n \in L(C)$ such that $\delta_n = \varphi$, and for each $i = 1, \ldots, n$:

- either $\delta_i \in \Gamma$; or

- there is a rule $\langle\{\varphi_1,\dots,\varphi_k\},\psi\rangle \in R$ and a schema C-substitution σ such that $\delta_i = \sigma(\psi)$ and $\{\sigma(\varphi_1),\dots,\sigma(\varphi_k)\} \subseteq \{\delta_1,\dots,\delta_{i-1}\}$.

When there is such a proof in $\mathcal{D}$ of φ from Γ, we write $\Gamma \vdash_{\mathcal{D}} \varphi$.

As usual we may also omit the set of premises when it is empty. Note that proofs are closed for substitutions: if $\Gamma \vdash_{\mathcal{D}} \delta$ then $\sigma(\Gamma) \vdash_{\mathcal{D}} \sigma(\delta)$, for any schema substitution σ. It is straightforward to check that $\vdash_{\mathcal{D}}$ is a finitary Tarskian consequence on $L(C)$. The associated logical system is $\langle L(C), \vdash_{\mathcal{D}}\rangle$.

Example. The deductive system for the *implicative fragment of intuitionistic propositional logic* is

$$IPL = \langle C^i, R^i\rangle$$

where:

- C^i is the intuitionistic signature defined above (introducing the symbols P and the binary connective $\to$);
- R^i contains the schema axioms

  I1: $\xi_1 \to (\xi_2 \to \xi_1)$
  I2: $(\xi_1 \to (\xi_2 \to \xi_3)) \to ((\xi_1 \to \xi_2) \to (\xi_1 \to \xi_3))$

  and the schema rule

  IMP:
  $$\frac{\xi_1 \quad (\xi_1 \to \xi_2)}{\xi_2}$$

Recall that the axioms **I1-2** and the rule of *modus ponens* **IMP** are eactly what one need to establish the *deduction metatheorem*, that is, $\Gamma, \varphi \vdash_{IPL} \psi$ if and only if $\Gamma \vdash_{IPL} \varphi \to \psi$.

Example. The deductive system for the *implicative fragment of classical propositional logic* is

$$CPL = \langle C^c, R^c\rangle$$

where:

- C^c is the signature defined above (introducing the symbols Q and the binary connective $\Rightarrow$);
- R^c contains the schema axioms

  C1: $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$
  C2: $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3))$
  C3: $((\xi_1 \Rightarrow \xi_2) \Rightarrow \xi_1) \Rightarrow \xi_1$

  and the schema rule

CMP:

$$\frac{\xi_1 \quad (\xi_1 \Rightarrow \xi_2)}{\xi_2}$$

The axioms **C1-2** and the *modus ponens* rule **CMP** are the immediate counterparts of axioms **I1-2** and of rule **IMP** for the classical implication. The well-known axiom **C3** is often called *Peirce's law*. Once again, the *deduction metatheorem* holds, $\Gamma, \varphi \vdash_{CPL} \psi$ if and only if $\Gamma \vdash_{CPL} \varphi \Rightarrow \psi$.

Note that we are using *IPL* and *CPL* to denote both the intuitionistic and classical interpretation systems and deductive systems. In the sequel, we will write $\vDash_i$ and $\vdash_i$ instead of $\vDash_{IPL}$ and $\vdash_{IPL}$, and also $\vDash_c$ and $\vdash_c$ instead of $\vDash_{CPL}$ and $\vdash_{CPL}$.

In many cases one is interested in having both a semantic and a deductive counterpart of a certain logical system. Let $\mathcal{I} = \langle C, \mathcal{M}, \alpha \rangle$ be an interpretation system and $\mathcal{D} = \langle C, R \rangle$ be a deductive system, both over a common signature C.

Definition 2.8. We say that $\mathcal{D}$ is *sound* with respect to $\mathcal{I}$ if $\Gamma \vdash_{\mathcal{D}} \varphi$ implies $\Gamma \vDash_{\mathcal{I}} \varphi$, for $\Gamma \cup \{\varphi\} \subseteq L(C)$. Conversely, we say that $\mathcal{D}$ is *complete* with respect to $\mathcal{I}$ if $\Gamma \vDash_{\mathcal{I}} \varphi$ implies $\Gamma \vdash_{\mathcal{D}} \varphi$.

Clearly, $\mathcal{D}$ is sound with respect to $\mathcal{I}$ provided that the C-structure $\mathbb{A}_m$ associated to each model m of $\mathcal{I}$ is *appropriate* for all ground instances of the rules of $\mathcal{D}$, that is, for each rule $\langle \Phi, \psi \rangle$ of $\mathcal{D}$ and each ground substitution ρ, if $[\![\rho(\Phi)]\!]_{\mathbb{A}_m} \subseteq T_m$ then $[\![\rho(\psi)]\!]_{\mathbb{A}_m} \in T_m$. Clearly, this is equivalent to saying that $\rho(\Phi) \vDash_{\mathcal{I}} \rho(\psi)$. Of course, only when both soundness and completeness hold can we be sure that $\vDash_{\mathcal{I}} = \vdash_{\mathcal{D}}$.

Example. Soundness and completeness hold for the systems of intuitionistic and classical logics defined above.

3. Fibring and the collapsing problem

Next, we present the notion of fibring of interpretation systems. We assume given two interpretation systems $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$ and denote by $\widehat{C}$ the common subsignature $C' \cap C''$. As usual, we will assume that the constructors in $\widehat{C}$ are shared by both systems and should be identified in their combination. Hence, in general, fibrings will be *constrained* by shared constructors. However, if $\widehat{C} = \emptyset$ we call the combination a *free* fibring.

Definition 3.1. The *fibring of $\mathcal{I}'$ and $\mathcal{I}''$ (constrained by sharing $\widehat{C}$)* is the interpretation system

$$\mathcal{I}' * \mathcal{I}'' = \langle C' \cup C'', \mathcal{M}' * \mathcal{M}'', \alpha_* \rangle$$

where:

- $\mathcal{M}' * \mathcal{M}''$ is the class of all pairs $\langle m', m'' \rangle$ such that:
  - $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$,

  - $A_{m'} = A_{m''}$, $T_{m'} = T_{m''}$, and $c_{m'} = c_{m''}$ if $c \in \widehat{C}$;
- $\alpha_*(\langle m', m'' \rangle) = \langle \mathbf{A}, T \rangle$ where:
  - $\mathbf{A}$ is the unique $C' \cup C''$-algebra with $A = A_{m'} = A_{m''}$ that extends both $\mathbf{A}_{m'}$ and $\mathbf{A}_{m''}$,
  - $T = T_{m'} = T_{m''}$.

As shown in [3, 5], inter alia, the operation of fibring two interpretation systems can be given a categorial characterization in terms of a universal construction.

We can illustrate fibred semantics, as well as the *collapsing problem*, by combining the intuitionistic and classical interpretation systems defined in the previous section.

Example. The *free fibring of the interpretation systems IPL and CPL*, assuming that $P \cap Q = \emptyset$, is the interpretation system

$$\mathit{IPL} * \mathit{CPL} = \langle C, \mathcal{M}, \alpha \rangle$$

defined as follows:

- $C_0 = P \cup Q$, $C_2 = \{\Rightarrow, \rightarrow\}$, and $C_1 = C_n = \emptyset$ for every $n > 2$;
- $\mathcal{M} = \{\langle \langle \bot, \top, v \rangle, \langle W, \leq, V \rangle \rangle | \, \mathcal{U}pp_{\leq} = \{\emptyset, W\}, \bot = \emptyset, \top = W\}$;
- for each $m \in \mathcal{M}$, $\mathbb{A}_m = \langle \langle \{\emptyset, W\}, \cdot_m \rangle, \{W\} \rangle$ where:
  - $p_m = V(p)$;
  - $q_m = v(q)$;
  - $\rightarrow_m$ and $\Rightarrow_m$ are given by the tables

$\rightarrow_m$	$\emptyset$	W
$\emptyset$	W	W
W	$\emptyset$	W

$\Rightarrow_m$	$\emptyset$	W
$\emptyset$	W	W
W	$\emptyset$	W

By definition of fibred semantics, the only pairs of models in the resulting interpretation system are formed by models whose algebras have the same carrier set. In this case, the algebras of *CPL* models have carrier sets with exactly two elements. So, we can only choose *IPL* models whose algebras have also two elements. This implies that the set of worlds of those *IPL* models must be a singleton. Indeed, if W is a singleton then $\mathcal{U}pp_{\leq} = \{\emptyset, W\}$. Otherwise, if W has at least two elements w_1 and w_2, then $\mathcal{U}pp_{\leq}$ will have, at least, the following three distinct elements: $\emptyset$, $\{w \in W : w_1 \leq w\}$ and $\{w \in W : w_2 \leq w\}$.

It is immediate to conclude that in $\mathit{IPL} * \mathit{CPL}$ intuitionistic implication collapses to classical implication.

Next, we define the fibring of deductive systems. We assume given two deductive systems $\mathcal{D}' = \langle C', R' \rangle$ and $\mathcal{D}'' = \langle C'', R'' \rangle$. Again, in general, the fibring will be constrained by the shared constructors in $\widehat{C}$.

Definition 3.2. The *fibring of $\mathcal{D}'$ and $\mathcal{D}''$ (constrained by sharing $\widehat{C}$)* is the deductive system

$$\mathcal{D}' * \mathcal{D}'' = \langle C' \cup C'', R' \cup R'' \rangle.$$

Again, as shown in [3, 5], inter alia, the operation of fibring two deductive systems can also be given a universal categorial characterization.

Example. The *fibring of the deductive systems IPL and CPL* is the deductive system

$$IPL * CPL = \langle C^c \cup C^i, R^c \cup R^i \rangle.$$

In fact, the signature of the fibred deductive system is precisely the same as the signature of the fibred interpretation system presented above. The schematic inference rules of $IPL * CPL$ are precisely all the rules of IPL and all the rules of CPL. For the sake of visualization we list them below:

- Schema axioms

  I1: $\xi_1 \to (\xi_2 \to \xi_1)$
  I2: $(\xi_1 \to (\xi_2 \to \xi_3)) \to ((\xi_1 \to \xi_2) \to (\xi_1 \to \xi_3))$
  C1: $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$
  C2: $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3))$
  C3: $((\xi_1 \Rightarrow \xi_2) \Rightarrow \xi_1) \Rightarrow \xi_1$

- Schema rules

  IMP:

  $$\frac{\xi_1 \quad (\xi_1 \to \xi_2)}{\xi_2}$$

  CMP:

  $$\frac{\xi_1 \quad (\xi_1 \Rightarrow \xi_2)}{\xi_2}$$

Note again that we are using $IPL * CPL$ to denote both the fibred interpretation system and the fibred deductive system for the combination of IPL and CPL. In the sequel, we will write $\vDash_{ic}$ and $\vdash_{ic}$ instead of $\vDash_{IPL*CPL}$ and $\vdash_{IPL*CPL}$.

It is clear that CPL is strictly stronger than IPL, each in its own language, in the sense that $\vdash_c ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ but $\nvdash_i ((\varphi \to \psi) \to \varphi) \to \varphi$. Hence, we might expect that in the combined language $\varphi \to \psi \vdash_{ic} \varphi \Rightarrow \psi$. On the other hand, both IPL and CPL are known to enjoy the *deduction metatheorem*. As observed by Gabbay in [14], the two implications would collapse in the combined deductive system if their deduction metatheorems were preserved. For instance, since $\varphi \Rightarrow \psi, \varphi \vdash_{ic} \psi$ by using **CMP**, the deduction metatheorem for the intuitionistic implication would yield $\varphi \Rightarrow \psi \vdash_{ic} \varphi \to \psi$. However... none of the above speculations is evident. Indeed, we will show that $\to$ and $\Rightarrow$ do not collapse in $IPL * CPL$!

At the light of this example, it becomes clear that there is a real mismatch between fibred semantics and fibred deduction. The two constructions are known to go hand-in-hand only in certain cases. Let us sumarize the main general results about preservation of soundness and completeness by fibring. The proofs of these results can be found in [3, 5], inter alia.

Proposition 3.3. *Soundness preservation.*
Let $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ be appropriate for all ground instances of rules of $\mathcal{D}' = \langle C', R' \rangle$, and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$ be appropriate for all ground instances of rules of $\mathcal{D}'' = \langle C'', R'' \rangle$. Then, $\mathcal{D}' * \mathcal{D}''$ is sound with respect to $\mathcal{I}' * \mathcal{I}''$.

The result states that if all the inference rules of each of the given deductive systems is satisfied by the corresponding interpretation system, which embodies the usual way of proving that $\mathcal{D}'$ is sound with respect to $\mathcal{I}'$ and $\mathcal{D}''$ is sound with respect to $\mathcal{I}''$, then soundness is preserved by fibring. When applied to our example, it yields the (expected) fact that $\Gamma \vdash_{ic} \varphi$ implies $\Gamma \vDash_{ic} \varphi$.

Completeness preservation is much harder, and does not hold in general. If indeed the two implications do not collapse in the fibred deductive system *IPL*∗*CPL*, the combination of intuitionistic and classical logics is just another counterexample. Still, it is possible to find general sufficient conditions for completeness to transfer. The most general ones rely on the notion of *fullness*, as proposed in [22]. An interpretation system $\mathcal{I} = \langle C, \mathcal{M}, \alpha \rangle$ is said to be *full* for $\mathcal{D} = \langle C, R \rangle$ if for every C-structure $\mathbb{A}$ that is appropriate for all the ground instances of the rules in R there exists $m \in M$ such that $\mathbb{A}_m = \mathbb{A}$. It is very easy to see that fullness implies completeness. Moreover, since fullness is preserved by fibring, completeness preservation becomes possible.

Proposition 3.4. *Completeness preservation.*
Let $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ be full for $\mathcal{D}' = \langle C', R' \rangle$, and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$ be full for $\mathcal{D}'' = \langle C'', R'' \rangle$. If $\widehat{C} = C' \cap C'' = \emptyset$, then $\mathcal{D}' * \mathcal{D}''$ is complete with respect to $\mathcal{I}' * \mathcal{I}''$.

Note that our example does not contradict this result. Indeed, although the fibring of *IPL* and *CPL* is free (assuming that $P \cap Q = \emptyset$), fullness does not hold. Note, for instance, that there are many models of classical logic that are not two-valued. Other general sufficient condition for the preservation of completeness are known, namely for the constrained case, when the systems share a well-behaved implication-like constructor. Still, it is clear that they will not apply to our example. The question is how to prove that $\rightarrow$ and $\Rightarrow$ do not collapse in *IPL*∗*CPL*. The obvious way would be to find a sound model for the combined deductive system that would falsify, for instance, the counterpart of Peirce's law for intuitionistic implication. But fibred semantics does not provide us with such a model, as we have seen.

4. Cryptofibred semantics

In this section, we propose a generalization of fibred semantics. The trick is to consider a different way of relating semantic structures across different signatures. In the categorial characterization of fibred semantics it happens that there always exists a certain kind of morphism between the structure associated to a fibred model $\langle m', m''\rangle$ and the structures associated to m' and m'' in the original interpretation systems. Namely, $\mathrm{id} : \mathbb{A}_{m'} \to \mathbb{A}_{\langle m',m''\rangle}|_{C'}$ and $\mathrm{id} : \mathbb{A}_{m'} \to \mathbb{A}_{\langle m',m''\rangle}|_{C''}$ establish bijective homomorphisms of C'-algebras and C''-algebras, respectively, and bijections on the designated truth-values (id stands, in either case, for the identity function). In the sequel, we will make this relationship less strict.

Definition 4.1. Given two signatures $C \subseteq C'$, a C-structure $\mathbb{A}$ and a C'-structure $\mathbb{A}'$, a *cryptomorphism* $h : \mathbb{A} \to \mathbb{A}'$ is a C-algebra homomorphism $h : \mathbf{A} \to \mathbf{A}'|_C$ such that $T_{\mathbb{A}} = h^{-1}(T_{\mathbb{A}'})$.

Note that $\mathbf{A}'|_C$ is the C-algebra where each constructor $c \in C$ is evaluated as $c_{\mathbf{A}'}$, and $\mathbb{A}'|_C = \langle \mathbf{A}'|_C, T_{\mathbf{A}'}\rangle$.

Definition 4.2. The *cryptofibring of* $\mathcal{I}' = \langle C', \mathcal{M}', \alpha'\rangle$ *and* $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha''\rangle$ *(constrained by sharing the common subsignature* $\widehat{C}$*)* is the interpretation system

$$\mathcal{I}' \circledast \mathcal{I}'' = \langle C' \cup C'', \mathcal{M}' \circledast \mathcal{M}'', \alpha_{\circledast}\rangle$$

where:

- $\mathcal{M}' \circledast \mathcal{M}''$ is the class of tuples $\langle \mathbb{A}, m', m'', h', h''\rangle$ such that:
  - $\mathbb{A} \in \mathit{Str}(C' \cup C'')$,
  - $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$,
  - $h' : \mathbb{A}_{m'} \to \mathbb{A}$ and $h'' : \mathbb{A}_{m''} \to \mathbb{A}$ are cryptomorphisms;
- $\alpha_{\circledast}(\langle \mathbb{A}, m', m'', h', h''\rangle) = \mathbb{A}$.

The above construction might seem complex but the key ingredients of cryptofibred models are:

- all operations in the signature of one of the given interpretation systems are extended to operate also on the values of the model of the other interpretation system;
- the interpretations of shared terms in each of the given models are identified;
- values of the combined model corresponding to values of some of the original models are designated if and only if they were already designated and, as a consequence, two values of the original models can only be identified if they are both designated or both not designated.

We illustrate the construction with some very simple examples.

Example. Let $\mathcal{I}' = \langle C', \mathcal{M}', \alpha'\rangle$ and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha''\rangle$ be interpretation systems such that C' and C'' are disjoint signatures with exactly one constant symbol, namely $\mathbf{t}'$ and $\mathbf{t}''$, respectively. Consider a C'-structure $\mathbb{A}_{m'} = \langle \mathbf{A}_{m'}, \{1\}\rangle$ such that $A_{m'} = \{0, 1\}$ and $\mathbf{t}'_{m'} = 1$, for some model $m' \in \mathcal{M}'$, and a C''-structure $\mathbb{A}_{m''} = \langle \mathbf{A}_{m''}, \{1\}\rangle$ such that $A_{m''} = \{0, 1\}$ and $\mathbf{t}''_{m''} = 1$, for some model $m'' \in \mathcal{M}''$.

In the (free) cryptofibring of $\mathcal{I}'$ and $\mathcal{I}'$, $\mathcal{M}' \circledast \mathcal{M}''$ contains, among others, the tuple $\langle \mathbb{A}, m', m'', h', h''\rangle$ where $\mathbb{A} = \langle \mathbf{A}, \{1', 1''\}\rangle$ is the 4-valued $C' \cup C''$-structure

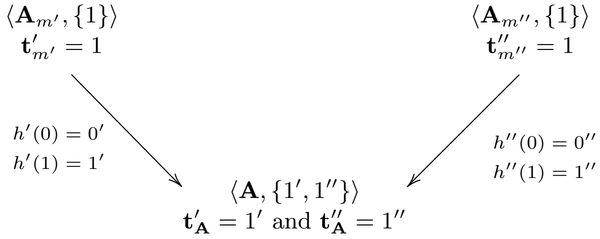

Figure 1. Example of free cryptofibring.

such that $A = \{0', 0'', 1', 1''\}$, $\mathbf{t}'_A = 1'$ and $\mathbf{t}''_A = 1''$, and h' and h'' are the obvious injections (see Figure 1).

Suppose now that C' and C'' are not disjoint, i.e, they share the constant symbol $\mathbf{t}$ (that is, $\mathbf{t}' = \mathbf{t}''$). Then, $\mathcal{M}' \circledast \mathcal{M}''$ contains, among others, the tuple $\langle \mathbb{A}, m', m'', h', h'' \rangle$ where $\mathbb{A} = \langle \mathbf{A}, \{1\} \rangle$ is the 3-valued $C' \cup C''$-structure such that $A = \{0', 0'', 1\}$ and $\mathbf{t}_A = 1$. Note that the $1'$ and the $1''$ of the model in Figure 1 must now be collapsed into a unique 1, because $\mathbf{t}$ is shared and so its interpretations in $\mathbb{A}_{m'}$ and $\mathbb{A}_{m''}$ must be identified. This implies that in order to define h' and h'', 1 must be designated in both $\mathbb{A}'$ and $\mathbb{A}''$, as in the present case (see Figure 2), or in none of them. If 1 would be designated in one of the structures but not in the other, then there would be no model in $\mathcal{M}' \circledast \mathcal{M}''$ corresponding to the pair of models m' and m''.

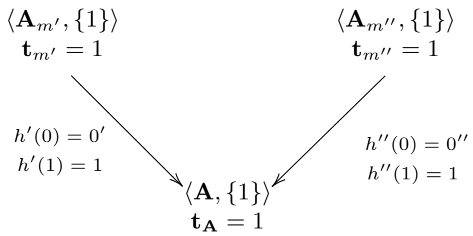

Figure 2. Example of constrained cryptofibring.

Note that in any of the cases above, $0'$ and $0''$ could have been collapsed. Although there is no explicit reason to impose this, such structures, as well suitable extensions, are also present in $\mathcal{M}' \circledast \mathcal{M}''$.

Example. Let $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ be an interpretation system such that C' is a signature with one constant symbol $\mathbf{t}$ and a unary function symbol $\neg$, and let $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$ be an interpretation system such that C'' is a signature with just the constant symbol $\mathbf{t}$. Consider a C'-structure $\mathbb{A}_{m'} = \langle \mathbf{A}_{m'}, \{1\} \rangle$ such that $A_{m'} = \{0, 1\}$, $\mathbf{t}_{m'} = 1$ and $\neg_{m'}(a) = 1 - a$, for some model $m' \in \mathcal{M}'$, and a C''-structure $\mathbb{A}_{m''} = \langle \mathbf{A}_{m''}, \{1\} \rangle$ such that $A_{m''} = \{0, 1\}$ and $\mathbf{t}_{m''} = 1$, for some model $m'' \in \mathcal{M}''$. In this case, $\mathcal{M}' \circledast \mathcal{M}''$ contains, among others, the tuple $\langle \mathbb{A}, m', m'', h', h'' \rangle$ such that $\mathbb{A} = \langle \mathbf{A}, T \rangle$ is the 3-valued $C' \cup C''$-structure such that $A = \{0', 0'', 1\}$, $\mathbf{t}_A = 1$ and $1 \in T$. There are several possible choices for the interpretation of $\neg$. There are two mandatory conditions: $\neg_A(0') = 1$ and $\neg_A(1) = 0'$, which are imposed by $\mathbb{A}'$. However, there are no restrictions on $\neg_A(0'')$. We can choose $0'$, $0''$ or 1.

We can even assume that there are other elements in the domain of A, designated or not, and $\neg_A(0'')$ can assume any of those new values (see Figure 3, where, for simplicity, $0'$ and $0''$ were renamed 0 and 2, respectively).

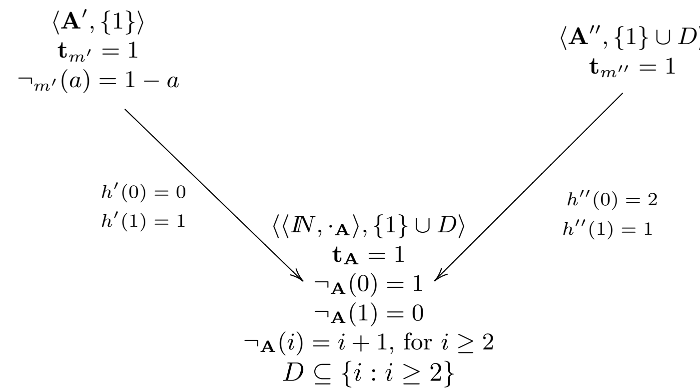

FIGURE 3. Example of constrained cryptofibring.

Cryptomorphisms and their capabilities for combining logical systems presented by semantic means were carefully studied in [9]. A nice universal characterization of cryptofibring can also be obtained, as explained in [8]. One thing is clear, though: all fibred models appear in the cryptofibring.

Proposition 4.3. *Fibred versus cryptofibred semantics.*
Let $\mathcal{I}'$ and $\mathcal{I}''$ be interpretation systems, $\mathcal{I}' * \mathcal{I}''$ their fibring and $\mathcal{I}' \circledast \mathcal{I}''$ their cryptofibring. Then, for every model $\langle m', m'' \rangle$ of $\mathcal{I}' * \mathcal{I}''$ there exists a model m of $\mathcal{I}' \circledast \mathcal{I}''$ such that $\mathbb{A}_{\langle m', m'' \rangle} = \mathbb{A}_m$.

Proof. Given a fibred model $\langle m', m'' \rangle$, one just needs to consider the cryptofibred model $\langle \mathbb{A}_{\langle m', m'' \rangle}, m', m'', \mathrm{id}, \mathrm{id} \rangle$. It is straightforward to check that $\mathrm{id} : \mathbb{A}_{m'} \to \mathbb{A}_{\langle m', m'' \rangle}$ and $\mathrm{id} : \mathbb{A}_{m'} \to \mathbb{A}_{\langle m', m'' \rangle}$ are cryptomorphisms. □

But there are in general many more cryptofibred models. In fact, there can be so many more models that even if we are given sound deductive systems $\mathcal{D}'$, with respect to $\mathcal{I}'$, and $\mathcal{D}''$, with respect to $\mathcal{I}''$, it may happen that some new cryptofibred models are not appropriate for the rules of $\mathcal{D}' * \mathcal{D}''$. For the sake of soundness preservation, the solution, already put forth in [8], is to restrict attention only to sound cryptofibred models.

Definition 4.4. Given $\mathcal{D}'$ sound with respect to $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ and $\mathcal{D}''$ sound with respect to $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$, the *sound cryptofibring of $\mathcal{I}'$ and $\mathcal{I}''$ (constrained by sharing the common subsignature $\widehat{C}$)* is the interpretation system

$$\mathcal{I}' \circledR \mathcal{I}'' = \langle C' \cup C'', \mathcal{M}' \circledR \mathcal{M}'', \alpha_{\circledR} \rangle$$

where:

- $\mathcal{M}' \circledR \mathcal{M}''$ is the class of all models in $\mathcal{M}' \circledast \mathcal{M}''$ that are appropriate for the rules of $\mathcal{D}' * \mathcal{D}''$;
- $\alpha_{\circledR}$ is the restriction of $\alpha_{\circledast}$ to $\mathcal{M}' \circledR \mathcal{M}''$.

Trivially, then, soundness is preserved. That is, $\mathcal{D}' * \mathcal{D}''$ is sound with respect to $\mathcal{I}' \circledR \mathcal{I}''$. More interestingly, though, in general, sound cryptofibring still encompasses many more models than fibring, which opens the way for obtaining more interesting completeness preservation results for cryptofibring. Still, the typical sufficient condition for completeness to be preserved by fibring, i.e. fullness, is so strong that it also applies to cryptofibring. Said, another way, the sound cryptofibring of full interpretation systems is not only sound but also full, and therefore complete.

Proposition 4.5. *Let $\mathcal{D}'$ be sound with respect to $\mathcal{I}'$ and $\mathcal{D}''$ be sound with respect to $\mathcal{I}''$. If $\mathcal{I}'$ is full for $\mathcal{D}'$ and $\mathcal{I}''$ is full for $\mathcal{D}''$ then, for every model m of $\mathcal{I}' \circledR \mathcal{I}''$ there exists a model $\langle m', m'' \rangle$ of $\mathcal{I}' * \mathcal{I}''$ such that $\mathbb{A}_{\langle m', m'' \rangle} = \mathbb{A}_m$.*

Proof. Given a cryptofibred sound model m, since it is appropriate for $\mathcal{D}' * \mathcal{D}''$ and $\mathcal{I}' * \mathcal{I}''$ is full with respect to $\mathcal{D}' * \mathcal{D}''$ then there exists a fibred model $\langle m', m'' \rangle$ such that $\mathbb{A}_{\langle m', m'' \rangle} = \mathbb{A}_m$. □

In any case, our purpose in this paper is to show another nice feature of cryptofibring. Namely, that among the sound cryptofibred models of *IPL* $\circledast$ *CPL* we can find a structure that settles the distinction between $\to$ and $\Rightarrow$ in the fibred deductive system.

Example. Recall from Section 3 that the combined signature C for (the implicative fragments of) intuitionistic and classical logic is such that:

- $C_0 = P \cup Q$, $C_2 = \{\Rightarrow, \to\}$, and $C_1 = C_n = \emptyset$ for every $n > 2$,

where we may further assume that the sets P and Q of intuitionistic and classical propositional symbols, respectively, are disjoint. Let $m = \langle W, \leq, V \rangle$, where $\langle W, \leq \rangle$ is the (intuitionistic) Kripke-frame such that:

- $W = \{a, b\}$;
- $\leq = \{\langle a, a \rangle, \langle b, b \rangle, \langle a, b \rangle\}$;

with $\mathcal{U}pp_{\leq} = \{\emptyset, \{b\}, \{a, b\}\}$, and $V : P \cup Q \to \mathcal{U}pp_{\leq}$ such that:

- $V(p') = \{b\}$ and $V(p'') = \emptyset$, with $p', p'' \in P$;
- $V(q) \in \{\emptyset, \{a, b\}\}$ for every classical propositional symbol $q \in Q$.

Consider also the C-structure $\mathbb{A}_m = \langle \langle \mathcal{U}pp_{\leq}, \cdot_m \rangle, \{W\} \rangle$ where:

- $p_m = V(p)$ and $q_m = V(q)$;

- $(X \rightarrow_m Y) = ((W \setminus X) \cup Y)^i$;
- $(X \Rightarrow_m Y) = ((W \setminus X) \cup Y)^c$,

where $Z^i = \{w \in W : \{w' : w \leq w'\} \subseteq Z\}$ and $Z^c = \{w \in W :$ there exists $z \in Z$ such that $z \leq w\}$. For instance, $\{a\}^i = \emptyset$ and so $(\{b\} \rightarrow_m \emptyset) = \{a\}^i = \emptyset$. The full truth-table for $\rightarrow_m$ is:

$\rightarrow_m$	$\emptyset$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\emptyset$	$\{a,b\}$	$\{a,b\}$
$\{a,b\}$	$\emptyset$	$\{b\}$	$\{a,b\}$

The operator $\cdot^c$ is less usual. For instance, $\{a\}^c = \{a,b\}$ and so $(\{b\} \Rightarrow_m \emptyset) = \{a\}^c = \{a,b\}$. This closure condition is essential to guarantee that the value of $\Rightarrow_m$ is in $\mathcal{U}pp_{\leq}$. The full truth-table for $\Rightarrow_m$ is:

$\Rightarrow_m$	$\emptyset$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{a,b\}$	$\emptyset$	$\{b\}$	$\{a,b\}$

It is straightforward to observe that the previous two tables are different. It is also routine to check that this structure satisfies all the axioms and inference rules of *IPL* and *CPL*. Just note that the interpretation of $\rightarrow$ is standard, and that although the interpretation of $\Rightarrow$ is not the usual, the truth-table obtained is Boolean. To show that we do not have the collapse of the two implications in the fibred deductive, we now must provide a formula that has different values when constructed with $\Rightarrow$ and $\rightarrow$. Consider the formulas $\varphi^c = ((p' \Rightarrow p'') \Rightarrow p') \Rightarrow p'$ and $\varphi^i = ((p' \rightarrow p'') \rightarrow p') \rightarrow p'$. It is not very difficult do observe that $[\![((p' \Rightarrow p'') \Rightarrow p') \Rightarrow p']\!]_m = \{a,b\}$. On the other hand,

$$
\begin{aligned}
[\![((p' \rightarrow p'') \rightarrow p') \rightarrow p']\!]_m &= (((\{b\} \rightarrow_m \emptyset) \rightarrow_m \{b\}) \rightarrow_m \{b\}) \\
&= ((\emptyset \rightarrow_m \{b\}) \rightarrow_m \{b\}) \\
&= (\{a,b\} \rightarrow_m \{b\}) \\
&= \{b\}.
\end{aligned}
$$

Hence, $[\![\varphi^c]\!]_m \in T_m$ and $[\![\varphi^i]\!]_m \notin T_m$ which proves that $\rightarrow$ does not collapse to $\Rightarrow$. In particular, Peirce's law does not hold for the intuitionistic implication.

Nevertheless, we still have to build the model in $\mathcal{M}^c \circledast \mathcal{M}^i$. Let $\langle \mathbb{A}, m^c, m^i, h^c, h^i \rangle$ be such that:

- $\mathbb{A}$ is the $C^c \cup C^i$-structure defined above;
- $m^i = \langle W, \leq, V|_P \rangle$;
- $m^c = \langle \emptyset, W, V|_Q \rangle$;
- $h^i : \mathbb{A}_{m^i} \rightarrow \mathbb{A}$ is the cryptomorphism such that $h^i = \text{id}$;

- $h^c : \mathbb{A}_{m^c} \to \mathbb{A}$ is the cryptomorphism such that $h^c(\emptyset) = \emptyset$ and $h^c(W) = W$.

The model built in the example above has a rather non-standard interpretation of classical implication in a Kripke-model. However, the usual interpretation of $\Rightarrow$ would not work, as it would not be persistent. In [12] del Cerro and Herzig have obtained a combined system of classical and intuitionistic logic precisely by adopting the standard interpretation of $\Rightarrow$, i.e. $(X \Rightarrow_m Y) = (W \setminus X) \cup Y$. However they have gone to the other extreme at the deductive level, by providing it with an axiomatization that makes thorough use of mixed and syntactically constrained axioms that could never be obtained by fibring deductive systems of intuitionistic and classical logic.

5. Conservativeness

We now investigate in more depth the question of conservativeness in the context of cryptofibring. As usual, given signatures $C' \subseteq C$ and logical systems $\mathcal{L}'$ and $\mathcal{L}$, respectively over C' and C, we say that $\mathcal{L}$ is a *conservative extension* of $\mathcal{L}$ whenever the following condition holds, for every $\Gamma \cup \{\varphi\} \subseteq L(C')$:

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ if and only if } \Gamma \vdash_{\mathcal{L}'} \varphi.$$

In the sequel, we will use the same terminology for interpretations systems. We will say that an interpretation system extends another one in a conservative way if that is the case for the logical systems induced by their semantic entailment, as introduced in Definition 2.4.

When the combination of two logical systems is not a conservative extension then strange phenomena like the collapsing problem become possible. In fact, the collapsing problem is just a particular case of this lack of conservativeness. In the case of fibred semantics, *CPL* $*$ *IPL* is not a conservative extension of the interpretation system for *IPL*. For instance, the formula $((p' \to p'') \to p') \to p' \in L(C^i)$ is satisfied by all fibred models (simply because they interpret the intuitionistic implication classically), but it is well-known not to be satisfied by all the models of the interpretation system for *IPL*. In the case of cryptofibring, we will show below that *CPL* $\circledast$ *IPL* is indeed a conservative extension of both interpretation systems *CPL* and *IPL*.

We start by having a look at how pairs of models of the interpretation systems being combined may give rise to cryptofibred models. We assume fixed two interpretation systems $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$, and models $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$.

Definition 5.1. The pair of models $\langle m', m'' \rangle$ is *represented* in $\mathcal{M}' \circledast \mathcal{M}''$ if there exist a $C' \cup C''$-structure $\mathbb{A}$ and cryptomorphisms h' and h'' such that $\langle \mathbb{A}, m', m'', h', h'' \rangle \in \mathcal{M}' \circledast \mathcal{M}''$.

Due to the properties of cryptomorphisms, we can now formulate and prove our sufficient condition for conservativeness along cryptofibring. Note that if $C' \subseteq$

C, $\mathbb{A}'$ is a C'-structure, $\mathbb{A}$ is a C-structure, and $h : \mathbb{A}' \to \mathbb{A}$ is a cryptomorphism then $h([\![\varphi]\!]_{\mathbb{A}'}) = [\![\varphi]\!]_{\mathbb{A}} \in T_{\mathbb{A}}$ if and only if $[\![\varphi]\!]_{\mathbb{A}'} \in T_{\mathbb{A}'}$, for every $\varphi \in L(C')$.

Proposition 5.2. *Assume that for every model $m' \in \mathcal{M}'$ there exists a model $m'' \in \mathcal{M}''$ such that $\langle m', m''\rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$, and vice-versa. Then $\mathcal{I}' \circledast \mathcal{I}''$ is a conservative extension of both $\mathcal{I}'$ and $\mathcal{I}''$.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq L(C')$.

Assume that $\Gamma \vDash_{\mathcal{I}'} \varphi$ and let $m = \langle \mathbb{A}, m', m'', h', h''\rangle$ be a cryptofibred model such that $m \Vdash_{\mathcal{I}' \circledast \mathcal{I}''} \Gamma$. Then $h'([\![\Gamma]\!]_{m'}) = [\![\Gamma]\!]_m \subseteq T_m$ and therefore $[\![\Gamma]\!]_{m'} \subseteq h'^{-1}(T_m) = T_{m'}$. Thus, we also have that $[\![\varphi]\!]_{m'} \in h'^{-1}(T_m) = T_{m'}$ and $h'([\![\varphi]\!]_{m'}) = [\![\varphi]\!]_m \in T_m$. Hence, $\Gamma \vDash_{\mathcal{I}' \circledast \mathcal{I}''} \varphi$.

Assume now, by absurd, that $\Gamma \vDash_{\mathcal{I}' \circledast \mathcal{I}''} \varphi$ but there exists a model m' of $\mathcal{I}'$ such that $m' \Vdash_{\mathcal{I}'} \Gamma$ and $m' \nVdash_{\mathcal{I}'} \varphi$, that is, $[\![\Gamma]\!]_{m'} \subseteq T_{m'}$ and $[\![\varphi]\!]_{m'} \notin T_{m'}$. By assumption, then, there exists $m'' \in \mathcal{M}''$ such that $m = \langle \mathbb{A}, m', m'', h', h''\rangle$ is a cryptofibred model for some $\mathbb{A}$, h' and h''. Immediately, $[\![\Gamma]\!]_m = h'([\![\Gamma]\!]_{m'}) \in h'(T_{m'}) \subseteq T_m$ and therefore $[\![\varphi]\!]_m = h'([\![\varphi]\!]_{m'}) \in T_m$. Thus, $[\![\varphi]\!]_{m'} \in h'^{-1}(T_m) = T_{m'}$, which is a contradiction. The proof for $\mathcal{I}''$ is similar. □

Although nice, this result is not very easy to apply. In general, it is unclear whether all models of each of the interpretation systems appear represented in the cryptofibring. To clarify this question, we will now establish a more usable characterization. The first lemma that we need states that if a pair of models is represented in the cryptofibring then the two models distinguish extactly the same shared formulas. Recall that shared formulas are those build over the common subsignature $\widehat{C} = C' \cap C''$.

Lemma 5.3. *If $\langle m', m''\rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$ then, for every $\varphi \in L(\widehat{C})$, $[\![\varphi]\!]_{m'} \in T_{m'}$ if and only if $[\![\varphi]\!]_{m''} \in T_{m''}$.*

Proof. Assume that $\langle m', m''\rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$ by $m = \langle \mathbb{A}, m', m'', h', h''\rangle$. Then, we have that $h'([\![\varphi]\!]_{m'}) = [\![\varphi]\!]_m = h''([\![\varphi]\!]_{m''})$. Hence, $[\![\varphi]\!]_{m'} \in T_{m'} = h'^{-1}(T_m)$ iff $h'([\![\varphi]\!]_{m'}) \in T_m$ iff $[\![\varphi]\!]_m \in T_m$ iff $h''([\![\varphi]\!]_{m''}) \in T_m$ iff $[\![\varphi]\!]_{m''} \in h''^{-1}(T_m) = T_{m''}$. □

When $\langle m', m''\rangle$ is represented in the cryptofibring, furthermore, the fact that there exist shared formulas imposes a certain regularity to the interpretation structures $\mathbb{A}_{m'}$ and $\mathbb{A}_{m''}$. Namely, if two shared formulas happen to have the same interpretation in one of the structures then that identification gives rise to a congruence on the other structure that must agree with the designation of truth-values. Consider the sequences $\equiv'_0 \subseteq \equiv'_1 \subseteq \equiv'_2 \subseteq \ldots$ of congruences over $\mathbf{A}_{m'}$ and $\equiv''_0 \subseteq \equiv''_1 \subseteq \equiv''_2 \subseteq \ldots$ of congruences over $\mathbf{A}_{m''}$ defined inductively as follows:

- $\equiv'_0$ and $\equiv''_0$ are the diagonal congruences;
- $\equiv'_{i+1}$ is the congruence generated by the identities

$$[\![\varphi]\!]_{m'} \equiv'_{i+1} [\![\psi]\!]_{m'} \text{ for } \varphi, \psi \in L(\widehat{C}) \text{ such that } [\![\varphi]\!]_{m''} \equiv''_i [\![\psi]\!]_{m''};$$

- analogously, $\equiv''_{i+1}$ is the congruence generated by the identities

$$[\![\varphi]\!]_{m''} \equiv''_{i+1} [\![\psi]\!]_{m''} \text{ for } \varphi,\psi \in L(\widehat{C}) \text{ such that } [\![\varphi]\!]_{m'} \equiv'_i [\![\psi]\!]_{m'}.$$

Let $\equiv' = (\bigcup_{i\in\mathbb{N}_0} \equiv'_i)$ and $\equiv'' = (\bigcup_{i\in\mathbb{N}_0} \equiv''_i)$. The following lemma guarantees that each of these two congruences is *compatible*, in the sense of [2], with the corresponding set of designated values, respectively $T_{m'}$ and $T_{m''}$. That is, if $a',b' \in \mathbf{A}_{m'}$ and $a' \equiv' b'$ then $a' \in T_{m'}$ if and only if $b' \in T_{m'}$, and symetrically, if $a'',b'' \in \mathbf{A}_{m''}$ and $a'' \equiv'' b''$ then $a'' \in T_{m''}$ if and only if $b'' \in T_{m''}$.

Lemma 5.4. *Let $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$. If $\langle m',m''\rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$ then $\equiv'$ is compatible with $T_{m'}$ and $\equiv''$ is compatible with $T_{m''}$.*

Proof. Assume that $\langle m',m''\rangle$ is represented in $\mathcal{M}'\circledast\mathcal{M}''$ by $m = \langle \mathbb{A}, m', m'', h', h''\rangle$. We first check that if $a' \equiv' b'$ then $h'(a') = h'(b')$, and that if $a'' \equiv'' b''$ then $h''(a'') = h''(b'')$. The proof follows, by mutual induction on the definitions of $\equiv'_i$ and $\equiv''_i$. If $i = 0$ the result is trivial because $a' = b'$ and $a'' = b''$. If $a' \equiv'_{i+1} b'$ then there must exist $\varphi,\psi \in L(\widehat{C})$ such that $a' = [\![\varphi]\!]_{m'}$, $b' = [\![\psi]\!]_{m'}$, and $[\![\varphi]\!]_{m''} \equiv''_i [\![\psi]\!]_{m''}$. Hence, by induction hypothesis, $h''([\![\varphi]\!]_{m''}) = h''([\![\varphi]\!]_{m''})$. But $h''([\![\varphi]\!]_{m''}) = [\![\varphi]\!]_m = h'(a')$ and $h''([\![\psi]\!]_{m''}) = [\![\psi]\!]_m = h'(b')$, and therefore $h'(a') = h'(b')$. The symmetric condition is analogous.

Now, let $a' \equiv' b'$. Then, $a' \in T_{m'}$ iff $a' \in h'^{-1}(T_m)$ iff $h'(a') \in T_m$ iff $h'(b') \in T_m$ iff $b' \in h'^{-1}(T_m)$ iff $b' \in T_{m'}$. The symmetric condition is analogous. $\square$

Finally, we can prove that not only represented pairs of models always imply these two properties, but also that these two properties guarantee that a pair of models is represented.

Proposition 5.5. *The pair of models $\langle m',m''\rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$ if and only if both the following conditions hold:*

A: *$[\![\varphi]\!]_{m'} \in T_{m'}$ if and only if $[\![\varphi]\!]_{m''} \in T_{m''}$, for every $\varphi \in L(\widehat{C})$;*
B: *$\equiv'$ is compatible with $T_{m'}$ and $\equiv''$ is compatible with $T_{m''}$.*

Proof. The left-to-right implication follows from Lemmas 5.3 and 5.4. We are left with proving the right-to-left implication, that is, the two conditions imply that $\langle m',m''\rangle$ is indeed represented in the cryptofibring. For the purpose, we will explicitly build a cryptofibred model $\langle \mathbb{A}, m', m'', h', h''\rangle$.

The $C' \cup C''$-structure $\mathbb{A} = \langle \mathbf{A}, T\rangle$ is defined by:

- $\mathbf{A} = \mathcal{F}_{C'\cup C''}(A_{m'} \uplus A_{m''})/_{\equiv}$ is the quotient of the free $C' \cup C''$-algebra built over the disjoint union of $A_{m'}$ and $A_{m''}$, with the congruence $\equiv$ generated by the identities of the following three kinds:
  1. $c'(a'_1,\ldots,a'_n) \equiv b'$ if $c'_{m'}(a'_1,\ldots,a'_n) = b'$, and
     $c''(a''_1,\ldots,a''_n) \equiv b''$ if $c''_{m''}(a''_1,\ldots,a''_n) = b''$;
  2. $a' \equiv b'$ if $a' \equiv' b'$, and
     $a'' \equiv b''$ if $a'' \equiv'' b''$,
  3. $[\![\varphi]\!]_{m'} \equiv [\![\varphi]\!]_{m''}$ if $\varphi \in L(\widehat{C})$;
- $T = (T_{m'} \uplus T_{m''})/_{\equiv}$.

Let $h' : A_{m'} \to A$ and $h'' : A_{m''} \to A$ be defined by $h'(a') = [a']_\equiv$ and $h''(a'') = [a'']_\equiv$. Then, the identities of kind 1 guarantee that $h' : \mathbf{A}_{m'} \to \mathbf{A}|_{C'}$ and $h'' : \mathbf{A}_{m''} \to \mathbf{A}|_{C''}$ are homomorphisms, respectively of C'-algebras and C''-algebras. To prove that $\langle \mathbb{A}, m', m'', h', h'' \rangle$ is a cryptofibred model, now, we just need to prove that $h' : \mathbb{A}_{m'} \to \mathbb{A}$ and $h'' : \mathbb{A}_{m''} \to \mathbb{A}$ are indeed cryptomorphisms, that is, $h'^{-1}(T) = T_{m'}$ and $h''^{-1}(T) = T_{m''}$.

If $a' \in T_{m'}$ then clearly $h'(a') = [a']_\equiv \in (T_{m'}/_\equiv) \subseteq T$. On the other hand, if $h'(a') = [a']_\equiv \in T$ then, either there exists $b' \in T_{m'}$ such that $a' \equiv b'$, the identities of kind 2 guarantee that $a' \equiv' b'$, and condition **B** guarantees that $a' \in T_{m'}$; or there exists $b'' \in T_{m''}$ such that $a' \equiv b''$, the identities of kind 3 guarantee that $a' = [\![\varphi]\!]_{m'}$ and $b'' = [\![\varphi]\!]_{m''}$ for some $\varphi \in L(\widehat{C})$, and condition **A** guarantees that $a' \in T_{m'}$. The other case is analogous. $\square$

Note that if there are no shared formulas, then both conditions **A** and **B** of the previous result are trivially satisfied. Namely, it is easy to see that both $\equiv'$ and $\equiv''$ will be the diagonal congruences. Of course, the absence of shared formulas does not necessarily mean that the shared signature $\widehat{C}$ is empty. However, when that is the case, certainly we will have no shared formulas.

Corollary 5.6. Every free cryptofibring is a conservative extension of both the interpretation systems being combined.

Example. By Corollary 5.6, we can assert that *IPL* $\circledast$ *CPL* is indeed a conservative extension of the interpretation systems of *IPL* and *CPL* provided that $P \cap Q = \emptyset$, that is, the intuitionistic and classical propositional symbols are disjoint.

If $P \cap Q \neq \emptyset$ then conservativeness may be lost. Assume that there are two shared propositional symbols p' and p'', and consider the intuitionistic model $m' = \langle W, \leq, V \rangle$ with $V(p') = \emptyset$ and $V(p'') = \{b\}$ over the intuitionistic Kripke frame $\langle W, \leq \rangle$ of the previous example. For this model, there is no classical model $m'' = \langle \bot, \top, v \rangle \in \mathcal{M}^c$ with $\bot = \emptyset$ and $\top = W$ such that $\langle m', m'' \rangle$ is represented in $\mathcal{M}^i \circledast \mathcal{M}^c$. For this to happen, by Proposition 5.5, m' and m'' must agree on all shared formulas. In this case, the shared formulas are precisely p' and p''. Since $p'_{m''}, p''_{m''} \notin T_{m''}$ then $p'_{m'}$ and $p''_{m'}$ cannot be in $T_{m'}$, that is, we must have $v(p') = v(p'') = \bot$. Then, by construction of $\equiv''$, it follows that $\emptyset \equiv'' \{b\}$ and so $(\{b\} \to_{m''} \{b\}) \equiv'' (\{b\} \to_{m''} \emptyset)$. But $(\{b\} \to_{m''} \{b\}) = \{a, b\} \in T_{m''}$ and $(\{b\} \to_{m''} \emptyset) = \emptyset \notin T_{m''}$, which means that $\equiv''$ is not compatible with $T_{m''}$.

Still, Proposition 5.2 does not allows us to conclude immediately that *CPL* $\circledast$ *IPL* is not a conservative extension. Therefore, we prove it directly. Consider the formula $((p' \to p'') \to p') \to p'$. For the above model m'', we have that $[\![((p' \to p'') \to p') \to p']\!]_{m''} = \{b\} \notin T_{m''}$. It is straightforward to check that, for any model $m \in \mathcal{M}^c \circledast \mathcal{M}^i$, $[\![((p' \to p'') \to p') \to p']\!]_m \in T_m$, thus proving the loss of conservativeness. Let $m = \langle \mathbb{A}, m', m'', h', h'' \rangle$ for arbitrary $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}^i$. We have to consider the four possible choices of classical values for p', p''. The interesting situation is when $p'_{m'}, p''_{m'} \notin T_{m'}$. In this case, $p'_{m'} = p''_{m'} = \bot$ which implies that $p'_m = p''_m$ and so $[\![((p' \to p'') \to p') \to p']\!]_m \in T_m$. Note that for

$((p' \to p'') \to p') \to p'$ not to hold in a model, we need to have two distinct truth values that are not designated such that the implication of these two values is also not designated. We have this situation in the above model where $\emptyset$, $\{b\}$ and $(\{b\} \to_{m''} \emptyset)$ are all non-designated, but we have lost it by imposing the sharing of the propositional symbols.

Note also that if $P \cap Q = \emptyset$, the cryptofibring will be conservative even if we share the two implications. Indeed, if we do not have any shared propositional symbols we cannot use the shared implication to build any shared formulas.

6. Concluding remarks

In this paper we have overviewed the main features of the powerful mechanism of fibring for combining logics. In particular, we have concentrated in understanding and solving the semantic collapse problem. For that purpose, we have introduced cryptofibring, an extension of fibred semantics that allows for a more relaxed relationship between the models of the logics being combined and the resulting logic. We have shown that cryptofibring avoids the collapsing problem, by proving a general result that establishes a sufficient condition for the cryptofibred system to be a conservative extension of the given logical systems. We illustrated the constructions and results by means of the traditional collapsing example: the combination of (the implicative fragments of) intuitionistic and classical logics. We leave it to the reader to verify that a similar strategy can be used to show that the full logics can also be combined without any unexpected collapse.

Further work should contemplate three distinct directions. First, a thorougher understanding of cryptofibring and its power is still necessary. Namely, due to the rich structure of its models, we envisage to obtain nicer completeness preservation results that avoid the strong assumption of fullness. At present, we do not know if the deductive system of $IPL * CPL$ is complete with respect to the sound cryptofibring of the interpretation systems. The detailed study of the relationship between cryptofibring and modulated fibring is also envisaged.

Second, the question of conservativeness is still not definitively settled. We conjecture, though, that the sufficient condition for conservativeness that we have formulated is also necessary, or very close to that. That is, together with some possible minor assumptions, cryptofibred semantics will extend the given logical systems in a conservative way if and only if such a conservative extension exists. Even more, the system obtained by cryptofibring will be as general an extension as possible. If something gets collapsed then it gets collapsed in every logical system that extends the two given systems. If additionally, one is interested in the conservativeness of sound cryptofibring, then the question seems to be much harder. Its connection with the issue of completeness preservation is, nevertheless, clear. Note that in this paper we have not addressed the combination of logical systems *per se*, but only the combination of deductive, or semantical, presentations of logical systems. The interested reader is directed to [3] for further details on fibring structural logical systems.

Finally, the system of combined intuitionistic and classical logic is interesting in its own right. Its completeness is an open issue, not only because cryptofibred models do not have an explicit definition, but mainly because the combined deductive system appears to lack the deduction metatheorem for both the implications. In [10], we will address the characterization of the class of models used for showing the absence of the collapsing problem, that is, models whose associated interpretation structure over the combined language is defined over the uppersets $\mathcal{U}pp_{\leq}$ of partially ordered Kripke frames $\langle W, \leq \rangle$ by:

- $V(p) \in \mathcal{U}pp_{\leq}$ for $p \in P$;
- $V(q) = \emptyset$ or $V(q) = W$ for $q \in Q$;
- $(X \to_m Y) = ((W \setminus X) \cup Y)^i$;
- $(X \Rightarrow_m Y) = ((W \setminus X) \cup Y)^c$.

Topological interpretations of this system, its possible connections to type systems associated to lambda-calculi, its algebraizability and its relationship to other systems that combine different implications, e.g. the logic of bunched implications BI of [17] or the BCSK system of [16], are also to be developed and understood.

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