Completeness Results for Fibred Parchments ⋆

Beyond the Propositional Base

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Abstract. In [6] it was shown that fibring could be used to combine institutions presented as **c**-parchments, and several completeness preservation results were established. However, their scope of applicability was limited to propositional-based logics. Herein, we extend these results to a broader class of logics, possibly including variables, terms and quantifiers. On the way, we need to consider an enriched notion of proof-calculus that deals explicitly with the substitution provisos that often appear in schematic inference rules. For illustration of the concepts, constructions and results, we shall adopt modal first-order logic as a working example.

1 Introduction

Working with several logics is the rule, in practice, to wit in knowledge representation and formal specification. Due to its intuitive simplicity and theoretical interest, the *fibring* mechanism for combining logics has deserved close attention [9, 3, 20, 25]. In [6], **c**-*parchments* were proposed for bringing fibring to the realm of institutions [10, 15, 11, 24], as an alternative to other approaches for combining institutions [16–18]. A major strength of fibring is the possibility to establish general transfer results from the logics being combined to the resulting fibred logic. Soundness and completeness preservation for propositional-based logics was also obtained in [6]. Herein, we extend these results beyond the propositional base.

Recall that **c**-parchments, signature-indexed categories of **c**-rooms, are an evolution of [10, 17, 18] designed to promote a smooth characterization of fibring [20, 25, 5]. They differ from the *model-theoretic parchments* of [18] by endowing the algebras of truth-values with a Tarskian closure operation, rather than just a set of designated values. As shown in [6], fibred **c**-parchments appear as colimits in the corresponding category. The proof-theoretic counterpart of **c**-rooms in [6] was played by a notion of calculus with schematic inference rules, fit for representing the Hilbert-style axiomatizations of propositional-based logics. Since these logics are usually structural, every instantiation of a schematic inference rule was allowed. However, if we want to represent more complex logics, we need to gain control over these instantiations. A paradigmatic example are the provisos in some of the axioms of first-order logic, e.g., requiring that a variable is not

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free in a formula. The idea of making these side conditions explicit is not new [21], but the technique we shall use is improved along [7, 22]. This fine control of instantiations also has an impact on fibring, again characterizable by colimits. These aspects settled, we can study soundness and completeness transfer results in a broader context. As before [5], soundness preservation is immediate, by definition of fibring. For completeness, we capitalize on the notion of *fullness* [25] for guaranteeing that the logics at hand have a sufficient amount of models. Under reasonable assumptions on the logics being fibred, their syntactic constructors and the properties of their proof-calculi, we also generalize the completeness preservation results of [6]. We illustrate fibring by providing a detailed analysis of modal first-order logic as a fibring of propositional modal logic and first-order logic, considering various choices for its semantics, and clarifying the importance of provisos and the applicability of the completeness results.

In Section 2 we set up logic-parchments by recalling the details of **c**-parchments and introducing an improved version of proof-calculus. Section 3 is dedicated to fibring. After an overview of fibred semantics, we proceed to the categorial characterization of fibred logics, by understanding fibred deduction in the presence of provisos. A general soundness preservation result is also established, and the fundamental notion of fullness is introduced. In Section 4, we study completeness preservation under meaningful fullness requirements and reasonable assumptions on the syntactic constructors and the proof-calculi of the logics being fibred. We conclude by discussing the results obtained, their limitations and future work.

2 Rooms and parchments

We consider, in turn, semantics, deduction, and finally logics.

2.1 Semantics

In the sequel, $\mathbf{AlgSig}_\phi$ is the category of many-sorted signatures $\Sigma = \langle S, O\rangle$, where S is a set (of *sorts*) and $O = \{O_u\}_{u\in S^+}$ is a family of sets (of *operators*) indexed by their type, with a distinguished sort $\phi \in S$ (for formulas) and morphisms preserving it. We denote by $\mathbf{Alg}(\Sigma)$ the category of Σ-algebras and homomorphisms, and by $\mathrm{cAlg}(\Sigma)$ the class of all *interpretation structures* $\langle \mathcal{A}, \mathbf{c}\rangle$ with $\mathcal{A}$ a Σ-algebra and $\mathbf{c}$ a closure operation on $|\mathcal{A}|_\phi$ (the carrier of sort ϕ, intuitively corresponding to the set of truth-values). Recall that a closure operation $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is extensive, monotonic and idempotent, i.e., $B \subseteq B^{\mathbf{c}}$, $B^{\mathbf{c}} \subseteq (B \cup B')^{\mathbf{c}}$ and $(B^{\mathbf{c}})^{\mathbf{c}} \subseteq B^{\mathbf{c}}$. We use $\mathcal{W}_\Sigma$ to denote the free Σ-algebra (the *word* algebra), Form_Σ to denote the set $|\mathcal{W}_\Sigma|_\phi$ of *formulas*, and $[\![\_]\!]_\mathcal{A}$ (for word *interpretation*) to denote the unique $\mathbf{Alg}(\Sigma)$-homomorphism from $\mathcal{W}_\Sigma$ to a given Σ-algebra $\mathcal{A}$. We use φ, ψ to denote formulas, and Φ, Ψ sets of formulas. Elements of $|\mathcal{W}_\Sigma|_s$ are referred to as *terms* and denoted by t. Every $\mathbf{AlgSig}_\phi$-morphism $h : \Sigma_1 \to \Sigma_2$ has an associated reduct functor $\_|_h : \mathbf{Alg}(\Sigma_2) \to \mathbf{Alg}(\Sigma_1)$. Note that $[\![t]\!]_{\mathcal{A}|_h} = [\![h(t)]\!]_\mathcal{A}$ for each $t \in |\mathcal{W}_{\Sigma_1}|_s$ and Σ_2-algebra $\mathcal{A}$. As usual, we overload the notation and write h for word *translation* instead of $[\![\_]\!]_{\mathcal{W}_{\Sigma_2}|_h}$ to denote the unique $\mathbf{Alg}(\Sigma_1)$-homomorphism from $\mathcal{W}_{\Sigma_1}$ to $\mathcal{W}_{\Sigma_2}|_h$.

Definition 1. A **c**-*room* is a pair $\mathcal{R} = \langle \Sigma, M \rangle$ with $\Sigma \in |\mathbf{AlgSig}_\phi|$ and $M \subseteq \mathrm{cAlg}(\Sigma)$. A *morphism of* **c**-*rooms* from $\mathcal{R}_1 = \langle \Sigma_1, M_1 \rangle$ to $\mathcal{R}_2 = \langle \Sigma_2, M_2 \rangle$ is an $\mathbf{AlgSig}_\phi$-morphism $h : \Sigma_1 \to \Sigma_2$ such that $\langle \mathcal{A}|_h, \mathbf{c} \rangle \in M_1$ for every $\langle \mathcal{A}, \mathbf{c} \rangle \in M_2$.

Clearly, **c**-rooms and morphisms set up a category **CPRoom**, from which the category **CPar** of **c**-*parchments* is obtained by a simple Grothendieck construction (see [6]). Namely, a **c**-parchment is a functor from any given category **Sig** of abstract signatures to **CPRoom**. Building on the cocompleteness of **CPRoom**, **CPar** is cocomplete [6]. A **c**-room $\mathcal{R} = \langle \Sigma, M \rangle$ induces an *entailment*[1] relation defined by $\Phi \vDash_{\mathcal{R}} \psi$ if $[\![\psi]\!]_{\mathcal{A}} \in \{[\![\varphi]\!]_{\mathcal{A}} : \varphi \in \Phi\}^{\mathbf{c}}$ for every $\langle \mathcal{A}, \mathbf{c} \rangle \in M$, where $\Phi \cup \{\psi\} \subseteq \mathrm{Form}_\Sigma$. The following property holds: if $h : \mathcal{R}_1 \to \mathcal{R}_2$ is a **c**-room morphism and $\Phi \vDash_{\mathcal{R}_1} \psi$ then $h(\Phi) \vDash_{\mathcal{R}_2} h(\psi)$. Given a **c**-parchment $R : \mathbf{Sig} \to \mathbf{CPRoom}$ and $\Omega \in |\mathbf{Sig}|$, we denote $\vDash_{R(\Omega)}$ simply by $\vDash_\Omega$.

Example 1. X is a fixed set of variables. The **c**-parchment of first-order logic with equality is the functor *FOLEq* defined from the category $\mathbf{Set}^{\mathbb{N}_0} \times \mathbf{Set}^{\mathbb{N}}$ of abstract signatures $\langle F, P \rangle$ of ranked function and predicate alphabets, by assigning to each $\langle F, P \rangle$ the **c**-room $\mathcal{R}_{FOLEq} = \langle \Sigma_{FOLEq}, M \rangle$ such that:

- $\Sigma_{FOLEq} = \langle \{\tau, \phi\}, O \rangle$ with $O_\tau = X \cup F_0$, $O_{\tau^n \tau} = F_n$ for $n > 0$, $O_{\tau^n \phi} = P_n$ for $n \neq 2$, $O_{\tau^2 \phi} = P_2 \cup \{\doteq\}$, $O_{\phi\phi} = \{\neg\} \cup \{\forall x : x \in X\}$, $O_{\phi^2 \phi} = \{\Rightarrow\}$;
- M contains all structures $\langle \mathcal{A}, \mathbf{c} \rangle$ obtained from $\langle F, P \rangle$-interpretations $\langle D, I \rangle$ with $D \neq \emptyset$, $f_I : D^n \to D$ for $f \in F_n$, $p_I \subseteq D^n$ for $p \in P_n$, by: $|\mathcal{A}|_\tau = D^{\mathrm{Asg}(X,D)}$, $|\mathcal{A}|_\phi = \wp(\mathrm{Asg}(X, D))$, where $\mathrm{Asg}(X, D) = D^X$ is the set of assignments μ; $x_{\mathcal{A}}(\mu) = \mu(x)$ for $x \in X$, $f_{\mathcal{A}}(\langle e_i \rangle)(\mu) = f_I(\langle e_i(\mu) \rangle)$ for $f \in F$, $p_{\mathcal{A}}(\langle e_i \rangle) = \{\mu : \langle e_i(\mu) \rangle \in p_I\}$ for $p \in P$, $\doteq_{\mathcal{A}} (e_1, e_2) = \{\mu : e_1(\mu) = e_2(\mu)\}$, $\neg_{\mathcal{A}}(v) = \mathrm{Asg}(X, D) \setminus v$, $\forall x_{\mathcal{A}}(v) = \{\mu : \mu[x/d] \in v \text{ for every } d \in D\}$, $\Rightarrow_{\mathcal{A}}(v_1, v_2) = (\mathrm{Asg}(X, D) \setminus v_1) \cup v_2$; $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is the cut closure induced by set inclusion: given $V \in \wp(|\mathcal{A}|_\phi)$, $V^{\mathbf{c}} = \{v \in |\mathcal{A}|_\phi : (\bigcap V) \subseteq v\}$ is the principal ideal determined by $(\bigcap V)$ on the complete lattice $\langle \wp(|\mathcal{A}|_\phi), \supseteq \rangle$.

The denotation of a formula is the set of all assignments for which it holds. First-order logic without equality can be obtained by omitting $\doteq$.

Example 2. The **c**-parchment K of propositional modal logic is defined from the category **Set** of abstract signatures, by mapping each set PS of propositional symbols to the **c**-room $\mathcal{R}_K = \langle \Sigma_K, M \rangle$ such that:

- $\Sigma_K = \langle \{\phi\}, O \rangle$ with $O_\phi = \mathrm{PS}$, $O_{\phi\phi} = \{\Box, \neg\}$, $O_{\phi^2\phi} = \{\Rightarrow\}$;
- M contains every $\langle \mathcal{A}, \mathbf{c} \rangle$ obtained from Kripke models $\langle W, R, \vartheta \rangle$ with $W \neq \emptyset$, $R \subseteq W^2$, $\vartheta : \mathrm{PS} \to \wp(W)$, by: $|\mathcal{A}|_\phi = \wp(W)$; $q_{\mathcal{A}} = \vartheta(q)$ for $q \in \mathrm{PS}$, $\Box_{\mathcal{A}}(U) = \{w : \{w' : w\, R\, w'\} \subseteq U\}$, $\neg_{\mathcal{A}}(U) = W \setminus U$, $\Rightarrow_{\mathcal{A}}(U_1, U_2) = (W \setminus U_1) \cup U_2$; $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is the cut closure induced by $\supseteq$.

The denotation of a formula is the set of all worlds where it holds.

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[1] This is a local entailment relation. A stronger, global, entailment can also be defined by letting $\Phi \;\; '_{\mathcal{R}} \psi$ if $[\![\psi]\!]_{\mathcal{A}} \in \emptyset^{\mathbf{c}}$ whenever $\{[\![\varphi]\!]_{\mathcal{A}} : \varphi \in \Phi\} \subseteq \emptyset^{\mathbf{c}}$ for every $\langle \mathcal{A}, \mathbf{c} \rangle \in M$. This terminology is borrowed from [5] and is reflected, below, by the separation of local and global rules in deduction-rooms.

2.2 Deduction

As noted in [21], to represent the Hilbert-calculi of non-propositional-based logics we need a notion of *schematic inference rule* with a *proviso* delimiting its possible instantiations. From now on ξ, with any decoration, stands for a *schema variable*. Given $\Sigma = \langle S, O\rangle$ and $s \in S$, we denote by Ξ_s the set $\{\xi_s^k : k \in \mathbb{N}_0\}$ and let $\Xi = \{\Xi_s\}_{s\in S}$. We define the set of *schema formulas* $\mathrm{Form}_\Sigma(\Xi)$ to be $|\mathcal{W}_\Sigma(\Xi)|_\phi$, the carrier of sort ϕ in the free algebra with generators Ξ. We use γ, δ to denote schema formulas, and Γ, Δ sets of schema formulas. Elements of $|\mathcal{W}_\Sigma(\Xi)|_s$ are *schema terms*, denoted by θ. A *schema substitution* is a family $\sigma = \{\sigma_s : \Xi_s \to |\mathcal{W}_\Sigma(\Xi)|_s\}_{s\in S}$, that extends freely to schema terms. We write $\theta\sigma$ for the corresponding instantiation. $\mathrm{SSub}(\Sigma)$ denotes the set of schema substitutions over Σ. If σ maps each schema variable to a term without schema variables, we call it a (ground) substitution and denote it by ρ. $\mathrm{Sub}(\Sigma)$ denotes the set of all substitutions. Given $h : \Sigma \to \Sigma'$, we use $h(\sigma)$ to denote $(h \circ \sigma) \in \mathrm{SSub}(\Sigma')$. In the sequel, $\mathbf{AlgSig}_\phi(\Sigma, \_)$ denotes the class of all morphisms with domain Σ.

Definition 2. A Σ-*proviso* is $\pi = \{\pi_h\}_{h\in\mathbf{AlgSig}_\phi(\Sigma,\_)}$, where $\pi_h \subseteq \mathrm{Sub}(\Sigma')$ for each $h : \Sigma \to \Sigma'$, such that $\rho \in \pi_h$ if and only if $h'(\rho) \in \pi_{h'\circ h}$.

Provisos make their behaviour explicit on signature changes, which is essential when inference rules are translated to a richer language [7]. We denote by *univ* the universal Σ-proviso, $univ_h = \mathrm{Sub}(\Sigma')$ for $h : \Sigma \to \Sigma'$. Given a Σ-proviso π we denote by π_Σ the component π_{id_Σ}. Given $h : \Sigma \to \Sigma'$, we denote by $h(\pi)$ the Σ'-proviso such that $h(\pi)_{h'} = \pi_{h'\circ h}$. Given $\sigma \in \mathrm{SSub}(\Sigma)$ we denote by $\pi\sigma$ the Σ-proviso defined by $(\pi\sigma)_h = \{\rho \in \mathrm{Sub}(\Sigma') : \rho \circ h(\sigma) \in \pi_h\}$. Note that for $\rho \in \mathrm{Sub}(\Sigma)$, $\pi\rho = univ$ if $\rho \in \pi_\Sigma$, and $\pi\rho = \emptyset$ if $\rho \notin \pi_\Sigma$. By analogy, we define $\sigma \in \pi_\Sigma$ if $\pi\sigma = univ$. Given Σ-provisos π_1 and π_2 we denote by $\pi_1 \cap \pi_2$ the Σ-proviso such that $(\pi_1 \cap \pi_2)_h = (\pi_1)_h \cap (\pi_2)_h$. We say that $\pi_1 \subseteq \pi_2$ iff $\pi_{1_h} \subseteq \pi_{2_h}$. A Σ-proviso π is said to be *insensitive* to ξ if for every $h : \Sigma \to \Sigma'$ and any $\rho, \rho' \in \mathrm{Sub}(\Sigma')$ that may only differ on ξ, $\rho \in \pi_h$ if and only if $\rho' \in \pi_h$.

Example 3. Recall Example 1 and let $x \in X$. We define the following provisos:

- $nfv(\xi_\phi^n, x)$: given h, $\rho \in nfv(\xi_\phi^n, x)_h$ if $h(x)$ does not occur free in $\rho(\xi_\phi^n)$[2];
- $fts(\xi_\phi^n, \xi_\phi^m, \xi_\tau^k, x)$: given h, $\rho \in fts(\xi_\phi^n, \xi_\phi^m, \xi_\tau^k, x)_h$ if $\rho(\xi_\tau^k)$ is free for $h(x)$ in $\rho(\xi_\phi^n)$ and $\rho(\xi_\phi^m)$ is $\rho(\xi_\phi^n)$ with all free occurrences of $h(x)$ replaced by $\rho(\xi_\tau^k)$;
- $eqrep(\xi_\phi^n, \xi_\phi^m, x, y)$: given h, $\rho \in eqrep(\xi_\phi^n, \xi_\phi^m, x, y)_h$ if $\rho(\xi_\phi^m)$ is $\rho(\xi_\phi^n)$ with some free occurrences of $h(x)$, out of the scope of $h(\forall y)$, replaced by $h(y)$.

Definition 3. A Σ-*rule* is a triple $\langle\Gamma, \delta, \pi\rangle$ with $\Gamma \cup \{\delta\} \subseteq \mathrm{Form}_\Sigma(\Xi)$ finite and π a Σ-proviso insensitive to all the schema variables not in $\Gamma \cup \{\delta\}$.

We represent $r = \langle\Gamma, \delta, \pi\rangle$ by $\frac{\Gamma}{\delta} : \pi$, or even $\frac{\gamma_1 \ldots \gamma_n}{\delta} : \pi$ if $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. If the set of premises Γ is empty the rule is identified with its conclusion δ and proviso π and called a *schema axiom*. The translation of r by $h : \Sigma \to \Sigma'$ is the Σ'-rule $h(r) = \langle h(\Gamma), h(\delta), h(\pi)\rangle$.

[2] We mean that $h(x)$ always occurs in the scope of $h(\forall x)$. If variables and quantifiers are maintained on translating via h, it means precisely that x occurs under $\forall x$.

As in [20, 25, 5, 6], we explicitly distinguish local from global inference rules (see Examples 4 and 5). The nature of this distiction shows up both on their diverse deductive roles and on their different soundness requirements, below.

Definition 4. A *deduction-room (d-room, for short)* is a triple $\mathcal{D} = \langle \Sigma, lR, gR \rangle$ where $lR \cup gR$ is a set of Σ-rules such that gR-rules have non-empty premises. A *morphism of d-rooms* from $\mathcal{D}_1 = \langle \Sigma_1, lR_1, gR_1 \rangle$ to $\mathcal{D}_2 = \langle \Sigma_2, lR_2, gR_2 \rangle$ is a morphism $h : \Sigma_1 \to \Sigma_2$ such that $h(lR_1) \subseteq lR_2$ and $h(gR_1) \subseteq gR_2$.

Deduction-rooms and morphisms set up a category **DRoom**, from which the category **DPar** of *deduction-parchments (d-parchments)* is obtained by a Grothendieck construction, its colimits built from colimits in **DRoom**, as in [6].

Each $\mathcal{D} = \langle \Sigma, lR, gR \rangle$ induces a *deducibility* relation, built on top of a notion of *theoremhood* (the global counterpart of (local) deducibility). Let $\Gamma \cup \{\delta\} \subseteq \mathrm{Form}_\Sigma(\Xi)$ and π be a Σ-proviso. We say that δ with proviso π is a *schema theorem* of $\mathcal{D}$ generated from Γ, $\Gamma \vdash^{\mathrm{thm}}_{\mathcal{D}} \delta : \pi$, if there exists a finite sequence $\langle \delta_1, \pi_1 \rangle, \ldots, \langle \delta_n, \pi_n \rangle$ with $\delta = \delta_n$ and $\pi \subseteq \pi_n$, such that for each i, either $\delta_i \in \Gamma$ and $\pi_i = \mathit{univ}$, or there exists a rule $\langle \Gamma_r, \delta_r, \pi_r \rangle \in lR \cup gR$ and $\sigma \in \mathrm{SSub}(\Sigma)$ with $\Gamma_r \sigma = \{\delta_{j_1}, \ldots, \delta_{j_k}\} \subseteq \{\delta_j : j < i\}$, $\delta_i = \delta_r \sigma$ and $\pi_i = (\pi_{j_1} \cap \cdots \cap \pi_{j_k}) \cap \pi_r \sigma$. To simplify, we write $\vdash^{\mathrm{thm}}_{\mathcal{D}} \delta : \pi$ if $\Gamma = \emptyset$, or $\Gamma \vdash^{\mathrm{thm}}_{\mathcal{D}} \delta$ if $\pi = \mathit{univ}$. Easily, for every $\sigma \in \mathrm{SSub}(\Sigma)$: if $\Gamma \vdash^{\mathrm{thm}}_{\mathcal{D}} \delta : \pi$ then $\Gamma\sigma \vdash^{\mathrm{thm}}_{\mathcal{D}} \delta\sigma : \pi\sigma$. In deductions, now, only lR-rules and schema theorems are allowed. We say that δ with proviso π is *deducible* in $\mathcal{D}$ from Γ, $\Gamma \vdash_{\mathcal{D}} \delta : \pi$, if there exist $\langle \delta_1, \pi_1 \rangle, \ldots, \langle \delta_n, \pi_n \rangle$ with $\delta = \delta_n$ and $\pi \subseteq \pi_n$, such that for each i, either $\delta_i \in \Gamma$ and $\pi_i = \mathit{univ}$, or $\vdash^{\mathrm{thm}}_{\mathcal{D}} \delta_i : \pi_i$, or there exists a rule $\langle \Gamma_r, \delta_r, \pi_r \rangle \in lR$ and $\sigma \in \mathrm{SSub}(\Sigma)$ such that $\Gamma_r \sigma = \{\delta_{j_1}, \ldots, \delta_{j_k}\} \subseteq \{\delta_j : j < i\}$, $\delta_i = \delta_r \sigma$ and $\pi_i = (\pi_{j_1} \cap \cdots \cap \pi_{j_k}) \cap \pi_r \sigma$. Simplified notation applies to $\vdash_{\mathcal{D}}$, and the following also holds: if $\Gamma \vdash_{\mathcal{D}} \delta : \pi$ then $\Gamma\sigma \vdash_{\mathcal{D}} \delta\sigma : \pi\sigma$. Given a d-parchment $D : \mathbf{Sig} \to \mathbf{DRoom}$, we denote each $\vdash_{D(\Omega)}$ by $\vdash_\Omega$. The next structurality result is straightforward.

Proposition 1. *Let $h : \mathcal{D} \to \mathcal{D}'$ be a d-room morphism. If $\Gamma \vdash^{\mathrm{thm}}_{\mathcal{D}} \delta : \pi$ then $h(\Gamma) \vdash^{\mathrm{thm}}_{\mathcal{D}'} h(\delta) : h(\pi)$, and if $\Gamma \vdash_{\mathcal{D}} \delta : \pi$ then $h(\Gamma) \vdash_{\mathcal{D}'} h(\delta) : h(\pi)$.*

Example 4. The d-parchment for first-order logic with equality is the functor $\mathit{FOLEq} : \mathbf{Set}^{\mathbb{N}_0} \times \mathbf{Set}^{\mathbb{N}} \to \mathbf{DRoom}$ that maps each $\langle F, P \rangle$ to the d-room $\mathcal{D}_{\mathit{FOLEq}} = \langle \Sigma_{\mathit{FOLEq}}, lR, gR \rangle$, with Σ_{FOLEq} defined as in Example 1 and:

*lR***:** $\xi^1_\phi \Rightarrow (\xi^2_\phi \Rightarrow \xi^1_\phi) : \mathit{univ}$

$(\xi^1_\phi \Rightarrow (\xi^2_\phi \Rightarrow \xi^3_\phi)) \Rightarrow ((\xi^1_\phi \Rightarrow \xi^2_\phi) \Rightarrow (\xi^1_\phi \Rightarrow \xi^3_\phi)) : \mathit{univ}$

$(\neg\xi^2_\phi \Rightarrow \neg\xi^1_\phi) \Rightarrow (\xi^1_\phi \Rightarrow \xi^2_\phi) : \mathit{univ}$

$(\forall x\ \xi^1_\phi) \Rightarrow \xi^2_\phi : \mathit{fts}(\xi^1_\phi, \xi^2_\phi, \xi^1_\tau, x)$

$(\forall x\ (\xi^1_\phi \Rightarrow \xi^2_\phi)) \Rightarrow (\xi^1_\phi \Rightarrow (\forall x\ \xi^2_\phi)) : \mathit{nfv}(\xi^1_\phi, x)$

$\forall x\ (x \doteq x) : \mathit{univ}$

$(x \doteq y) \Rightarrow (\xi^1_\phi \Rightarrow \xi^2_\phi) : \mathit{eqrep}(\xi^1_\phi, \xi^2_\phi, x, y)$

$\frac{\xi^1_\phi \quad \xi^1_\phi \Rightarrow \xi^2_\phi}{\xi^2_\phi} : \mathit{univ}$;

*gR***:** $\frac{\xi^1_\phi}{\forall x\ \xi^1_\phi} : \mathit{univ}$.

The shape of the fourth axiom is unusual. The usual notation that replaces ξ^2_ϕ by $\xi^1_\phi(x := t)$ and requires t to be free for x is fine, informally, but we make it precise with *fts*. Let us deduce $\forall x\, p(x) \vdash_{\mathcal{D}_{FOLEq}} p(x)$, with p a unary predicate.

1. $\langle \forall x\, p(x), univ\rangle$ — Hypothesis
2. $\langle (\forall x\, p(x)) \Rightarrow p(x), univ\rangle$ — Axiom 4
3. $\langle p(x), univ\rangle$ — MP rule:1,2

In step 2, we used the fourth axiom with substitution $\rho_2(\xi^1_\phi) = \forall x\, p(x)$, $\rho_2(\xi^2_\phi) = p(x)$ and $\rho_2(\xi^1_\tau) = x$. Easily, $\rho_2 \in fts(\xi^1_\phi, \xi^2_\phi, \xi^1_\tau, x)_{\Sigma_{FOLEq}}$, and so $fts(\xi^1_\phi, \xi^2_\phi, \xi^1_\tau, x)\rho_2 = univ$. In step 3, we used MP with $\rho_3(\xi^1_\phi) = \forall x\, p(x)$ and $\rho_3(\xi^2_\phi) = (\forall x\, p(x)) \Rightarrow p(x)$.

Example 5. The d-parchment for modal logic is $K : \mathbf{Set} \to \mathbf{DRoom}$, mapping each PS to $\mathcal{D}_K = \langle \Sigma_K, lR, gR\rangle$, with Σ_K defined as in Example 2 and:

***lR*:** $\xi^1_\phi \Rightarrow (\xi^2_\phi \Rightarrow \xi^1_\phi) : univ$
$(\xi^1_\phi \Rightarrow (\xi^2_\phi \Rightarrow \xi^3_\phi)) \Rightarrow ((\xi^1_\phi \Rightarrow \xi^2_\phi) \Rightarrow (\xi^1_\phi \Rightarrow \xi^3_\phi)) : univ$
$(\neg\xi^2_\phi \Rightarrow \neg\xi^1_\phi) \Rightarrow (\xi^1_\phi \Rightarrow \xi^2_\phi) : univ$
$\Box(\xi^1_\phi \Rightarrow \xi^2_\phi) \Rightarrow (\Box\xi^1_\phi \Rightarrow \Box\xi^2_\phi) : univ$
$\frac{\xi^1_\phi \quad \xi^1_\phi \Rightarrow \xi^2_\phi}{\xi^2_\phi} : univ$

***gR*:** $\frac{\xi^1_\phi}{\Box\xi^1_\phi} : univ.$

2.3 Logics

Often, we shall consider a $\mathbf{c}$-parchment $R : \mathbf{Sig} \to \mathbf{CPRoom}$ together with a d-parchment $D : \mathbf{Sig} \to \mathbf{DRoom}$ such that, for each $\Omega \in |\mathbf{Sig}|$, $R(\Omega)$ and $D(\Omega)$ share the same signature Σ_Ω. With $\Phi \cup \{\psi\}$ a set of formulas, we define:

- D is *sound* for R if $\Phi \vdash_\Omega \psi$ implies $\Phi \vDash_\Omega \psi$, for all Ω;
- D is *weakly complete* for R if $\vDash_\Omega \varphi$ implies $\vdash_\Omega \varphi$, for all Ω;
- D is *finitely complete* for R if Φ finite and $\Phi \vDash_\Omega \psi$ imply $\Phi \vdash_\Omega \psi$, for all Ω;
- D is *complete* for R if $\Phi \vDash_\Omega \psi$ implies $\Phi \vdash_\Omega \psi$, for all Ω.

A Σ-rule $r = \langle \Gamma, \delta, \pi\rangle$ is said to be *locally sound* for $\langle \mathcal{A}, \mathbf{c}\rangle \in \mathrm{cAlg}(\Sigma)$ if $[\![\delta\rho]\!]_\mathcal{A} \in \{[\![\gamma\rho]\!]_\mathcal{A} : \gamma \in \Gamma\}^\mathbf{C}$, for every $\rho \in \pi_\Sigma$, and *globally sound* if $[\![\delta\rho]\!]_\mathcal{A} \in \emptyset^\mathbf{C}$ whenever $\{[\![\gamma\rho]\!]_\mathcal{A} : \gamma \in \Gamma\} \subseteq \emptyset^\mathbf{C}$, for every $\rho \in \pi_\Sigma$. If all the rules of a d-room $\mathcal{D}$ are sound for all the structures of a $\mathbf{c}$-room $\mathcal{R}$, i.e., the rules in *lR* are locally sound and the rules in *gR* are globally sound, we say that the rules of $\mathcal{D}$ are sound for $\mathcal{R}$. Obviously, if the rules of $D(\Omega)$ are sound for $R(\Omega)$ for every Ω, then the d-parchment D is sound for the $\mathbf{c}$-parchment R [20, 25, 5].

Definition 5. A *logic-room (l-room, for short)* is a tuple $\mathcal{L} = \langle \Sigma, M, lR, gR\rangle$ where $\mathcal{R}(\mathcal{L}) = \langle \Sigma, M\rangle$ is a $\mathbf{c}$-room and $\mathcal{D}(\mathcal{L}) = \langle \Sigma, lR, gR\rangle$ is a d-room with rules sound for $\mathcal{R}(\mathcal{L})$. A *morphism of l-rooms* from $\mathcal{L}_1$ to $\mathcal{L}_2$ is an $\mathbf{AlgSig}_\phi$-morphism $h : \Sigma_1 \to \Sigma_2$ such that $h : \mathcal{R}(\mathcal{L}_1) \to \mathcal{R}(\mathcal{L}_2)$ is a morphism of $\mathbf{c}$-rooms and $h : \mathcal{D}(\mathcal{L}_1) \to \mathcal{D}(\mathcal{L}_2)$ a morphism of d-rooms.

Logic-rooms and morphisms set up a cocomplete category **LRoom**. The category **LPar** of *logic-parchments (l-parchments)* is again obtained by a Grothendieck construction, and is also cocomplete (see [6]). We write $\models_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}$ for $\models_{\mathcal{R}(\mathcal{L})}$ and $\vdash_{\mathcal{D}(\mathcal{L})}$, and say that $\mathcal{L}$ is (weakly/finitely) complete when $\mathcal{D}(\mathcal{L})$ is, for $\mathcal{R}(\mathcal{L})$. By definition, all l-rooms are sound. Given a l-parchment $L : \mathbf{Sig} \to \mathbf{LRoom}$ and $\Omega \in |\mathbf{Sig}|$, we denote $\models_{L(\Omega)}$ and $\vdash_{L(\Omega)}$ simply by $\models_{\Omega}$ and $\vdash_{\Omega}$.

In [20, 25] we have noted that fibring is very sensitive to the way logics are presented, leading sometimes to the so-called *collapsing problem* [23]. Deductively, this difficulty can be dealt with by an appropriate use of provisos. Semantically, a way to deal with the possible trivialization of fibred logics is to require certain *fullness* conditions, guaranteeing that the logics have "enough" models [25]. Given $\Sigma \in |\mathbf{AlgSig}_{\phi}|$, let $\mathcal{I} \subseteq \mathrm{cAlg}(\Sigma)$ be a class of *intended* structures.

Definition 6. A l-room $\mathcal{L} = \langle \Sigma, M, lR, gR \rangle$ is *full* for $\mathcal{I}$ if M contains every $\langle \mathcal{A}, \mathbf{c} \rangle \in \mathcal{I}$ for which the rules of $\mathcal{D}(\mathcal{L})$ are sound.

Although fullness seems to be a fairly strong requirement, making a l-room full for $\mathcal{I}$ is a well-behaved operation. Given $\mathcal{L} = \langle \Sigma, M, lR, gR \rangle$, its full version is $\overline{\mathcal{L}} = \langle \Sigma, \overline{M}, lR, gR \rangle$ with $\overline{M} = M \cup \{\langle \mathcal{A}, \mathbf{c} \rangle \in \mathcal{I} : \mathcal{D}(\mathcal{L}) \text{ is sound for } \langle \mathcal{A}, \mathbf{c} \rangle\}$. This definition easily extends to an endofunctor in **LPar**. More important is the fact that soundness and completeness carry over from $\mathcal{L}$ to $\overline{\mathcal{L}}$. In general, we have that $\vdash_{\mathcal{L}} = \vdash_{\overline{\mathcal{L}}} \subseteq \models_{\overline{\mathcal{L}}} \subseteq \models_{\mathcal{L}}$, meaning that $\models_{\overline{\mathcal{L}}}$ can be weaker than $\models_{\mathcal{L}}$ but if that happens $\overline{\mathcal{L}}$ is closer to being complete. Later, in the context of fibring, we shall consider several interesting choices of intended structures. For now, we just consider the unrestricted class of all interpretation structures.

Example 6. The l-parchment *FOLEq* of first-order logic with equality maps each $\langle F, P \rangle$ to the room $\mathcal{L}_{FOLEq} = \langle \Sigma_{FOLEq}, M, lR, gR \rangle$, with $\mathcal{R}_{FOLEq} = \langle \Sigma_{FOLEq}, M \rangle$ and $\mathcal{D}_{FOLEq} = \langle \Sigma_{FOLEq}, lR, gR \rangle$ as in Examples 1 and 4, well known to be sound and complete [14]. Its full unrestricted version $\overline{FOLEq}$ considers, instead, the room $\mathcal{L}_{\overline{FOLEq}} = \langle \Sigma_{FOLEq}, \overline{M}, lR, gR \rangle$ where $\overline{M}$ contains all structures making $\mathcal{D}_{FOLEq}$ sound, and not just the usual structures of first-order logic already in M. By construction, $\overline{FOLEq}$ is also sound and complete.

Example 7. The l-parchment K of modal logic assigns $\mathcal{L}_K = \langle \Sigma_K, M, lR, gR \rangle$, with $\mathcal{R}_K = \langle \Sigma_K, M \rangle$ and $\mathcal{D}_K = \langle \Sigma_K, lR, gR \rangle$ as in Examples 2 and 5, to each PS and is sound and complete [12]. The full unrestricted version $\overline{K}$ considers, instead, the room $\mathcal{L}_{\overline{K}} = \langle \Sigma_K, \overline{M}, lR, gR \rangle$ where $\overline{M}$ is the class of all structures for which $\mathcal{D}_K$ is sound. Besides the Kripke structures in M, $\overline{M}$ also contains, for instance, all modal algebras. $\overline{K}$ is also sound and complete.

3 Fibring

We already know that colimits of parchments are built from colimits of rooms. As in [6], we characterize fibring using colimits, and so we concentrate just on rooms. Obviously, all the characterizations to follow can be immediately lifted to parchments. When considering two rooms with signatures $\Sigma_1 = \langle S_1, O_1\rangle$ and $\Sigma_2 = \langle S_2, O_2\rangle$, we assume that their fibring is *constrained* by sharing the sorts and constructors in their largest common subsignature $\Sigma_0 = \langle S_0, O_0\rangle$, with $S_0 = S_1 \cap S_2$ (it always includes ϕ) and $O_{0,u} = O_{1,u} \cap O_{2,u}$, for $u \in S_0^+$, according to the corresponding inclusions $h_1 : \Sigma_0 \to \Sigma_1$ and $h_2 : \Sigma_0 \to \Sigma_2$. Below, $\mathcal{R}_0 = \langle \Sigma_0, M_0\rangle$ with $M_0 = \text{cAlg}(\Sigma_0)$, $\mathcal{D}_0 = \langle \Sigma_0, \emptyset, \emptyset\rangle$ and $\mathcal{L}_0 = \langle \Sigma_0, M_0, \emptyset, \emptyset\rangle$.

The envisaged combined signature is $\Sigma_1 \circledast \Sigma_2 = \langle S, O\rangle$ such that $S = S_1 \cup S_2$, with inclusions $f_i : S_i \to S$, and $O_u = O_{1,u} \cup O_{2,u}$ if $u \in S_0^+$, $O_u = O_{i,u}$ if $u \in S_i^+ \setminus S_0^+$, with inclusions $g_i : O_i \to O$. Easily, $\Sigma_1 \circledast \Sigma_2$ is a pushout of $\{h_i : \Sigma_0 \to \Sigma_i\}_{i\in\{1,2\}}$ in $\mathbf{AlgSig}_\phi$, with inclusions $\langle f_i, g_i\rangle : \Sigma_i \to \Sigma_1 \circledast \Sigma_2$. When $S_0 = \{\phi\}$ and $O_0 = \emptyset$ we say that the fibring is *unconstrained*, and the construction corresponds to a coproduct in $\mathbf{AlgSig}_\phi$. The fibring of two $\mathbf{c}$-rooms $\mathcal{R}_1 = \langle \Sigma_1, M_1\rangle$ and $\mathcal{R}_2 = \langle \Sigma_2, M_2\rangle$ is $\mathcal{R}_1 \circledast \mathcal{R}_2 = \langle \Sigma_1 \circledast \Sigma_2, M_1 \circledast M_2\rangle$, where $M_1 \circledast M_2$ is the class of all structures $\langle \mathcal{A}, \mathbf{c}\rangle \in \text{cAlg}(\Sigma_1 \circledast \Sigma_2)$ such that both $\langle \mathcal{A}|_{\langle f_1,g_1\rangle}, \mathbf{c}\rangle \in M_1$ and $\langle \mathcal{A}|_{\langle f_2,g_2\rangle}, \mathbf{c}\rangle \in M_2$, i.e., $M_1 \circledast M_2$ is obtained by joining together $\langle \mathcal{A}_1, \mathbf{c}_1\rangle \in M_1$ and $\langle \mathcal{A}_2, \mathbf{c}_2\rangle \in M_2$ with $|\mathcal{A}_1|_s = |\mathcal{A}_2|_s = |\mathcal{A}|_s$ for $s \in S_0$, $o_{\mathcal{A}_1} = o_{\mathcal{A}_2} = o_{\mathcal{A}}$ for $o \in O_{0,u}$ with $u \in S_0^+$, and $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}$. The fibring $\mathcal{R}_1 \circledast \mathcal{R}_2$ is a pushout of $\{h_i : \mathcal{R}_0 \to \mathcal{R}_i\}_{i\in\{1,2\}}$ in **CPRoom**, as proved in [6], where a similar characterization for propositional-based proof-calculi, without provisos, was also given. To generalize the characterization, let $\mathcal{D}_1 = \langle \Sigma_1, lR_1, gR_1\rangle$ and $\mathcal{D}_2 = \langle \Sigma_2, lR_2, gR_2\rangle$ be d-rooms and $\mathcal{D}_1 \circledast \mathcal{D}_2$ their fibring.

Definition 7. $\mathcal{D}_1 \circledast \mathcal{D}_2 = \langle \Sigma_1 \circledast \Sigma_2, lR_1 \circledast lR_2, gR_1 \circledast gR_2\rangle$ where $lR_1 \circledast lR_2 = \langle f_1, g_1\rangle(lR_1) \cup \langle f_2, g_2\rangle(lR_2)$ and $gR_1 \circledast gR_2 = \langle f_1, g_1\rangle(gR_1) \cup \langle f_2, g_2\rangle(gR_2)$.

Proposition 2. *$\mathcal{D}_1 \circledast \mathcal{D}_2$ is a pushout of $\{h_i : \mathcal{D}_0 \to \mathcal{D}_i\}_{i\in\{1,2\}}$ in* **DRoom**.

Fibred l-rooms capitalize on the characterizations above but, first, we need to note that soundness of rules is preserved. Let $\mathcal{L}_1 = \langle \Sigma_1, M_1, lR_1, gR_1\rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, M_2, lR_2, gR_2\rangle$ be l-rooms and $\mathcal{L}_1 \circledast \mathcal{L}_2$ their fibring.

Theorem 1. *The rules of $\mathcal{D}(\mathcal{L}_1) \circledast \mathcal{D}(\mathcal{L}_2)$ are sound for $\mathcal{R}(\mathcal{L}_1) \circledast \mathcal{R}(\mathcal{L}_2)$.*

It is now safe to state the definition of fibred l-room.

Definition 8. $\mathcal{L}_1 \circledast \mathcal{L}_2 = \langle \Sigma_1 \circledast \Sigma_2, M_1 \circledast M_2, lR_1 \circledast lR_2, gR_1 \circledast gR_2\rangle$.

Proposition 3. *$\mathcal{L}_1 \circledast \mathcal{L}_2$ is a pushout of $\{h_i : \mathcal{L}_0 \to \mathcal{L}_i\}_{i\in\{1,2\}}$ in* **LRoom**.

To see fibring interact with fullness, consider a system of intended structures $\mathcal{I}_1 \subseteq \text{cAlg}(\Sigma_1)$, $\mathcal{I}_2 \subseteq \text{cAlg}(\Sigma_2)$, $\mathcal{I} \subseteq \text{cAlg}(\Sigma_1 \circledast \Sigma_2)$ satisfying a coherence requirement: if $\langle \mathcal{A}, \mathbf{c}\rangle \in \mathcal{I}$ then $\langle \mathcal{A}|_{\langle f_1,g_1\rangle}, \mathbf{c}\rangle \in \mathcal{I}_1$ and $\langle \mathcal{A}|_{\langle f_2,g_2\rangle}, \mathbf{c}\rangle \in \mathcal{I}_2$. The result below is an immediate consequence of the definition of fibring.

Proposition 4. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for $\mathcal{I}_1$ and $\mathcal{I}_2$ then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is full for $\mathcal{I}$.*

Example 8. We obtain a l-room $\mathcal{L}_{\overline{FOLEq}} \circledast \mathcal{L}_{\overline{K}}$ for modal first-order logic by fibring the full versions of modal and first-order logic of Examples 7 and 6. The combined signature $\Sigma_{FOLEq} \circledast \Sigma_K = \langle S, O\rangle$ has:

– $S = \{\tau, \phi\}$, $O_\tau = X \cup F_0$, $O_{\tau^n\tau} = F_n$ for $n > 0$, $O_\phi = \text{PS}$, $O_{\tau^2\phi} = P_2 \cup \{\doteq\}$, $O_{\tau^n\phi} = P_n$ for $n > 0$ and $n \neq 2$, $O_{\phi\phi} = \{\neg, \Box\} \cup \{\forall x : x \in X\}$, $O_{\phi^2\phi} = \{\Rightarrow\}$.

Consider the usual interpretation structures of modal first-order logic with constant domain and rigid interpretation of symbols, i.e., the first-order component does not change from one world to the other. It corresponds to considering both a *FOLEq* interpretation $\langle D, I\rangle$ and a Kripke model $\langle W, R, \vartheta\rangle$. It is easy to see that among the structures of $\mathcal{L}_{\overline{FOLEq}} \circledast \mathcal{L}_{\overline{K}}$ we can find the $\langle \mathcal{A}, \mathbf{c}\rangle$ such that:

– $|\mathcal{A}|_\tau = D^{W \times \text{Asg}(X,D)}$ and $|\mathcal{A}|_\phi = \wp(W \times \text{Asg}(X,D))$; $x_\mathcal{A}(w,\mu) = \mu(x)$, $f_\mathcal{A}(\langle e_i\rangle)(w,\mu) = f_I(\langle e_i(w,\mu)\rangle)$, $q_\mathcal{A} = \vartheta(q) \times \text{Asg}(X,D)$, $p_\mathcal{A}(\langle e_i\rangle) = \{\langle w,\mu\rangle : \langle e_i(w,\mu)\rangle \in p_I\}$, $\neg_\mathcal{A}(b) = (W \times \text{Asg}(X,D)) \setminus b$, $\Rightarrow_\mathcal{A}(b_1,b_2) = ((W \times \text{Asg}(X,D)) \setminus b_1) \cup b_2$, $\forall x_\mathcal{A}(b) = \{\langle w,\mu\rangle : \langle w, \mu[x/d]\rangle \in b \text{ for every } d \in D\}$, and $\Box_\mathcal{A}(b) = \{\langle w,\mu\rangle : \{\langle w',\mu\rangle : wRw'\} \subseteq b\}$; $\mathbf{c} : \wp(|\mathcal{A}|_\phi) \to \wp(|\mathcal{A}|_\phi)$ is the cut closure operation induced by $\supseteq$.

This structure makes all the rules of $\mathcal{D}_{FOLEq} \circledast \mathcal{D}_K$ sound, but other usual modal first-order semantic structures could be considered (see Example 9). But now, the reason why we considered the full versions is obvious: the structure above is in $\mathcal{L}_{\overline{FOLEq}} \circledast \mathcal{L}_{\overline{K}}$ but certainly not in $\mathcal{L}_{FOLEq} \circledast \mathcal{L}_K$.

4 Completeness

Again we concentrate on l-rooms, since everything can be lifted to l-parchments. The results in this section generalize [6] and make thorough use of the notion of fullness. The first result applies to l-rooms full for the class of all structures.

Proposition 5. *If $\mathcal{L}$ is full for all structures then $\mathcal{L}$ is complete.*

Proof. The rules of $\mathcal{D}(\mathcal{L})$ are sound for $\langle \mathcal{W}_\Sigma, \mathbf{c}\rangle$ with $\mathbf{c} = \vdash_\mathcal{L}$. Thus, by fullness, the structure $\langle \mathcal{W}_\Sigma, \mathbf{c}\rangle$ belongs to $\mathcal{L}$. Suppose now that $\Phi \nvdash_\mathcal{L} \varphi$. To show that $\Phi \not\models_\mathcal{L} \varphi$ it is enough to note that $[\![\_]\!]_{\mathcal{W}_\Sigma}$ is the identity on formulas. □

The system $\mathcal{I}_1 = \text{cAlg}(\Sigma_1)$, $\mathcal{I}_2 = \text{cAlg}(\Sigma_2)$ and $\mathcal{I} = \text{cAlg}(\Sigma_1 \circledast \Sigma_2)$ of intended structures trivially satisfies the necessary coherence requirement.

Proposition 6. *Fullness for all structures is preserved by fibring.*

The first completeness transfer result follows from Propositions 5 and 6.

Theorem 2. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for all structures then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is complete.*

Although this result is a bit too syntactic (see the structure in Proposition 5), its proof is enlightening. Reusing the technique of [25], we have shown that if a l-room has certain properties then it is complete (Proposition 5), and also that the relevant properties are preserved by fibring (Proposition 6). All the subsequent completeness preservation results follow the same pattern.

We start by considering a reasonable syntactic restriction. A signature $\Sigma = \langle S, O\rangle$ is said to be *plain* if for every $o \in O_{us}$ with $s \neq \phi$, $u \in (S \setminus \{\phi\})^*$. Plainhood of signatures prevents us from building terms using formulas.

Proposition 7. *If Σ_1 and Σ_2 are plain signatures then so is $\Sigma_1 \circledast \Sigma_2$.*

A closure operation $\langle A, \mathbf{c}\rangle$ is said to be *elementary* if for every $a_1, a_2 \in A$, $a_1 \in \{a_2\}^{\mathbf{c}}$ and $a_2 \in \{a_1\}^{\mathbf{c}}$ imply $a_1 = a_2$. Clearly, the structure in Proposition 5 is not elementary. Let us take as intended the class of all structures whose closure is elementary. The corresponding system of intended structures clearly fulfills the coherence requirement and fullness for this class is preserved by fibring.

Proposition 8. *Fullness for elementary structures is preserved by fibring.*

The following characterizations are a simple reformulation from [25, 6, 5]. A d-room $\mathcal{D}$ over $\Sigma = \langle S, O\rangle$ is said to have *implication* if there exists $\Rightarrow \in O_{\phi^2\phi}$ satisfying: (i) $\vdash_{\mathcal{D}} \xi^1_\phi \Rightarrow \xi^1_\phi$, (ii) $\xi^1_\phi, \xi^1_\phi \Rightarrow \xi^2_\phi \vdash_{\mathcal{D}} \xi^2_\phi$, (iii) $\xi^2_\phi \vdash_{\mathcal{D}} \xi^1_\phi \Rightarrow \xi^2_\phi$, (iv) for each $\langle \Gamma, \delta, \pi\rangle \in lR$ and ξ^n_ϕ not in $\Gamma \cup \{\delta\}$, $\{\xi^n_\phi \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{\mathcal{D}} \xi^n_\phi \Rightarrow \delta : \pi$.

In the sequel, $\mathcal{D}$ is said to be *formula-congruent* if it has implication and we have that, for every $o \in O_{s_1 \ldots s_n \phi}$ and $s_i = \phi$, $\xi^i_\phi \Rightarrow \xi^0_\phi, \xi^0_\phi \Rightarrow \xi^i_\phi \vdash^{\text{thm}}_{\mathcal{D}}$ $o(\xi^1_{s_1}, \ldots, \xi^{i-1}_{s_{i-1}}, \xi^i_\phi, \xi^{i+1}_{s_{i+1}}, \ldots, \xi^n_{s_n}) \Rightarrow o(\xi^1_{s_1}, \ldots, \xi^{i-1}_{s_{i-1}}, \xi^0_\phi, \xi^{i+1}_{s_{i+1}}, \ldots, \xi^n_{s_n})$. The next result follows from Proposition 1.

Proposition 9. *If $\mathcal{D}_1$ and $\mathcal{D}_2$ are formula-congruent and share an implication then $\mathcal{D}_1 \circledast \mathcal{D}_2$ is formula-congruent.*

We are now able to present the following completeness result.

Proposition 10. *If $\mathcal{L}$ is full for elementary structures, has a plain signature, and $\mathcal{D}(\mathcal{L})$ is formula-congruent then $\mathcal{L}$ is complete.*

Proof. Let Σ be the signature of $\mathcal{L}$. Easily, $\equiv_\phi$ defined on $|\mathcal{W}_\Sigma|_\phi$ by $\varphi_1 \equiv_\phi \varphi_2$ if $\{\varphi_1\} \vdash_{\mathcal{L}} \varphi_2$ and $\{\varphi_2\} \vdash_{\mathcal{L}} \varphi_1$ is an equivalence. Since $\mathcal{D}(\mathcal{L})$ is formula-congruent and Σ is plain, $\equiv = \{\equiv_s\}_{s \in S}$ with $\equiv_s$ the identity if $s \neq \phi$ is a congruence on $\mathcal{W}_\Sigma$. Let us consider the Lindenbaum-Tarski structure $\langle \mathcal{W}_\Sigma/\equiv, \mathbf{c}\rangle$, with $\mathbf{c}$ defined by $\{[\psi] : \psi \in \Psi\}^{\mathbf{c}} = \{[\psi'] : \Psi \vdash_{\mathcal{L}} \psi'\}$. Clearly $\mathcal{D}(\mathcal{L})$ is sound for $\langle \mathcal{W}_\Sigma/\equiv, \mathbf{c}\rangle$, which is elementary and, by fullness, belongs to $\mathcal{L}$. Suppose that, $\Phi \not\vdash_{\mathcal{L}} \varphi$. The structure just built shows that $\Phi \not\vDash_{\mathcal{L}} \psi$. □

Theorem 3. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for elementary structures, Σ_1 and Σ_2 are plain, and $\mathcal{D}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ are formula-congruent and share an implication then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is complete.*

Let us try to improve this result. In logic, algebras of truth-values are often partially ordered. Every partial-order $\langle A, \leq\rangle$ induces two polarities $\text{Upp}(B) = \{a \in A : b \leq a \text{ for every } b \in B\}$ and $\text{Low}(B) = \{a \in A : a \leq b \text{ for every } b \in B\}$, and a cut closure operation on A defined by $B^{\mathbf{c}} = \text{Upp}(\text{Low}(B))$ [2]. *Partial-order* structures are precisely those whose closure operation fulfills this condition.

Proposition 11. *Fullness for partial-order structures is preserved by fibring.*

We can now present the following preservation results.

Proposition 12. *If $\mathcal{L}$ is full for partial-order structures, has a plain signature, and $\mathcal{D}(\mathcal{L})$ is formula-congruent then $\mathcal{L}$ is weakly complete.*

Proof. The proof is similar to Proposition 10, but with a different closure. Note that $[\varphi_1] \leq [\varphi_2]$ if $\vdash_{\mathcal{L}} \varphi_1 \Rightarrow \varphi_2$ defines a partial-order on $|\mathcal{W}_\Sigma/{\equiv}|_\phi$. Consider the structure $\langle \mathcal{W}_\Sigma/{\equiv}, \mathbf{c}\rangle$ where $\mathbf{c}$ is the cut closure induced by $\leq$. Let us check, in this less trivial case, that the structure makes the rules of $\mathcal{D}(\mathcal{L})$ sound. Consider a rule $r = \frac{\gamma_1 \dots \gamma_n}{\delta} : \pi$ and fix a substitution $\rho \in \pi_\Sigma$. Assume that r is an l-rule. Since $[\![\_]\!]_{\mathcal{W}_\Sigma/\equiv} = [\_]$, we need to show that $[\delta\rho] \in \{[\gamma_1\rho], \dots, [\gamma_n\rho]\}^{\mathbf{c}}$. Let φ be a formula such that $[\varphi] \leq [\gamma_i\rho]$ for $i = 1, \dots, n$. This means that $\vdash_{\mathcal{L}} \varphi \Rightarrow \gamma_i\rho$ for each i. Using requirement (iv) of implication, $\{\xi^n_\phi \Rightarrow \gamma_i : i = 1, \dots, n\} \vdash_{\mathcal{L}} \xi^n_\phi \Rightarrow \delta : \pi$ for ξ^n_ϕ not in r. Consider $\rho' \in \mathrm{Sub}(\Sigma)$ such that ρ' equals ρ, except that $\rho'(\xi^n_\phi) = \varphi$. Clearly, $\gamma_i\rho = \gamma_i\rho'$ and $\delta\rho = \delta\rho'$. Moreover, π is insensitive to ξ^n_ϕ and $\rho' \in \pi_\Sigma$. By the structurality of deducibility and the fact that $\pi\rho' = \mathit{univ}$, $\{\varphi \Rightarrow \gamma_i\rho : i = 1, \dots, n\} \vdash_{\mathcal{L}} \varphi \Rightarrow \delta\rho$. So $\vdash_{\mathcal{L}} \varphi \Rightarrow \delta\rho$, or equivalently, $[\varphi] \leq [\delta\rho]$, and the l-rule is sound. Assume now that r is a g-rule. We need to show that $\{[\gamma_1\rho], \dots, [\gamma_n\rho]\} \subseteq \emptyset^{\mathbf{c}}$ implies $[\delta\rho] \in \emptyset^{\mathbf{c}}$. Easily, $\emptyset^{\mathbf{c}}$ has precisely one element, the equivalence class of formulas ψ such that $\vdash_{\mathcal{L}} \psi$. If $\vdash_{\mathcal{L}} \gamma_i\rho$ for each i, then by using r, we conclude that $\vdash_{\mathcal{L}} \delta\rho$ and the g-rule is sound. By fullness, the structure belongs to $\mathcal{R}(\mathcal{L})$. So, if $\not\vdash_{\mathcal{L}} \psi$ this structure shows that $\not\vDash_{\mathcal{L}} \psi$. □

Theorem 4. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for partial-order structures, Σ_1 and Σ_2 are plain, and $\mathcal{D}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ are formula-congruent and share an implication then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is weakly complete.*

A little improvement is still possible. A d-room $\mathcal{D}$ over $\Sigma = \langle S, O\rangle$ is said to have *conjunction* if there exists $\wedge \in O_{\phi^2\phi}$ such that: (i) $\xi^1_\phi \wedge \xi^2_\phi \vdash_{\mathcal{D}} \xi^1_\phi$, (ii) $\xi^1_\phi \wedge \xi^2_\phi \vdash_{\mathcal{D}} \xi^2_\phi$, (iii) $\xi^1_\phi, \xi^2_\phi \vdash_{\mathcal{D}} \xi^1_\phi \wedge \xi^2_\phi$.

Proposition 13. *If $\mathcal{D}_1$ or $\mathcal{D}_2$ have conjunction then so has $\mathcal{D}_1 \circledast \mathcal{D}_2$.*

Proposition 14. *If $\mathcal{L}$ is full for partial-order structures, has a plain signature, and $\mathcal{D}(\mathcal{L})$ is formula-congruent and has conjunction then $\mathcal{L}$ is finitely complete.*

Proof. Consider the structure of Proposition 12, and suppose $\{\varphi_1, \dots, \varphi_n\} \not\vdash_{\mathcal{L}} \psi$. With $\varphi = \varphi_1 \wedge \dots \wedge \varphi_n$, it is trivial that $\vdash_{\mathcal{L}} \varphi \Rightarrow \varphi_i$ for each i. Easily, it is also the case that $\not\vdash_{\mathcal{L}} \varphi \Rightarrow \psi$ and the structure shows that $\{\varphi_1, \dots, \varphi_n\} \not\vDash_{\mathcal{L}} \psi$. □

Theorem 5. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for partial-order structures, Σ_1 and Σ_2 are plain, $\mathcal{D}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ are formula-congruent, share an implication, and one of them has conjunction then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is finitely complete.*

All the previous results are still valid if we concentrate only on structures providing a standard interpretation of equality, when it exists. A signature $\Sigma = \langle S, O\rangle \in |\mathbf{AlgSig}_\phi|$ is said to have a *system of equalities* if, for every $s \in S \setminus \{\phi\}$, there exists $\doteq \in O_{s^2\phi}$. The existence of equality symbols is preserved by fibring.

Proposition 15. *If Σ_1 and Σ_2 have systems of equalities then so has $\Sigma_1 \circledast \Sigma_2$.*

If Σ has a system of equalities, then $\langle \mathcal{A}, \mathbf{c}\rangle \in \mathrm{cAlg}(\Sigma)$ is said to be *standard* for equality if: (i) if $a_1 \doteq_{\mathcal{A}} a_2 \in \emptyset^{\mathbf{c}}$ then $a_1 = a_2$, (ii) for $T \subseteq |\mathcal{A}|_\phi$, the congruence $\equiv_T$ on $\mathcal{A}$ generated by $R_T = \{\langle a_1, a_2\rangle : a_1 \doteq_{\mathcal{A}} a_2 \in T^{\mathbf{c}}\}$ is such that $\equiv_{T,s} = R_T \cap (|\mathcal{A}|_s \times |\mathcal{A}|_s)$ for $s \neq \phi$. The conditions mean that $\mathbf{c}$ captures precisely the congruence imposed by the equalities.

Proposition 16. *Fullness for standard structures is preserved by fibring.*

Now, of course, we should require a similar standard treatment of equality at the deductive level. A d-room $\mathcal{D} = \langle \Sigma, lR, gR \rangle$ is said to have *equality* if Σ has a system of equality symbols and the following hold: (i) $\vdash_{\mathcal{D}} \xi^1_s \doteq \xi^1_s$, (ii) $\xi^1_s \doteq \xi^2_s \vdash_{\mathcal{D}} \xi^2_s \doteq \xi^1_s$, (iii) $\xi^1_s \doteq \xi^2_s, \xi^2_s \doteq \xi^3_s \vdash_{\mathcal{D}} \xi^1_s \doteq \xi^3_s$, (iv) for every $o \in O_{s_1 \dots s_n s}$ with $s \neq \phi$ and every $i \in \{1, \dots, n\}$ with $s_i \neq \phi$, $\xi^i_{s_i} \doteq \xi^0_{s_i} \vdash_{\mathcal{D}} o(\xi^1_{s_1}, \dots, \xi^{i-1}_{s_{i-1}}, \xi^i_{s_i}, \xi^{i+1}_{s_{i+1}}, \dots, \xi^n_{s_n}) \doteq o(\xi^1_{s_1}, \dots, \xi^{i-1}_{s_{i-1}}, \xi^0_{s_i}, \xi^{i+1}_{s_{i+1}}, \dots, \xi^n_{s_n})$, and last but not least (v) for every $o \in O_{s_1 \dots s_n \phi}$ and every $i \in \{1, \dots, n\}$ with $s_i \neq \phi$, $\xi^i_{s_i} \doteq \xi^0_{s_i}, o(\xi^1_{s_1}, \dots, \xi^{i-1}_{s_{i-1}}, \xi^i_{s_i}, \xi^{i+1}_{s_{i+1}}, \dots, \xi^n_{s_n}) \vdash_{\mathcal{D}} o(\xi^1_{s_1}, \dots, \xi^{i-1}_{s_{i-1}}, \xi^0_{s_i}, \xi^{i+1}_{s_{i+1}}, \dots, \xi^n_{s_n})$.

Proposition 17. *If $\mathcal{D}_1$ and $\mathcal{D}_2$ have equality then so has $\mathcal{D}_1 \circledast \mathcal{D}_2$.*

We can now state the following completeness result.

Proposition 18. *If $\mathcal{L}$ is full for standard elementary structures, has a plain signature, and $\mathcal{D}(\mathcal{L})$ is formula-congruent and has equality then $\mathcal{L}$ is complete.*

Proof. For each $s \in S \backslash \{\phi\}$, $\equiv_s$ such that $t_1 \equiv_s t_2$ if $\vdash_{\mathcal{L}} t_1 \doteq t_2$ is an equivalence on $|\mathcal{W}_\Sigma|_s$. Considering $\equiv = \{\equiv_s\}_{s \in S}$ with $\equiv_\phi$ as in Proposition 10, and noting that $\mathcal{D}(\mathcal{L})$ has equality, we conclude that $\equiv$ is a congruence on $\mathcal{W}_\Sigma$. Consider $\langle \mathcal{W}_\Sigma/{\equiv}, \mathbf{c} \rangle$, with $\mathbf{c}$ defined as in Proposition 10. The structure makes $\mathcal{D}(\mathcal{L})$ sound and is elementary. We now prove that it is standard for equality. Given $[t_1], [t_2] \in |\mathcal{W}_\Sigma/{\equiv}|_s$, if $[t_1] \doteq_{\mathcal{W}_\Sigma/\equiv} [t_2] \in \emptyset^{\mathbf{c}}$ then $\vdash_{\mathcal{L}} t_1 \doteq t_2$ and, by definition of $\equiv_s$, $t_1 \equiv_s t_2$ and $[t_1] = [t_2]$. Given $T = \{[\psi] : \psi \in \Psi\} \subseteq |\mathcal{W}_\Sigma/{\equiv}|_\phi$ and $R_T = \{\langle a_1, a_2 \rangle : a_1 \doteq_{\mathcal{W}_\Sigma/\equiv} a_2 \in T^{\mathbf{c}}\}$, let $R_{T,s} = R_T \cap (|\mathcal{A}|_s \times |\mathcal{A}|_s)$ for each $s \neq \phi$ and recall that $T^{\mathbf{c}} = \{[\psi'] : \Psi \vdash_{\mathcal{D}_1 \circledast \mathcal{D}_2} \psi'\}$. Consider the congruence $\equiv_T$ generated by R_T. By definition, $R_{T,s} \subseteq \equiv_{T,s}$. Since $\mathcal{D}(\mathcal{L})$ has equality, it is easy to see that $R_{T,s}$ is an equivalence and $\langle [o(t_1, \dots, t_i, \dots, t_n)], [o(t_1, \dots, t'_i, \dots, t_n)] \rangle \in R_{T,s}$ for each $o \in O_{s_1 \dots s_n s}$ with $s \neq \phi$ (and, since Σ is plain, each $s_i \neq \phi$) whenever $\langle [t_i], [t'_i] \rangle \in R_{T,s_i}$. So, $\equiv_{T,s} \subseteq R_{T,s}$, $\langle \mathcal{W}_\Sigma/{\equiv}, \mathbf{c} \rangle$ is standard and belongs to $\mathcal{L}$, by fullness. As before, if $\Phi \not\vdash_{\mathcal{L}} \varphi$, $\langle \mathcal{W}_\Sigma/{\equiv}, \mathbf{c} \rangle$ clearly shows that $\Phi \not\vDash_{\mathcal{L}} \psi$. ☐

Completeness preservation results for elementary, or partial-order, structures, assuming herein that they are also standard for equality, easily follow.

Theorem 6. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for standard elementary structures, Σ_1 and Σ_2 are plain, and $\mathcal{D}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ are formula-congruent, with equality and a shared implication then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is complete.*

Theorem 7. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for standard partial-order structures, Σ_1 and Σ_2 are plain, and $\mathcal{D}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ are formula-congruent, with equality and a shared implication then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is weakly complete.*

Theorem 8. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are full for standard partial-order structures, Σ_1 and Σ_2 are plain, $\mathcal{D}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ are formula-congruent, with equality, a shared implication, and one of them has conjunction then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is finitely complete.*

Example 9. In Example 8 we obtained a system of modal first-order logic by fibring full versions of propositional modal logic and first-order logic. It is well known (e.g., [12]) that the structures considered therein make both the Barcan formula $(\forall x(\Box \xi^n_\phi)) \Rightarrow (\Box(\forall x \xi^n_\phi))$ and its converse $(\Box(\forall x \xi^n_\phi)) \Rightarrow (\forall x(\Box \xi^n_\phi))$ sound. However, although the latter is deducible from $\mathcal{D}_{FOLEq} \circledast \mathcal{D}_K$, the former is not. Since our completeness preservation results apply, $\mathcal{L}_{\overline{FOLEq}} \circledast \mathcal{L}_{\overline{K}}$ is complete and must contain structures where the Barcan formula fails. This is the case for the expanding domains interpretations of [12], with an extra component $Q : W \to \wp(D)$ that assigns to each world a domain of interpretation such that if wRw' then $Q(w) \subseteq Q(w')$. We denote by D_w the set $Q(w)$, by $\mathrm{Asg}(X, D)_w$ the set D_w^X of assignments in D_w, and by U the set $\{\langle w, \mu\rangle : w \in W \text{ and } \mu \in \mathrm{Asg}(X, D)_w\}$. Then, $\langle \mathcal{A}, \mathbf{c}\rangle$ is defined just as in Example 8, considering $\mathcal{A}_\tau = D^U$, $\mathcal{A}_\phi = \wp(U)$ and $\forall x_{\mathcal{A}}(b) = \{\langle w, \mu\rangle : \{\langle w, \mu[x/d]\rangle : d \in D_w\} \subseteq b\}$.

One important aspect of the structures considered so far is that the interpretation of symbols is rigid. However, if we consider flexible symbols, we must proceed with caution. As noted for instance in [22], some axioms of *FOLEq* do not behave well in the presence of flexible symbols. Consider the following instance of the fourth *FOLEq* axiom, $(\forall x\ ((c \doteq x) \Rightarrow \Diamond(c > x))) \Rightarrow ((c \doteq c) \Rightarrow \Diamond(c > c))$, where c is a flexible symbol and $>$ is an irreflexive ordering. It is easy to find a structure that falsifies the formula. The problem arises when we try to replace a variable by a flexible term (in this case c) in the scope of a modality. One way to avoid this problem is to strengthen the proviso as follows:

– $fts(\xi^n_\phi, \xi^m_\phi, \xi^k_\tau, x)$ is such that, given $h : \Sigma_{FOLEq} \to \Sigma'$, $\rho \in fts(\xi^n_\phi, \xi^m_\phi, \xi^k_\tau, x)_h$ iff $\rho(\xi^k_\tau)$ free for $h(x)$ in $\rho(\xi^n_\phi)$ and $\rho(\xi^m_\phi)$ results from $\rho(\xi^n_\phi)$ by replacing by $\rho(\xi^k_\tau)$ the free occurrences of $h(x)$. Plus, if $\rho(\xi^k_\tau)$ is a Σ_{FOLEq}-term, then no free occurrence of $h(x)$ can appear in the scope of a $\Sigma' \setminus h(\Sigma_{FOLEq})$-symbol.

In our example, this means that no term of Σ_{FOLEq} may be replaced in the scope of a modality. This change has no impact whatsoever on first-order logic *per se*, but makes a huge difference when we combine it with modal logic. Likewise, the fifth axiom in $\mathcal{D}_{FOLEq}$ must also be changed to prevent $\rho(\xi^n_\phi)$ to contain modalities. If we consider equality, *eqrep* must also be changed so that [13]: "if x occurs free in the scope of a modal operator, then either all or no occurrence of x may be replaced by y". With these changes, the corresponding fibred system includes structures $\langle \mathcal{A}, \mathbf{c}\rangle$ defined from $\langle W, R, \vartheta, D, Q, I\rangle$ with $f_I = \{f_{I,w} : D^n_w \to D_w\}_{w\in W}$ for $f \in F_n$, and $p_I = \{p_{I,w}\}_{w\in W}$ with $p_{I,w} \subseteq D^n_w$ for $p \in P_n$, by letting:

– $f_{\mathcal{A}}(\langle e_i\rangle)(w, \mu) = f_{I,w}(\langle e_i(w, \mu)\rangle)$, $p_{\mathcal{A}}(\langle e_i\rangle) = \{\langle w, \mu\rangle : \langle e_i(w, \mu)\rangle \in p_{I,w}\}$.

However, after changing the provisos, the converse Barcan formula is no longer deducible (see [13]). According to the completeness results, the class of fibred models must now contain structures falsifying it. The structures of [22] are general enough to provide such counterexamples. Consider $\langle W, R, \vartheta, D, I\rangle$ where $R = \{R_\mu\}_{\mu\in D^X}$ with each $R_\mu \subseteq W^2$, $f_I = \{f_{I,w} : D^n \to D\}_{w\in W}$ for $f \in F_n$, and $p_I = \{p_{I,w}\}_{w\in W}$ with $p_{I,w} \subseteq D^n$ for $p \in P_n$. Letting $U = W \times \mathrm{Asg}(X, D)$, and for each $b \subseteq U$, $b_w = \{w : \langle w, \mu\rangle \in b$ for some $\mu\}$ and $b_\mu = \{\mu : \langle w, \mu\rangle \in b$ for some $w\}$ we define $\langle \mathcal{A}, \mathbf{c}\rangle$ by:

– $|\mathcal{A}|_\tau = D^U$, $|\mathcal{A}|_\phi = \wp(U)$; $x_\mathcal{A}(w, \mu) = \mu(x)$, $f_\mathcal{A}(\langle e_i\rangle)(w, \mu) = f_{I,w}(\langle e_i(w, \mu)\rangle)$, $q_\mathcal{A} = \vartheta(q) \times \mathrm{Asg}(X, D)$, $p_\mathcal{A}(\langle e_i\rangle) = \{\langle w, \mu\rangle : \langle e_i(w, \mu)\rangle \in p_{I,w}\}$, $\forall x_\mathcal{A}(b) = \bigcup_{w\in b_w}\{\langle w, \mu\rangle : \mu[x/d] \in b_\mu$ for every $d \in D\}$, $\Box_\mathcal{A}(b) = \bigcup_{\mu\in b_\mu}\{\langle w, \mu\rangle : \{w' : wR_\mu w'\} \subseteq b_w\}$.

Of course, these are just examples of structures obtained in the fibred system. Due to fullness, it should contain many others.

5 Conclusion

We have extended the restricted propositional-based setting of [6] to the fibring of logics also admitting variables, terms and quantifiers. Along with the semantic dimension provided by **c**-parchments, we have adopted an improved notion of Hilbert-style calculus with explicit control of schema rule instantiations, following [7, 22]. Besides a detailed account of modal first-order logic as a fibring, we have also reused the technique of fullness from [25] to provide a smooth generalization of the completeness preservation results of [6] to this more general context. The techniques used include congruences and Lindenbaum-Tarski algebras, together with assumptions on the existence of suitable logic constructors.

A word is due on the relationship between our work and Pawlowski's [19]. Indeed, the context information provided by his inference systems presentations can be seen as an alternative way to achieve the same kind of control over schema rule instantiations, leading to a setting where schema variable substitutions are restricted accordingly. Pawlowski's approach is certainly more abstract and systematic, namely in the sense that it tends to treat logical variables and schema variables in a uniform way. However, his claim that, thanks to context information, he can express and manipulate inference rules "without referring to binding operators or requirements" is a little misleading. Context information is certainly present in his framework from the very beginning, but his inference rules are still decorated with additional relevant context information regarding schema variables. This information is strongly related to our provisos. Of course, the provisos seem to be more complex since they have to carry over on signature changes. However, this is due to the fact that all the context information is placed exactly where we really need it: the inference rules. Moreover, note that our provisos are sufficiently general to take into account the scope of modal operators (c.f., Example 9). Binding operators like modalities, very different in nature from quantifiers (namely on the absence of any explicit reference to logical variables), seem to be a good challenge to Pawlowski's notion of context, which is directly built around variables. Last but not least, the incomplete deductive system for first-order logic with equality that he obtains by combining first-order logic and equational logic is certainly to be expected and does not contradict the completeness preservation results presented herein. Note that, on the one hand, equational logic does not come with an implication connective (thus barring the application of Theorems 3 to 8), and on the other hand, fullness would require considering also semantic structures for equational logic whose interpretation of equality would not be the standard (also ruling out Theorem 2). In fact, we could as well have mentioned this example to motivate the difficulties involved in preserving completeness, and to stress the importance of obtaining non-trivial sufficient conditions for completeness preservation as the ones we have presented.

Despite all the results obtained so far, the challenge of combining logics is still far from over. Regarding fibring, specifically, one interesting line of research to pursue is a comprehensive comparison to Diaconescu's Grothendieck institutions [8]. Another important subject that needs further investigation is the *collapsing problem*. In this paper we have avoided the problem by making a careful use of fullness. However, in general, fibring logics of very distinct nature can give rise to trivialities. In [23] *modulated fibring* was presented as a first solution to this problem, using adjunctions between orders on truth-values. Work already in progress aims at solving the same problem using simpler machinery, via the novel notion of *cryptofibring*. Other interesting lines of research require a deep understanding of the process of algebraization of logics, putting in context the notion of fullness and the role that it plays in the completeness results, bringing us closer to the rich field of algebraic logic [4, 1]. We are also interested in studying the representation of fibring in logical frameworks, by capitalizing on the theory of general logics [15]. Finally, future work must also cover transfer results for other relevant properties, like decidability, complexity or interpolation.

References

1. H. Andréka, I. Németi, and I. Sain. Algebraic logic. In *Handbook of Philosophical Logic, 2nd Edition*, volume 2. Kluwer Academic Publishers, 2001.
2. G. Birkhoff. *Lattice Theory*. AMS Colloquium Publications, 1967.
3. P. Blackburn and M. Rijke. Why combine logics? *Studia Logica*, 59(1):5–27, 1997.
4. W. Blok and D. Pigozzi. *Algebraizable Logics*, volume 77. Memoires AMS, 1989.
5. C. Caleiro. *Combining Logics*. PhD thesis, Universidade Técnica de Lisboa, 2000.
6. C. Caleiro, P. Mateus, J. Ramos, and A. Sernadas. Combining logics: Parchments revisited. In *Recent Trends in Algebraic Development Techniques*, volume 2267 of *LNCS*, pages 48–70. Springer, 2001.
7. M.E. Coniglio, A. Sernadas, and C. Sernadas. Fibring logics with topos semantics. *Journal of Logic and Computation*, in print.
8. R. Diaconescu. Grothendieck institutions. *App. Cat. Str.*, 10(4):383–402, 2002.
9. D. Gabbay. Fibred semantics and the weaving of logics: part 1. *Journal of Symbolic Logic*, 61(4):1057–1120, 1996.
10. J. Goguen and R. Burstall. A study in the foundations of programming methodology: specifications, institutions, charters and parchments. In *Category Theory and Computer Programming*, volume 240 of *LNCS*, pages 313–333. Springer, 1986.
11. J. Goguen and R. Burstall. Institutions: abstract model theory for specification and programming. *Journal of the ACM*, 39(1):95–146, 1992.
12. G. Hughes and M. Cresswell. *A New Introduction to Modal Logic*. Routledge, 1996.
13. M. Kracht and O. Kutz. The semantics of modal predicate logic I: Counter-part frames. In *Advances in Modal Logic, Volume 3*. CSLI, 2002.

14. E. Mendelson. *Introduction to Mathematical Logic*. D. van Nostrand, 1979.
15. J. Meseguer. General logics. In *Proceedings of the Logic Colloquium'87*, pages 275–329. North-Holland, 1989.
16. T. Mossakowski. Using limits of parchments to systematically construct institutions of partial algebras. In *Recent Trends in Data Type Specification*, volume 1130 of *LNCS*, pages 379–393. Springer, 1996.
17. T. Mossakowski, A. Tarlecki, and W. Pawłowski. Combining and representing logical systems. In *Category Theory and Computer Science 97*, volume 1290 of *LNCS*, pages 177–196. Springer, 1997.
18. T. Mossakowski, A. Tarlecki, and W. Pawłowski. Combining and representing logical systems using model-theoretic parchments. In *Recent Trends in Algebraic Development Techniques*, volume 1376 of *LNCS*, pages 349–364. Springer, 1998.
19. W. Pawłowski. Presenting and combining inference systems. In *Recent Trends in Algebraic Development Techniques*, LNCS. Springer, this volume.
20. A. Sernadas, C. Sernadas, and C. Caleiro. Fibring of logics as a categorial construction. *Journal of Logic and Computation*, 9(2):149–179, 1999.
21. A. Sernadas, C. Sernadas, C. Caleiro, and T. Mossakowski. Categorial fibring of logics with terms and binding operators. In *Frontiers of Combining Systems 2*, pages 295–316. Research Studies Press, 2000.
22. A. Sernadas, C. Sernadas, and A. Zanardo. Fibring modal first-order logics: Completeness preservation. *Logic Journal of the IGPL*, 10(4):413–451, 2002.
23. C. Sernadas, J. Rasga, and W.A. Carnielli. Modulated fibring and the collapsing problem. *Journal of Symbolic Logic*, 67(4):1541–1569, 2002.
24. A. Tarlecki. Moving between logical systems. In *Recent Trends in Data Type Specification*, volume 1130 of *LNCS*, pages 478–502. Springer, 1996.
25. A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *Journal of Symbolic Logic*, 66(1):414–439, 2001.