Cryptofibring*

(extended abstract)

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1 Introduction

Fibring [3, 5, 6] is recognized as one of the main mechanisms for combining logics, namely because of the general preservation results that have been established for different metatheoretic properties, eg. completeness [8, 1]. However, fibring suffers from an anomaly usually known as "the collapsing problem" [2, 4]. Indeed, ever since the first accounts of fibring, it could be noticed that fibring the semantics of classical with intuitionistic logic would collapse into just classical logic. In [7], modulated fibring has been introduced and shown to avoid these collapses, by means of a very careful use of adjunctions between lattice structured models. Cryptofibring is a new structurally simpler alternative to solve the semantic collapse problem, by adopting a generalization of fibred semantics using cryptomorphisms. In particular, cryptofibring encompasses the original definition of fibred model, while admiting also amalgamated models that can be used to show that the above mentioned collapses are no longer present. In this presentation we focus only on propositional based logics, but cryptofibring can be smoothly generalized to cover a wider universe of logics.

2 Cryptofibred semantics

A *signature* is an $\mathbb{N}$-indexed family C. The elements of each C_k are known as *constructors* or *connectives* of arity k. Given a signature C the generated set of formulae is the carrier $L(C)$ of the free C-algebra. A *signature morphism* $g : C \to C'$ is an $\mathbb{N}$-indexed family of maps where each $g_k : C_k \to C'_k$.

The *denotation* $[\![\varphi]\!]_{\mathbf{A}}$ of $\varphi \in L(C)$ in a given a C-algebra $\mathbf{A} = \langle A, \cdot_{\mathbf{A}} \rangle$ is inductively defined, as usual, by $[\![c(\varphi_1, \ldots, \varphi_k)]\!]_{\mathbf{A}} = c_{\mathbf{A}}([\![\varphi_1]\!]_{\mathbf{A}}, \ldots, [\![\varphi_k]\!]_{\mathbf{A}})$. A C-*structure* is a pair $\mathbb{A} = \langle \mathbf{A}, T_{\mathbb{A}} \rangle$ where $\mathbf{A} = \langle A, \cdot_{\mathbf{A}} \rangle$ is a C-algebra and

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$T_{\mathbb{A}} \subseteq A$. The elements of A are called *truth-values* and those in $T_{\mathbb{A}}$ are known as *designated* truth-values. In the sequel, we write $[\![\varphi]\!]_{\mathbb{A}}$ for the denotation of φ in the underlying algebra. We denote the class of all C-structures by $Str(C)$. Given a signature morphism $g : C \to C'$ and a C'-structure $\mathbb{A}' = \langle \mathbf{A}', T_{\mathbb{A}'} \rangle$, the *reduct* of $\mathbb{A}'$ by g is the C-structure $\mathbb{A}'|_g = \langle \mathbf{A}'|_g, T_{\mathbb{A}'} \rangle$ where $\mathbf{A}'|_g = \langle A', \cdot_{\mathbf{A}'} \circ g \rangle$. Observe that $[\![\varphi]\!]_{\mathbb{A}'|_g} = [\![g(\varphi)]\!]_{\mathbb{A}'}$ for any $\varphi \in L(C)$. Given two C-structures $\mathbb{A}$ and $\mathbb{B}$, a *C-homomorphism* $h : \mathbb{A} \to \mathbb{B}$ is a C-algebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ such that $T_{\mathbb{A}} = h^{-1}(T_{\mathbb{B}})$. Given a signature morphism $g : C \to C'$, a C-structure $\mathbb{A}$ and a C'-structure $\mathbb{A}'$, a *g-cryptomorphism* $h : \mathbb{A} \to \mathbb{A}'$ is a C-homomorphism $h : \mathbb{A} \to \mathbb{A}'|_g$.

An *interpretation system* is a tuple $\mathcal{I} = \langle C, \mathcal{M}, \alpha \rangle$ where C is a signature, $\mathcal{M}$ is a class and $\alpha : \mathcal{M} \to Str(C)$. The elements of $\mathcal{M}$ are known as *models*. In the sequel, we write $\mathbb{A}_m = \langle \mathbf{A}_m, T_m \rangle$ for $\alpha(m)$, $\cdot_m$ for $\cdot_{\mathbf{A}_m}$, and $[\![\varphi]\!]_m$ for $[\![\varphi]\!]_{\alpha(m)}$. Given $\Psi \subseteq L(C)$ and $\varphi \in L(C)$, we say that Ψ *entails* φ in $\mathcal{I}$, written $\Psi \vDash_{\mathcal{I}} \varphi$, if, for every $m \in \mathcal{M}$, $[\![\varphi]\!]_m \in T_m$ whenever $[\![\Psi]\!]_m \subseteq T_m$.

Let $\mathcal{I} = \langle C, \mathcal{M}, \alpha \rangle$ and $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ be interpretation systems. An *interpretation system cryptomorphism* $f : \mathcal{I} \to \mathcal{I}'$ is a triple $f = \langle g, \mu, h \rangle$ where:

- $g : C \to C'$ is a signature morphism;
- $\mu : \mathcal{M}' \to \mathcal{M}$ is a map;
- $h = \{h_{m'}\}_{m' \in \mathcal{M}'}$ with each $h_{m'} : \mathbb{A}_{\mu(m')} \to \mathbb{A}_{m'}$ being a g-cryptomorphism.

Interpretation systems and cryptomorphisms constitute a category **cInt**.

We assume given two interpretation systems $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$. Furthermore, we denote by C_0 the signature $C' \cap C''$ and we consider the interpretation system $\mathcal{I}_0 = \langle C_0, Str(C_0), Id \rangle$ endowed with the obvious cryptomorphisms $i'_0 : \mathcal{I}_0 \to \mathcal{I}'$ and $i''_0 : \mathcal{I}_0 \to \mathcal{I}''$. Then, the *cryptofibring* of $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$ *constrained by sharing* C_0, is the interpretation system $\mathcal{I}' \circledast \mathcal{I}'' = \langle C' \cup C'', \mathcal{M}' \circledast \mathcal{M}'', \alpha_{\circledast} \rangle$ where:

- $C' \cup C''$ is the union of C' and C'', and $i' : C' \to C' \cup C''$ and $i'' : C'' \to C' \cup C''$ are the obvious inclusion morphisms;
- $\mathcal{M}' \circledast \mathcal{M}''$ is the class of tuples $\langle \mathbb{A}, m', m'', h', h'' \rangle$ such that:
  - $\mathbb{A} \in Str(C' \cup C'')$;
  - $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$;
  - $h' : \mathbb{A}_{m'} \to \mathbb{A}$ is a i'-cryptomorphism;
  - $h'' : \mathbb{A}_{m''} \to \mathbb{A}$ is a i''-cryptomorphism;
- $\alpha_{\circledast}(\langle \mathbb{A}, m', m'', h', h'' \rangle) = \mathbb{A}$.

When C_0 is the empty signature we say that the cryptofibring is *unconstrained*. Observe that $\mathcal{I}' \circledast \mathcal{I}''$ includes those structures that appear in the original notion of fibring: the class of all structures $\mathbb{A}$ such that (i) $\mathbb{A}|_{i'} \in \alpha'(\mathcal{M}')$ and

(ii) $\mathbb{A}|_{i''} \in \alpha''(\mathcal{M}'')$. But $\mathcal{I}' \circledast \mathcal{I}''$ is much richer. It also includes amalgamated structures as described next. Given a C'-algebra $\mathbf{A}'$ and a C''-algebra $\mathbf{A}''$, their *amalgamation* is the $C' \cup C''$-algebra $\mathbf{A} = \mathbf{A}' \oplus \mathbf{A}'' = \mathcal{F}_{C' \cup C''}(A' \uplus A'')/_{\equiv}$ where $\equiv$ is the least congruence such that:

- $c'(a'_1, \ldots, a'_k) \equiv c'_{\mathbb{A}'}(a'_1, \ldots, a'_k)$ for $a'_1, \ldots, a'_k \in A'$ and $c' \in C'_k$;
- $c''(a''_1, \ldots, a''_k) \equiv c''_{\mathbb{A}''}(a''_1, \ldots, a''_k)$ for $a''_1, \ldots, a''_k \in A''$ and $c'' \in C''_k$.

This is however not a colimit in **cInt** as shown below.

In the sequel, we use the obvious amalgamation injections $j' : A' \to A$ and $j'' : A'' \to A$. Note that we can set up a flattened category **cAlg** of algebras and cryptomorphisms as follows. Each object is a pair $\langle C, \mathbf{A} \rangle$ where $\mathbf{A}$ is a C-algebra. Each morphism $\langle g, h \rangle : \langle C, \mathbf{A} \rangle \to \langle C', \mathbf{A}' \rangle$ is composed of a signature morphism $g : C \to C'$ and a C-algebra homomorphism $h : \mathbf{A} \to \mathbf{A}'|_g$. In this category, the pair $\langle C' \cup C'', \mathbf{A}' \oplus \mathbf{A}'' \rangle$ endowed with $\langle i', j' \rangle$ and $\langle i'', j'' \rangle$ is a coproduct of $\langle C', \mathbf{A}' \rangle$ and $\langle C'', \mathbf{A}'' \rangle$. Given two interpretation systems $\mathcal{I}' = \langle C', \mathcal{M}', \alpha' \rangle$ and $\mathcal{I}'' = \langle C'', \mathcal{M}'', \alpha'' \rangle$, their *amalgamation* is the interpretation system $\mathcal{I}' \oplus \mathcal{I}'' = \langle C' \cup C'', \mathcal{M}' \oplus \mathcal{M}'', \alpha_\oplus \rangle$ where:

- $\mathcal{M}' \oplus \mathcal{M}''$ is the class of tuples $\langle m', m'', T \rangle$ such that:
  - $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$;
  - T is a subset of the carrier set of $\mathbf{A}_{m'} \oplus \mathbf{A}_{m''}$ such that:
    - $j'^{-1}(T) = T_{m'}$;
    - $j''^{-1}(T) = T_{m''}$;

    where j' and j'' are the underlying amalgamation injections;
- $\alpha_\oplus(\langle m', m'', T \rangle) = \langle \mathbf{A}_{m'} \oplus \mathbf{A}_{m''}, T \rangle$.

Note that $\mathcal{M}' \oplus \mathcal{M}''$ includes, among others, the "minimal" models of the form $\langle m', m'', j'(T_{m'}) \cup j''(T_{m''}) \rangle$. It also includes models with more designated values, as long as they are chosen outside $j'(A_{m'}) \cup j''(A_{m''})$.

Proposition 2.1 $\mathcal{I}' \oplus \mathcal{I}''$ is a pushout of $\{i'_0 : \mathcal{I}_0 \to \mathcal{I}', i''_0 : \mathcal{I}_0 \to \mathcal{I}''\}$ in **cInt**.

However, cryptofibring of interpretation systems enjoys the following nice relationship with amalgamation.

Proposition 2.2 $\vDash_{\mathcal{I}' \circledast \mathcal{I}''}$ coincides with $\vDash_{\mathcal{I}' \oplus \mathcal{I}''}$.

Cryptofibred semantics, *per se*, may not be suitable as it encompasses so many models that soundness with respect to possible deduction systems associated with the interpretation systems at hand can easily be lost. Next, we adapt the construction to a suitable notion of logic system encompassing both a semantic and a deductive component, together with a soundness condition.

3 Cryptofibring

Assume given once and for all a set Ξ of *schema variables.* Given a signature C, the generated set of schema formulae is the carrier $SL(C)$ of the free C-algebra with generators Ξ. A *schema C-substitution* is a function $\sigma : \Xi \to SL(C)$. Given an schema formula δ, the *instance* of δ by the schema substitution σ is denoted by $\delta\sigma$ and is the result of simultaneously replacing each schema variable ξ in δ by $\sigma(\xi)$. A *(ground) C-substitution* is a function $\rho : \Xi \to L(C)$. The *instance* of δ by the substitution ρ is denoted by $\delta\rho$ and is the result of simultaneously replacing each schema variable ξ in δ by $\rho(\xi)$.

A *C-rule* is a pair $\langle \Upsilon, \eta \rangle$, where $\Upsilon \cup \{\eta\} \subseteq SL(C)$. A rule is said to be *finitary* when Υ is finite and is said to be *axiomatic* when Υ is empty. A *deduction system* is a pair $\mathcal{D} = \langle C, R \rangle$ where C is a signature and R is a set of finitary C-rules. The notion of proof is the usual. When there is a proof in $\mathcal{D}$ of δ from Γ, we write $\Gamma \vdash_{\mathcal{D}} \delta$. As usual we may also omit the set of premises when it is empty. Note that proofs are closed for substitutions: if $\Gamma \vdash_{\mathcal{D}} \delta$ then $\Gamma\sigma \vdash_{\mathcal{D}} \delta\sigma$, for any schema substitution σ. The *image* of a C-rule $r = \langle \Upsilon, \eta \rangle$ by a signature morphism $g : C \to C'$ is the C'-rule $g(r) = \langle g(\Upsilon), g(\eta) \rangle$. A *deduction system morphism* $g : \mathcal{D} \to \mathcal{D}'$ is a signature morphism $g : C \to C'$ such that $g(\Upsilon) \vdash_{\mathcal{D}'} g(\eta)$ for every $\langle \Upsilon, \eta \rangle \in R$.

We put the two components together into logic systems. A *logic system* is a tuple $\mathcal{L} = \langle C, R, \mathcal{M}, \alpha \rangle$ where $\mathcal{D}_{\mathcal{L}} = \langle C, R \rangle$ is a deduction system, and $\mathcal{I}_{\mathcal{L}} = \langle C, \mathcal{M}, \alpha \rangle$ is an interpretation system. In the sequel, we may write $\vdash_{\mathcal{L}}$ for $\vdash_{\mathcal{D}_{\mathcal{L}}}$ and $\vDash_{\mathcal{L}}$ for $\vDash_{\mathcal{I}_{\mathcal{L}}}$. A C-structure $\mathbb{A}$ is said to be *appropriate* for a set R of C-rules iff, for every $\langle \Upsilon, \eta \rangle \in R$ and C-substitution ρ, if $[\![\Upsilon\rho]\!]_{\mathbb{A}} \subseteq T_{\mathbb{A}}$ then $[\![\eta\rho]\!]_{\mathbb{A}} \in T_{\mathbb{A}}$. We denote by $App(R)$ the class of C-structures that are appropriate for R. A logic system $\mathcal{L} = \langle C, R, \mathcal{M}, \alpha \rangle$ is said to be:

- *sound* iff $\Psi \vDash_{\mathcal{L}} \varphi$ whenever $\Psi \vdash_{\mathcal{L}} \varphi$ for $\Psi \cup \{\varphi\} \subseteq L(C)$;
- *full* iff $\alpha(\mathcal{M}) = App(R)$;
- *complete* iff $\Psi \vdash_{\mathcal{L}} \varphi$ whenever $\Psi \vDash_{\mathcal{L}} \varphi$ for $\Psi \cup \{\varphi\} \subseteq L(C)$.

Clearly, a logic system is sound iff all its structures are appropriate for its rules. Therefore, every full logic system is sound. Furthermore, every full logic system is complete[1].

Let $\mathcal{L} = \langle C, R, \mathcal{M}, \alpha \rangle$ and $\mathcal{L}' = \langle C', R', \mathcal{M}', \alpha' \rangle$ be logic systems. A *logic system cryptomorphism* $f : \mathcal{L} \to \mathcal{L}'$ is a triple $f = \langle g, \mu, h \rangle$ such that:

1. $g : \mathcal{D}_{\mathcal{L}} \to \mathcal{D}_{\mathcal{L}'}$ is a deduction system morphism;
2. $f : \mathcal{I}_{\mathcal{L}} \to \mathcal{I}_{\mathcal{L}'}$ is an interpretation system cryptomorphism;
3. for every $m' \in \mathcal{M}'$, $\mathbb{A}_{m'} \in App(g(R))$ whenever $\mathbb{A}_{\mu(m')} \in App(R)$.

[1] Fullness underlies the completeness techniques used, for instance, in [8], which are in fact also applicable to cryptofibring since it includes all fibred models. We shall not dwell on completeness preservation by cryptofibring here, but the fact that cryptofibred semantics is richer opens the way to obtaining more general sufficient conditions for completeness preservation.

Condition 3 above is a reasonable requirement that guarantees the preservation of soundness by cryptomorphisms in the following sense: if $\mathcal{L} = \langle C, R, \mathcal{M}, \alpha \rangle$ is sound, so is $g(\mathcal{L}) = \langle C', g(R), \mathcal{M}', \alpha' \rangle$. Otherwise, valid rules when embedded in a larger language context might become unsound (see for instance [1]).

Logic systems and cryptomorphisms constitute a category **cLog**.

We assume given two deduction systems $\mathcal{D}' = \langle C', R' \rangle$ and $\mathcal{D}'' = \langle C'', R'' \rangle$, and again we denote by C_0 the subsignature $C' \cap C''$, and by $\mathcal{D}_0$ the canonical deduction system $\langle C_0, \emptyset \rangle$ endowed with the obvious morphisms $i' : \mathcal{D}_0 \to \mathcal{D}'$ and $i'' : \mathcal{D}_0 \to \mathcal{D}''$. We start by defining the cryptofibring of deduction systems: the *cryptofibring* of $\mathcal{D}'$ and $\mathcal{D}''$ constrained by sharing C_0 is the deduction system $\mathcal{D}' \circledast \mathcal{D}'' = \langle C' \cup C'', R' \cup R'' \rangle$.

We can finally define the notion of cryptofibring of logic systems. We assume given two logic systems $\mathcal{L}' = \langle C', R', \mathcal{M}', \alpha' \rangle$ and $\mathcal{L}'' = \langle C'', R'', \mathcal{M}'', \alpha'' \rangle$. The *cryptofibring* of $\mathcal{L}'$ and $\mathcal{L}''$ *constrained by sharing* C_0 is the logic system $\mathcal{L}' \circledR \mathcal{L}'' = \langle C' \cup C'', R' \cup R'', \mathcal{M}' \circledR \mathcal{M}'', \alpha_{\circledR} \rangle$ where:

- $\mathcal{M}' \circledR \mathcal{M}''$ is composed of every $\langle \mathbb{A}, m', m'', h', h'' \rangle$ in $M' \circledast M''$ such that:
  - $\mathbb{A} \in App(R' \cup R'')$ whenever $\mathbb{A}_{m'} \in App(R')$;
  - $\mathbb{A} \in App(R' \cup R'')$ whenever $\mathbb{A}_{m''} \in App(R'')$;
- $\alpha_{\circledR}(\langle \mathbb{A}, m', m'', h', h'' \rangle) = \mathbb{A}$.

Expectedly, this is not a colimit in **cLog**. As before, we define the amalgamation of logic systems. The *amalgamation* of $\mathcal{L}'$ and $\mathcal{L}''$ is $\mathcal{L}' \oplus_R \mathcal{L}'' = \langle C' \cup C'', R' \cup R'', \mathcal{M}' \oplus_R \mathcal{M}'', \alpha_{\oplus_R} \rangle$ where:

- $\mathcal{M}' \oplus_R \mathcal{M}''$ is composed of every $\langle m', m'', T \rangle$ in $\mathcal{M}' \oplus \mathcal{M}''$ such that:
  - T is closed for all ground instances of rules in $R' \cup R''$;
- $\alpha_{\oplus_R}(\langle m', m'', T \rangle) = \langle \mathbf{A}_{m'} \oplus \mathbf{A}_{m''}, T \rangle$.

Then, the following result holds.

Proposition 3.1 $\mathcal{L}' \oplus_R \mathcal{L}''$ is a pushout of $\{i'_0 \colon \mathcal{L}_0 \to \mathcal{L}', i''_0 \colon \mathcal{L}_0 \to \mathcal{L}''\}$ in **cLog**.

Fortunately, the strong relationship between the two entailments still holds.

Proposition 3.2 $\vDash_{\mathcal{L}' \circledR \mathcal{L}''}$ coincides with $\vDash_{\mathcal{L}' \oplus_R \mathcal{L}''}$.

4 Combining classical and intuitionistic logics

In this section, we use cryptofibring to combine the implicative fragments of classical propositional logic (*CPL*) and intuitionistic propositional logic (*IPL*), and show that the resulting logic does not collapse to *CPL*. To avoid any constraint in the combination we shall assume that *CPL* and *IPL* are based on disjoint denumerable sets P and Q, respectively, of propositional symbols.

The implicative fragment of *CPL* can be easily presented as a logic system with the usual semantics based on bivaluations $v : P \to \{0,1\}$ with 1 designated, and the set of rules:

- $\langle \emptyset, (\xi_1 \Rightarrow^c (\xi_2 \Rightarrow^c \xi_1))\rangle$
  $\langle \emptyset, ((\xi_1 \Rightarrow^c (\xi_2 \Rightarrow^c \xi_3)) \Rightarrow^c ((\xi_1 \Rightarrow^c \xi_2) \Rightarrow^c (\xi_1 \Rightarrow^c \xi_3)))\rangle$
  $\langle \emptyset, (((\xi_1 \Rightarrow^c \xi_2) \Rightarrow^c \xi_1) \Rightarrow^c \xi_1)\rangle$
  $\langle \{\xi_1, (\xi_1 \Rightarrow^c \xi_2)\}, \xi_2\rangle$.

The implicative fragment of *IPL* can be presented with the usual Kripke semantics and the set of rules:

- $\langle \emptyset, (\xi_1 \Rightarrow^i (\xi_2 \Rightarrow^i \xi_1))\rangle$
  $\langle \emptyset, ((\xi_1 \Rightarrow^i (\xi_2 \Rightarrow^i \xi_3)) \Rightarrow^i ((\xi_1 \Rightarrow^i \xi_2) \Rightarrow^i (\xi_1 \Rightarrow^i \xi_3)))\rangle$
  $\langle \{\xi_1, (\xi_1 \Rightarrow^i \xi_2)\}, \xi_2\rangle$.

Note that the only difference between the calculi is the absence in *IPL* of *Peirce's law*, the third rule of *CPL*.

In order to show that *CPL* ® *IPL* does not collapse into *CPL*, we have to find a model over the combined language that satisfies all the rules from both logics, together with cryptomorphisms from suitable models of each of the logics, in such a way that, for instance, the analogous of Peirce's rule for intuitionistic implication is not valid. We shall consider the model obtained by a natural extension of the usual Kripke semantics for intuitionistic logic. Given a partially-ordered Kripke frame $\langle W, \leq\rangle$, and letting $B_\leq$ be the set of all upper-subsets of W, any corresponding model $m = \langle W, \leq, V\rangle$ with $V : P \cup Q \to B_\leq$ induces an interpretation structure $\langle\langle B_\leq, \cdot_m\rangle, \{W\}\rangle$ with:

- $p_m = V(p)$ and $q_m = V(q)$;
- $\Rightarrow^c_m(b_1, b_2) = ((W \setminus b_1) \cup b_2)^c$;
- $\Rightarrow^i_m(b_1, b_2) = ((W \setminus b_1) \cup b_2)^i$,

where $X^c = \{w \in W : \text{there exists } x \in X \text{ such that } x \leq w\}$ and $X^i = \{w \in W : \{w' : w \leq w'\} \subseteq X\}$, given $X \subseteq W$. In the particular case when $W = \{u, v, w\}$ and $u \leq v$, only, and V is such that $V(p)$ is either $\emptyset$ or W, and $V(q_1) = \{v\}$, $V(q_2) = \emptyset$, the structure we obtain satisfies all the rules but $(((q_1 \Rightarrow_i q_2) \Rightarrow_i q_1) \Rightarrow_i q_1)$ does not hold. Indeed, $[\![(((q_1 \Rightarrow_i q_2) \Rightarrow_i q_1) \Rightarrow_i q_1)]\!]_m = \{v, w\}$. However, the reduct of this structure to $\Rightarrow_i$ and Q is a structure of *IPL* with the obvious cryptomorphism. Moreover, it is also possible to map the *CPL* structure based on the bivaluation $v : P \to \{0,1\}$ such that $v(p) = 0$ if $V(p) = \emptyset$ and $v(p) = 1$ if $V(p) = W$, along the cryptomorphism that sends 0 to $\emptyset$ and 1 to W.

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