Equipollent logical systems

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Abstract. When can we say that two distinct logical systems are, nevertheless, essentially the "same"? In this paper we discuss the notion of "sameness" between logical systems, bearing in mind the expressive power of their associated spaces of theories, but without neglecting their syntactical dimension. Departing from a categorial analysis of the question, we introduce the new notion of *equipollence* between logical systems. We use several examples to illustrate our proposal and to support its comparison to other proposals in the literature, namely homeomorphisms [7], and translational equivalence (or synonymity) [6].

Keywords. Logical system, theory space, equipollence.

1. Introduction

When we talk about classical propositional logic (CPL), for example, we are most often not referring to just a particular entity but rather to a family of (possibly very) different logical systems that do all present essentially the "same" CPL. But what do we mean when we say that two logical systems are the "same"? Our goal is to find a satisfactory answer to this question and to show how our proposal, *equipollence*, relates to earlier proposals in the literature [7, 6, 1].

Certainly there is no point in even discussing this question without first agreeing on what a logical system is. Still, we do not wish to dwell on such a delicate subject here. The interested reader can find a very complete discussion of this theme in [4], along with a myriad of different viewpoints and possible definitions. For what we are concerned in this paper, we will restrict our attention to Tarski-style consequence operators over a structured language. This choice is certainly not without controversy, but most logicians should at least agree that it covers a wide range of well known examples.

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Methodologically, we shall adopt a category-theoretical perspective, where logical systems will constitute our main category of interest. An adequate notion of morphism between logical systems will however be essential, since we are not only interested in logical systems as objects by themselves, but we are particularly interested in the way different logical systems relate to each other. Morphisms between logical systems will be uniform translations between the structured languages that preserve the corresponding consequence operators. We will also take into account that logical systems generate spaces of theories, which in their turn can also be endowed with a suitable categorial structure, in a functorial way. This categorial setup is directly inspired by [8, 2].

As a first attempt to attack the problem, we shall analyze the most obvious idea, that is, to try and characterize "sameness" between logical systems using the built-in notion of isomorphism in the corresponding category. It turns out that isomorphisms between logical systems are very closely related to Pollard's homeomorphisms, as presented in [7], and thus suffer from very similar merits and shortcomings. In fact, with the support of suitable examples, we can claim that, although meaningful, this notion is syntactically too strict as a definition of "sameness". To overcome this strictness, we shall then take a brief look at the theory spaces generated by these logical systems, thus abstracting away from syntactical fine details. At the level of logical systems, individual formulas play a fundamental role, while at the level of their associated theory spaces, the expressive power of isolated formulas is not so much important. It turns out, however, that the built-in notion of isomorphism between theory spaces is now too broad, as we also illustrate. The correct notion of "sameness" seems therefore to lie somewhere in between these two notions of isomorphism. By capturing the right amount of interplay between the two, and taking into account the functorial relationship between the categories of logical systems and theory spaces, we finally manage to isolate our notion of *equipollence*, in a way similar to the one developed in [2]. It is interesting to note that the ideas underlying *equipollence* are very closely related to those that stand behind Pelletier and Urquhart's proposal [6] of translational equivalence (or synonymity). *Equipollence* can nevertheless be shown to be more widely applicable, although the two notions coincide under mild assumptions, that we make explicit.

We begin by introducing, in section 2, the categories **Log** of logical systems and **Tsp** of theory spaces, that will be used throughout the paper. In section 3, we analyze the notions of isomorphism in these categories, and we provide examples that help us to conclude that none of them, per se, is satisfactory as a definition of "sameness" between logical systems. We also show how isomorphisms in **Log** relate to the homeomorphisms of [7]. Our notion of *equipollence* is then introduced and analyzed in section 4. We illustrate our proposal with a few examples, that will also enhance its comparison with the notion of synonymity of [6]. We conclude, in section 5, with an overview of our proposal and a discussion of its adequation.

2. From logical systems to theory spaces

In this section we introduce the precise definitions of logical systems and theory spaces, along with their associated categorial structure. We should however start with the more syntactical details. We will consider logical languages that are freely generated from a given signature including constructors of different arities, as is most often the case.

Definition 2.1. A *signature* is an indexed set $\Sigma = \{\Sigma^n\}_{n\in\mathbb{N}}$, where each Σ^n is the set of n-ary constructors.

We consider that the set of *propositional variables* is included in Σ^0.

Definition 2.2. The *language* over a given a signature Σ, which we denote by L_Σ, is build inductively in the usual way:

- $\Sigma^0 \subseteq L_\Sigma$;
- If $n \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_n \in L_\Sigma$ and $c \in \Sigma^n$ then $c(\varphi_1, \ldots, \varphi_n) \in L_\Sigma$.

We call Σ-*formulas* to the elements of L_Σ, or simply *formulas* when Σ is clear from the context.

Definition 2.3. A *logical system* is a pair $\mathcal{L} = \langle \Sigma, \vdash \rangle$, where Σ is a signature and $\vdash$ is a consequence operator on L_Σ (in the sense of Tarski, cf. eg. [9]), that is, $\vdash : 2^{L_\Sigma} \to 2^{L_\Sigma}$ is a function that satisfies the following properties, for every $\Gamma, \Phi \subseteq L_\Sigma$:

Extensiveness: $\Gamma \subseteq \Gamma^\vdash$;
Monotonicity: If $\Gamma \subseteq \Phi$ then $\Gamma^\vdash \subseteq \Phi^\vdash$;
Idempotence: $(\Gamma^\vdash)^\vdash \subseteq \Gamma^\vdash$.

For the sake of generality, we do not require here the consequence operator to be finitary, or even structural.

Since we will need to talk about the expressive power of the language of a given logical system, we will need to refer to its connectives (primitive or derived). For the purpose, we consider fixed once and for all a set $\Xi = \{\xi_i\}_{i\in\mathbb{N}^+}$ of *metavariables*. Then, given a signature Σ and $k \in \mathbb{N}$, we can consider the set L^k_Σ defined inductively by:

- $\{\xi_1, \ldots, \xi_k\} \subseteq L^k_\Sigma$;
- $\Sigma^0 \subseteq L^k_\Sigma$;
- If $n \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_n \in L^k_\Sigma$ and $c \in \Sigma^n$ then $c(\varphi_1, \ldots, \varphi_n) \in L^k_\Sigma$.

Clearly, we have that $L_\Sigma = L^0_\Sigma$. We can also consider the set $L^\omega_\Sigma = \bigcup_{n\in\mathbb{N}} L^n_\Sigma$. Given $\varphi \in L^k_\Sigma$ we will write $\varphi(\xi_1 \setminus \psi_1, \ldots, \xi_k \setminus \psi_k)$ to denote the formula that is obtained from φ by simultaneously replacing each occurrence of ξ_i in φ by ψ_i, for every $i \leq k$.

A *derived connective* of arity $k \in \mathbb{N}$ is a λ-term $d = \lambda\xi_1 \ldots \xi_k.\varphi$ where $\varphi \in L^k_\Sigma$. We denote by DC^k_Σ the set of all derived connectives of arity k over Σ. Note that, if $c \in \Sigma_k$ is a primitive connective, it can also be considered as the derived connective $c = \lambda\xi_1 \dots \xi_k.c(\xi_1, \dots, \xi_k)$. Given a derived connective $d = \lambda\xi_1 \dots \xi_n.\varphi$ we will often write $d(\psi_1, \dots, \psi_n)$ instead of $\varphi(\xi_1 \setminus \psi_1, \dots, \xi_n \setminus \psi_n)$.

Different languages generated from different signatures can be translated according to the following notion of morphism, where primitive connectives from one signature are mapped to derived connectives from another signature, while preserving the corresponding arities.

Definition 2.4. Given signatures Σ_1 and Σ_2, a *signature morphism* $h : \Sigma_1 \to \Sigma_2$ is an $\mathbb{N}$-indexed family of functions $h = \{h^n : \Sigma_1^n \to DC^n_{\Sigma_2}\}_{n \in \mathbb{N}}$.

Given a signature morphism $h : \Sigma_1 \to \Sigma_2$, we can define its free extensions $h : L^k_{\Sigma_1} \to L^k_{\Sigma_2}$ for $k \in \mathbb{N}$, and $h : L^\omega_{\Sigma_1} \to L^\omega_{\Sigma_2}$ inductively, as follows:

- $h(\xi_i) = \xi_i$ if $\xi_i \in \Xi$;
- $h(c) = h^0(c)$ if $c \in \Sigma_1^0$;
- $h(c(\varphi_1, \dots, \varphi_n)) = h^n(c)(h(\varphi_1), \dots, h(\varphi_n))$ if $c \in \Sigma_1^n$.

A translation function h that satisfies the above requirements will be dubbed *uniform.*

Signatures and their morphisms constitute a category **Sig** with identities $id_\Sigma : \Sigma \to \Sigma$ such that $id^n_\Sigma(c) = \lambda\xi_1 \dots \xi_n.c(\xi_1, \dots, \xi_n)$ for every $n \in \mathbb{N}$ and $c \in \Sigma^n$, and the composition of signature morphisms $f : \Sigma_1 \to \Sigma_2$ and $g : \Sigma_2 \to \Sigma_3$ defined to be $g \circ f : \Sigma_1 \to \Sigma_3$ such that $(g \circ f)^n(c) = \lambda\xi_1 \dots \xi_n.g(\varphi)$, assuming that $f^n(c) = \lambda\xi_1 \dots \xi_n.\varphi$.

We can now take advantage of uniform translations to put forth the notion of morphism between logical systems. Given a function $h : L_{\Sigma_1} \to L_{\Sigma_2}$ and $\Phi \subseteq L_{\Sigma_1}$ we can consider the set $h[\Phi] = \{h(\varphi) : \varphi \in \Phi\}$.

Definition 2.5. Let $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ be logical systems. A *logical system morphism* $h : \mathcal{L}_1 \to \mathcal{L}_2$ is a signature morphism $h : \Sigma_1 \to \Sigma_2$ such that $h[\Phi^{\vdash_1}] \subseteq h[\Phi]^{\vdash_2}$ for every $\Phi \subseteq L_{\Sigma_1}$.

Logical systems and their morphisms constitute a concrete category **Log**, over **Sig**. The following is a well known useful lemma.

Lemma 2.6. *Let $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ be logical systems, and $h : \mathcal{L}_1 \to \mathcal{L}_2$ a* **Log***-morphism. Then, $h[\Phi^{\vdash_1}]^{\vdash_2} = h[\Phi]^{\vdash_2}$ for every $\Phi \subseteq L_{\Sigma_1}$.*

Proof. Clearly, by the extensiveness of $\vdash_1$, $\Phi \subseteq \Phi^{\vdash_1}$. Therefore, $h[\Phi] \subseteq h[\Phi^{\vdash_1}]$ and by the monotonicity of $\vdash_2$ we get that $h[\Phi]^{\vdash_2} \subseteq h[\Phi^{\vdash_1}]^{\vdash_2}$. On the other hand, since h is a morphism, we have that $h[\Phi^{\vdash_1}] \subseteq h[\Phi]^{\vdash_2}$. Thus, by the monotonicity of $\vdash_2$ it follows that $h[\Phi^{\vdash_1}]^{\vdash_2} \subseteq (h[\Phi]^{\vdash_2})^{\vdash_2}$. Now, by the idempotence of $\vdash_2$ we get that $(h[\Phi]^{\vdash_2})^{\vdash_2} \subseteq h[\Phi]^{\vdash_2}$. Therefore, we have $h[\Phi^{\vdash_1}]^{\vdash_2} \subseteq h[\Phi]^{\vdash_2}$. □

As usual, a *theory* of a logical system $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is a set $\Phi \subseteq L_\Sigma$ such that $\Phi^\vdash = \Phi$. We denote by $Th(\mathcal{L})$ the set of all theories of $\mathcal{L}$. It is well known that the structure of the set $Th(\mathcal{L})$ under the inclusion ordering is very important. Namely, it is always a complete lattice.

Definition 2.7. A *theory space* is a complete lattice $tsp = \langle Th, \leq \rangle$, that is, a partial order $\leq$ on the set Th such that every $T \subseteq Th$ has a least upper-bound (or join) $\bigvee T$.

In particular, given a logical system $\mathcal{L} = \langle \Sigma, \vdash \rangle$, $tsp_{\mathcal{L}} = \langle Th(\mathcal{L}), \subseteq \rangle$ is always a theory space (cf. eg. [9]). Moreover, the language translations associated to logical system morphisms always act on the consequence operators in such a way that joins are preserved in the corresponding theory spaces.

Definition 2.8. Let $tsp_1 = \langle Th_1, \leq_1 \rangle$ and $tsp_2 = \langle Th_2, \leq_2 \rangle$ be theory spaces. A *theory spaces morphism* $h : tsp_1 \to tsp_2$ is a function $h : Th_1 \to Th_2$ such that $h(\bigvee_1 T) = \bigvee_2 h[T]$ for every $T \subseteq Th_1$.

We prove now a straightforward but useful property of theory spaces morphisms.

Lemma 2.9. *Let $tsp_1 = \langle Th_1, \leq_1 \rangle$ and $tsp_2 = \langle Th_2, \leq_2 \rangle$ be theory spaces and $h : tsp_1 \to tsp_2$ a theory spaces morphism. Then h is order preserving, that is, for every $\Phi, \Gamma \in Th_1$, if $\Phi \leq_1 \Gamma$ then $h(\Phi) \leq_2 h(\Gamma)$.*

Proof. Clearly, if $\Phi \leq_1 \Gamma$ then $\bigvee_1 \{\Phi, \Gamma\} = \Gamma$. Therefore, since h preserves joins, $h(\Gamma) = h(\bigvee_1 \{\Phi, \Gamma\}) = \bigvee_2 \{h(\Phi), h(\Gamma)\}$. Consequently, $h(\Phi) \leq_2 h(\Gamma)$. □

Theory spaces and their morphisms constitute the category **Tsp**, with the usual identity and composition of functions. What is more, the definition of the space of theories induced by a logical system can be extended to a functor.

Definition 2.10. The maps

- $Th(\mathcal{L}) = tsp_{\mathcal{L}}$;
- $Th(h : \mathcal{L}_1 \to \mathcal{L}_2) : tsp_{\mathcal{L}_1} \to tsp_{\mathcal{L}_2}$, with $Th(h)(\Phi) = h[\Phi]^{\vdash_2}$ if $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$, for every $\Phi \in Th(\mathcal{L}_1)$,

constitute a functor $Th : \mathbf{Log} \to \mathbf{Tsp}$.

Indeed, it can be shown that Th is an adjoint functor, although we shall not need to use this fact here. What is important, however, is that Th does not reflect isomorphisms from **Tsp** to **Log**. Recall however that, as a simple consequence of the functoriality of Th, isomorphisms in **Log** are preserved along Th to isomorphisms in **Tsp**.

3. Isomorphisms

In this section we analyze the notions of isomorphism in the categories **Log** and **Tsp**. These would certainly be the first obvious ways of measuring the degree of "sameness" of two given logical systems. We will see, however, with the help of some examples, that none of these notions is fully satisfactory.

First of all let us recall that an *isomorphism* in an arbitrary category **C** is a morphism $f : C_1 \to C_2$ for which there exists a morphism $g : C_2 \to C_1$ such that $g \circ f = id_{C_1}$ and $f \circ g = id_{C_2}$. In this case, the morphism g is also an isomorphism, and is usually referred to as the *inverse* of f, and denoted by f^{-1} due to its uniqueness. Of course, it is also the case that $f = g^{-1}$. Two **C**-objects C_1 and C_2 are *isomorphic* provided that there exists an isomorphism between them.

Since it will be useful, we first present a characterization of isomorphism in the category **Sig** of signatures.

Proposition 3.1. *Two signatures Σ_1 and Σ_2 are isomorphic if and only if there exists a family of bijections $h = \{h^n : \Sigma_1^n \to \Sigma_2^n\}_{n\in\mathbb{N}}$.*

Proof. Suppose first that Σ_1 and Σ_2 are isomorphic in **Sig**. Then there exist signature morphisms $f : \Sigma_1 \to \Sigma_2$ and $g : \Sigma_2 \to \Sigma_1$ such that $g \circ f = id_{\Sigma_1}$ and $f \circ g = id_{\Sigma_2}$. Given $n \in \mathbb{N}$ and $c \in \Sigma_1^n$, let us first prove that $f^n(c) = \lambda\xi_1 \dots \xi_n.c'(\xi_1, \dots, \xi_n)$, where $c' \in \Sigma_2^n$. Assume by absurd that $f^n(c) = \lambda\xi_1 \dots \xi_n.\varphi$ where φ had more than one constructor. Then, clearly, by the uniformity condition, $g(f^n(c))$ would also have more than one connective, and so, $g(f^n(c))$ could not be $\lambda\xi_1 \dots \xi_n.c(\xi_1, \dots, \xi_n)$, which would contradict the fact that $g \circ f = id_{\Sigma_1}$. Thus, we can define $h^n(c) = c'$. The fact that, for each $n \in \mathbb{N}$, the function h^n must then be a bijection follows immediately.

Suppose now that there exists a family of bijections $h = \{h^n : \Sigma_1^n \to \Sigma_2^n\}_{n\in\mathbb{N}}$. Then we can build signature morphisms $f : \Sigma_1 \to \Sigma_2$ and $g : \Sigma_2 \to \Sigma_1$ defined by $h^n(c) = \lambda\xi_1 \dots \xi_n.f^n(c)(\xi_1, \dots, \xi_n)$ for every $c \in \Sigma_1^n$, and $g^n(c') = \lambda\xi_1 \dots \xi_n.c(\xi_1, \dots, \xi_n)$ for every $c' = f^n(c) \in \Sigma_2^n$, respectively. It is straightforward that $g \circ f = id_{\Sigma_1}$ and $f \circ g = id_{\Sigma_2}$. □

It should be clear that signature isomorphisms induce bijective translations between the generated languages.

We can now characterize isomorphisms in **Log**.

Proposition 3.2. *Two logical systems $\mathcal{L}_1 = \langle\Sigma_1, \vdash_1\rangle$ and $\mathcal{L}_2 = \langle\Sigma_2, \vdash_2\rangle$ are isomorphic if and only if there exists a signature isomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h[\Phi^{\vdash_1}] = h[\Phi]^{\vdash_2}$ for every $\Phi \subseteq L_{\Sigma_1}$.*

Proof. Suppose first that $h : \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphism in **Log**. That is, there exists a **Log**-morphism $g : \mathcal{L}_2 \to \mathcal{L}_1$ such that $g \circ h = id_{\mathcal{L}_1}$ and $h \circ g = id_{\mathcal{L}_2}$. Both h and g are also signature isomorphisms $h : \Sigma_1 \to \Sigma_2$ and $g : \Sigma_2 \to \Sigma_1$ that also satisfy $h[\Phi^{\vdash_1}] \subseteq h[\Phi]^{\vdash_2}$ and $g[\Gamma^{\vdash_2}] \subseteq g[\Gamma]^{\vdash_1}$, for every $\Phi \subseteq L_{\Sigma_1}$ and every $\Gamma \subseteq L_{\Sigma_2}$. But then we have that $g[h[\Phi]^{\vdash_2}] \subseteq g[h[\Phi]]^{\vdash_1} = \Phi^{\vdash_1}$, and so it follows that $h[\Phi]^{\vdash_2} = h[g[h[\Phi]^{\vdash_2}]] \subseteq h[\Phi^{\vdash_1}]$, thus rendering $h[\Phi^{\vdash_1}] = h[\Phi]^{\vdash_2}$.

Suppose now that there exists a signature isomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h[\Phi^{\vdash_1}] = h[\Phi]^{\vdash_2}$ for every $\Phi \subseteq L_{\Sigma_1}$. Then there exists a **Sig**-morphism $g : \Sigma_2 \to \Sigma_1$ such that $g \circ h = id_{\Sigma_1}$ and $h \circ g = id_{\Sigma_2}$. Clearly $h : \mathcal{L}_1 \to \mathcal{L}_2$ is also a **Log**-morphism, but so is $g : \mathcal{L}_2 \to \mathcal{L}_1$. In fact, $h[g[\Gamma]^{\vdash_1}] = h[g[\Gamma]]^{\vdash_2} = \Gamma^{\vdash_2}$, and so $g[\Gamma]^{\vdash_1} = g[h[g[\Gamma]^{\vdash_1}]] = g[\Gamma^{\vdash_2}]$. Therefore, $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic. □

This notion of isomorphism in the category **Log** is very closely related to Pollard´s notion of *homeomorphism*, as introduced in [7]. Indeed, the only difference is that homeomorphism does only require the language translation function $h : L_{\Sigma_1} \to L_{\Sigma_2}$ to be a bijection, but not necessarily one that is uniform with respect to the structure of formulas. The same applies, of course, to its inverse $h^{-1} : L_{\Sigma_2} \to L_{\Sigma_1}$. In this respect, we should make clear that we find Pollard's notion slightly odd. Either one does not require the language to bear any structure at all, in which case a simple translation function would make perfect sense, or else one should not neglect this structure when translating formulas across logical systems. Still, if we choose to simply ignore the way L_Σ is build from the signature Σ, then we can as well assume that the language is build from a new signature Ω where all the relevant formulas come now without any structure whatsoever, that is, $\Omega^0 = L_\Sigma$ and $\Omega^n = \emptyset$ for every $n > 0$. In doing this transformation we would get $L_\Omega = L_\Sigma$, and it is clear that isomorphisms in **Log** and homeomorphisms would now coincide. Nevertheless, we maintain that translations across logical systems should be effective, to a certain extent, and therefore some kind of uniformity must be required (even if in a more general form than the one we are considering here).

Isomorphisms in **Log** are, however, too strict as a definition of "sameness". Many times logical systems differ just in the number of equivalent sentences they possess, while still exhibiting essentially the same closure properties. The following example illustrates this fact, and applies also to homeomorphisms.

Example. Consider the following logical systems:

- $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ where $\Sigma_1^0 = \{\top\}$ and $\Sigma_1^n = \emptyset$ for $n > 0$; and $\vdash_1$ is such that $\emptyset^{\vdash_1} = \{\top\}$; and
- $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ where $\Sigma_2^0 = \{\top_1, \top_2\}$ and $\Sigma_2^n = \emptyset$ for $n > 0$; and $\vdash_2$ is such that $\emptyset^{\vdash_2} = \{\top_1, \top_2\}$.

Clearly, $\mathcal{L}_1$ and $\mathcal{L}_2$ have exactly the same expressive power and there is absolutely no reason why they should not be considered the "same". But it is also clear that they are not isomorphic, nor homeomorphic.

Even if the set of formulas of each logical system is not finite, isomorphism remains a too strong condition. We present one example where the two given logical systems should clearly be the "same", but there cannot exist an isomorphism between them. They are homeomorphic, though, although in a non-uniform way (even if we allow a more general definition of uniformness).

Example. Consider the following logical systems:

- $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ where $\Sigma_1^0 = \{p\}$, $\Sigma_1^1 = \{\neg\}$ and $\Sigma_1^n = \emptyset$ for $n > 1$; and $\vdash_1$ is such that $\neg$ behaves like classical negation; and
- $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ where $\Sigma_2^0 = \{p\}$, $\Sigma_2^1 = \{\neg_1, \neg_2\}$ and $\Sigma_2^n = \emptyset$ if $n > 1$; and $\vdash_2$ is such that both $\neg_1$ and $\neg_2$ behave like classical negation.

Clearly these two logics must be the "same", since $\mathcal{L}_2$ is just $\mathcal{L}_1$ with two copies of $\neg$. However, it is obvious that they are not isomorphic.

Despite the existence of a homeomorphism between $\mathcal{L}_1$ and $\mathcal{L}_2$, it cannot be made uniform (in our sense nor in any broader sense). It is clear that $Th(\mathcal{L}_1) = \{\{\neg^{2n}p : n \in \mathbb{N}\}, \{\neg^{2n+1}p : n \in \mathbb{N}\}\}$, and that $Th(\mathcal{L}_2) = \{\{\{\neg_1, \neg_2\}^{2n}p : n \in \mathbb{N}\}, \{\{\neg_1, \neg_2\}^{2n+1}p : n \in \mathbb{N}\}\}$. Assume by absurd that there existed a uniform homeomorphism $h : \mathcal{L}_2 \to \mathcal{L}_1$. Being uniform, h would have to be presented inductively. So, for some $i \in \mathbb{N}$, there would exist distinct sequences $u, v \in \{\neg_1, \neg_2\}^i$ such that $h(u\varphi) = \neg^k h(\varphi)$ and $h(v\varphi) = \neg^j h(\varphi)$, for some $k, j \in \mathbb{N}$ and every formula φ. Then, $h(uv\varphi) = \neg^k h(v\varphi) = \neg^{k+j}h(\varphi)$ and $h(vu\varphi) = \neg^j h(u\varphi) = \neg^{j+k}h(\varphi)$. So we would have $h(uv\varphi) = h(vu\varphi)$ which, together with the bijectivity of h, would contradict the fact that $uv\varphi \neq vu\varphi$.

In both these examples, the one fact that stands out in support of the "sameness" of the logical systems involved is the fact that their theory spaces have exactly the same structure. This fact should certainly be, at least, a necessary condition for considering the logical systems to be the "same". Of course, an isomorphism in **Log** is always mapped by the functor Th to an isomorphism in **Tsp**. Let us then present a characterization of isomorphisms in **Tsp**, which should be interesting to analyze, even if we have reasons to believe that having isomorphic theory spaces may not be enough as a criterion for dubbing two logical systems the "same".

Proposition 3.3. *Two theory spaces $tsp_1 = \langle Th_1, \leq_1\rangle$ and $tsp_2 = \langle Th_2, \leq_2\rangle$ are isomorphic if and only if there exists a bijection $h : Th_1 \to Th_2$ such that, for every $\Phi, \Gamma \in Th_1$, $\Phi \leq_1 \Gamma$ if and only if $h(\Phi) \leq_2 h(\Gamma)$.*

Proof. Suppose that $h : tsp_1 \to tsp_2$ is an isomorphism in **Tsp**, that is, there exists a **Tsp**-morphism $g : tsp_2 \to tsp_1$ such that $g \circ h = id_{tsp_1}$ and $h \circ g = id_{tsp_2}$. Clearly $h : Th_1 \to Th_2$ must be a bijection and $g = h^{-1}$. If $\Phi \leq_1 \Gamma$, since h is a **Tsp**-morphism, lemma 2.9 implies that $h(\Phi) \leq_2 h(\Gamma)$. On the other hand, since g is also a **Tsp**-morphism, if $h(\Phi) \leq_2 h(\Gamma)$, lemma 2.9 implies that $\Phi = h^{-1}(h(\Phi)) \leq_1 h^{-1}(h(\Gamma)) = \Gamma$.

Assume now that $h : Th_1 \to Th_2$ is a bijection, and $\Phi \leq_1 \Gamma$ if and only if $h(\Phi) \leq_2 h(\Gamma)$, for every $\Phi, \Gamma \in Th_1$. We first show that $h : tsp_1 \to tsp_2$ is a **Tsp**-morphism. If $T \subseteq Th_1$ and $\Phi \in T$, then $\Phi \leq_1 \bigvee_1 T$. Therefore $h(\Phi) \leq_2 h(\bigvee_1 T)$, and consequently $\bigvee_2 h[T] \leq_2 h(\bigvee_1 T)$. On the other hand, clearly $h(\Phi) \leq_2 \bigvee_2 h[T]$. Since h is a bijection, we can rewrite this to $h(\Phi) \leq_2 h(h^{-1}(\bigvee_2 h[T]))$. Therefore, $\Phi \leq_1 h^{-1}(\bigvee_2 h[T])$ and we conclude that $\bigvee_1 T \leq_1 h^{-1}(\bigvee_2 h[T])$. Thus $h(\bigvee_1 T) \leq_2 \bigvee_2 h[T]$ and h is indeed a **Tsp**-morphism. Analogously, we can show that $h^{-1} : tsp_2 \to tsp_1$ is also a **Tsp**-morphism. □

Saying that two logical systems, $\mathcal{L}_1$ and $\mathcal{L}_2$, are the "same" if $tsp_{\mathcal{L}_1}$ and $tsp_{\mathcal{L}_2}$ are isomorphic in **Tsp** is expectedly not very satisfactory. Namely, this is due to the fact that theory space morphisms are not guided by syntax, that is, they neglect the expressive power of isolated formulas, by translating directly theories to theories. Although a necessary condition, too many logical systems that we would not want to consider the "same" end up having isomorphic theory spaces. Below, we present two such examples.

Example. Consider the following logical systems:

- $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ where $\Sigma_1^0 = \{a, b\}$ and $\Sigma_1^n = \emptyset$ for $n > 0$; and $\vdash_1$ is the identity; and
- $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ where $\Sigma_2^0 = \{a, b, ab\}$ and $\Sigma_2^n = \emptyset$ for $n > 0$; and $\vdash_2$ extends $\vdash_1$ by letting $ab \in \Phi^{\vdash_2}$ if and only if $\{a, b\} \subseteq \Phi^{\vdash_2}$.

Clearly, $Th(\mathcal{L}_1) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $Th(\mathcal{L}_2) = \{\emptyset, \{a\}, \{b\}, \{a, b, ab\}\}$ are isomorphic. However, $\mathcal{L}_1$ and $\mathcal{L}_2$ should not be considered the "same". In particular, $\mathcal{L}_2$ contains the formula ab that can be seen as a bottom particle since $\{ab\}^{\vdash_2} = L_{\Sigma_2}$. It is also clear that, in $\mathcal{L}_1$, no such formula exists.

A more interesting example is that of linear temporal logic.

Example. Let P be a set of propositional variables. Consider the following two fragments of discrete linear temporal logic (LTL), eg. as in [5], where X stands for "in the next instant" and G for "always in the future":

- $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ where $\Sigma_1^0 = P$, $\Sigma_1^1 = \{\neg, X\}$, $\Sigma_1^2 = \{\Rightarrow\}$, and $\Sigma_1^n = \emptyset$ for $n > 2$; and
- $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ where $\Sigma_2^0 = P$, $\Sigma_2^1 = \{\neg, X, G\}$, $\Sigma_2^2 = \{\Rightarrow\}$, and $\Sigma_2^n = \emptyset$ for $n > 2$; and

both $\vdash_1$ and $\vdash_2$ are the corresponding fragments of the consequence operator $\vdash$ of full LTL.

Most notably, it turns out that $\{G\varphi\}^{\vdash} = \{X^n\varphi : n > 0\}^{\vdash}$, which is of course also true in $\mathcal{L}_2$. Therefore, it is straightforward to verify that $tsp_{\mathcal{L}_1}$ and $tsp_{\mathcal{L}_2}$ are isomorphic. However, in $\mathcal{L}_1$ there is no single formula with the same expressive power of $G\varphi$. This is certainly a very good reason not to dub these two logical systems the "same".

4. Equipollence

At this point, it seems clear that the notion of "sameness" between logical systems must lie somewhere in between the notions of isomorphism in **Log** and **Tsp**. For the reasons already discussed, two logical systems should indeed have isomorphic theory spaces whenever they are to be called the "same". This isomorphism should however be based on a formula by formula translation, which must also be uniform on the structure of formulas. Isomorphisms in **Log** do satisfy this constraint, although they seem to be too sensitive to differences in the cardinality of the sets of formulas and constructors of the two logical systems. In the end, what we seem to need is a back and forth translation between the logical systems, that may perhaps not constitute an isomorphism in **Log**, but which induces an isomorphism between the corresponding theory spaces. Figure 1 depicts the idea behind our notion of *equipollence*, that is rigorously formulated below.

Definition 4.1. Two logical systems $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ are *equipollent* if there exist **Log**-morphisms $h : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_1$ such that $Th(h)$ and $Th(g)$ establish an isomorphism of $tsp_{\mathcal{L}_1}$ and $tsp_{\mathcal{L}_2}$ with $Th(h) = Th(g)^{-1}$.

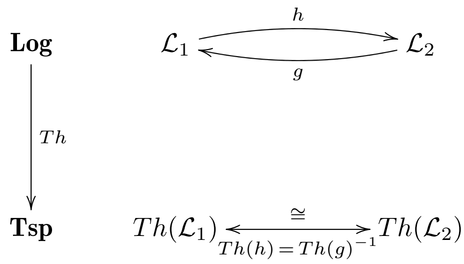

FIGURE 1. Equipollent logical systems.

As we intended, it is trivial to check that isomorphisms in **Log** constitute a very special case of equipollence. Furthermore, as expected, equipollent logical systems are always required to have isomorphic theory spaces.

We shall now provide an alternative, more appealing, characterization of equipollence in terms of the internal notion of logical equivalence provided by each logical system $\mathcal{L} = \langle \Sigma, \vdash \rangle$. Recall that two formulas $\varphi, \gamma \in L_\Sigma$ are said to be *logically equivalent* in $\mathcal{L}$ if both $\varphi \in \{\gamma\}^\vdash$ and $\gamma \in \{\varphi\}^\vdash$, or equivalently if $\{\varphi\}^\vdash = \{\gamma\}^\vdash$. We denote this fact by $\varphi \equiv_\mathcal{L} \gamma$. The following lemma shows that the theories of $\mathcal{L}$ are in fact independent, modulo logically equivalent formulas, of the way they are presented.

Lemma 4.2. *Let $\Phi, \Gamma \subseteq L_\Sigma$. Then $\Phi^\vdash = \Gamma^\vdash$ whenever the following two conditions are satisfied:*

- *for every $\varphi \in \Phi$ there exists $\varphi' \in \Gamma$ such that $\varphi \equiv_\mathcal{L} \varphi'$;*
- *for every $\gamma \in \Gamma$ there exists $\gamma' \in \Phi$ such that $\gamma \equiv_\mathcal{L} \gamma'$.*

Proof. Let us assume that both conditions hold. If $\varphi \in \Phi$ then $\varphi' \in \Gamma$. But $\varphi \equiv_\mathcal{L} \varphi'$ and therefore $\varphi \in \{\varphi'\}^\vdash \subseteq \Gamma^\vdash$, using also the monotonicity of $\vdash$. Analogously, we can show that $\Gamma \subseteq \Phi^\vdash$, and conclude that $\Phi^\vdash = \Gamma^\vdash$ by using the monotonicity and idempotence of $\vdash$. □

Next we state the aimed alternative characterization of the notion of equipollence. Besides its simplicity, the characterization will also be useful in order to compare equipollence with the notion of translational equivalence (or synonymity) due to Pelletier and Urquhart [6].

Proposition 4.3. *Let $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ be logical systems. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ are equipollent if and only if there exist* **Log***-morphisms $h : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_1$ such that the following two conditions hold:*

- *$\varphi \equiv_{\mathcal{L}_1} g(h(\varphi))$ for every $\varphi \in L_{\Sigma_1}$;*
- *$\gamma \equiv_{\mathcal{L}_2} h(g(\gamma))$ for every $\gamma \in L_{\Sigma_2}$.*

Proof. Assuming that $\mathcal{L}_1$ and $\mathcal{L}_2$ are equipollent, let $h : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_1$ be **Log**-morphisms such that $Th(h)$ and $Th(g)$ are isomorphisms, inverse of each other. Hence, given $\varphi \in L_{\Sigma_1}$, it must be the case that $\{\varphi\}^{\vdash_1} =$

$Th(g)(Th(h)(\{\varphi\}^{\vdash_1}))$. However, $Th(h)(\{\varphi\}^{\vdash_1}) = h[\{\varphi\}^{\vdash_1}]^{\vdash_2} = \{h(\varphi)\}^{\vdash_2}$, just using lemma 2.6. Similarly, $Th(g)(\{h(\varphi)\}^{\vdash_2}) = g[\{h(\varphi)\}^{\vdash_2}]^{\vdash_1} = \{g(h(\varphi))\}^{\vdash_1}$, thus implying that $\{\varphi\}^{\vdash_1} = \{g(h(\varphi))\}^{\vdash_1}$, or equivalently that $\varphi \equiv_{\mathcal{L}_1} g(h(\varphi))$. Analogously, we can prove that $\gamma \equiv_{\mathcal{L}_2} h(g(\gamma))$ for every $\gamma \in L_{\Sigma_2}$.

Assume now that $h : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_1$ are **Log**-morphisms satisfying the two conditions stated above. If we have that $\Phi \in Th(\mathcal{L}_1)$, then $Th(g)(Th(h)(\Phi)) = Th(g \circ h)(\Phi) = g[h[\Phi]]^{\vdash_1}$. Hence, $\Phi = \Phi^{\vdash_1}$ and $g[h[\Phi]]^{\vdash_1}$ are in the conditions of proposition 4.2 and we can conclude that $g[h[\Phi]]^{\vdash_1} = \Phi$. Using an analogous argument, we can conclude that also $Th(h)(Th(g)(\Gamma)) = \Gamma$ for every $\Gamma \in Th(\mathcal{L}_2)$. □

Let us illustrate the notion of equipollence with a meaningful example.

Example. Let P be a set of propositional variables. Consider the following two fragments of CPL:

- $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ where $\Sigma_1^0 = P$, $\Sigma_1^1 = \{\neg\}$, $\Sigma_1^2 = \{\Rightarrow\}$ and $\Sigma_1^n = \emptyset$ for $n > 2$; and
- $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ where $\Sigma_2^0 = P$, $\Sigma_2^1 = \{\neg\}$, $\Sigma_2^2 = \{\vee, \wedge\}$ and $\Sigma_1^n = \emptyset$ for $n > 2$.

Clearly we can define the following **Log**-morphisms $h : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_1$:

- $h^0(p) = p$ for every $p \in P$, $h^1(\neg) = \lambda\xi_1.\neg\xi_1$, and $h^2(\Rightarrow) = \lambda\xi_1\xi_2.(\neg\,\xi_1) \vee \xi_2$; and
- $g^0(p) = p$ for every $p \in P$, $g^1(\neg) = \lambda\xi_1.\neg\xi_1$, $g^2(\vee) = \lambda\xi_1\xi_2.((\neg\xi_1) \Rightarrow \xi_2)$, and $g^2(\wedge) = \lambda\xi_1\xi_2.\neg(\xi_1 \Rightarrow (\neg\xi_2))$.

Using proposition 4.3, it is an immediate consequence of well known facts about CPL that the logical systems $\mathcal{L}_1$ and $\mathcal{L}_2$ are equipollent. It is also easy to see that $\mathcal{L}_1$ and $\mathcal{L}_2$ are not isomorphic.

In [6], Pelletier and Urquhart have proposed to capture "sameness" of logical systems using a notion of translational equivalence, that turns out to be very closely related to our notion of equipollence. Indeed, translational equivalence is stated exactly as our alternative characterization of equipollence in proposition 4.3, but using a biconditional connective instead of logical equivalence. According to [6], in order to be translationally equivalent, the two logical systems must both be equivalential in the sense of [3], and with respect to precisely the same biconditional connective, which must therefore be expressible in both. Pelletier and Urquhart also show that this notion of translational equivalence turns out to be equivalent to requiring that both logical systems share a common definitional extension, a notion to which they call synonymity, and which has been considered before by several other logicians (cf. [6]).

One immediate observation that we can make is that equipollence is certainly more widely applicable than translational equivalence, since the logical systems at hand are not required to be equivalential, even less with the same biconditional. Still, equipollence and translational equivalence will obviously coincide if the two logical systems at hand are equivalential with a shared biconditional, that furthermore satisfies the deduction theorem in both. If that is not the case, then, translational equivalence always implies equipollence but nothing can be said about the converse.

The conditions underlying translational equivalence seem therefore very restrictive, since many logical systems fail to be equivalential. The next example illustrates equipollence between two logical systems where no reasonable biconditional connective can even be defined.

Example. Let P be a set of propositional variables. Consider the following two logical systems:

- $\mathcal{L}_1 = \langle \Sigma_1, \vdash_1 \rangle$ where $\Sigma_1^0 = P$, $\Sigma_1^1 = \emptyset$, $\Sigma_1^2 = \{\vee\}$ and $\Sigma_1^n = \emptyset$ for $n > 2$; and $\vdash_1$ is the corresponding restriction of the CPL consequence operator; and
- $\mathcal{L}_2 = \langle \Sigma_2, \vdash_2 \rangle$ where $\Sigma_2^0 = P$, $\Sigma_2^1 = \Sigma_2^2 = \emptyset$, $\Sigma_2^3 = \{\mathbb{V}\}$ and $\Sigma_1^n = \emptyset$ for $n > 3$; and $\vdash_2$ behaves classically with respect to $\mathbb{V}$, understood as ternary disjunction.

Clearly we can define the following **Log**-morphisms $h : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_1$:

- $h^0(p) = p$ for every $p \in P$, and $h^2(\vee) = \lambda \xi_1 \xi_2 . \mathbb{V}(\xi_1, \xi_2, \xi_2)$; and
- $g^0(p) = p$ for every $p \in P$, and $g^3(\mathbb{V}) = \lambda \xi_1 \xi_2 \xi_3 . (\xi_1 \vee \xi_2) \vee \xi_3$.

Using proposition 4.3, it is an immediate consequence of well known facts about CPL that the logical systems $\mathcal{L}_1$ and $\mathcal{L}_2$ are equipollent.

5. Conclusion

In this paper we have discussed the notion of "sameness" between logics. By adopting a categorial approach to the problem, and keeping an eye on previous proposals [6, 7, 1], we end up proposing the definition of equipollence. Two logical systems are equipollent whenever there exist uniform translations between the two logical languages that induce an isomorphism on the corresponding theory spaces. Several examples of equipollence and non-equipollence are presented along with the exposition.

We have shown, and illustrated with examples, that, as a notion of "sameness", equipollence is more accurate than Pollard's notion of homeomorphism [7]. Indeed, contrarily to our proposal, homeomorphisms are not even required to preserve the structure of formulas. Moreover, even if we ignore this fact, homeomorphisms (just like logical system isomorphisms) are too sensitive to cardinality issues, and end up distinguishing logical systems that are equipollent, and should in our opinion be considered the "same".

Equipollence is also comparable, with advantage, to Pelletier and Urquhart's notion of translational equivalence (or synonymity) [6]. Although very similar in spirit, equipollence is first of all much more widely applicable than translational equivalence, since the logical systems at hand are not bound to being equivalential, even less regarding the same shared biconditional connective. Still, once these very strong conditions are fulfilled, the two notions simply coincide if we further require the deduction theorem to hold in both systems. Nevertheless, there is a gap in this relationship that we have not been able to fill in. Under the conditions for the applicability of the notion of translational equivalence, if the deduction theorem fails in some of the logical systems, it is still the case that translational equivalence implies equipollence, as an immediate consequence of detachment. The converse, however, may not hold, but we were unable to find any meaningful example where two logical systems would, under these conditions, be equipollent but fail to be translational equivalent.

We conclude with a remark on the notion of postmodern equivalence put forth by Béziau, de Freitas and Viana in [1], which is indeed too lax as a proposal to capture the "sameness" of logical systems. As suspected by its very authors, postmodern equivalence indeed seems to mix "bananas with tomatoes", simply because it tries to solve "too many problems". Perhaps with the exception of any postmodernist joker, there is certainly no one willing to defend that propositional and first-order classical logic are the "same".

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