Non-Truth-Functional Fibred Semantics

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Abstract ***Until recently, truth-functionality has been considered essential to the mechanism for combining logics known as*** *fibring*. ***Following the first efforts towards extending fibred semantics to logics with non-truth-functional operators, this paper aims to clarify the subject at the light of ideas borrowed from the theory of general logics as institutions and the novel notion of*** *non-truth-functional room*. ***Besides introducing the relevant concepts and constructions, the paper presents a detailed worked example combining classical first-order logic with the paraconsistent propositional system $\mathcal{C}_1$, for which a meaningful semantics is obtained. The possibility of extending this technique to build first-order versions of further logics of formal inconsistency is also discussed.***

Keywords: Fibring, non-truth-functional semantics, paraconsistency, first-order.

1 Introduction

Recently, the problem of combining logics has been deserving much attention. The practical impact of combining logics is clear. In the fields of artificial intelligence and software engineering, the need for working with several formalisms at the same time is widely recognized. Besides, combinations of logics are also of great theoretical interest [3]. Among the different combination techniques, both *fibring* [11, 16] and combinations of parchments [15] deserve close attention, as well as [7] as far as non-truth-functionality is concerned. Moreover, after the work in [5], it seems that the theory of fibring can also deal with logics endowed with non-truth-functional semantics, including a wide class of paraconsistent logics.

To clarify the subject we adopt the general context of institutions [13, 14], and introduce the novel notion of *non-truth-functional (ntf) room*. These can be seen, in fact, as the basic constituents of *ntf parchments*, an algebraic-oriented view on presentations of logics as institutions [12], from where we borrow the terminology. Following the tradition of institutions, we consider a logic to consist of an indexing functor to a suitable category of logic systems. In our case, the logic systems of interest are *ntf rooms*. For simplicity, we shall only work at this level of abstraction. As shown in [4], everything can be smoothly lifted to the fully fledged indexed case.

This seems to provide the adequate setting for widening the work reported in [5] to a larger class of non-truth-functional logics, by providing a neat separation between interpretation structures and interpretation maps and, altogether, a sharp delimitation of truth-functionality. Our *ntf rooms* essentially extend the rooms for model-theoretic parchments of [15], as in the layered rooms of [6], by endowing the algebras of truth-values with more than just a set of designated values. In fact, we require the set of truth-values to be structured according to a Tarskian closure operation as in [4], recovering an early proposal of Smiley [18]. On the other hand, we shall also extend these, following the ideas in [5], to cope with the possible non-truth-functionality of operators.

The paper is organized as follows: in Section 2 the concept of *ntf room* and related notions are introduced and illustrated with representations of both classical first-order logic and the paraconsistent propositional system $\mathcal{C}_1$; in Section 3, after establishing the morphisms of *ntf rooms* and using them to characterize fibring, we explore the fibred semantics obtained by combining classical first-order logic with $\mathcal{C}_1$; Section 4 concludes the paper by hinting at the possible systematization of the process of obtaining first-order versions of paraconsistent systems, and by discussing the possible extension of the completeness preservation results that are known for the truth-functional case.

2 Non-truth-functional logics

In the sequel, $\mathbf{AlgSig}_\varphi$ denotes the category of algebraic many sorted signatures $\Sigma = \langle S, O\rangle$, where S is the set of *sorts* and $O = \{O_w\}_{w\in S^+}$ is the family of sets of *operators* indexed by their type, with a distinguished sort φ (for formulae) and morphisms preserving it. Given one such signature, we denote by $\mathbf{Alg}(\Sigma)$ the category of Σ-algebras and Σ-algebra homomorphisms, and by $\mathrm{cAlg}(\Sigma)$ the class of all pairs $\langle \mathcal{A}, \mathbf{c}\rangle$ with $\mathcal{A}$ a Σ-algebra and $\mathbf{c}$ a closure operation on $|\mathcal{A}|_\varphi$ (the carrier of sort φ, that we can see as the set of truth-values). We shall use $\mathcal{T}_\Sigma$ to denote the initial Σ-algebra (the *term* algebra), and $[\![\_]\!]^{\mathcal{A}}$ (for term *interpretation*) to denote the unique $\mathbf{Alg}(\Sigma)$-homomorphism from $\mathcal{T}_\Sigma$ to any given Σ-algebra $\mathcal{A}$. Also recall that every $\mathbf{AlgSig}_\varphi$-morphism $\sigma : \Sigma_1 \to \Sigma_2$ has an associated reduct functor $\_|_\sigma : \mathbf{Alg}(\Sigma_2) \to \mathbf{Alg}(\Sigma_1)$. As usual, we shall preferrably write $\widehat{\sigma}$ (for term *translation*) instead of $[\![\_]\!]^{\mathcal{T}_{\Sigma_2}|_\sigma}$ to denote the unique $\mathbf{Alg}(\Sigma_1)$-homomorphism from $\mathcal{T}_{\Sigma_1}$ to $\mathcal{T}_{\Sigma_2}|_\sigma$.

Definition 2.1 An *ntf room* is a tuple $R = \langle \Sigma, T, \mathcal{I}, \mathcal{S}, \mathcal{H}\rangle$ where:

- $\Sigma = \langle S, O\rangle \in |\mathbf{AlgSig}_\varphi|$ is a signature (of *syntactic operators*);
- $\Sigma^{\mathrm{t}} = \langle S, T\rangle \in |\mathbf{AlgSig}_\varphi|$ is a subsignature of Σ (the *truth-functional part*), with $\iota : \Sigma^{\mathrm{t}} \to \Sigma$ the corresponding $\mathbf{AlgSig}_\varphi$-inclusion morphism;
- $\mathcal{I}$ is a class (of *interpretation structures*);
- $\mathcal{S} : \mathcal{I} \to \mathrm{cAlg}(\Sigma^{\mathrm{t}})$ is a map (assigning truth-functional *interpretation algebras* to interpretation structures);
- $\mathcal{H} = \{\mathcal{H}_I\}_{I\in\mathcal{I}}$, where each $\mathcal{H}_I \subseteq \hom_{\mathbf{Alg}(\Sigma^{\mathrm{t}})}(\mathcal{T}_\Sigma|_\iota, \mathcal{A})$ is a class (of *interpretation maps*), letting $\mathcal{S}(I) = \langle \mathcal{A}, \mathbf{c}\rangle$.

In the sequel, whenever clear from the context, we shall denote $\mathcal{S}(I)$ by $\langle \mathcal{A}_I, \mathbf{c}_I\rangle$.

Of course, the possible non-truth-functionality of an interpretation map regarding the whole syntax given by Σ follows from the fact that it is only required to be homomorphic over the truth-functional part Σ^{t}. For instance, an operator $o \in O_{\varphi\varphi}$ not in T can be non-truth-functional in that the value of $h(o(\gamma))$ may not be a function of $h(\gamma)$ for some interpretation I and $h \in \mathcal{H}_I$. However, if C and T coincide, ι is the identity, and we recover the plain old truth-functional case by letting each $\mathcal{H}_I$ contain the unique possible homomorphism $[\![\_]\!]^{\mathcal{A}_I}$.

A *global entailment system* can be extracted from an ntf room by considering, for each interpretation structure I, the set $\emptyset^{\mathbf{c}_I} \subseteq |\mathcal{A}_I|_\varphi$ of designated values. If we recognize $|\mathcal{T}_\Sigma|_\varphi$ (the carrier of sort φ in the initial Σ-algebra) as the set of formulae and $\mathcal{M}^g = \{\langle I, h\rangle : I \in \mathcal{I}, h \in \mathcal{H}_I\}$ as the family of *global models*, we can define the corresponding *global satisfaction relation* $\Vdash^g$ between models and formulae by:

- $\langle I, h\rangle \Vdash^g \gamma$ if $h(\gamma) \in \emptyset^{\mathbf{c}_I}$,

and obtain the induced *global consequence relation* $\vDash^g$ between sets of formulae and formulae, as expected, by defining:

- $\Gamma \vDash^g \delta$ if $\langle I, h\rangle \Vdash^g \delta$ whenever $\langle I, h\rangle \Vdash^g \Gamma$, for every $\langle I, h\rangle \in \mathcal{M}^g$.

Now, by further exploring the closure operation on truth-values and freely varying the admitted set of distinguished values, we can also define a *local entailment system*. *Local models* are set to be $\mathcal{M}^l = \{\langle I, h, T\rangle : \langle I, h\rangle \in \mathcal{M}^g, T^{\mathbf{c}_I} \subseteq T \subseteq |\mathcal{A}_I|_\varphi\}$ and the *local satisfaction relation* $\Vdash^l$ is defined by:

- $\langle I, h, T\rangle \Vdash^l \gamma$ if $h(\gamma) \in T$.

The *local consequence relation* $\vDash^l$ is defined as expected from $\Vdash^l$, and can be easily seen to coincide with:

- $\Gamma \vDash^l \delta$ if $h(\delta) \in \{h(\gamma) : \gamma \in \Gamma\}^{\mathbf{c}_I}$, for every $\langle I, h\rangle \in \mathcal{M}^g$.

In general, $\vDash^l_\Sigma$ is weaker than $\vDash^g_\Sigma$, but we always have that $\emptyset \vDash^l_\Sigma \gamma$ iff $\emptyset \vDash^g_\Sigma \gamma$.

For the sake of illustration we develop two examples. The first one, naturally just truth-functional, is classical first-order logic. The second, where negation is an essentially non-truth-functional operator, is the paraconsistent propositional logic $\mathcal{C}_1$ of da Costa [9].

Example 2.2 *Classical first-order logic.*
Let $F = \{F_n\}_{n\in\mathbb{N}}$ and $P = \{P_n\}_{n\in\mathbb{N}}$ be families of sets of function and predicate symbols, respectively, with the given arities, and X a denumerable set of variables. The *first-order* ntf room over $\langle F, P, X\rangle$ consists of:

- $S = \{\tau, \varphi\}$, where τ is the sort of terms;
- $O = T$ (all operators are truth-functional) is such that:
  - $O_\tau = X \cup F_0$,
  - $O_{\tau^n\tau} = F_n$, $n > 0$,
  - $O_{\tau^n\varphi} = P_n$, $n \in \mathbb{N}$,
  - $O_{\varphi\varphi} = \{\sim\} \cup \{\forall x, \exists x : x \in X\}$,
  - $O_{\varphi^2\varphi} = \{\wedge, \vee, \supset\}$;
- $\mathcal{I}$ is the class of all $\langle F, P\rangle$-interpretations $I = \langle D, \_{}_I\rangle$ with $D \neq \emptyset$ a set, $f_I : D^n \to D$ for $f \in F_n$, and $p_I \subseteq D^n$ for $p \in P_n$;
- each $\mathcal{S}(I) = \langle\mathcal{A}, \mathbf{c}\rangle$ with $|\mathcal{A}|_\tau = D^{\mathrm{Asg}(X,D)}$ and $|\mathcal{A}|_\varphi = \wp(\mathrm{Asg}(X, D))$, where $\mathrm{Asg}(X, D) = D^X$ is the set of assignments ρ to variables, and:
  - $x_\mathcal{A}(\rho) = \rho(x)$, $x \in X$;
  - $f_\mathcal{A}(e_1, \ldots, e_n)(\rho) =$
    $f_I(e_1(\rho), \ldots, e_n(\rho))$, $f \in F_n$;
  - $p_\mathcal{A}(e_1, \ldots, e_n) =$
    $\{\rho : \langle e_1(\rho), \ldots, e_n(\rho)\rangle \in p_I\}$, $p \in P_n$;
  - $\sim_\mathcal{A}(r) = \mathrm{Asg}(X, D) \setminus r$;
  - $\forall x_\mathcal{A}(r) =$
    $\{\rho : \rho[x/d] \in r \text{ for every } d \in D\}$;
  - $\exists x_\mathcal{A}(r) =$
    $\{\rho : \rho[x/d] \in r \text{ for some } d \in D\}$;
  - $\wedge_\mathcal{A}(r_1, r_2) = r_1 \cap r_2$;
  - $\vee_\mathcal{A}(r_1, r_2) = r_1 \cup r_2$;
  - $\supset_\mathcal{A}(r_1, r_2) = (\mathrm{Asg}(X, D) \setminus r_1) \cup r_2$;

  endowed with the cut closure operation induced by set inclusion, that is, for every $R \subseteq \wp(\mathrm{Asg}(X, D))$, $R^\mathbf{c} = \{r \subseteq \mathrm{Asg}(X, D) : (\bigcap R) \subseteq r\}$ (the principal ideal determined by $(\bigcap R)$ on the complete lattice $\langle\wp(\mathrm{Asg}(X, D), \supseteq\rangle)$;
- each $\mathcal{H}_I = \{[\![\_]\!]^{\mathcal{A}_I}\}$.

In all cases, $\emptyset^{\mathbf{c}_I} = \{\mathrm{Asg}(X, D)\}$ and global satisfaction at I corresponds to truth for all assignments, leading to the corresponding global entailment. Local entailment, instead, corresponds to consequence over a fixed assignment. Note that $\{\gamma\} \vDash^g (\forall x\, \gamma)$ but $\{\gamma\} \nvDash^l (\forall x\, \gamma)$, hinting to the well-known fact that generalization holds globally but not locally.

Example 2.3 *Paraconsistent propositional system $\mathcal{C}_1$.*
Let Π be a set of propositional symbols. The $\mathcal{C}_1$ ntf room over Π consists of:

- $S = \{\varphi\}$;
- O is such that:
  - $O_\varphi = \Pi$;
  - $O_{\varphi\varphi} = \{\neg\}$;
  - $O_{\varphi^2\varphi} = \{\wedge, \vee, \supset\}$,

  whereas T does not include $\neg$.
- $\mathcal{I}$ is the class of all pairs $I = \langle\mathcal{B}, \vartheta\rangle$ where $\mathcal{B} = \langle B, \sqcap, \sqcup, -, \top, \bot\rangle$ is a Boolean algebra and $\vartheta : \Pi \to B$ is a valuation;
- each $\mathcal{S}(I) = \langle\mathcal{A}, \mathbf{c}\rangle$ with $|\mathcal{A}|_\varphi = B$, and:
  - $\pi_\mathcal{A} = \vartheta(\pi)$, $\pi \in \Pi$;

* $\wedge_{\mathcal{A}}(b_1, b_2) = b_1 \sqcap b_2$;
* $\vee_{\mathcal{A}}(b_1, b_2) = b_1 \sqcup b_2$;
* $\supset_{\mathcal{A}} (b_1, b_2) = -b_1 \sqcup b_2$;

endowed with the cut closure operation induced by the usual order on B defined by $b_1 \leq b_2$ iff $b_1 \sqcup b_2 = b_2$, that is, for every $X \subseteq B$, $X^{\mathbf{c}} = \{b \in B : b_1 \leq b \text{ if } b_1 \leq x \text{ for every } x \in X\}$ (the least closed ideal of $\langle B, \geq \rangle$ that contains X);

- each $\mathcal{H}_I$ is the class of all $\mathbf{Alg}(\Sigma^{\mathrm{t}})$-homomorphisms $h : \mathcal{T}_\Sigma|_\iota \to \mathcal{A}_I$ such that:

  * $-h(\gamma) \leq h(\neg\,\gamma)$;
  * $h(\neg\,\neg\,\gamma) \leq h(\gamma)$;
  * $(h(\gamma^\circ) \sqcap h(\gamma) \sqcap h(\neg\,\gamma)) = \bot$;
  * $(h(\gamma^\circ) \sqcap h(\delta^\circ)) \leq h((\gamma \wedge \delta)^\circ)$;
  * $(h(\gamma^\circ) \sqcap h(\delta^\circ)) \leq h((\gamma \vee \delta)^\circ)$;
  * $(h(\gamma^\circ) \sqcap h(\delta^\circ)) \leq h((\gamma \supset \delta)^\circ)$,

  where γ° is the usual $\mathcal{C}_1$ abbreviation of $\neg(\gamma \wedge \neg\,\gamma)$.

In all cases, $\emptyset^{\mathbf{c}_I} = \{\top\}$, and therefore global satisfaction at I corresponds to truth, leading to the corresponding global entailment. It is also easy to see that, in this case, local and global entailments coincide.

Although in the $\mathcal{C}_1$ system negation is not truth-functional, the possible interpretations of $\neg$ are restricted according to the previous 6 conditions. Obviously, we would end up exactly with classical propositional logic if we replaced the last 5 conditions by just $(h(\gamma) \sqcap h(\neg\,\gamma)) = \bot$. This last condition is clearly a form of *Pseudo-Scotus* and would immediately lead to $h(\neg\,\gamma) = -h(\gamma)$. However, as it is, the third condition still embodies a controlled form of explosion in the presence of *consistency* (as expressed by the γ° abbreviation).

3 Fibring

Morphisms of ntf rooms are specially tailored for *fibring*. Let us consider fixed two arbitrary ntf rooms $R_1 = \langle \Sigma_1, T_1, \mathcal{I}_1, \mathcal{S}_1, \mathcal{H}_1 \rangle$ and $R_2 = \langle \Sigma_2, T_2, \mathcal{I}_2, \mathcal{S}_2, \mathcal{H}_2 \rangle$.

Definition 3.1 A *morphism* from R_1 to R_2 is a pair $\langle \sigma, \theta \rangle$ where $\sigma : \Sigma_1 \to \Sigma_2$ is an $\mathbf{AlgSig}_\varphi$-morphism and $\theta : \mathcal{I}_2 \to \mathcal{I}_1$ is a map such that:

- $\sigma(T_1) \subseteq T_2$, inducing an $\mathbf{AlgSig}_\varphi$-morphism $\sigma^{\mathrm{t}} : \Sigma_1^{\mathrm{t}} \to \Sigma_2^{\mathrm{t}}$ that satisfies $(\iota_2 \circ \sigma^{\mathrm{t}}) = (\sigma \circ \iota_1)$;
- if $\mathcal{S}_2(I) = \langle \mathcal{A}, \mathbf{c} \rangle$ then $\mathcal{S}_1(\theta(I)) = \langle \mathcal{A}|_{\sigma^{\mathrm{t}}}, \mathbf{c} \rangle$;
- if $h \in \mathcal{H}_{2,I}$ then $(h|_{\sigma^{\mathrm{t}}} \circ \widehat{\sigma}|_{\iota_1}) \in \mathcal{H}_{1,\theta(I)}$.

Easily, ntf rooms and their morphisms constitute a category **NTFRoom**, where we can characterize *fibring* via colimits as in [16, 4, 5, 6, 19]. Extending these previous characterizations of fibring to this level, we shall just concentrate on the particular cases of colimit defining fibring constrained by sharing of symbols. Thus, when fibring R_1 and R_2, we shall assume that the required sharing of operators is specified by means of the largest common subsignature $\Sigma_0 = \langle S_0, O_0 \rangle$ of both Σ_1 and Σ_2, that is $S_0 = S_1 \cap S_2$ (it always includes at least φ) and $O_{0,w} = O_{1,w} \cap O_{2,w}$ for $w \in S_0^+$. For simplicity, since it serves our current purpose, we shall just dwell on the case where all the shared operators are truth-functional on both R_1 and R_2, that is, we assume that $T_0 = O_0$ is contained in both T_1 and T_2. We denote by $\sigma_k : \Sigma_0 \to \Sigma_k$ the corresponding signature inclusions and by R_0 the ntf room $\langle \Sigma_0, T_0, \mathcal{I}_0, \mathcal{S}_0, \mathcal{H}_0 \rangle$ where $\mathcal{I}_0 = \mathrm{cAlg}(\langle S_0, O_0 \rangle)$, $\mathcal{S}_0$ is the identity on $\mathcal{I}_0$ and each $\mathcal{H}_{0,\langle \mathcal{A}, \mathbf{c} \rangle} = \{[\![\_]\!]^{\mathcal{A}}\}$. In the simplest possible case when $S_0 = \{\varphi\}$ and $O_0 = T_0 = \emptyset$ we say that the fibring is *free* or *unconstrained*.

Definition 3.2 The *fibring* of R_1 and R_2 *constrained by sharing* Σ_0 is the ntf room $R = \langle \langle S, O \rangle, T, \mathcal{I}, \mathcal{S}, \mathcal{H} \rangle$ such that:

- $S = S_1 \cup S_2$, with inclusions $f_k : S_k \to S$;
- $O_w = O_{1,w} \cup O_{2,w}$ if $w \in S_0^+$, $O_w = O_{k,w}$ if $w \in S_k^+ \setminus S_0^+$ and $O_w = \emptyset$ otherwise, with inclusions $g_k : O_k \to O$;

- $T_w = T_{1,w} \cup T_{2,w}$ if $w \in S_0^+$, $T_w = T_{k,w}$ if $w \in S_k^+ \setminus S_0^+$ and $T_w = \emptyset$ otherwise;

- $\mathcal{I}$ is the class of all pairs $\langle I_1, I_2 \rangle \in \mathcal{I}_1 \times \mathcal{I}_2$ such that $|\mathcal{A}_{I_1}|_s = |\mathcal{A}_{I_2}|_s$ for every $s \in S_0$, $\mathbf{c}_{I_1} = \mathbf{c}_{I_2}$ and $o_{\mathcal{A}_{I_1}} = o_{\mathcal{A}_{I_2}}$ for every $w \in S_0^+$ and $o \in T_{0,w}$;

- each $\mathcal{S}(\langle I_1, I_2 \rangle) = \langle \mathcal{A}, \mathbf{c} \rangle$, where $\mathcal{A}$ is the unique $\langle S, T \rangle$-algebra such that $\mathcal{S}_1(I_1) = \langle \mathcal{A}|_{\langle f_1, g_1 \rangle^{\mathrm{t}}}, \mathbf{c} \rangle$ and $\mathcal{S}_2(I_2) = \langle \mathcal{A}|_{\langle f_2, g_2 \rangle^{\mathrm{t}}}, \mathbf{c} \rangle$;

- each $\mathcal{H}_{\langle I_1, I_2 \rangle}$ consists of all $\mathbf{Alg}(\langle S, T \rangle)$-homomorphisms $h : \mathcal{T}_{\langle S, C \rangle}|_\iota \to \mathcal{A}$ such that $(h|_{\langle f_1, g_1 \rangle^{\mathrm{t}}} \circ \widehat{\langle f_1, g_1 \rangle}|_{\iota_1}) \in \mathcal{H}_{1, I_1}$ and $(h|_{\langle f_2, g_2 \rangle^{\mathrm{t}}} \circ \widehat{\langle f_2, g_2 \rangle}|_{\iota_2}) \in \mathcal{H}_{2, I_2}$.

Note that the fibred interpretation algebras are precisely those $\langle \mathcal{A}, \mathbf{c} \rangle$ obtained by joining together any two given $\langle \mathcal{A}_1, \mathbf{c}_1 \rangle$ and $\langle \mathcal{A}_2, \mathbf{c}_2 \rangle$ that are compatible on the shared syntax, and that the fibred interpretation maps h are obtained by extending any two given h_1 and h_2.

Proposition 3.3 *The constrained fibring of layered rooms R_1 and R_2 by sharing Σ_0 is a pushout of $\{\langle \sigma_k, \theta_k \rangle : R_0 \to R_k\}_{k \in \{1,2\}}$ in* **NTFRoom**, *where each $\theta_k(I) = \langle \mathcal{A}|_{\sigma_k^t}, \mathbf{c} \rangle$ if $\mathcal{S}_k(I) = \langle \mathcal{A}, \mathbf{c} \rangle$.*

As a corollary, unconstrained fibring is a coproduct in **NTFRoom**. Let us now analyze in some detail the application of this construction to the combination of classical first-order logic and the propositional system $\mathcal{C}_1$.

Example 3.4 *Paraconsistent first-order system $\mathcal{C}_1^*$.*
By fibring classical first-order logic over $\langle F, P, X \rangle$ and the paraconsistent propositional system $\mathcal{C}_1$ (in the particular case when $\Pi = \emptyset$) while sharing the classical operators $\wedge$, $\vee$ and $\supset$ via a corresponding pushout in **NTFRoom**, we obtain the following ntf room:

- $S = \{\tau, \varphi\}$, where τ is the sort of terms;

- O is such that:

  * $O_\tau = X \cup F_0$;
  * $O_{\tau^n \tau} = F_n$, $n > 0$;
  * $O_{\tau^n \varphi} = P_n$, $n \in I\!N$;
  * $O_{\varphi\varphi} = \{\neg, \sim\} \cup \{\forall x, \exists x : x \in X\}$;
  * $O_{\varphi^2 \varphi} = \{\wedge, \vee, \supset\}$,

  whereas T does not include $\neg$;

- $\mathcal{I}$ is the class of all $\langle F, P \rangle$-interpretations $I = \langle D, \_{}_I \rangle$, as in the classical case, since $\wp(\mathrm{Asg}(X, D))$ is always a Boolean algebra with $\sqcap = \cap$, $\sqcup = \cup$, $- = (\mathrm{Asg}(X, D) \setminus \_)$, $\top = \mathrm{Asg}(X, D)$ and $\perp = \emptyset$;

- $\mathcal{S}(I)$ also coincides with the classical case;

- each $\mathcal{H}_I$ is the class of all $\mathbf{Alg}(\Sigma^{\mathrm{t}})$-homomorphisms $h : \mathcal{T}_\Sigma|_\iota \to \mathcal{A}_I$ such that:

  * $\mathrm{Asg}(X, D) \setminus h(\gamma) \subseteq h(\neg \gamma)$;
  * $h(\neg \neg \gamma) \subseteq h(\gamma)$;
  * $(h(\gamma^\circ) \cap h(\gamma) \cap h(\neg \gamma)) = \emptyset$;
  * $(h(\gamma^\circ) \cap h(\delta^\circ)) \subseteq h((\gamma \wedge \delta)^\circ)$;
  * $(h(\gamma^\circ) \cap h(\delta^\circ)) \subseteq h((\gamma \vee \delta)^\circ)$;
  * $(h(\gamma^\circ) \cap h(\delta^\circ)) \subseteq h((\gamma \supset \delta)^\circ)$.

As expected, $\emptyset^{\mathbf{c}_I} = \{\mathrm{Asg}(X, D)\}$, and local and global entailments again reflect reasoning with or without fixing an assignment. What is more, if we restrict the interpretation maps a little further in order to encompass also the following conditions:

* $\forall x_{\mathcal{A}}(h(\gamma^\circ)) \subseteq h((\forall x\, \gamma)^\circ)$;
* $\forall x_{\mathcal{A}}(h(\gamma^\circ)) \subseteq h((\exists x\, \gamma)^\circ)$;
* $\exists x_{\mathcal{A}}(h(\neg \gamma)) = h(\neg \forall x\, \gamma)$;
* $\forall x_{\mathcal{A}}(h(\neg \gamma)) = h(\neg \exists x\, \gamma)$,

we obtain precisely the paraconsistent first-order system $\mathcal{C}_1^*$ of [9], but with a semantics that is richer than the bivalued semantics proposed in [2], in the sense that local and global reasoning are still distinguished (vide generalization). Note also that classical negation $\sim$ is indeed definable in terms of the paraconsistent negation $\neg$. Namely, $\sim \gamma$ is interpreted precisely as $(\neg \gamma) \wedge \gamma^\circ$.

Adding explosiveness back to $\mathcal{C}_1$, one obtains simply the classical propositional logic. But, as mentioned before, $\mathcal{C}_1$ indeed contains a qualified form of explosion: a contradiction γ and $\neg\gamma$ implies anything else as soon as we are sure that γ is *consistent*, as indicated in $\mathcal{C}_1$ by the validity of γ°. This fact characterizes $\mathcal{C}_1$ as a particular case of a *logic of formal inconsistency*, in fact, a **C**-*system* based on classical propositional logic [8]. A promissing next step, in this line of investigation, would be the application of the above techniques to paraconsistent logics in general, or at least to larger classes of **C**-systems, and logics of formal inconsistency.

4 Conclusions

By adopting the general setting of the theory of institutions [12, 13] and the novel notion of ntf room, we have given a rigorous categorial characterization of fibring of logics with possible non-truth-functional semantics, in a way that abstracts away from the previous attempt reported in [5] and also extends it to deal with logics that are not propositionally based. Moreover, we have illustrated the capabilities of the proposed framework by obtaining a meaningful fibred semantics for the paraconsistent first-order system $\mathcal{C}_1^*$ of [9]. Although just an example, which by the way could not even be dealt with in the context of [5], we think that its implicit general character is worth exploring on the way to systematizing the process of *first-orderfying* a logic, namely at the light of Gabbay's original ideas on the potentialities of the idea of fibring [11].

While the hub of paraconsistent logic – namely, avoiding the explosive character of inferences in the presence of contradictions – is in general completely identified already at the propositional level, it is often mathematically interesting to count on first-order versions of these logics. In fact, according to the third requisite set forth by da Costa [9], which would be responsible later on for making some authors identify da Costa as the "true founder of paraconsistent logic" (see for instance [10]), all paraconsistent logics should be first-orderfiable. This study opens the way to the first-orderfying of a paraconsistent logic to become something more than a craftsman job.

Beyond this goal, we also aim at exploring the non-truth-functional representation of other many-valued logics, and in particular the possibility of building, for instance, fibred logics that are simultaneously paraconsistent and paracomplete, such as the logic of bilattices [1], or the systematization of the process of *fuzzyfying* a logic [11]. Other interesting applications of fibring, in a truth-functional setting, have been explored elsewhere and include, for instance, the interplay between modalities and quantifiers [17] and a treatment of partiality in the context of equational logic [6].

Moreover, and most importantly, we intend to study the extension to this general setting of the soundness and completeness preservation results already obtained for truth-functional fibring [19, 4, 6], and also for a much more restricted non-truth-functional setting [5], within the context of Hilbert-style proof calculi and on a propositional basis. With respect to soundness, everything is expected to work smoothly, according to the general results in [4]. The completeness results, on their turn, use techniques involving either Lindenbaum-Tarski constructions [4, 6], Henkin style constructions [19] or encodings in conditional equational logic as a meta-logic [5] and also seem to be easily adaptable if we keep the propositional base restriction. However, the key ideas towards results also encompassing logics with terms and quantification are already being developed in the recent paper [17].

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