Cryptomorphisms at Work*

Carlos Caleiro and Jaime Ramos

CLC - Departamento de Matemática, IST - UTL
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Abstract. We show that the category proposed in [5] of logic system presentations equipped with *cryptomorphisms* gives rise to a category of parchments that is both complete and translatable to the category of institutions, improving on previous work [15]. We argue that limits in this category of parchments constitute a very powerful mechanism for combining logics.

1 Introduction

The importance of studying combined logics and, specially, general mechanisms for combining logics is widely recognized [1]. This happens not only because of the theoretical interest and technical difficulties of the subject, but also for practical reasons. In many fields, the need for working with several logics at the same time is the rule rather than the exception. Among the various approaches to the combination of logics, two deserve our close attention. One has been developed within the general theory of institutions [12, 18], and focuses on the categorial combination of parchments [11, 13–15]. Another, very successful, approach is fibring [8, 7, 9, 16, 21, 2, 17]. The two approaches have also met each other in [4, 3], where some of the very general preservation results already identified for fibring have been brought to the level of parchments.

In [5], cryptofibring was proposed as an extension of fibring and shown to keep its general soundness and completeness preservation while also attacking the so-called "collapsing problem" [8, 6]. In [17] another variant of fibring, modulated fibring, has been introduced and shown to avoid these collapses by means of a very careful use of adjunctions between lattice structured models. However, cryptofibring presents a structurally simpler independent solution to the problem, interesting in its own right, and that encompasses the original definition of fibred model but also admits amalgamated models that can be used to show that the above mentioned collapses are no longer present. Cryptofibring is characterized categorially as a special kind of pushout in a suitable category of logic system presentations. Its objects are simple algebraic presentations of both the syntax and semantics of logic systems. The main novelty concerns its morphisms, that have been called *cryptomorphisms*, from where cryptofibring borrows its name.

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It turns out that this category is, modulo presentation details, precisely half-way between the categories of *rooms* used in [15] to build the categories of *model-theoretic* parchments and its *logical* large subcategory. The main aim of its authors was to obtain a framework for combining logics using limits of parchments, and a smooth way of presenting them as institutions [12], following earlier work [11, 13, 14]. However, they could prove that model-theoretic parchments form a complete category but fail in general to present institutions, whereas logical model-theoretic parchments present institutions in a smooth way but do not constitute a complete category. The counterexample used in [13–15], with minor variations, is the nonexistence of a certain limit combining total equational logic and partiality, whose intended result should help to grasp the meaning of equations involving undefined terms.

Our goal in this paper is to show that cryptomorphisms really work. Not only they extend fibring as is already known from [5], but they also give rise to a category of parchments which is simultaneously complete and easily translatable to institutions. Indeed, both these properties follow directly from properties of cryptomorphisms: they always fulfill the necessary *satisfaction condition*, and they constitute a cocomplete category of logic system presentations. Parchments based on cryptomorphisms are therefore an extremely powerful tool for combining logics. As an application of cryptomorphism, we shall revisit the partial equational logic example and show that the corresponding colimit encompasses models that are compatible with each possible interpretation of equality involving undefinedness, be it strong, weak, existential, three-valued, or even other. We proceed as follows. In Section 2 we introduce the category of logic system presentations with cryptomorphisms and explore its relationship to the categories of rooms used to build model-theoretic parchments and logical model-theoretic parchments. Then, in Section 3, we show that cryptomorphisms indeed build up a cocomplete category, and highlight a few differences with respect to the other cases. The Section 4 is devoted to presenting the details of the example and discussing its result. We conclude in Section 5 with a discussion of the results obtained and an outline of future work.

2 Cryptomorphisms

To combine logics and achieve a meaningful interplay between them we need to work with presentations that pinpoint the fine details of the logic's syntax and semantics. A usual approach, underlying the notion of parchment [11] as well as an essential dimension of fibring [16], is to consider some kind of algebraic presentation a logic. Another common feature of both approaches is to adopt a categorial setting where the combination mechanisms should be characterized as universal constructions [10]. Before we proceed to the definition of our working category of logic system presentations and *cryptomorphisms*, we start by recalling, or introducing, some notions and notation.

In the sequel, **AlgSig** is the category of many-sorted signatures, and $\mathbf{AlgSig}_\phi$ its subcategory whose signatures have a distinguished sort $\phi \in S$ (for *formulas*) and whose morphisms preserve ϕ. We denote by **Alg** the flat category of many-sorted algebras and homomorphisms, and by $\mathbf{Alg}(\Sigma)$ the category of Σ-algebras and Σ-homomorphisms, for each signature Σ. We use $\mathcal{W}_\Sigma$ to denote the free Σ-algebra (the word algebra), and $[\![\_]\!]_\mathcal{A}$ (for word *interpretation*) to denote the unique Σ-homomorphism from $\mathcal{W}_\Sigma$ to a given Σ-algebra $\mathcal{A}$. Elements of $|\mathcal{W}_\Sigma|_s$ are referred to as *terms* and denoted by t. Every **AlgSig**-morphism $\sigma : \Sigma_1 \to \Sigma_2$ has an associated reduct functor $\_|_\sigma : \mathbf{Alg}(\Sigma_2) \to \mathbf{Alg}(\Sigma_1)$. Note that $[\![t]\!]_{\mathcal{A}|_\sigma} = [\![\sigma(t)]\!]_\mathcal{A}$ for each $t \in |\mathcal{W}_{\Sigma_1}|_s$ and Σ_2-algebra $\mathcal{A}$. As usual, we overload the notation and write σ for word *translation* instead of $[\![\_]\!]_{\mathcal{W}_{\Sigma_2}|_\sigma}$ to denote the unique Σ_1-homomorphism from $\mathcal{W}_{\Sigma_1}$ to $\mathcal{W}_{\Sigma_2}|_\sigma$. If Σ has a distinguished sort ϕ, Form_Σ stands for the set $|\mathcal{W}_\Sigma|_\phi$ of *formulas*. We use φ to denote a formula.

Definition 1. A *logic system presentation* is a triple $\langle \Sigma, M, \mathbb{A}\rangle$ where $\Sigma \in |\mathbf{AlgSig}_\phi|$, M is a class (of *models*), and $\mathbb{A}$ associates to each $m \in M$ a Σ-*interpretation structure* $\mathbb{A}(m) = \langle \mathcal{A}_m, T_m\rangle$, where $\mathcal{A}_m$ is a Σ-algebra and $T_m \subseteq |\mathcal{A}_m|_\phi$ (the *designated* subset of the set of *truth-values*).

Given a model $m \in M$, we shall simply write $[\![\_]\!]_m$ instead of $[\![\_]\!]_{\mathcal{A}_m}$.

This kind of interpretation structure, featuring a set of designated truth-values, is commonly known as a *logical matrix* in the logic literature (see, for instance, [20]). We can define the usual *satisfaction* of a formula $\varphi \in \mathrm{Form}_\Sigma$ by $m \Vdash \varphi$ if $[\![\varphi]\!]_m \in T_m$.

The authors of [15] considered parchments built over a category **MPRoom** whose objects can be seen as logic system presentations, modulo presentation details that we shall ignore. They also considered a *logical* version of these parchments, which can similarly be built over a large subcategory **LogMPRoom** of **MPRoom**. We recall the precise definition of their morphisms.

Definition 2. A *morphism* $\langle\sigma,\mu,\eta\rangle : \langle \Sigma_1, M_1, \mathbb{A}_1\rangle \to \langle \Sigma_2, M_2, \mathbb{A}_2\rangle$ of logic system presentations consists of an $\mathbf{AlgSig}_\phi$-morphism $\sigma : \Sigma_1 \to \Sigma_2$, a map $\mu : M_2 \to M_1$, and a family $\eta = \{\eta_m : \mathbb{A}_1(\mu(m)) \to \mathbb{A}_2(m)\}_{m\in M_2}$ where each η_m is a Σ_1-homomorphism from $\mathcal{A}_{\mu(m)}$ to $\mathcal{A}_m|_\sigma$ that preserves designated values, that is, $\eta_m(T_{\mu(m)}) \subseteq T_m$.

A morphism is said to be *closed* if each η_m also reflects designated values, that is, $\eta_m^{-1}(T_m) = T_{\mu(m)}$ for every $m \in M_2$.

A closed morphism is said to be *logical* if η_m is injective for every $m \in M_2$.

Let $\langle\sigma,\mu,\eta\rangle : \langle \Sigma_1, M_1, \mathbb{A}_1\rangle \to \langle \Sigma_2, M_2, \mathbb{A}_2\rangle$ be a morphism, $\varphi \in \mathrm{Form}_{\Sigma_1}$ and $m \in M_2$. In general, it is clear that if $\mu(m) \Vdash_1 \varphi$ then $m \Vdash_2 \sigma(\varphi)$. However the converse does not hold, in general, since designated values are preserved but may not be reflected. For closed morphisms, however, we obtain the usual *satisfaction condition*: $\mu(m) \Vdash_1 \varphi$ if and only if $m \Vdash_2 \sigma(\varphi)$.

With the obvious definitions of identity and composition, logic system presentations and morphisms constitute the category **MPRoom**. **LogMPRoom** is the subcategory of **MPRoom** with only logical morphisms. But what is more, closed morphisms are precisely what have been called *cryptomorphisms* in [5]. Requiring the weakest possible condition that ensures the satisfaction condition was indeed the main reason for their precise formulation. In the remainder of the paper, we shall call **Crypt** to the corresponding category. It is worthwhile recalling that **Crypt** was proposed in order to characterize cryptofibring, a generalization of fibring aimed at solving an anomaly known as the "collapsing problem". Indeed, at this level of abstraction, cryptofibring can be seen to extend fibring in exactly the same proportion as the notion of cryptomorphism extends the notion of arrow between logic system presentations used to characterize fibring. For fibring, we just have $\langle \sigma, \mu \rangle : \langle \Sigma_1, M_1, \mathbb{A}_1 \rangle \to \langle \Sigma_2, M_2, \mathbb{A}_2 \rangle$ and require that $\mathcal{A}_{\mu(m)} = \mathcal{A}_m|_\sigma$ and $T_{\mu(m)} = T_m$. In this context, it is easy to understand why they were named cryptomorphisms. Each η_m is precisely a "homomorphism" between $\mathcal{A}_{\mu(m)}$ and $\mathcal{A}_m$, algebras over distinct signatures, mediated by the signature morphism σ, as in **Alg**. This kind of "homomorphism" has been called a cryptomorphism before (see, for instance, [19]).

If one wishes to work with logics at the level of institutions it is essential that one considers parchments [11]. Fixing a base category $\mathbf{B}$ as a category of *rooms*, we can build up a corresponding category of $\mathbf{B}$-*parchments*. We just define a $\mathbf{B}$-parchment to be a functor $P : \mathbf{Sig} \to \mathbf{B}$, where $\mathbf{Sig}$ is some category of abstract signatures. A *morphism of* $\mathbf{B}$-*parchments* from $P_1 : \mathbf{Sig}_1 \to \mathbf{B}$ to $P_2 : \mathbf{Sig}_2 \to \mathbf{B}$ is now just a pair $\langle \Phi, \alpha \rangle$ where $\Phi : \mathbf{Sig}_1 \to \mathbf{Sig}_2$ is a functor and $\alpha : P_2 \circ \Phi \to P_1$ is a natural transformation. Clearly, using **MPRoom** and **LogMPRoom** as a basis we obtain precisely the categories of model-theoretic parchments and logical model-theoretic parchments of [15]. These constructions mimic precisely the construction of the category of institutions and institution morphisms using (the dual of) the category of twisted relations [12] as a base. There are two very interesting features that these categories of parchments may enjoy: one is the possibility of setting up a functor to institutions, thus showing that the parchments at hand are indeed good ways of representing logics; the other is the possibility of combining logics using limits of parchments, when they exist. It is a straightforward property of the general construction of these categories of parchments that a translation to institutions can be immediately obtained from a translation of the base category considered to (the dual of) the category of twisted relations. In the case of our logic system presentations, this ammounts to choosing a notion of arrow that fulfils the satisfaction condition mentioned before. Moreover, well known results on indexed categories [19] show that this construction always yields a complete category of parchments if the base category considered is cocomplete.

Despite its problem with the satisfaction condition, **MPRoom** is a cocomplete category. However, the combinations obtained often feature combined models with a diversity of newly generated truth-values corresponding to the previously unknown result of applying an operation of one of the logics being combined to a value of another. These were called "junk" values and considered harmful in [13–15], which led its authors to restrict attention not to closed, but directly to logical morphisms. The category **LogMPRoom** is however not cocomplete (as usual, injectivity does not go along too well with coequalizers), as they have shown in a very interesting example. The category **Crypt** is already known to have some colimits, at least precisely those used for characterizing cryptofibring. Our job, in the next section, will be to show that **Crypt** is indeed cocomplete. While doing that we shall see that the previously mentioned "junk" values are an essential ingredient of the envisaged free interplay of concepts, if measured along the differences between colimits in **Crypt** and **MPRoom**, that we shall pinpoint.

3 Cocompleteness

Our task now is to prove the cocompleteness of **Crypt**, while highlighting the main differences between its colimits and those that can be obtained in **MPRoom**.

As argued in [19], many categories arising in computer science can be seen as indexed categories. Our categories of interest for now, **MPRoom** and **Crypt**, but also **LogMPRoom**, are good examples of that. We could explore this fact and the well known results about indexed categories to attack the cocompleteness of **Crypt** and compar its colimits with those of **MPRoom**. Indeed, at a first level, both can be seen as categories indexed by $\mathbf{AlgSig}_\phi$, and at a second level, the component categories of each of them can be seen as categories of interpretation structures indexed by the dual of the category of classes. However, there is a major difference between the two cases: whereas in the case of **MPRoom** each category of Σ-interpretation structures ($\mathbf{Str}(\Sigma)$ in the terminology of [15]) is also cocomplete, the relevant categories of structures in the case of **Crypt** (of course, the subcategories $\mathbf{CryptStr}(\Sigma)$ of each $\mathbf{Str}(\Sigma)$ with homomorphisms that do not only preserve but also reflect designated values) are not cocomplete. This fact implies that, at the second level, the sufficient conditions for the cocompleteness of indexed categories of [19] apply to **MPRoom** but not to **Crypt**. Still we do not start from scratch. In the sequel, we shall capitalize on two well known results (see, for instance, [19]): the cocompleteness of **Alg**, and the fact that the forgetful functor from **Alg** to **AlgSig** preserves colimits. Although we do not want to get into the fine details of the cocompleteness of **Alg** here, we shall at least have a brief look at their essential aspects by analyzing some very simple but critical examples. This exercise will not only provide further insight to the forthcoming construction of colimits in **Crypt**, but also help us in making the contrast between **Crypt** and **MPRoom**, by emphasizing the differences between $\mathbf{CryptStr}(\Sigma)$ and $\mathbf{Str}(\Sigma)$.

Colimits in **Alg** have two base pillars: colimits in each category $\mathbf{Alg}(\Sigma)$, and the existence of a left adjoint F_σ of the reduct functor $\_|_\sigma : \mathbf{Alg}(\Sigma_2) \to \mathbf{Alg}(\Sigma_1)$ associated to each signature morphism $\sigma : \Sigma_1 \to \Sigma_2$. Coproducts in $\mathbf{Alg}(\Sigma)$, for a given signature Σ, are built by taking the free Σ-algebra over the disjoint union of all the carrier sets of the different algebras, and then making its quotient under the congruence generated by the interpretation of terms in each of the algebras. We next consider two contrasting situations.

Example 1. Let $\Sigma \in |\mathbf{AlgSig}_\phi|$ be the signature with only the sort ϕ and the constant operation $c : \phi$, and consider the Σ-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively with $|\mathcal{A}|_\phi = \{*\}$ and $c_\mathcal{A} = *$, and $|\mathcal{B}|_\phi = \{0, 1\}$ and $c_\mathcal{B} = 1$. Their coproduct $\mathcal{A} \coprod \mathcal{B}$ in $\mathbf{Alg}(\Sigma)$ is (up to isomorphism) $\mathcal{B}$ itself, along with the homomorphisms $h : \mathcal{A} \to \mathcal{B}$ such that $h(*) = 1$ and $id_\mathcal{B} : \mathcal{B} \to \mathcal{B}$: just put together $*$, 0 and 1 with the only Σ-term c, and collapse $*$, 1 and c by noting that $[\![c]\!]_\mathcal{A} = *$ and $[\![c]\!]_\mathcal{B} = 1$.

Consider now the Σ-structures $\langle \mathcal{A}, \emptyset \rangle$ and $\langle \mathcal{B}, \{1\} \rangle$. In $\mathbf{Str}(\Sigma)$ there is a canonical way of endowing the coproduct $\mathcal{B}$ of the algebras with a set T of designated values that makes both homomorphisms preserve designated values, that is, $h(\emptyset) \subseteq T$ and $id_\mathcal{B}(\{1\}) \subseteq T$. Just take the minimal choice $T = \{1\}$ because any (bigger) choice will certainly preserve it. Indeed, the coproduct $\langle \mathcal{A}, \emptyset \rangle \coprod \langle \mathcal{B}, \{1\} \rangle$ in $\mathbf{Str}(\Sigma)$ is precisely $\langle \mathcal{B}, \{1\} \rangle$, along with h and $id_\mathcal{B}$.

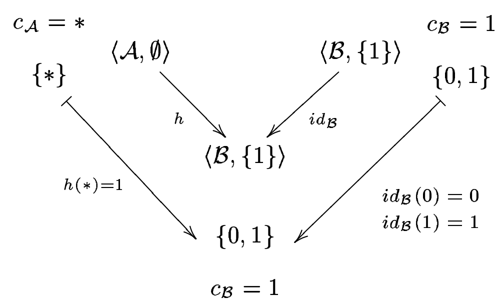

Fig. 1. Coproduct in $\mathbf{Str}(\Sigma)$.

However, while $id_\mathcal{B} : \langle \mathcal{B}, \{1\} \rangle \to \langle \mathcal{B}, \{1\} \rangle$ also reflects designated values, $h : \langle \mathcal{A}, \emptyset \rangle \to \langle \mathcal{B}, \{1\} \rangle$ does not because $*$ is not designated but $h(*) = 1$ is. Actually, there is no possible choice of T that makes both homomorphisms preserve and reflect designated values. This is the reason why a coproduct $\langle \mathcal{A}, \emptyset \rangle \coprod \langle \mathcal{B}, \{1\} \rangle$ in $\mathbf{CryptStr}(\Sigma)$ does not exist. △

Example 2. Now, let $\Sigma \in |\mathbf{AlgSig}_\phi|$ be the signature with only the sort ϕ and a binary operation $f : \phi \times \phi \to \phi$, and consider the Σ-algebras $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}|_\phi = \{0\}$ and $f_\mathcal{A}(0, 0) = 0$, and $|\mathcal{B}|_\phi = \{1\}$ and $f_\mathcal{B}(1, 1) = 1$. The coproduct $\mathcal{A} \coprod \mathcal{B}$ in $\mathbf{Alg}(\Sigma)$ is in this case (up to isomorphism) the free Σ-algebra over $\{0, 1\}$ but with $f(0, 0) \approx 0$ and $f(1, 1) \approx 1$, let us call it $\mathcal{C}$, along with the injection homomorphisms $h_\mathcal{A} : \mathcal{A} \to \mathcal{C}$ and $h_\mathcal{B} : \mathcal{B} \to \mathcal{C}$. Clearly, $|\mathcal{C}|_\phi$ is infinite and contains 0, 1, $f(0, 1)$, $f(1, 0)$, $f(0, f(0, 1))$, and so on, but not $f(0, 0)$ nor $f(1, 1)$.

Consider now the Σ-structures $\langle \mathcal{A}, \emptyset \rangle$ and $\langle \mathcal{B}, \{1\} \rangle$. In $\mathbf{Str}(\Sigma)$ there are now many ways of endowing the coproduct $\mathcal{C}$ of the algebras with a set T of designated values that makes both homomorphisms preserve designated values. Indeed any $T \subseteq |\mathcal{C}|_\phi$ such that $1 \in T$ will do. However, the minimal choice $T = \{1\}$ is canonical. Indeed, the coproduct $\langle \mathcal{A}, \emptyset \rangle \coprod \langle \mathcal{B}, \{1\} \rangle$ in $\mathbf{Str}(\Sigma)$ is precisely $\langle \mathcal{C}, \{1\} \rangle$, along with $h_\mathcal{A}$ and $h_\mathcal{B}$.

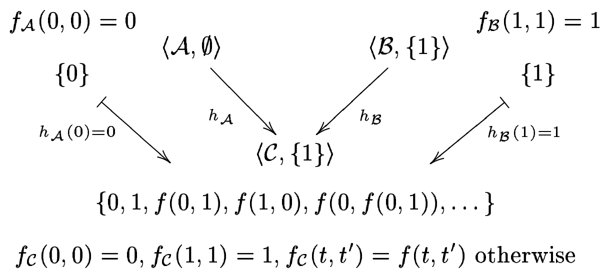

Fig. 2. Coproduct in $\mathbf{Str}(\Sigma)$.

Easily, now, both $h_\mathcal{A}$ and $h_\mathcal{B}$ also reflect designated values. However, the choice of $T = \{1\}$ is not canonical in $\mathbf{CryptStr}(\Sigma)$ because any (bigger) choice will preserve but not reflect it. In fact, none of the possible choices of T, that must not include 0, is canonical. This is the reason why a coproduct $\langle \mathcal{A}, \emptyset \rangle \coprod \langle \mathcal{B}, \{1\} \rangle$ in $\mathbf{CryptStr}(\Sigma)$ does not exist. $\triangle$

It turns out that coequalizers do not raise these problems. They always exist in $\mathbf{CryptStr}(\Sigma)$ and coincide with those obtained in $\mathbf{Str}(\Sigma)$, by capitalizing on coequalizers in $\mathbf{Alg}(\Sigma)$ which are simple quotients. Absolutely similar situations reappear, however, if we consider the left adjoint F_σ of the reduct functor $\_|_\sigma : \mathbf{Alg}(\Sigma_2) \to \mathbf{Alg}(\Sigma_1)$ associated to a signature morphism $\sigma : \Sigma_1 \to \Sigma_2$. We do not show any examples here, due to space limitations, but we can just recall that given a Σ_1-algebra $\mathcal{A}$, $F_\sigma(\mathcal{A})$ is the Σ_2-algebra built by taking the free Σ_2-algebra over the disjoint union of all the carrier sets of $\mathcal{A}$ that σ maps to each sort, and then making its quotient under the congruence generated by the interpretation of terms in $\mathcal{A}$ translated by σ.

Despite all of this, we can still prove that coequalizers and arbitrary coproducts exist in **Crypt**. We start with coproducts.

Proposition 1. Crypt *has arbitrary coproducts.*

Proof. Let I be a set and $\{\langle \Sigma_i, M_i, \mathbb{A}_i \rangle\}_{i \in I}$ a family of logic system presentations. For each tuple of models $m = \langle m_i \rangle_{i \in I} \in \prod_{i \in I} M_i$, we need to combine the family $\{\langle \Sigma_i, \mathcal{A}_{i,m_i} \rangle\}_{i \in I}$ in **Alg**. Since we need to share the distinguished sort ϕ, we can consider the canonical signature $\Sigma^\phi = \langle \{\phi\}, \emptyset \rangle$ together with the corresponding word algebra $\emptyset$, and the corresponding injections $\langle \sigma_i^\phi, \emptyset \rangle$:

$\langle \Sigma^\phi, \emptyset \rangle \to \langle \Sigma_i, \mathcal{A}_{i,m_i} \rangle$. Since **Alg** is cocomplete, we can consider the family $\{\langle \sigma_i^\phi, \emptyset \rangle\}_{i \in I}$ and build its pushout $\{\langle \sigma_i, \eta_{i,m} \rangle : \langle \Sigma_i, \mathcal{A}_{i,m_i} \rangle \to \langle \Sigma, \mathcal{A}_m \rangle\}_{i \in I}$. Obviously, we can assume without loss of generality that the signature Σ and the signature morphisms σ_i are the same for every m. Now we need to consider all possible compatible choices of designated values in $\mathcal{A}_m$, and define $M = \{\langle m, T \rangle : m \in \prod_{i \in I} M_i, T \subseteq |\mathcal{A}_m|_\phi, \eta_{i,m}^{-1}(T) = T_{i,m_i} \text{ for every } i \in I\}$, $\mu_i : M \to M_i$ such that $\mu_i(m, T) = m_i$, and $\mathbb{A}(m, T) = \langle \mathcal{A}_m, T \rangle$. We claim that $\{\langle \sigma_i, \mu_i, \eta_i \rangle : \langle \Sigma_i, M_i, \mathbb{A}_i \rangle \to \langle \Sigma, M, \mathbb{A} \rangle\}_{i \in I}$ is a coproduct of $\{\langle \Sigma_i, M_i, \mathbb{A}_i \rangle\}_{i \in I}$ in **Crypt**. The facts that $\langle \Sigma, M, \mathbb{A} \rangle$ is a well defined logic system presentation and each $\langle \sigma_i, \mu_i, \eta_i \rangle$ a well defined cryptomorphism are straightforward. Note that the sharing of Σ^ϕ via the family of signature morphisms $\{\sigma_i^\phi\}_{i \in I}$ is essential to guarantee that Σ has a distinguished sort ϕ to which the distinguished sort of each Σ_i is mapped by σ_i. We are left with proving the corresponding universal property.

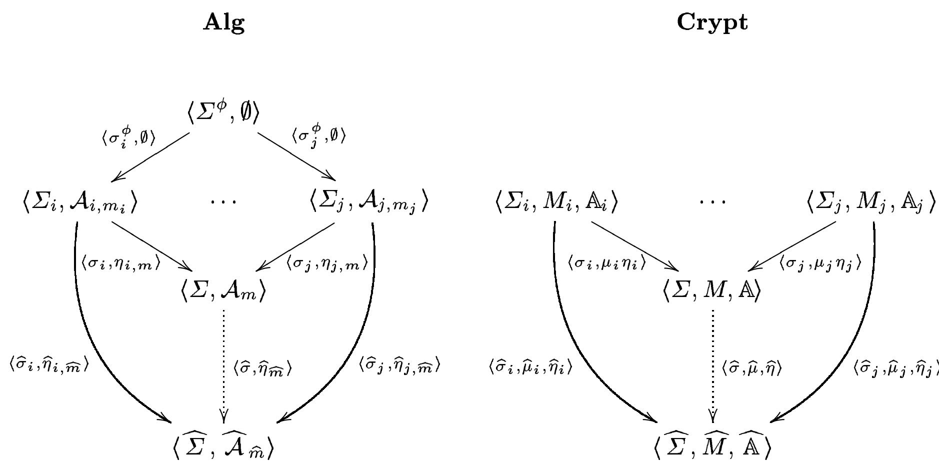

Fig. 3. Coproducts in **Crypt**.

Assume given a logic system presentation $\langle \widehat{\Sigma}, \widehat{M}, \widehat{\mathbb{A}} \rangle$ and a family of cryptomorphisms $\{\langle \widehat{\sigma}_i, \widehat{\mu}_i, \widehat{\eta}_i \rangle : \langle \Sigma_i, M_i, \mathbb{A}_i \rangle \to \langle \widehat{\Sigma}, \widehat{M}, \widehat{\mathbb{A}} \rangle\}_{i \in I}$. For each $\widehat{m} \in \widehat{M}$, let $m = \langle m_i \rangle_{i \in I}$ with each $m_i = \widehat{\mu}_i(\widehat{m})$. Clearly, it must also be the case that each composition $\widehat{\sigma}_i \circ \sigma_i^\phi$ maps ϕ to the distinguished sort of $\widehat{\Sigma}$. Thus, the universal property of the pushout in **Alg** guarantees the existence of a unique morphism $\langle \widehat{\sigma}, \widehat{\eta}_{\widehat{m}} \rangle : \langle \Sigma, \mathcal{A}_m \rangle \to \langle \widehat{\Sigma}, \widehat{\mathcal{A}}_{\widehat{m}} \rangle$ such that $\widehat{\sigma} \circ \sigma_i = \widehat{\sigma}_i$ and $\widehat{\eta}_{\widehat{m}} \circ \eta_{i,m} = \widehat{\eta}_{i,\widehat{m}}$ for each $i \in I$. Again, it is obvious that the signature morphism $\widehat{\sigma}$ is the same for every $\widehat{m}$. We can now define $\widehat{\mu} : \widehat{M} \to M$ by $\widehat{\mu}(\widehat{m}) = \langle m, T \rangle$ with $T = \widehat{\eta}_{\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}})$. This is clearly well defined because each $\eta_{i,m}^{-1}(\widehat{\eta}_{\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}})) = (\widehat{\eta}_{\widehat{m}} \circ \eta_{i,m})^{-1}(\widehat{T}_{\widehat{m}}) =$

$\widehat{\eta}_{i,\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}}) = T_{i,m}$. So, it is immediate that $\langle\widehat{\sigma}, \widehat{\mu}, \widehat{\eta}\rangle$ constitutes a cryptomorphism and composes with each $\langle\sigma_i, \mu_i, h_i\rangle$ into $\langle\widehat{\sigma}_i, \widehat{\mu}_i, \widehat{\eta}_i\rangle$. Uniqueness follows from the fact that $T = \widehat{\eta}_{\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}})$ is the unique possible choice that fulfils the closedness condition for each $\widehat{\eta}_{\widehat{m}}$. □

We can now turn to coequalizers. The construction is a little simpler because there is no need to share the canonical signature Σ^{ϕ}: we start with a pair of parallel arrows in $\mathbf{AlgSig}_{\phi}$ that already preserve ϕ.

Proposition 2. Crypt *has coequalizers.*

Proof. Let $\langle\Sigma_1, M_1, \mathbb{A}_1\rangle$ and $\langle\Sigma_2, M_2, \mathbb{A}_2\rangle$ be logic system presentations, and consider a pair $\langle\sigma', \mu', \eta'\rangle, \langle\sigma'', \mu'', \eta''\rangle : \langle\Sigma_1, M_1, \mathbb{A}_1\rangle \to \langle\Sigma_2, M_2, \mathbb{A}_2\rangle$ of cryptomorphisms. For each model $m_2 \in M_2$ such that $\mu'(m_2) = \mu''(m_2) = m_1$, we can form in **Alg** the pair of arrows $\langle\sigma', \eta'_{m_2}\rangle, \langle\sigma'', \eta''_{m_2}\rangle : \langle\Sigma_1, \mathcal{A}_{1,m_1}\rangle \to \langle\Sigma_2, \mathcal{A}_{2,m_2}\rangle$ and take their coequalizer $\langle\sigma, \eta_{m_2}\rangle : \langle\Sigma_2, \mathcal{A}_{2,m_2}\rangle \to \langle\Sigma, \mathcal{A}_{m_2}\rangle$. Obviously, we can assume without loss of generality that the signature Σ and the signature morphism σ are the same for every m_2. Now we need to consider all possible compatible choices of designated values, and define $M = \{\langle m_2, T\rangle : m_2 \in M_2, \mu'(m_2) = \mu''(m_2), T \subseteq |\mathcal{A}_{m_2}|_{\phi}, \eta_{m_2}^{-1}(T) = T_{2,m_2}\}$, $\mu : M \to M_2$ such that $\mu(m_2, T) = m_2$, and $\mathbb{A}(m_2, T) = \langle\mathcal{A}_{m_2}, T\rangle$. We claim that $\langle\sigma, \mu, \eta\rangle : \langle\Sigma_2, M_2, \mathbb{A}_2\rangle \to \langle\Sigma, M, \mathbb{A}\rangle$ is a coequalizer of $\langle\sigma', \mu', \eta'\rangle$ and $\langle\sigma'', \mu'', \eta''\rangle$ in **Crypt**. The facts that $\langle\Sigma, M, \mathbb{A}\rangle$ is a well defined logic system presentation and $\langle\sigma, \mu, \eta\rangle$ a well defined cryptomorphism are straightforward. Checking that $\langle\sigma, \mu, \eta\rangle$ indeed coequalizes $\langle\sigma', \mu', \eta'\rangle$ and $\langle\sigma'', \mu'', \eta''\rangle$ is also routine. We are left with proving the corresponding universal property.

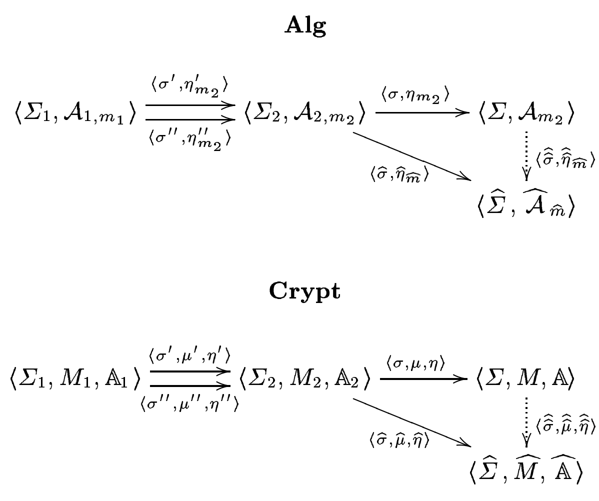

Fig. 4. Coequalizers in **Crypt**.

Assume that $\langle \widehat{\sigma}, \widehat{\mu}, \widehat{\eta} \rangle : \langle \Sigma_2, M_2, \mathbb{A}_2 \rangle \to \langle \widehat{\Sigma}, \widehat{M}, \widehat{\mathbb{A}} \rangle$ also coequalizes $\langle \sigma', \mu', \eta' \rangle$ and $\langle \sigma'', \mu'', \eta'' \rangle$. For each $\widehat{m} \in \widehat{M}$, let $m_2 = \widehat{\mu}(\widehat{m})$. It is clear that $\mu'(m_2) = \mu''(m_2)$. Since it must also be the case that $\langle \widehat{\sigma}, \widehat{\eta}_{\widehat{m}} \rangle : \langle \Sigma_2, \mathcal{A}_{2,m_2} \rangle \to \langle \widehat{\Sigma}, \widehat{\mathcal{A}}_{\widehat{m}} \rangle$ coequalizes $\langle \sigma', \eta'_{m_2} \rangle$ and $\langle \sigma'', \eta''_{m_2} \rangle$ in **Alg**, there exists a unique morphism $\langle \widehat{\widehat{\sigma}}, \widehat{\widehat{\eta}}_{\widehat{m}} \rangle : \langle \Sigma, \mathcal{A}_{m_2} \rangle \to \langle \widehat{\Sigma}, \widehat{\mathcal{A}}_{\widehat{m}} \rangle$ such that $\widehat{\widehat{\sigma}} \circ \sigma = \widehat{\sigma}$ and $\widehat{\widehat{\eta}}_{\widehat{m}} \circ \eta_{m_2} = \widehat{\eta}_{\widehat{m}}$. Again, it is obvious that the signature morphism $\widehat{\widehat{\sigma}}$ is the same for every $\widehat{m}$. We can now define $\widehat{\widehat{\mu}} : \widehat{M} \to M$ by $\widehat{\widehat{\mu}}(\widehat{m}) = \langle m_2, T \rangle$ with $T = \widehat{\widehat{\eta}}_{\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}})$. This is clearly well defined because $\eta_{m_2}^{-1}(\widehat{\widehat{\eta}}_{\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}})) = (\widehat{\widehat{\eta}}_{\widehat{m}} \circ \eta_{m_2})^{-1}(\widehat{T}_{\widehat{m}}) = \widehat{\eta}_{\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}}) = T_{2,m_2}$. So, it is immediate that $\langle \widehat{\widehat{\sigma}}, \widehat{\widehat{\mu}}, \widehat{\widehat{\eta}} \rangle$ constitutes a cryptomorphism and composes with $\langle \sigma, \mu, \eta \rangle$ into $\langle \widehat{\sigma}, \widehat{\mu}, \widehat{\eta} \rangle$. Uniqueness follows from the fact that $T = \widehat{\widehat{\eta}}_{\widehat{m}}^{-1}(\widehat{T}_{\widehat{m}})$ is the unique possible choice that fulfils the closedness condition for each $\widehat{\widehat{\eta}}_{\widehat{m}}$. □

Finally we can state the desired result.

Theorem 1. Crypt *is cocomplete.*

Of course the proofs above are not too informative with respect to the concrete result obtained in specific examples. To provide a better understanding of the power of the colimit construction in **Crypt** and work out a meaningful application of cryptomorphisms we shall revisit the example of [13–15] in the next Section. However, we can already analyze the essential differences between colimits in **Crypt** and **MPRoom**, at the light of Examples 1 and 2.

In **MPRoom** it is always possible to obtain one canonical combined model: it features the minimal possible set of designated values, possibly at the expense of designating a value that was previously not designated. In **Crypt** it depends: no previously undesignated values can become designated by the construction, which means that in some cases the combined structure must simply be ignored; still, if that is not the case, any choice of newly generated values will provide a relevant set of designated values. We claim that this is precisely where the free interaction of the logics being combined emerges, as a result of the absolutely fundamental role played by the "junk" values. If we are combining logics that share a formula that is valid in one of them but not in the other then we are (and should be) in trouble, and the combination will trivialize. However, if no such inconsistencies exist, the resulting combined logic will encompass models that correspond to any possible choice of new validities among the new combined formulas, while still keeping intact the validities of each of the given logics.

4 Partial equational logic

We shall now borrow the example from [13, 14], used in [15] precisely to show that **LogMPar** was not complete. We choose this example not only because it was developed in this context, but also because it concerns the relevant question of assigning a meaning to equations in the presence of undefined operations. Last but not least, the end result is an excellent illustration of the power of cryptomorphisms.

Of course, we recast the example at the level of rooms, and not of parchments as in the original formulation, and work it out in the category **Crypt**. Therefore, we consider fixed a many-sorted signature with partial operations, that is, a triple $\langle S, TO, PO\rangle$ such that both $\langle S, TO\rangle$ and $\langle S, PO\rangle$ are many-sorted signatures, respectively of total and partial operations, with $TO \cap PO = \emptyset$. The logic system presentation ALG of *total equational logic without equations* has sorts $S \cup \{\phi\}$ and operations TO, its models are precisely the $\langle S, TO\rangle$-algebras, and each $\langle S, TO\rangle$-algebra $\mathcal{A}$ is endowed with the structure $\langle \mathcal{A}^2, \{1\}\rangle$ where $|\mathcal{A}^2|_\phi = \{0, 1\}$ and $|\mathcal{A}^2|_s = |\mathcal{A}|_s$ for $s \in S$, with $f_{\mathcal{A}^2} = f_\mathcal{A}$ for each $f \in TO$. Although not very interesting *per se*, this logic system presentation is the common part of two other logic system presentations: ALG($\approx$) for *total equational logic*, and PALG for *partial equational logic without equations*. The idea is precisely to obtain a free combined semantics for *partial equational logic*. The example is particularly well set since the colimit of ALG($\approx$) and ALG while sharing ALG focuses precisely on the missing bit: the interpretation of equations involving undefined values.

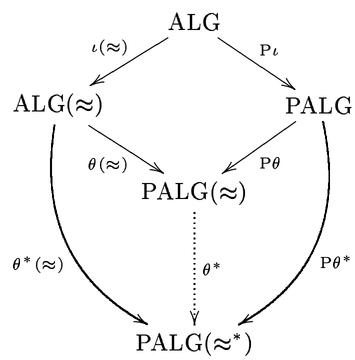

Fig. 5. The partial equational logic pushout.

Now, ALG($\approx$) has sorts $S \cup \{\phi\}$ and operations TO plus $\approx$: $s \times s \to \phi$ for each $s \in S$, the models are also the $\langle S, TO\rangle$-algebras, and each $\langle S, TO\rangle$-algebra $\mathcal{A}$ is endowed with the structure $\langle \mathcal{A}^2_\approx, \{1\}\rangle$ where $\mathcal{A}^2_\approx$ extends $\mathcal{A}^2$ by

$$\approx_{\mathcal{A}^2_\approx} (x, y) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases} .$$

Clearly, $\mathcal{A} \Vdash_{\mathrm{ALG}(\approx)} t \approx t'$ if and only if $[\![t]\!]_\mathcal{A} = [\![t']\!]_\mathcal{A}$. We denote by $\iota(\approx)$: ALG $\to$ ALG($\approx$) the obvious cryptomorphism that injects the signatures, is the identity on models, and also the identity on each structure.

On its turn, PALG has sorts $S \cup \{\phi\}$ and operations $TO \cup PO$, the models are also precisely the $\langle S, TO, PO\rangle$-partial algebras, and each $\langle S, TO, PO\rangle$-partial algebra $\mathcal{B}$ is endowed with the structure $\langle \mathcal{B}^2, \{1\}\rangle$ where $|\mathcal{B}^2|_\phi = \{0, 1\}$ and

$|\mathcal{B}^2|_s = |\mathcal{B}|_s \uplus \{\perp_s\}$ for $s \in S$, with

$$f_{\mathcal{B}^2}(\overrightarrow{x}) = \begin{cases} f_{\mathcal{B}}(\overrightarrow{x}) & \text{if} \quad \text{all } x_i \neq \perp_{s_i} \text{ and } f_{\mathcal{B}}(\overrightarrow{x}) \downarrow \\ \perp_s & \text{otherwise} \end{cases}$$

for each $f : s_1 \times \cdots \times s_n \to s$ in $TO \cup PO$. We denote by $\mathrm{P}\iota : \mathrm{ALG} \to \mathrm{PALG}$ the obvious cryptomorphism that injects the signatures, forgets the partial operations, and injects the corresponding structures.

The desired result should therefore correspond to the pushout of $\iota(\approx)$ and $\mathrm{P}\iota$ in **Crypt**, which is actually not very difficult to compute. First of all, we have to combine the signatures of ALG($\approx$) and PALG while sharing their common subsignature ALG. We end up with sorts $S \cup \{\phi\}$ and operations $TO \cup PO$ plus $\approx: s \times s \to \phi$ for each $s \in S$. Let us consider a pair of models, $\mathcal{A}$ from ALG($\approx$) and $\mathcal{B}$ from PALG that coincide when mapped to ALG, that is, $\mathcal{A}$ is precisely the restriction of $\mathcal{B}$ to the total operations. The corresponding combined algebra $\mathcal{C}$ will include, besides $\{0,1\}$, a whole new set of freely generated truth-values corresponding to the new denotations of $\approx$ involving undefined values, that is, $V = \bigcup_{s \in S} \{x \approx \perp_s : x \in |\mathcal{B}|_s\} \cup \{\perp_s \approx x : x \in |\mathcal{B}|_s\} \cup \{\perp_s \approx \perp_s\}$. In detail, $\mathcal{C}$ is such that $|\mathcal{C}|_s = |\mathcal{B}|_s \uplus \{\perp_s\}$ for $s \in S$, $|\mathcal{C}|_\phi = \{0,1\} \cup V$, $f_{\mathcal{C}} = f_{\mathcal{B}^2}$ and

$$\approx_{\mathcal{C}} (x,y) = \begin{cases} 1 & \text{if} \quad x = y \text{ and } x \neq \perp_s \text{ and } y \neq \perp_s \\ 0 & \text{if} \quad x \neq y \text{ and } x \neq \perp_s \text{ and } y \neq \perp_s \\ x \approx y & \text{otherwise} \end{cases}.$$

Therefore, the combined models can be seen as pairs $\langle \mathcal{B}, T \rangle$ with $T \subseteq V$ representing any possible choice of new designated values. The structure associated to each pair $\langle \mathcal{B}, T \rangle$ is precisely $\langle \mathcal{C}, \{1\} \cup T \rangle$. It is straightforward to set up the inclusion cryptomorphisms $\theta(\approx) : \mathrm{ALG}(\approx) \to \mathrm{PALG}(\approx)$ and $\mathrm{P}\theta : \mathrm{PALG} \to \mathrm{PALG}(\approx)$: $\theta(\approx)$ is the inclusion on signatures, maps each model $\langle \mathcal{B}, T \rangle$ to the restriction $\mathcal{A}$ of $\mathcal{B}$ to the total operations, and then injects $\mathcal{A}^2_\approx$ into $\mathcal{C}$; $\mathrm{P}\theta$ is also the inclusion on signatures, maps each model $\langle \mathcal{B}, T \rangle$ to $\mathcal{B}$, and then injects $\mathcal{B}^2$ into $\mathcal{C}$. It is clear that $\theta(\approx) \circ \iota(\approx) = \mathrm{P}\theta \circ \mathrm{P}\iota$.

Proposition 3. *The logic system presentation PALG($\approx$) together with the cryptomorphisms $\theta(\approx)$ and Pθ constitutes a pushout of $\iota(\approx)$ and Pι in* **Crypt**.

The universal property of the construction of PALG($\approx$) can be interpreted as follows. Choose your favourite interpretation of partial equations, and define with it a logic system presentation PALG($\approx^*$). One can of course imagine very strange situations, but one can impose as a minimal requirement that the choice is at least based on partial algebras, and that it extends the usual interpretation of total equations. In that case, it should be routine to define two cryptomorphisms $\theta^*(\approx) : \mathrm{ALG}(\approx) \to \mathrm{PALG}(\approx^*)$ and $\mathrm{P}\theta^* : \mathrm{PALG} \to \mathrm{PALG}(\approx^*)$ such that $\theta^*(\approx) \circ \iota(\approx) = \mathrm{P}\theta^* \circ \mathrm{P}\iota$. Therefore, the construction guarantees that there exists precisely one compatible cryptomorphism $\theta^* : \mathrm{PALG}(\approx) \to \mathrm{PALG}(\approx^*)$, which means that the chosen interpretation of partial equations corresponds to a particular choice of models in PALG($\approx$). Let us see how this goes for some of the most common interpretations of partial equations.

Example 3. The logic system presentations $\mathrm{PALG}(\approx^*)$, with $* \in \{w, s, e, 3\}$, of *weak, strong, existential, and strict three-valued partial equational logic*, respectively, all have sorts $S \cup \{\phi\}$ and operations $TO \cup PO$ plus $\approx: s \times s \to \phi$ for each $s \in S$, and the $\langle S, TO, PO\rangle$-partial algebras as models. They differ from $\mathrm{PALG}(\approx)$, and between each other, on the interpretation structures associated to each model $\mathcal{B}$. In the sequel, $\mathcal{A}$ always stands for the restriction of $\mathcal{B}$ to the partial operations.

In the weak case, $\mathcal{B}$ is endowed with $\langle \mathcal{B}^{2,w}_{\approx}, \{1\}\rangle$ where $\mathcal{B}^{2,w}_{\approx}$ extends $\mathcal{B}^2$ by

$$\approx_{\mathcal{B}^{2,w}_{\approx}}(x,y) = \begin{cases} 1 & \text{if} \quad x = y \text{ or } x = \perp_s \text{ or } y = \perp_s \\ 0 & \text{otherwise} \end{cases} .$$

Clearly, $\mathcal{B} \Vdash_{\mathrm{PALG}(\approx^w)} t \approx t'$ if and only if $[\![t]\!]_{\mathcal{B}} = [\![t']\!]_{\mathcal{B}}$ or at least one of them is undefined. The cryptomorphisms $\theta^w(\approx)$ and $\mathrm{P}\theta^w$ simply inject $\mathcal{A}^2_{\approx}$ and $\mathcal{B}^2$ into $\mathcal{B}^{2,w}_{\approx}$. The unique compatible cryptomorphism θ^w maps each $\mathcal{B}$ to $\langle \mathcal{C}, \{1\} \cup V\rangle$, and then all the values in V, from $\mathcal{C}$, to 1 in $\mathcal{B}^{2,w}_{\approx}$.

In the strong case, the structure is $\langle \mathcal{B}^{2,s}_{\approx}, \{1\}\rangle$ where $\mathcal{B}^{2,s}_{\approx}$ extends $\mathcal{B}^2$ by

$$\approx_{\mathcal{B}^{2,s}_{\approx}}(x,y) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{otherwise} \end{cases} .$$

Clearly, $\mathcal{B} \Vdash_{\mathrm{PALG}(\approx^s)} t \approx t'$ if and only if $[\![t]\!]_{\mathcal{B}} = [\![t']\!]_{\mathcal{B}}$ or both are undefined. The cryptomorphisms $\theta^s(\approx)$ and $\mathrm{P}\theta^s$ simply inject $\mathcal{A}^2_{\approx}$ and $\mathcal{B}^2$ into $\mathcal{B}^{2,s}_{\approx}$. The cryptomorphism θ^s maps each $\mathcal{B}$ to $\langle \mathcal{C}, \{1\} \cup \{\perp_s \approx \perp_s : s \in S\}\rangle$, and then all the values in $V \setminus \{\perp_s \approx \perp_s : s \in S\}$ to 0 in $\mathcal{B}^{2,w}_{\approx}$, and $\{\perp_s \approx \perp_s : s \in S\}$ to 1.

In the existential case, the structure is $\langle \mathcal{B}^{2,e}_{\approx}, \{1\}\rangle$ where $\mathcal{B}^{2,e}_{\approx}$ extends $\mathcal{B}^2$ by

$$\approx_{\mathcal{B}^{2,e}_{\approx}}(x,y) = \begin{cases} 1 & \text{if} \quad x = y \text{ and } x \neq \perp_s \text{ and } y \neq \perp_s \\ 0 & \text{otherwise} \end{cases} .$$

Clearly, $\mathcal{B} \Vdash_{\mathrm{PALG}(\approx^e)} t \approx t'$ if and only if $[\![t]\!]_{\mathcal{B}} = [\![t']\!]_{\mathcal{B}}$ with both values defined. The cryptomorphisms $\theta^e(\approx)$ and $\mathrm{P}\theta^e$ again simply inject $\mathcal{A}^2_{\approx}$ and $\mathcal{B}^2$ into $\mathcal{B}^{2,e}_{\approx}$. The unique cryptomorphism θ^e now maps each $\mathcal{B}$ to $\langle \mathcal{C}, \{1\}\rangle$, and then all the values in V to 0 in $\mathcal{B}^{2,e}_{\approx}$.

The strict three-valued case is slightly different. Each $\mathcal{B}$ is now endowed with the structure $\langle \mathcal{B}^3_{\approx}, \{1\}\rangle$ where $|\mathcal{B}^3_{\approx}|_\phi = \{0,1\} \uplus \{\perp\}$ and $|\mathcal{B}^3_{\approx}|_s = |\mathcal{B}|_s \uplus \{\perp_s\}$ for $s \in S$, with $f_{\mathcal{B}^3_{\approx}} = f_{\mathcal{B}^2}$ for each $f \in TO \cup PO$ and

$$\approx_{\mathcal{B}^3_{\approx}}(x,y) = \begin{cases} 1 & \text{if} \quad x = y \text{ and } x \neq \perp_s \text{ and } y \neq \perp_s \\ 0 & \text{if} \quad x \neq y \text{ and } x \neq \perp_s \text{ and } y \neq \perp_s \\ \perp & \text{otherwise} \end{cases} .$$

Clearly, $\mathcal{B} \Vdash_{\mathrm{PALG}(\approx^3)} t \approx t'$ if and only if $\mathcal{B} \Vdash_{\mathrm{PALG}(\approx^e)} t \approx t'$. The cryptomorphisms $\theta^3(\approx)$ and $\mathrm{P}\theta^3$ inject $\mathcal{A}^2_{\approx}$ and $\mathcal{B}^2$ into $\mathcal{B}^3_{\approx}$. The cryptomorphism θ^3 maps each $\mathcal{B}$ to $\langle \mathcal{C}, \{1\}\rangle$, and then all the values in V from $\mathcal{C}$ to $\perp$ in $\mathcal{B}^3_{\approx}$. △

Note however that the combination obtained is so absolutely free that less orthodox choices are also possible, namely asymmetric ones, or choices that consider different solutions for each sort.

5 Conclusion

We have shown that cryptomorphisms really work, in the sense that they set up a category of logic system presentations that is cocomplete, together with the fact that they always fulfill the usual satisfaction condition. This implies that cryptomorphisms give rise to a complete category of parchments that easily translates to the category of institutions. Therefore, limits in this category of parchments always exist, and constitute a very powerful mechanism for combining logics that extends fibring along the lines of [5]. Not only the syntaxes of the given logics are freely combined, but also their semantics. Undesired collapses are also avoided, as long as shared formulas have a uniform semantics in the logics being combined. We have also put cryptomorphisms in context with the notions of morphism and logical morphism arising from the work on model-theoretic parchments, and explained the absolutely fundamental role played by "junk" values in the freeness of the colimits obtained using cryptomorphisms, in contrast with the logicality constraints advocated in [15]. The power of the approach was illustrated using a meaningful partial equational logic example, whose result encompasses models that are compatible with every possible interpretation of equality involving undefinedness, even if less standard.

Nevertheless, we agree that the proliferation of truth-values can be seen, at least, as annoying. Moreover, the freeness of the construction really takes advantage of this fact in allowing possibly less orthodox choices of newly designated values. But there are certainly other ways to prevent unorthodox choices, given any reasonable notion of orthodoxy. The subsequent use of congruence relations, as in [13], is one of them. As usual, each uniform congruence on the resulting combined structures can be seen as the outcome of a corresponding cryptomorphism. The cryptomorphisms θ^* do precisely that in the example of the previous Section. Still, there are more interesting possibilities. One of them, certainly worth pursuing, is to consider representations of all the logics involved in a *universal* logic, as proposed in [14, 15]. In alternative, we can work along with calculi associated to each of the logics, and require their soundness as a minimal requirement, as done in [5] for cryptofibring. This last possibility also opens the way to incorporating and extending the soundness and completeness preservation results well-known for fibring to this wider context.

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