On the algebraization of valuation semantics* (extended abstract)

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In 1989, W. Blok and D. Pigozzi proposed a precise mathematical definition of the notion of *algebraizable logic* [1], which generalizes the traditional Lindenbaum-Tarski method. Nevertheless, many interesting logics fall out of the scope of this approach. It is the case of the so-called *non-truth-functional logics*, and in particular of the paraconsistent systems of da Costa [4]. The major problem with these logics is the lack of congruence for some connective(s), the key ingredient in the algebraization process. Our goal is to generalize the Blok-Pigozzi approach by dropping the assumption that formulas should be homomorphically evaluated over algebras of truth-values of the same type. As an example we shall consider da Costa's system $\mathcal{C}_1$, whose non-algebraizability has been studied in [7, 6].

As in the Blok-Pigozzi case, we shall focus on logic systems $\mathcal{L} = \langle L, \vdash \rangle$ which are tarskian and structural, in the sense that L is freely generated by a signature Σ from a set of propositional variables and $\vdash$ is invariant under substitutions. As observed by J. Czelakowski and R. Jansana in [3], an equivalent characterization of the algebraizability of $\mathcal{L}$ can be obtained in terms of the existence of a mutual interpretability between $\mathcal{L}$ and unsorted equational logic. Hence $\mathcal{L}$ is represented over the unsorted equational logic $Eqn(\Sigma, K)$, where Σ is precisely the signature of $\mathcal{L}$ and K is a class of Σ-algebras. Each Σ-algebra $A \in K$ can thus be seen as an interpretation for $\mathcal{L}$ along the unique homomorphism $[\![\_]\!]_A : L \to A$. Our generalization proceeds by replacing unsorted equational logic with another suitable base logic. Following the idea in [2], we will work with a two-sorted equational logic with sorts ϕ and τ of formulas and truth values, respectively, plus an operation v from ϕ to τ, that represents the valuation map. Moreover, the two-sorted signature Σ_2 will have the operations of Σ on sort ϕ, but can have an arbitrarily chosen signature Σ' of operations on truth-values. We shall represent $\mathcal{L}$ over the two-sorted equational logic $Eqn(\Sigma_2, K_2)$, where K_2 is a class of Σ_2-algebras. In this case we say that $\mathcal{L}$ is $Eqn(\Sigma_2, K_2)$-able. Here, each $A \in K_2$ can be seen as an interpretation for $\mathcal{L}$ along the valuation map v_A over the set of truth-values A_τ. The crucial observation is that v does not have to be an homomorphism, as advocated in valuation semantics [5]. Of course, we can recover the Blok-Pigozzi case by choosing $\Sigma' = \Sigma$ and K_2 as the class of all algebras whose ϕ-fragment is L and whose τ-fragment is in K, and such that v satisfies the homomorphism conditions $v(c(y_1, \ldots, y_n)) = c(v(y_1), \ldots, v(y_n))$ for every n-ary constructor $c \in \Sigma$.

Under this new approach, we show that $\mathcal{C}_1$ is algebraizable using the class K_2 of two-sorted algebras whose truth-values form a Boolean algebra, in such a way that the valuation map v fulfills the homomorphism conditions for every connective, except for the paraconsistent negation. Every formula $\varphi \in L$ is translated to the τ-equation $v(\varphi) = \top$, and every τ-equation $t_1 = t_2$ is translated to a formula $t_1^* \equiv t_2^*$. Here, t^* is obtained by taking advantage of the usual representation of classical negation in $\mathcal{C}_1$.

One of the most important tools of the Blok-Pigozzi approach is the *Leibniz operator* Ω that maps each theory of $\mathcal{L}$ to the largest congruence on L that is compatible with the theory. In

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our case, since the valuation map v is not necessarily an homomorphism, we may end up having contexts δ (formulas in one variable) such that $v(\varphi) = v(\psi)$ does not imply $v(\delta(\varphi)) = v(\delta(\psi))$. If the previous implication holds we call $\delta(\_)$ a *congruent context.* An equivalence relation $\sim$ on L is then called a *semi-congruence* if $\varphi \sim \psi$ implies $\delta(\varphi) \sim \delta(\psi)$ for every congruent context $\delta(\_)$. In analogy, we can now also define an operator "Ω" that maps each theory of $\mathcal{L}$ to the largest semi-congruence on L compatible with the theory. Using "Ω" we can generalize the notion of protoalgebraization. We say that $\mathcal{L}$ is proto-$Eqn(\Sigma_2, K_2)$-able if $\langle\varphi, \psi\rangle \in$ "Ω"(Γ) implies $\Gamma \cup \{\varphi\} \dashv\vdash \Gamma \cup \{\psi\}$.

We can now prove that the general characterization properties of the Blok-Pigozzi approach with respect to the Leibniz operator carry over to our more general setting.

Theorem 1 *$\mathcal{L}$ is proto-$Eqn(\Sigma_2, K_2)$-able if and only if "Ω" is monotone.*

When a logic is $Eqn(\Sigma_2, K_2)$-able we can relate, in a strong sense, the operator "Ω" with the corresponding representation map $\theta : \mathcal{L} \to Eqn(\Sigma_2, K_2)$. In fact, for any theory Γ of $\mathcal{L}$, we can show that "Ω"$(\Gamma) = \{\langle\varphi, \psi\rangle : \theta(\Gamma) \vDash_{Eqn(\Sigma_2,K_2)} v(\varphi) = v(\psi)\}$.

Theorem 2 *$\mathcal{L}$ is $Eqn(\Sigma_2, K_2)$-able if and only if "Ω" is injective and sup-preserving.*

Since sup-preservation implies monotonicity, we also have that every $Eqn(\Sigma_2, K_2)$-able logic is proto-$Eqn(\Sigma_2, K_2)$-able.

In this work we generalized the Blok-Pigozzi theory of algebraization of logics and got some similar results. Still there are many open problems. In particular, there are two very important open questions. Given a logic $\mathcal{L}$ is it always possible to find a truth-values signature Σ' such that $\mathcal{L}$ is $Eqn(\Sigma_2, K_2)$-able? If we fix a truth-values signature Σ' and the valuation axioms, is there a proto-$Eqn(\Sigma_2, K_2)$-able logic that is not $Eqn(\Sigma_2, K_2)$-able? Nevertheless this work leaves good perspectives for further generalizations, namely by choosing other interesting base logics to replace the role played by unsorted equation logic in the Blok-Pigozzi case. We have already some preliminary work on abstracting the relevant properties of unsorted equational logic that are essential to the algebraization process.

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