# Suszko's Thesis and dyadic semantics

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**Abstract**

A well-known result by Wójcicki-Lindenbaum shows that any tarskian logic is many-valued, and another result by Suszko shows how to provide 2-valued semantics to these very same logics. This paper investigates the question of obtaining 2-valued semantics for many-valued logics, including paraconsistent logics, in the lines of the so-called "Suszko's Thesis". We set up the bases for developing a general algorithmic method to transform any truth-functional finite-valued semantics satisfying reasonable conditions into a computable quasi tabular 2-valued semantics, that we call dyadic. We also discuss how "Suszko's Thesis" relates to such a method, in the light of truth-functionality, while at the same time we reject an endorsement of Suszko's philosophical views about the misconception of many-valued logics.

## 1 Introduction

A widespread abstract way of regarding the notion of logic (which is at the same time mathematically clear and well-founded, and philosophically appealing) involves defining a *logic* $\mathcal{L}$ as a set of formulas endowed with a consequence relation respecting reflexivity, monotonicity and transitivity. Although some meaningful examples are not covered, logics with these characteristics embrace a very large class of interesting cases, and are called *tarskian*. The following well-known reductive results are well-known to apply:

*Wójcicki's Reduction*
(WR): Every tarskian logic $\mathcal{L} = \langle \mathbb{L}, \models \rangle$ is $n$-valued, for some $n \leq |\mathbb{L}|$.

*Suszko's Reduction*
(SR): Every tarskian $n$-valued logic can also be characterized as 2-valued.

Wójcicki's Reduction does not mean that the resulting $n$-valued logic is finite-valued. Still, applying also Suszko's Reduction it may come as a surprise that logics like Łukasiewicz's turn out to be 2-valued. This fact resulted in a philosophical standpoint, often misunderstood, known as *Suszko's Thesis*, according to which "there are but two logical values, true and false". We argue that the whole point essentially concerns the opposition between algebraic (truth-functional) and logical (possibly non-truth-functional) valuations. Taking into account that truth-functionality is an essential characteristic of classical logic, the argument of misconception involved in many-valued reasoning that some read in Suszko's Thesis loses impact if we regard many-valuedness as a technique to recover truth-functionality. In Suszko's own words, one only finds the surprise, blamed on Łukasiewicz: "how was it possible that the humbug of many logical values persisted over the last fifty years?"; "after 50 years we still face an illogical paradise of many truths and falsehoods" (cf. [26]). See Sections 2, 3 and 4 for a more detailed discussion of the matter.

Sections 5 and 6 discuss decidability, which is completely neglected by the non-constructive character of both Wójcicki's and Suszko's Reductions, and proposes a suitable notion of computable quasi tabular 2-valued semantics, that we call dyadic, taking advantage of a generalization of the idea of truth-functionality. The companion paper [7] deals with the question of effectively obtaining dyadic semantics for truth-functional finite-valued logics whose truth-values can be separated using formulas of its language.

## 2 Many values

Let $\mathbb{L}$ denote a non-empty set of *formulas* and let $\mathcal{V}$ denote a non-empty set of *truth-values*. We assume $\mathcal{V} = \mathcal{D} \cup \mathcal{U}$ for suitable disjoint sets $\mathcal{D} = \{d_1, d_2, \ldots\}$ and $\mathcal{U} = \{u_1, u_2, \ldots\}$ of *designated* and *undesignated* values. A *semantics* over $\mathbb{L}$ and $\mathcal{V}$ is said to be any set $\mathsf{sem}$ of mappings $\S : \mathbb{L} \to \mathcal{V}$, called *valuations* (we also call them $|\mathcal{V}|$-valuations, where $|\mathcal{V}|$ denotes the cardinality of $\mathcal{V}$, or else append the corresponding suffix for each $|\mathcal{V}|$: uni-/bi-/tri-, and so on). Given some *$n$-valued* semantics $\mathsf{sem}$ and some formula $\varphi \in \mathbb{L}$, we say that we have a *model* for $\varphi$ when there is some $\S \in \mathsf{sem}$ such that $\S(\varphi) \in \mathcal{D}$; when this is true for every $\S \in \mathsf{sem}$ we say that $\varphi$ is *validated*. A canonical notion of *entailment* given by a *consequence relation* $\vDash_{\mathsf{sem}} \subseteq \mathsf{Pow}(\mathbb{L}) \times \mathbb{L}$ associated to the semantics $\mathsf{sem}$ is then defined by saying that a formula $\varphi \in \mathbb{L}$ *follows from* a set of formulas $\Gamma \subseteq \mathbb{L}$ whenever all models of all formulas of $\Gamma$ are also models of $\varphi$, that is:

$$\Gamma \vDash_{\mathsf{sem}} \varphi \quad \text{iff} \quad \S(\varphi) \in \mathcal{D} \text{ whenever } \S(\Gamma) \subseteq \mathcal{D}. \tag{DER}$$

We will omit the semantics in the index whenever it results clear from the context. Note that we write clauses like $\Gamma, \varphi, \Delta \vDash \psi$ to denote $\langle \Gamma \cup \{\varphi\} \cup \Delta, \psi \rangle \in \vDash$; we call such clauses *inferences*. It was early remarked by Tarski (cf. [27]) that the above notion of consequence relation might be axiomatized as follows:

(CR1) $\Gamma, \varphi, \Delta \vDash \varphi$; (reflexivity)
(CR2) If $\Delta \vDash \varphi$, then $\Gamma, \Delta \vDash \varphi$; (monotonicity)
(CR3) If $\Gamma, \varphi \vDash \psi$ and $\Delta \vDash \varphi$, then $\Gamma, \Delta \vDash \psi$. (transitivity) or (cut)

A *logic* $\mathcal{L}$ will be given by a set of formulas together with a consequence relation defined over it. Logics respecting axioms (CR1)–(CR3) will be called *tarskian*.

A *theory* will be any subset of formulas of a logic. A logic given by some convenient set of formulas and a consequence relation $\Vdash$ is said to be *sound* with respect to some given semantics $\mathsf{sem}$ whenever $\Gamma \Vdash \varphi$ implies $\Gamma \vDash_{\mathsf{sem}} \varphi$ (that is, $\Vdash \subseteq \vDash_{\mathsf{sem}}$); and is said to have a *complete* semantics when $\Gamma \vDash_{\mathsf{sem}} \varphi$ implies $\Gamma \Vdash \varphi$ (that is, $\Vdash \supseteq \vDash_{\mathsf{sem}}$). A semantics which is both sound and complete for a given logic is often called *adequate*. The cut property allows one to take the *closure* $\overline{\Gamma}$ of a given theory $\Gamma \subseteq \mathbb{L}$, by setting $\alpha \in \overline{\Gamma}$ iff $\Gamma \vDash \alpha$; any theory $\Gamma$ such that $\overline{\Gamma} = \mathbb{L}$ is said to be *trivial*. In case a logic $\mathcal{L}$ is *characterized* by some $n$-valued semantics, we will dub it a *(general) $n$-valued* logic, where $n = |\mathcal{V}|$; $\mathcal{L}$ will be called *finite-valued* if $n < \aleph_0$, otherwise it will be called *infinite-valued*. Note that, as a logic may have different semantical presentations, the '$n$' in $n$-valued is not necessarily unique. Given a family of logics $\{\mathcal{L}_i\}_{i \in I}$, where each $\mathcal{L}_i$ is $n_i$-valued, we will also say that the logic given by $\mathcal{L} = \cap_{i \in I} \mathcal{L}_i$ is $\mathsf{Max}_{i \in I}(n_i)$-valued (an arbitrary intersection of tarskian logics is still tarskian).

One might say that the above definition of semantics was too broad. Indeed, it is easy to see that the existence of only one truth-value, be it distinguished or undistinguished, gives rise to trivial uninteresting logics. So, considering several 'degrees of truth' only starts making sense when there are also some 'degrees of falsity' around, and vice-versa. However, the principle of *semantical bivalence*, (SB), implies that there are exactly two truth-values, one designated and the other undesignated. Why should a logic be required to have a 2-valued semantics? General $n$-valued logics do not necessarily respect (SB) in that matter, though they keep a shadow of bivalence in the opposition between *designated* $\times$ *undesignated* values. Besides, there is some metalinguistical bivalence that one will not easily get rid off: *either* an inference obtains *or* it does not, but *not both.* Why should a logic be expected to be more 'bivalent' than that? We will come back to this issue. By now you should only recall that there is an influential yet widely misunderstood philosophical standpoint known as *Suszko's Thesis*, (ST) (cf. [12], [16], [28] and [26]), according to which "there are but two logical values, true and false." How should (ST) be understood? This might be debatable, but one could start by arguing that (ST) seems to be implied by (SB). Besides, we cannot forget about two well-known *reductive* theorems:

**Theorem 2.1** *Wójcicki's Reduction*
(WR): Every tarskian logic $\mathcal{L} = \langle \mathbb{L}, \vDash \rangle$ is $n$-valued, for some $n \leq |\mathbb{L}|$.

**Proof:** Cf. [29]. The proof uses the fact that the so-called *Lindenbaum matrix* $\mathcal{L}_\Gamma = \langle \mathbb{L}, \mathcal{V}, \mathcal{D} \rangle$, with $\mathcal{V} = \mathbb{L}$ and $\mathcal{D} = \overline{\Gamma}$, for any $\Gamma \subseteq \mathbb{L}$, determines a consequence relation $\vDash_\Gamma$, where general morphisms play the role of valuations. Now one can show that $\vDash \; = \; \cap_{\Gamma \subseteq \mathbb{L}} \vDash_\Gamma$. QED

**Theorem 2.2** *Suszko's Reduction*
(SR): Every tarskian $n$-valued logic can also be characterized as 2-valued.

**Proof:** Cf. [16, 4]. For any $n$-valuation § of a given semantics $\mathsf{sem}(n)$, and every consequence relation based on $\mathcal{V}_n$ and $\mathcal{D}_n$, define $\mathcal{V}_2 = \{T, F\}$ and $\mathcal{D}_2 = \{T\}$

and set the characteristic total function $b_{\S} : \mathbb{L} \to \mathcal{V}_2$ to be such that $b_{\S}(\varphi) = T$ iff $\S(\varphi) \in \mathcal{D}$. Now, collect all such bivaluations $b_{\S}$'s into a new semantics $\mathsf{sem}(2)$, and notice that $\Gamma \vDash_{\mathsf{sem}(2)} \varphi$ iff $\Gamma \vDash_{\mathsf{sem}(n)} \varphi$. QED

We have, up to now, totally disconsidered the nature of the sets of formulas and of truth-values. This was done for the sake of generality, but is probably unfair, in practice. Most logics we work with do bring some profitable built-in structure —their formulas include atomic sentences, connectives and so on; their truth-values often form some kind of algebra where operators correspond to the connectives, and so forth. This is how it usually goes, in the propositional case.

Let $ats = \{p_1, p_2, \ldots\}$ be a denumerable set of *atomic sentences*, or simply *atoms*, and let $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ be a propositional signature, where each $\Sigma_n$ is a set of *connectives* of arity $n$. Let $cct = \bigcup_{n\in\mathbb{N}} \Sigma_n$ be the set of connectives. The set of formulas $\mathbb{L}$ is then defined as the algebra freely generated by $ats$ over $\Sigma$. Thus, $p_k \in \mathbb{L}$, for any atomic sentence $p_k \in ats$, and $\otimes(\varphi_1, \ldots, \varphi_m) \in \mathbb{L}$, for any $m$-ary connective $\otimes \in cct$, and any formulas $\varphi_1, \ldots, \varphi_m \in \mathbb{L}$. The *(canonical)* measure of *complexity* of a formula $\varphi$ is then set to be the output of the mapping $l : \mathbb{L} \to \mathbb{N}^+$, where $l(p_k) = 1$ and $l(\otimes(\varphi_1, \ldots, \varphi_m)) = 1 + l(\varphi_1) + \ldots + l(\varphi_m)$. The set of atomic sentences *on* which a formula $\varphi$ *depends* is the output of $var : \mathbb{L} \to \mathsf{Pow}(ats)$, where $var(p_k) = \{p_k\}$ and $var(\otimes(\varphi_1, \ldots, \varphi_m)) = var(\varphi_1) \cup \ldots \cup var(\varphi_m)$; the mapping $var$ and the corresponding definition of dependence on atomic sentences is extended to sets of formulas by taking $var(\Gamma) = \cup\{var(\varphi) : \varphi \in \Gamma\}$. Any two sets or formulas which depend on no common sentences are said to be *disconnected*. The set of *subformulas* of a formula $\varphi$ is the output of $sb : \mathbb{L} \to \mathsf{Pow}(\mathbb{L})$, where $sb(p_k) = \{p_k\}$ and $sb(\otimes(\varphi_1, \ldots, \varphi_m)) = sb(\varphi_1) \cup \ldots \cup sb(\varphi_m) \cup \{\otimes(\varphi_1, \ldots, \varphi_m)\}$; the *proper* subformulas of a formula $\varphi$, denoted by $psb(\varphi)$, will be given by $sb(\varphi) \setminus \{\varphi\}$. Obviously, atomic sentences have no proper subformulas. *Immediate* subformulas of a formula $\otimes(\varphi_1, \ldots, \varphi_m)$ are exactly the proper subformulas $\varphi_1, \ldots, \varphi_m$. In accordance, we shall sometimes denote by $\varphi(p_1, \ldots, p_n)$ a formula $\varphi$ which depends only on the atomic sentences $var(\varphi) \subseteq \{p_1, \ldots, p_n\}$, and by $\varphi[\alpha]$ a formula $\varphi$ having a formula $\alpha$ as subformula.

Let a truth-value *assignment* be any mapping $\pounds : ats \to \mathcal{V}$, and let $\mathsf{asg}$ be the set of all possible assignments, each of them representing a different *state of affairs*. Obviously, any given valuation $\S : \mathbb{L} \to \mathcal{V}$ of a given semantics $\mathsf{sem}$ can be restricted to the domain of an assignment $\pounds_{\S}$, by taking $\S|_{ats}$; conversely, any assignment $\pounds \in \mathsf{asg}$ can be extended into a valuation $\S_{\pounds}$, in a variety of ways. Given an atom $p$ and a semantics $\mathsf{sem}$, let's denote by $\mathsf{asg}[p]$ the set $\{\pounds(p) : \pounds_{\S} \in \mathsf{asg} \text{ and } \S \in \mathsf{sem}\}$ of all values that it can assume, in all available states of affairs; in general, for any given formula $\varphi$, denote by $\mathsf{sem}[\varphi]$ the set $\{\S(\varphi) : \S \in \mathsf{sem}\}$. Now, one reasonable initial requirement for a 'representative' semantics $\mathsf{sem}$ might be that the 'complexification' of a given formula should bring no novelty, as discussed in the following (R?)-conditions. First of all, (R1): a non-atomic formula $\varphi$ should not be allowed to assume any truth-value which could not have already been assumed by an atomic sentence $p$, that is, $\mathsf{sem}[\varphi] \subseteq \mathsf{asg}[p]$, for some atom $p$. Moreover, (R2): given the algebraic character of $\mathbb{L}$, one should presumably expect each choice of state of affairs to coherently affect the correlated 'state of the world.' Thus, following this line of reasoning, if you take a non-atomic formula $\varphi$ and substitute all occurrences of an atom $p_k \in var(\varphi)$ for some other formula $\alpha$, (R2) says you should not expect any *new* situation to obtain: Intuitively, $\alpha$ should not be expected to assume any value that the atom it substitutes already could, and accordingly $\varphi$ should not be 'less true' then it already was. Let's denote by $\varphi(p_1, \ldots, p_k/\alpha, \ldots, p_m)$ the above described performance, to be called *(uniform) substitution*. From the point of view of the free algebra of formulas, a uniform substitution is simply a(n endomorphism) $\varepsilon : \mathbb{L} \to \mathbb{L}$, which uniquely extends some $sbs : ats \to \mathbb{L}$ into a homomorphism of the the whole set of formulas into itself, that is, a mapping such that $\varepsilon(\otimes(\varphi_1, \ldots, \varphi_m)) = \otimes(\varepsilon(\varphi_1), \ldots, \varepsilon(\varphi_m))$. Now, a semantics $\mathsf{sem}$ will be called *laplacian* in case it allows for all possible assignments to be 'realizable', that is, if $\mathsf{sem}|_{ats} = \mathsf{asg}$ (each atomic sentence is allowed, by some assignment, to take any available truth-value); furthermore, $\mathsf{sem}$ will be called *representative* exactly in case it turns out to be 'resistant' to substitutions, in the sense of (R1) and (R2), that is, if $\mathsf{sem}[\varphi] \supseteq \varepsilon(\mathsf{sem}[\varphi])$, for any formula $\varphi$ and any substitution $\varepsilon$. In case an $n$-valued representative semantics $\mathsf{sem}$ is non-laplacian, then there certainly are 'redundant' truth-values in it. In case representativity fails for $\mathsf{sem}$ already at the atomic level, that is, clause (R1) above is falsified, and thus $\mathsf{sem}[p] \not\supseteq \varepsilon(\mathsf{sem}[p])$, for some $p$ and $\varepsilon$, then $\mathsf{sem}$ is surely non-laplacian. In practice, representative semantics seem to be extremely common, and laplacian semantics are general rule. When a given semantics is representative, it in fact just allows for substitutions to preserve entailment. This corresponds to the following property of the associated consequence relation (cf. [14]):

(CR4) If $\Gamma \vDash \varphi$ then $\varepsilon(\Gamma) \vDash \varepsilon(\varphi)$, for any endomorphism $\varepsilon$. (structurality)

There are many ways of exploiting the structurality presented by the algebra of formulas. For instance, the map defining canonical complexity of a formula is all but *schematic*. Its definition did not depend on the particular atomic sentences involved, but only on the format of the formulas whose complexity was being measured. Complexity is one of the many meta-theoretical properties associated to a structural logic which can be said to be invariant under substitutions. It has been argued that structurality is exactly what makes logic a science of forms and patterns, rather than of content or meaning; even logics which propose to deal with content and meaning are, as a rule, structural (cf. [6]). Non-structural logics *do* exist, of course, but seem to be pretty rare in the current literature.

Let us now go back to Suszko's Thesis. There are of course several ways of understanding (or misunderstanding) the statement of (ST). We have already remarked that (ST) initially seems to be no more than a particular consequence of semantical bivalence, but there are some further details which should be stressed. First of all, (ST) should be sharply discriminated (cf. [12]) from Suszko's Reduction, (SR), given in Theorem 2.2. Indeed, just to start with, note that (SR) was 'only' shown to apply in the case of tarskian logics. Surely, this is not to say that it does *not* apply, in a way or another, for other things one might want to call 'logic.' Suppose that one starts from a set of truth-values partitioned into designated and undesignated values, as before, but now take from the latter a subset of *rejected* values. Forget now about (DER) and define a new notion of consequence relation according to which designated values follow from non-rejected ones (in the spirit of an original idea of Łukasiewicz, cf. [17, 23]). In case 'rejected' does not coincide with 'undesignated', the most natural reduction that one can make, following the trick in (SR), makes use of a 3-valued characteristic function. Such logics thus do not seem to be 'logically 2-valued', in the sense Suszko would have liked them to be. Nevertheless, their consequence relation can be axiomatized by taking monotonicity together with:

(CR5) If $\Gamma, \varphi \vDash \psi$ and $\Gamma \vDash \varphi$, then $\Gamma \vDash \psi$. (cautious cut) or (weak transitivity)

Such *quasi consequence* relations characterizing logics which are resistant to a 2-valued reduction à la Suszko were carefully studied in [15]. Curiously, Suszko was not only known to held that "obviously, any multiplication of logical values is a mad idea" (cf. [26]), but he also advanced in [24] (sup.II, p.221) that such 'logical many-valuedness' was "probably, beyond my [his] comprehension."

Clearly, from (WR) (Theorem 2.1) and (SR) it follows that tarskian logics, whether structural or not, can be endowed with bivalued semantics which capitalize on the opposition between designated $\times$ undesignated values. This might have led to some confusion in the literature in associating Suszko's 2-valued reduction to structural logics only (as in ch.10 of [16], or all along [28]). In fact, Suszko did help in creating that imbroglio, when affirming, in accordance with the Polish tradition which he helped inaugurating (cf. [14]), that to assume structurality is to put "almost no restriction at all" (cf. [24], p.228). He also maintained that, when a new non-classical logic is proposed, "your logic is good only if Wójcicki's method [(WR)] leads to nice models" (id., p.191). Finally, Wójcicki also assumed structurality in presenting his reduction (cf. [29, 31]), though it is easy to see that this assumption is strictly unnecessary for the proof. But then, Suszko himself seems never to have suggested anything about the *need* of structurality in the reduction in spite of his canonical example having always been that of a structural logic, namely Łukasiewicz's $Ł_3$. But again, those historical points are really tricky, as there seems to be no paper where Suszko explicitly formulates (SR) in full generality! (By the way, the so-called "Suszko's bivalued semantics for Łukasiewicz's $Ł_3$", as formulated in [16], p.73, does *not* appear in [25] as claimed, for instance, in [16, 12, 28, 5]. It is true though that one can find in [25] at least a sketch of Theorem 2.2, but only for the particular case of $Ł_3$.) At any rate, reductive theorems similar in spirit to (SR) have in fact been independently proposed in the 70s by other authors, such as Newton da Costa and Dana Scott.

# 3 Valuations

So far we have said no word about *how* assignments should be extended into valuations. Some will find it reasonable to suppose, though, that one should be able to find exact semantical counterparts not only for the atomic sentences, but also for the connectives. The value of a complex formula $\otimes(\varphi_1, \ldots, \varphi_m)$ under some valuation would in such a case perhaps be expected to depend functionally —preferentially recursively— on the value of its immediate subformulas. In case this is true for all formulas of some structural logic $\mathcal{L}$, we will say that this logic has a *(truth-functional) tabular semantics*. This allows us to extend a given assignment into a valuation by induction on the canonical complexity of the formulas. What we are doing, in effect, is to suppose that the truth-values $\mathcal{V}$ come embedded on an algebra $\mathbb{V}$, of same similarity type as the algebra of $\mathbb{L}$. In particular, each $m$-ary connective $c \in con$ has a corresponding $m$-ary operator $c : \mathcal{V}^m \to \mathcal{V}$ in $\mathbb{V}$ (we use here the same name for both the connective and the corresponding operator). It should be clear that this inductive tabular procedure gives neither the only way of building a matrix nor the only way of counting on truth-functionality in such a construction. We have already seen in (WR) the use of another notion of matrix which does not necessarily involve a counterpart for each connective; one could also think that the value of a complex formula could depend functionally on any subset of the subformulas involved in each case; one could even change the way one measures complexity (check the 'evaluative position' of formulas, in Section 5), and then expect the operators to depend functionally on richer sets of related formulas (for instance, an implication could depend on its immediate subformulas, but also on their negations); the operators could alternatively depend on some more involved aspects of the corresponding algebras (generalizations of truth-functionality have been explored, for instance, in [8]). Once we have two algebras with the same signature, the algebra of formulas and the algebra of truth-values, it is immediate to define a valuation as the unique extension of an assignment into a homomorphism from $\mathbb{L}$ into $\mathbb{V}$.

Any logic having some characterizing $n$-valued semantics can be given another characterizing $m$-valued semantics, for $m > n$ —one just has to introduce $m - n$ dummy truth-values to mock the first $n$ given truth-values. Given some characterizing matrix with $m$ truth-values, and given some $0 < n < m$, it is not always the case that a characterizing matrix with $n$ truth-values can be attained by erasing $m - n$ 'redundant' truth-values from the first set of matrices. To simplify notation and avoid 'non-standard' models we will say thus that a logic $\mathcal{L}$ is *genuinely $n$-valued* if it can be characterized by some $n$-valued tabular semantics given by the set of all possible valuations as homomorphisms, but no other $m$-valued tabular semantics will do the same job, for $m < n$; evidently, any logic with a tabular semantics is genuinely $n$-valued, for some $n$. Equivalently, consider a $|\mathcal{V}|$-valued semantics $\mathsf{s1}$ defining a consequence relation $\vDash_{\mathsf{s1}}$; then, $x \in \mathcal{V}$ will be called a *redundant* truth-value of $\vDash_{\mathsf{s1}}$ if there is some other truth-value $y \in \mathcal{V}$ such that the semantics $\mathsf{s2}$ given by exchanging $x$ for $y$ everywhere is such that $\Gamma \vDash_{\mathsf{s1}} \varphi$ iff $\Gamma \vDash_{\mathsf{s2}} \varphi$. Obviously, a logic has a genuinely $n$-valued semantics only if it has no redundant truth-values. The consequence relations of genuinely $n$-valued logics are axiomatized as structural tarskian relations, i.e., those respecting (CR1)–(CR4), plus the following additional property (cf. [22], where it is called 'cancellation'):

(CR6) $\Gamma \cup \{\delta_i\}_{i \in I} \vDash \varphi$ implies $\Gamma \vDash \varphi$, whenever all the theories (uniformity)

$\Gamma \cup \{\varphi\}$ and $\{\delta_i\}_{i \in I}$ are disconnected two by two,
and no $\delta_i$ is trivial.

In the particular case of an atomic sentence $p$, uniformity postulates that $p$ follows from some theory $\Gamma$ whenever it follows from some richer theory $\Gamma \cup \{\Delta_i\}_{i \in I}$ and all the theories $\Gamma$ and $\{\Delta_i\}_{i \in I}$ are disconnected two by two, no $\Delta_i$ depends on $p$, and no $\Delta_i$ is trivial. Now, it is quite often the case that the condition on non-triviality can be deleted as soon as $\Gamma$ itself is already known to depend on $p$, i.e., as soon as the following property also holds:

(CR7) $\Gamma \cup \{\delta_i\}_{i \in I} \vDash p$ implies $\Gamma \vDash p$, whenever all the theories $\Gamma$ and $\{\delta_i\}_{i \in I}$ are disconnected two by two, and $\Gamma$ depends on $p$. (segregability)

Remember that Wójcicki's method (WR), from Theorem 2.1, showed how to represent any tarskian logic as the $n$-valued logic given by the intersection of all of its Lindenbaum matrices (which need not be structural). Now, one can in fact prove more (cf. [30], where the above property is called 'absolute separability'): That a (structural) tarskian logic can be characterized by a *single* matrix if, and only if, its consequence relation is both uniform and segregable. The logics to which our reductive method [7] applies will not only come endowed with a tabular semantics, but in fact a segregable one (that is, a tabular semantics which defines a segregable consequence relation). Moreover, we will be dealing exclusively with finite-valued logics. But then, one can easily check that logics with truth-functional tabular semantics are characterized by (cf. [21]):

(CR8) $\Gamma \vDash \varphi$ only if $\Gamma^{\mathbf{f}} \vDash \varphi$, for some finite $\Gamma^{\mathbf{f}} \subseteq \Gamma$. (compacity)

Perhaps one can now better understand the apparent disparity which constrasts (WR) and (SR) in the case of structural tarskian logics, specially of uniform ones. Indeed, if a logic has a genuinely $n$-valued semantics, for some $n > 2$, how can it also be characterized by bivaluations? The point is that the 2-valued reduction of a logic whose genuine tabular semantics has more than two truth-values will simply make it lose truth-functionality on the way. The reduction will obviously make the logic no less tarskian, structural or uniform than before (it is simply a different characterization of the *same* logic). In one way or another, it appears that Suszko did not fully comprehend the generality of his method (SR) (a similar observation could be made about (WR)). Extrapolating on the structurality of all logics actually studied by the Polish school (under the force of [14]), Suszko talked always about *algebraic valuations* given as homomorphisms being substituted by *logical valuations* which are homomorphisms "only in some exceptional cases" (cf. [26], p.378). Notice though that this means much more than to assume the (pretty harmless) algebraic structure of the set of formulas, which allows for substitution, but it involves also a strong assumption about the algebraic structure of the set of truth-values and the corresponding uniformity of the associated consequence relation, accompanied by an assumption about the most 'logical' choice for its cardinality, namely 2.

## 4 Algebraization

Given some tarskian logic $\mathcal{L} = \langle \mathbb{L}, \vDash \rangle$, let's say that two formulas $\alpha$ and $\beta$ are *logically equivalent*, and denote it by $\alpha \approx \beta$, if both $\alpha \vDash \beta$ and $\beta \vDash \alpha$. By transitivity we see that the logical equivalence of $\alpha$ and $\beta$ means that these formulas have the same set of consequences, that is, $\overline{\{\alpha\}} = \overline{\{\beta\}}$. Suppose now that $\mathcal{L}$ is a structural tarskian logic, and let $\varphi[\alpha]$ be one of its formulas (to be sure, a 'contextual' formula $\varphi$ having a formula $\alpha$ as subformula). We shall denote by $\varphi[\alpha/\beta]$ the result of replacing in $\varphi$ one or more occurrences of its subformula $\alpha$ by the formula $\beta$. Now, should it be the case that logically equivalent formulas have exactly the same effect wherever they occur? When the answer is positive for any choice of formulas-as-contexts of a tarskian logic $\mathcal{L}$, we say that $\mathcal{L}$ allows for *replacement*. Tarskian consequence relations allowing for replacement may be axiomatized by the following property (cf. [31], p.381–2):

(CR9) $\alpha \approx \beta$ implies $\varphi[\alpha] \approx \varphi[\alpha/\beta]$. (self-extensionality)

In a tarskian logic $\mathcal{L}$, logical equivalence obviously defines an equivalence relation. In case this same logic is also structural and self-extensional, then we say that $\approx$ defines a *congruence* over the logic. In any $n$-valued logic $\mathcal{L}$ with a congruence $\approx$, congruent formulas turn out to be semantically indistinguishable. Consider a mapping $\alpha \mapsto [\alpha]_{/\approx}$ taking each formula $\alpha$ into the equivalence class $[\alpha]_{/\approx} = \{\beta : \beta \approx \alpha\}$ to which it belongs with respect to $\approx$ (and extend this mapping to sets of formulas by applying it to each element of the set); let $\mathbb{L}_{/\approx} = \{[\alpha]_{/\approx} : \alpha \in \mathbb{L}\}$ be the set of such classes, and let $\vDash_{/\approx}$ be defined accordingly, by setting $[\Gamma]_{/\approx} \vDash_{/\approx} [\varphi]_{/\approx}$ iff $\Gamma \vDash \varphi$. One can then greatly simplify the work over $\mathcal{L} = \langle \mathbb{L}, \vDash \rangle$ by working instead over the simpler *quotient logic* $\mathcal{L}_{/\approx} = \langle \mathbb{L}_{/\approx}, \vDash_{/\approx} \rangle$ (for instance, in the quotient all valid formulas will work as one and only). This procedure of taking some structural tarskian logic, checking whether it is also self-extensional, that is, whether its relation of logical equivalence defines a congruence over it, and then going into the canonical quotient logic defines what is known as *Lindenbaum-Tarski algebraization* of that logic. A Lindenbaum-Tarski algebra is not sensible to differences among designated values or to differences among undesignated ones: Two formulas of the original logic which differ, from the point of view of a given semantics, only in having some valuation taking them to distinct designated / undesignated truth-values will share the same algebraic class (in particular, while in the case of a logic with a self-extensional $n$-valued tabular semantics, each formula $\varphi$ with $v = |var(\varphi)|$ atomic sentences would have theoretically a maximum number of $n^{v^n}$ valuations —see comments on the matrix decision procedure in the next section—, the number of algebraic classes it can belong to would be no bigger than $n^{v^2}$). The quotient logic corresponding to an $n$-valued logic will thus be 2-valued; in fact, its most natural semantics will be the one given by Suszko's Reduction. Self-extensionality is the keystone to modern *modal logic* à la Kripke. Indeed, it can be shown (cf. [31], ch.5) that a structural tarskian logic $\mathcal{L}$ is self-extensional iff it has a class of adequate 2-valued 'frame interpretations'.

It is interesting to advance here a bit of the (quite negative) trade-off to be negotiated between self-extensionality and the scope of application of our reductive method [7]. The failure of self-extensionality means that it is possible to find a context-formula that is capable of 'separating' two designated truth-values, or two undesignated ones, of a given semantics. Our method of 'dyadic semantics' applies exactly in the cases in which *each* pair of designated / undesignated truth-values can be thus separated. The no-man's land in between 'modal' self-extensional finite-valued logics and those finite-valued logics allowing for a 'dyadic semantics' resists up to now the formulation of a general method for the recursive construction of a pragmatically valuable 2-valued reduction. This in fact relates to the trouble of algebraization. It turns out that the pre-conditions for self-extensionality and the existence of a Lindenbaum-Tarski algebra for a given logic $\mathcal{L}$ are quite sensitive to the linguistic resources of that logic.

# 5 Decidability

A most striking feature of genuinely finite-valued logics is their *decidability*. Let sem be a finite tabular semantics (one with a finite number of truth-values), and suppose that one wants to test whether $\Gamma \vDash_{\mathsf{sem}} \varphi$, for some finite theory $\Gamma$ and some formula $\varphi$. An exhaustive decision procedure would be simply that of looking at the values assumed by all the formulas in $\Gamma \cup \{\varphi\}$ at all possible states of affairs, and then use the very definition (DER) of $\vDash_{\mathsf{sem}}$.

The procedure can in fact be promptly generalized one little step further so as to reckon with many other structural tarskian logics. A typical tabular decision procedure † for a logic $\mathcal{L}$ will have the following elements. First, let the *evaluative position* of a formula $\varphi$ with respect to † be set as the output of some schematic mapping $l : \mathbb{L} \to \mathbb{N}^+$ such that $l(p_k) = 1$, for each atom $p_k$. Now write the elements of $var(\Gamma \cup \{\varphi\})$ successively on different columns. Next, write under those same columns all the possible assignments of truth-values which can be associated to those atomic sentences, possibly with a finite number of repetitions (to be dictated by † at some stage of the process). Then, † has to tell you how to draw the succeeding columns of the table in such a way as to arrive in the end at columns having as headers each of the formulas in $\Gamma \cup \{\varphi\}$ and filled with truth-values written in the lines below them, to be calculated according to instructions laid down by †. The only restrictions on the introduction of new columns are the following. Suppose a formula $\alpha$ in the header has some of the truth-values to be written under it calculated in terms of the truth-values of some other formula $\beta$. Then, it should not occur that $l(\alpha) \leq l(\beta)$; moreover, for each $l(\alpha) > l(\beta)$, we should make sure that $\beta$ has already appeared as the header of a previous column. When some of the truth-values under $\alpha$ do not depend on the truth-values assumed by any other formula, then they should accordingly be allowed to assume any of the (finite number of) available truth-values —that's why we have allowed for repetition of assignments from the start. These instructions will obviously generate a finite table. The lines drawn by † in the end are supposed to represent all possible valuations, thus, if the method is adequate, it is sufficient to have a look at those lines to see whether $\Gamma \vDash \varphi$ or not, according to (DER). Any such † will be said to delineate a *quasi matrix procedure* to test the finite inferences of $\mathcal{L}$. In case: (i) we can take the canonical complexity of the formulas to give their evaluative position, (ii) there is no need for repetition of assignments at the start, and (iii) each value written under each header depends entirely on the value of the immediate subformulas of the header: then we can say that we simply have a *matrix procedure*, and the logic will obviously have a finite tabular semantics —each line of a matrix testing whether $\Gamma \vDash \varphi$ represents in fact the restriction to $var(\Gamma \cup \{\varphi\})$ of some of the possible homomorphisms from the whole algebra of formulas into the similar algebra of truth-values, and all homomorphisms are thereby represented.

Suppose now we are given a decidable (compact) logic $\mathcal{L}$. Then, it is easy to see that $\mathcal{L}$ will have an $n$-valued tabular semantics iff there is some quasi matrix procedure † for it such that any finite inference $\Gamma \vDash \varphi$ can be tested using no more than $n^{var(\Gamma\cup\{\varphi\})}$ lines. If one is now willing to look for a bivalued semantics equivalent to a given genuinely finite-valued tabular semantics, it seems interesting then to look for a set of bivaluations which might immediately suggest a decision procedure, so as not to block the up to now easy access to decidability. Given that one is bound to lose truth-functionality in the 2-valued reduction of a genuinely $n$-valued logic, for some finite $n > 2$, then one could at least expect to learn an automatic way of substituting the original $n$-valued matrices of its decision procedure by similarly intuitive and useful 2-valued quasi matrices. Suszko's Reduction, (SR), is pretty general in that it works for any $n$-valued logic, but even in case this logic is decidable (SR) gives us no hint on how to construct a decision procedure for the bivalued semantics thereby produced, if not by erratically trying to build quasi matrices with repeated assignments so as to mock the original associated matrices. (SR) is in fact, just like (WR), an entirely non-constructive result. It seems only reasonable to require however, given a logic with an elegant matrix decision procedure, that the bivalued semantics for it should be automatically defined in as many cases as possible *and* that it should bring together a new decision procedure. It is shown in [7] that, whenever our method applies, for some given genuinely $n$-valued logic $\mathcal{L}$, it is possible to find a new way of spelling the evaluative position of its formulas (usually not following their canonical complexity, but some other convenient ordering), so as to indicate a new 'bivalent' decision procedure for $\mathcal{L}$.

How hard is it to automate the process of 2-valued reduction for genuinely finite-valued logics, if it is possible at all? And, taking for granted that some bivalued semantics will be more suggestive than others, how can we find the 'nice' ones? G. Malinowski pointed out (cf. [16], p.73) that "usually, the degree of complexity of the many-valued logic description increases with the quantity of values". This seems more or less expectable, and is hard to contest, anyway. But then, Malinowski also submitted that, given an arbitrary relation $\vDash_C$, "it seems that giving a general method for the recursive description of these valuations without knowing precisely the structure of the class $K$ of matrices adequate for $C$ is hardly possible. At the same time, however, even for simple relations of inference the conditions defining valuations are illegible" (id., ibid.).

Here we contend. In fact, it happens that, as it can be seen by way of our method in [7], under reasonable conditions on the separability of logical values —to which the great majority of the well-known many-valued logics conform—, there *is* a general recursive method requiring very little more knowledge about the structure of the matrices than some information about their expressibility, more specifically about the range of (unary) operators that they can define. Moreover, the axioms governing the bivaluations attained through our method define particular instances of what we call 'dyadic semantics', which have the advantage of immediately offering a proof-theoretical formulation which recovers and displays their decision procedure in a very elegant fashion. General methods for inter-relating bivaluations and sequent systems have already been carefully investigated by Béziau in [3, 2]: As a practical example of application of those methods, the initial "illegibility" of the bivalued semantics offered for $Ł_3$ in [16], p.73, was fixed and a sequent-style formulation of that logic was offered in [5]. In the companion paper [7], we hint at how traditional 'bivalent' tableau rules can be formulated so as to correspond to the axioms of any dyadic semantics produced by our constructive reductive method. This makes things even more convenient and legible than the mere formulation of the often clumsy quasi matrices. So, on this point we should insist: There are of course some things to be lost when passing from the tabular semantics of some given logics to its 2-valued (dyadic) semantics —one is truth-functionality, another is the subformula property. But then, there are many advantages and compensations as well. The tableaux obtained from the dyadic semantics are also quite 'classical', in that they contain no labels nor other aliens, and no semantical intromissions: They simply explore the bivalued mechanism from good ol' classical logic. Moreover —and here we tread in fact in non-truth-functional terrain, one of the many gains of radical non-classicality—, there are many other logics, including many well-known paraconsistent logics, which have a dyadic semantics, in spite of not possessing any characterizing genuinely finite-valued semantics. For those logics we can also obtain the same kind of tableaux, using the proof-theoretical result which turns out to be more general than it could have been initially assumed. We exhibit an example of such a logic in the last section of this paper.

Our work in this paper and in [7] has a few ancestors. One of those is a paper by D. Batens (cf. [1]). The method he proposes for the 2-valued reduction makes use of a sort of 'binary print' of the truth-values: each truth-value of a given $n$-valued semantics is to be put into a one-to-one correspondence with one element of a set of conveniently long 'equivalent' sequences of 0's and 1's (one will need of course at least $\mathsf{Ceiling}(\log_2 n)$ values in those sequences, where $\mathsf{Ceiling}(x)$ gives the smallest integer greater than or equal to $x$), and he exemplifies that again for $Ł_3$. This is in fact extremely similar to what had been proposed by D. Scott about a decade before, for the whole hierarchy $Ł_n, 2 < n$ (cf. [19, 20]). To get those sequences, for each given finite-valued logic, Batens and Scott try to find and use resources of this very same logic —formulas which generate the mentioned sequences by exploiting the 'bivalent' opposition between designated $\times$ undesignated values. Once these sequences have been found, it is not difficult to formulate the corresponding bivalued semantics. In view of that, Batens boldly asserts that the axiomatization of the original logic "is an easy exercise to any student who took a decent logic course" (cf. [1], p.321). Batens does not assert that his method is universally applicable, but only that, when it is applicable at all, its application is "effective", and when it is not, it is "of great heuristic value" (id., ibid.) He does certainly require though, one must concede, some ingenuity of the reader in applying it.

Our own method (also inspired by [18]) is applicable in quite the same circumstances, as we also use resources of the logic in producing the appropriate binary prints for each of the initial truth-values. But then we put an emphasis in the formulas producing the separation of the truth-values, and use them to automatically produce a corresponding set of bivaluation axioms in a special format —that of a dyadic semantics—, from which the corresponding tableau rules will later on follow quite naturally. We contribute this way to perfect Suszko's reductionist intuition, without any need to partake his philosophy.

# 6 Dyadic semantics

Using the assumptions and ideas from the previous sections, we will now proceed to obtain a formal definition of computable quasi tabular 2-valued semantics; they will appear as the 2-valued output of the $n$-valued semantics obtained by means of the reductive method described in [7]. Those 2-valued semantics are described by certain axioms which follow a very specific format, characterizing what we shall call 'dyadic semantics'. To that effect, we shall be making use of an appropriate equational language, made explicit in the following.

**Def. 6.1** A *gentzenian semantics* for a logic $\mathcal{L}$ is an adequate (sound and complete) set of 2-valued valuations $b : \mathbb{L} \to \{T, F\}$ given by conditional clauses $(\Phi \to \Psi)$ where both $\Phi$ and $\Psi$ are (meta)formulas of the form $\top$ (top), $\perp$ (bottom) or:

$$b(\varphi_1^1) = w_1^1, \ldots, b(\varphi_1^{n_1}) = w_1^{n_1} \mid \ldots \mid b(\varphi_m^1) = w_m^1, \ldots, b(\varphi_m^{n_m}) = w_m^{n_m}. \quad (G)$$

Here, $w_i^j \in \{T, F\}$, each $\varphi_i^j$ is a formula of $\mathcal{L}$, commas "," represent conjunctions, and bars "|" represent disjunctions. The (meta)logic governing these clauses is FOL, First-Order Classical Logic. We can alternatively write a clause of the form $(G)$ as $\bigvee_{1\leq k\leq m} \bigwedge_{1\leq s\leq n_m} b(\varphi_k^s) = w_k^s$.

Now, a dyadic semantics will be just a specialization of gentzenian semantics, in a deliberate intent to capture the computable class of such semantics, as follows. Please take into account that not all decidable 2-valued semantics will come with a built-in gentzenian presentation. Moreover, as shown in Example 6.6, there are many logics characterizable by gentzenian or even by dyadic semantics, but not by any genuinely finite-valued semantics.

**Remark 6.2** (i) Given an algebra of formulas $\mathbb{L}$, we recall that an appropriate measure of complexity of some $\varphi \in \mathbb{L}$ is defined as the output of a schematic mapping $l : \mathbb{L} \to \mathbb{N}^+$, with the restriction that $l(p_k) = 1$, for each $p_k \in ats$. As a particular case, the canonical measure of complexity of $\varphi = \otimes(\varphi_1, \ldots, \varphi_m)$ has the additional restriction that $l(\varphi) = 1 + l(\varphi_1) + \ldots + l(\varphi_m)$, for each $\otimes \in cct$.

(ii) Given an algebra of formulas $\mathbb{L}$, denote by $\mathbb{L}[n]$, for $n \geq 1$, the set $\mathbb{L}[n] = \{\varphi \in \mathbb{L} : var(\varphi) = \{p_1, \ldots, p_n\}\}$. Also, there are surely non-empty (and possibly finite) families of formulas $\{\psi_i\}_{i \in I}$, for some $I = \{1, 2, \ldots\} \subseteq \mathbb{N}^+$, and there are $1 \leq n_i \leq \aleph_0$, for each $i \in I$, with $\psi_i \in \mathbb{L}[n_i]$, which cover the whole set of formulas up to some substitution, that is, such that $\mathbb{L} = \bigcup_{i \in I}\{\varepsilon(\psi_i) : \varepsilon$ is a substitution$\}$. A minimal example of such a covering family is given by $\{\otimes(p_1, \ldots, p_n) : \otimes \in \Sigma_n$ and $n \in \mathbb{N}\}$.

**Def. 6.3** A logic $\mathcal{L}$ is said to be *quasi tabular* in case:
(i) There is some measure of complexity $l$ and there is some covering family of formulas $\{\psi_i\}_{i \in I}$, with $\psi_i \in \mathbb{L}[n_i]$, for some (possibly finite) set $I = \{1, 2, \ldots\} \subseteq \mathbb{N}^+$ such that for each $\psi_i$ there is a finite sequence $\langle \phi^i_s \rangle_{s=1,\ldots,k_i}$ of formulas such that $var(\phi^i_s) \subseteq \{p_1, \ldots, p_{n_i}\}$, and $l(\phi^i_s) < l(\psi_i)$, for $1 \leq s \leq k_i$;
(ii) there is an adequate $|\mathcal{V}|$-valued set of valuations $\S : \mathbb{L} \to \mathcal{V}$ for $\mathcal{L}$, for some finite set of truth-values $\mathcal{V}$, such that for each $i \in I$ there is some recursive function $\lceil . \rceil_i : \mathcal{V}^{k_i} \to \mathcal{V}$ according to which, if $\phi = \varepsilon(\psi_i)$ for some substitution $\varepsilon$, then $\S(\phi) \bowtie_i \lceil \S(\varepsilon(\phi^i_1)), \ldots, \S(\varepsilon(\phi^i_{k_i})) \rceil_i$ for every $\S$, where $\bowtie_i$ is one of the following (partial ordering) relations defined on $\mathcal{V}$: $=$, $\leq$, or $\geq$.

Note that quasi tabular (structural) logics can be given quasi matrix decision procedures, as spelled out in Section 5.

**Def. 6.4** A quasi tabular logic is called *tabular* in case $l$ can be taken as the canonical measure of complexity and, accordingly, for each $i \in I$, one can take $\langle \phi^i_s \rangle_{s=1,\ldots,k_i}$ as the immediate subformulas of $\psi_i$. In that case, also, the covering set $\{\psi_i\}_{i \in I}$ can be taken to be the minimal one (check Remark 6.2(ii)), and each $\bowtie_i$ can be limited to the equality symbol $=$.

Tabular logics can be given matrix decision procedures. Indeed, such logics are always genuinely $n$-valued, for some $1 \leq n \leq |\mathcal{V}|$.

**Def. 6.5** A quasi tabular logic $\mathcal{L}$ is said to have *dyadic semantics* in case the set $\mathcal{V}$ of Def. 6.3(ii) is $\{T, F\}$, and additionally $\mathcal{L}$ has an adequate gentzenian semantics.

Suszko's 2-valued reduction result is quite general —it applies to every tarskian logic, or, equivalently, to any logic having an $n$-valued semantics, be it truth-functional or not. Our reductive method (cf. [7]) applies to any logic characterized by finite matrices which are expressive enough, so as to allow for the definition of appropriate operators displaying the distinctions between the possibly many truth-values.

The class of quasi tabular logics is very wide: Genuinely finite-valued logics are but a very special case of them, and the former class in fact coincides with the class of logics which can be given a so-called 'society semantics with complex base' (cf. [13]). It even includes logics that cannot be characterized as genuinely finite-valued, as the following example shows.

**Example 6.6** Consider the paraconsistent logic $\mathcal{C}_1$ (cf. [10]). It is well-known for long that this logic has no genuinely finite-valued characterizing semantics, though it *can* be decided by quasi matrices (cf. [11]). In fact, a dyadic semantics for $\mathcal{C}_1$ is prompty available (cf. [9]). Just recall that $\alpha^\circ$ abbreviates $\neg(\alpha \wedge \neg\alpha)$ in $\mathcal{C}_1$, and consider the following bivaluation axioms:

(6.6.1) $b(\neg\alpha) \geq -b(\alpha)$;
(6.6.2) $b(\neg\neg\alpha) \leq b(\alpha)$;
(6.6.3) $b(\alpha \wedge \beta) = b(\alpha) \sqcap b(\beta)$;
(6.6.4) $b(\alpha \vee \beta) = b(\alpha) \sqcup b(\beta)$;
(6.6.5) $b(\alpha \Rightarrow \beta) = -b(\alpha) \sqcup b(\beta)$;
(6.6.6) $b(\alpha^\circ) = -b(\alpha) \sqcup -b(\neg\alpha)$;
(6.6.7) $b((\alpha \otimes \beta)^\circ) \geq (-b(\alpha) \sqcup -b(\neg\alpha)) \sqcap (-b(\beta) \sqcup -b(\neg\beta))$,
for $\otimes \in \{\wedge, \vee, \Rightarrow\}$.

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